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The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations

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Abstract. In this paper, we study the existence of critical points for the following functional constrained on $S_c = \{u \in H^1(\mathbb{R}^N) | \, |u|_2 = c\}$:

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{N}{2N+8}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}},$$

where $N = 1, 2, 3$ and $a, b > 0$ are constants. The constraint problem is L^2-critical. We showed that $I(u)$ has a constraint critical point with a mountain pass geometry on S_c if $c > c^* := (2^{-1}b|Q|_2^{\frac{8}{N}})^{\frac{N}{8-2N}}$, where Q is the unique positive radial solution of $-2\Delta Q + (\frac{4}{N}-1)Q = |Q|^{\frac{8}{N}}Q$ in $\mathbb{R}^N$. For $0 < c < c^*$, $I(u)$ has no critical point on S_c, and we proved the existence of minimizers for a new perturbation functional on S_c:

$$E_{a,b}(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{1}{4}\int_{\mathbb{R}^N} V(x)|u|^4 - \frac{N}{2N+8}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}.$$

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1. Introduction and main result

In the past years, the following nonlinear Kirchhoff equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u - |u|^{p-2}u = \lambda u, \quad x \in \mathbb{R}^N, \ \lambda \in \mathbb{R} \tag{1.1}$$

has attracted considerable attention, where $N \leq 3$, a, $b > 0$ are constants and $p \in (2, 2^*)$, $2^* = 6$ if $N = 3$ and $2^* = +\infty$ if $N = 1, 2$.

Equation (1.1) is a nonlocal one as the appearance of the term $\int_{\mathbb{R}^N} |\nabla u|^2$ implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.1) particularly interesting, see [1,2,4,6,7,13,17,20] and the references therein. The first line to study (1.1) is to consider the case where λ is a fixed and assigned parameter, see [9,12,14–16,19,22,25]. In such direction, the critical point theory is used to look for nontrivial solutions of (1.1); however, nothing can be given a priori on the L^2-norm of the solutions. Recently, since the physicists are often interested in "normalized

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solutions," solutions with prescribed L^2-norm are considered. Such solutions are obtained by looking for critical points of the following C^1 functional

$$I_p(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \tag{1.2}$$

constrained on the L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S_c = \{u \in H^1(\mathbb{R}^N)|\ |u|_2 = c,\ c > 0\}.$$

Set

$$I_{p,c^2} := \inf_{u\in S_c} I_p(u). \tag{1.3}$$

We see that minimizers of I_{p,c^2} are critical points of $I_p|_{S_c}$ and the parameter λ is not fixed any longer but appears as an associated Lagrange multiplier.

By the L^2-preserving scaling and the well-known Gagliardo–Nirenberg inequality with the best constant [23]: Let $p \in [2, \frac{2N}{N-2})$ if $N \geq 3$ and $p \geq 2$ if $N = 1, 2$, then

$$|u|_p^p \leq \frac{p}{2|Q_p|_2^{p-2}} |\nabla u|_2^{\frac{N(p-2)}{2}} |u|_2^{p-\frac{N(p-2)}{2}}, \tag{1.4}$$

with equality only for $u = Q_p$, where Q_p is, up to translations, the unique positive ground state solution of

$$-\frac{N(p-2)}{4}\Delta Q + \left(1 + \frac{p-2}{4}(2-N)\right) Q = |Q|^{p-2}Q,\ x \in \mathbb{R}^N, \tag{1.5}$$

it has been shown in [26] that $p = \frac{2N+8}{N}$ is L^2-critical exponent for (1.3), namely for all $c > 0$, $I_{p,c^2} > -\infty$ if $p \in (2, \frac{2N+8}{N})$ and $I_{p,c^2} = -\infty$ if $p \in (\frac{2N+8}{N}, 2^*)$.

When $p = \frac{2N+8}{N}$, for simplicity, we use $I(u)$ and I_{c^2} to denote $I_{\frac{2N+8}{N}}(u)$ and $I_{\frac{2N+8}{N},c^2}$, respectively. Then, we have the following existing results:

Lemma 1.1. ([26], Theorem 1.1 and Theorem 1.2) *For* $p = \frac{2N+8}{N}$, *then*

(1) $I_{c^2} = \begin{cases} 0, & 0 < c \leq c^* := (2^{-1}b|Q_{\frac{2N+8}{N}}|_2^{\frac{8}{N}})^{\frac{N}{8-2N}}, \\ -\infty & c > c^*. \end{cases}$

(2) I_{c^2} *has no minimizer for all* $c > 0$.

(3) $I(u)$ *has no critical point on the constraint* S_c *for all* $0 < c \leq c^*$.

In this paper, we look for critical points restricted to S_c for the L^2-critical case $p = \frac{2N+8}{N}$. To the best of our knowledge, there is no paper on this respect. In the past years, there have been some works studying similar L^2-critical problems related to Schrödinger operators by adding a nonnegative perturbation term $\frac{1}{2}\int_{\mathbb{R}^2} V(x)|u|^2$ to get minimizers, see e.g., [8], where $V(x)$ is locally bounded satisfying $V(x) \geq 0$ and $\lim\limits_{|x|\to+\infty} V(x) = +\infty$.

In our setting, when $c \in (0, c^*]$, by Lemma 1.1 (3), in order to get critical points, we try to add a nonpositive and lower perturbation term to the right-hand side of (1.2). Considering the effect of the nonlocal term, we study the following perturbation functional

$$E_{a,b}(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{1}{4}\int_{\mathbb{R}^N} V(x)|u|^4 - \frac{N}{2N+8}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}, \tag{1.6}$$

where $V(x) \in L^\infty_{loc}(\mathbb{R}^N)$ satisfies that

$$V(x) \geq 0 \text{ and } \lim_{|x|\to+\infty} V(x) = 0. \tag{V}$$

Then, we consider the associated minimization problem

$$e_{a,b}(c^2) = \min_{u\in S_c} E_{a,b}(u). \tag{1.7}$$

It easily sees that $e_{a,b}(c^2) \in (-\infty, 0]$. To state our main result, for $a > 0$ fixed, we introduce an equivalent norm on $H^1(\mathbb{R}^N)$:

$$\|u\| = \left(\int\limits_{\mathbb{R}^N} (a|\nabla u|^2 + u^2) \right)^{\frac{1}{2}}, \ \forall\ u \in H^1(\mathbb{R}^N),$$

which is induced by the corresponding inner product on $H^1(\mathbb{R}^N)$. We recall in [21] that the following minimum problem:

$$\lambda_1 := \inf \left\{ \left(\int\limits_{\Omega} |\nabla u|^2 \right)^2 \mid u \in H_0^1(\Omega), \int\limits_{\Omega} V(x)|u|^4 = 1 \right\} > 0 \tag{1.8}$$

is achieved by some $\phi \in H_0^1(\Omega)$ with $\int_\Omega V(x)|\phi|^4 = 1$ and $\phi > 0$ a.e. in Ω, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $V(x) \not\equiv 0$ a.e. in Ω. Then, we get our main result as follows:

Theorem 1.2. *If $0 < b \le \frac{1}{\lambda_1}$, then there exists $a^* > 0$ small such that $e_{a,b}(c^2) < 0$ for all $a \in (0, a^*)$ and $c \in (0, c^*)$, where c^* is given in Lemma 1.1. Moreover, there exists a couple of solution $(u_c, \lambda_c) \in S_c \times \mathbb{R}^-$ satisfying the following equation*

$$-\left(a + b \int\limits_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u - V(x)u^3 - |u|^{\frac{8}{N}} u = \lambda_c u, \ x \in \mathbb{R}^N$$

with $E_{a,b}(u_c) = e_{a,b}(c^2)$.

For $c > c^*$, by Lemma 1.1 (2), it is impossible to look for minimizers of I on S_c. We must look for a critical point of I with a minimax characterization. As far as we know, there is no paper to study L^2-critical constrained problems in this way, which was used in [3,10,26] to deal with L^2 subcritical cases. To this end, by Lemma 1.1 (1), we shall show that I possesses a kind of mountain pass geometry on S_c (see e.g., [10]): There exists $K(c) > 0$ depending on c such that

$$\gamma(c) = \inf_{h\in\Gamma(c)} \max_{\tau\in[0,1]} I(h(\tau)) > \max_{h\in\Gamma(c)} \max\{I(h(0)), I(h(1))\},$$

where

$$\Gamma(c) = \{h \in C([0,1], S_c) |\ h(0) \in B_{K(c)}, I(h(1)) < 0\} \tag{1.9}$$

and $B_{K(c)} = \{u \in S_c |\ |\nabla u|_2^2 \le K(c)\}$. Then, we will look for critical points of I on S_c at the level $\gamma(c)$:

Theorem 1.3. *For $p = \frac{2N+8}{N}$ and $c > c^*$, then problem (1.1) has at least one couple of solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}^-$ with $|u_c|_2 = c$ and $I(u_c) = \gamma(c)$.*

We give the main idea in the proof of Theorems 1.2 and 1.3.

To prove Theorem 1.2, we see that $E_{a,b}(u)$ is coercive on S_c if $c < c^*$. The main difficulty now is to deal with a possible lack of compactness for minimizing sequences of $e_{a,b}(c^2)$. Motivated by Lions [18], we try to reduce our problem to the classical vanishing–dichotomy–compactness situation to check the compactness and prove Theorem 1.2. To do so, a necessary step is to show that

$$e_{a,b}(c^2) < 0, \tag{1.10}$$

which can be proved by using the minimizers for (1.9) and restricting the ranges of a and b. We succeeded in excluding the vanishing case with the help of (1.10) and the decay property of $V(x)$ at infinity. To avoid the dichotomy case, we need to obtain a strong version of subadditivity inequality

$$e_{a,b}(c^2) < e_{a,b}(\alpha^2) + e_{a,b}(c^2 - \alpha^2), \quad \forall\, 0 < \alpha < c. \tag{1.11}$$

The scaling argument used in [26] to get the strong version of subadditivity inequality does not work here since $E_{a,b}(u)$ is no more an autonomous functional. To overcome this difficulty, we note that for $u \in S_c$ and $\theta > 1$ the only scaling: $u_\theta := \theta u$ can be used in our case, which also tells us why we add the perturbation term $\frac{1}{4}\int_{\mathbb{R}^N} V(x)u^4$ but not $\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2$. By using (1.10) and such a scaling, we finally prove that (1.11) holds, which requires more careful analysis.

Let us underline the difficulties in proving Theorem 1.3. First, since the mountain pass geometry on S_c does not guarantee the boundedness of the Palais–Smale sequence, no minimax type theorem can be directly used. To overcome this difficulty, we adopt the method introduced by Jeanjean in [10] to consider an auxiliary functional $\widetilde{I}(u,t) : H^1(\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{I}(u,t) = \frac{ae^{2t}}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 + \frac{be^{4t}}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{Ne^{4t}}{2N+8}\int\limits_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}.$$

With the help of $\widetilde{I}$, we succeeded in constructing a bounded $(PS)_{\gamma(c)}$ sequence $\{v_n\} \subset S_c$ for $I|_{S_c}$ satisfying that $G(v_n) \to 0$, where

$$G(u) := a\int\limits_{\mathbb{R}^N} |\nabla u|^2 + b\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{2N}{N+4}\int\limits_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}.$$

Secondly, since we look for critical points on S_c, we have to show that $\{v_n\}$ strongly converges in $H^1(\mathbb{R}^N)$. We try to use the concentration-compactness principle to do so. As $\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2$ and $\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}$ behave the same under L^2-preserving scaling of u, it seems impossible to show that there exists $t > 0$ such that $I(u^t) < 0$ for all $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, where $u^t(x) = t^{\frac{N}{2}}u(tx)$. Then, it makes that the usual arguments allowed us to benefit from "natural" constraints $\{u \in S_c |\ G(u) = 0\}$ (see [3,10,26]) cannot be applied here. To overcome this difficulty, we indeed notice that for any $u \in H^1(\mathbb{R}^N)$ with $G(u) < 0$, there exists a unique $t_0 \in (0,1)$ such that $G(u^{t_0}) = 0$ and $t_1 > t_0$ such that $I(u^{t_1}) < 0$ and if we assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, then

$$\int\limits_{\mathbb{R}^N} |\nabla v_n|^2 \to \int\limits_{\mathbb{R}^N} |\nabla v|^2 \Longleftrightarrow v_n \to v \quad \text{in } L^2(\mathbb{R}^N).$$

Hence, if by contradiction we just assume that $\int_{\mathbb{R}^N} |\nabla v|^2 < \lim\limits_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla v_n|^2$, it must lead to $G(v) < 0$, by which we get a path in $\Gamma(|v|_2)$ to show that $\gamma(|v|_2) < \gamma(c)$. However, the function $c \mapsto \gamma(c)$ is indeed nondecreasing on $(c^*, +\infty)$, which induces a contradiction and completes the proof of the theorem. Our method to show the monotonicity of $c \mapsto \gamma(c)$ is quite different from [3,10,26] since no suitable manifold can be used here. We succeeded in doing so by a more careful analysis that $K(c)$ given in (1.9) is nonincreasing on $(c^*, +\infty)$ and $K(c) \to 0$ as $c \to +\infty$ and by an L^2-preserving scaling of ∇u.

We also obtain a supplementary result to [26] in the spacial case $p = 2$, which will complete the proof of the existence of constrained minimizers on S_c related to Kirchhoff equations.

Theorem 1.4. *For any $c > 0$, then $I_{2,c^2} = -\frac{c^2}{2}$ and there is no minimizer for I_{2,c^2}.*

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^N$. $L^p := L^p(\mathbb{R}^N)$ $(1 \le p \le +\infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use " $\to$" and " $\rightharpoonup$" to denote the strong and weak convergence in the related function space, respectively. C will denote a positive constant unless specified. We use " $:=$" to denote definitions and $B_r(x) := \{y \in \mathbb{R}^N \,|\, |x - y| < r\}$. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows. In Sect. 2, we prove Theorem 1.2. In Sect. 3, we prove Theorem 1.3. In Sect. 4, we prove Theorem 1.4.

2. Proof of Theorem 1.2

In this section, we consider the case $c \in (0, c^*)$ and the constrained minimization problem (1.7), where c^* is given in Lemma 1.1.

By the assumption (V), $V(x)$ is bounded a.e. in $\mathbb{R}^N$, i.e., $0 \le V(x) \le V_0 := |V(x)|_\infty$ a.e. in $\mathbb{R}^N$.

Lemma 2.1. *For all $c \in (0, c^*)$, then*

(1) *$E_{a,b}(u)$ is bounded from below and coercive on S_c.*

(2) *$e_{a,b}(c^2) \le 0$.*

Proof. (1) For any $c \in (0, c^*)$ and $u \in S_c$, then by (1.4) there exists $C > 0$ such that

$$\frac{1}{4}\int\limits_{\mathbb{R}^N} V(x)|u|^4 \le V_0 C \left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{N}{2}}$$

and

$$\frac{N}{2N+8}\int\limits_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}} \le \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}} \frac{b}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^2.$$

Hence,

$$E_{a,b}(u) \ge \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - V_0 C|\nabla u|_2^N,$$

which implies that $E_{a,b}(u)$ is bounded from below and coercive on S_c.

(2) For any $u \in S_c$ and $t > 0$, set $u^t(x) := t^{\frac{N}{2}}u(tx)$, then $u^t \in S_c$ and

$$E_{a,b}(u^t) = \frac{at^2}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 + \frac{bt^4}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{t^N}{4}\int\limits_{\mathbb{R}^N} V\left(\frac{x}{t}\right)|u|^4 - \frac{Nt^4}{2N+8}\int\limits_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}.$$

Hence, by (V), we see that $e_{a,b}(c^2) \le E_{a,b}(u^t) \to 0$ as $t \to 0^+$. □

Lemma 2.2. *If $b \in (0, \frac{1}{\lambda_1}]$, then there exists $a^* > 0$ such that $e_{a,b}(c^2) < 0$ for all $a \in (0, a^*)$ and $c \in (0, c^*)$.*

Proof. For each $c \in (0, c^*)$, let $\phi_c = \frac{c\phi}{|\phi|_2}$, where ϕ is given in (1.8). Then, $\phi_c \in S_c$. Hence, we have

$$\begin{aligned} E_{0,b}(\phi_c) &= \frac{c^4}{4|\phi|_2^4}\left[b\left(\int\limits_{\mathbb{R}^N}|\nabla\phi|^2\right)^2 - \int\limits_{\mathbb{R}^N}V(x)|\phi|^4\right] - \frac{N\left(\frac{c}{|\phi|_2}\right)^{\frac{2N+8}{N}}}{2N+8}\int\limits_{\mathbb{R}^N}|\phi|^{\frac{2N+8}{N}} \\ &= \frac{c^4}{4|\phi|_2^4}\left[b\left(\int\limits_{\Omega}|\nabla\phi|^2\right)^2 - \int\limits_{\Omega}V(x)|\phi|^4\right] - \frac{N\left(\frac{c}{|\phi|_2}\right)^{\frac{2N+8}{N}}}{2N+8}\int\limits_{\Omega}|\phi|^{\frac{2N+8}{N}} \\ &= \frac{(b\lambda_1-1)c^4}{4|\phi|_2^4} - \frac{N(\frac{c}{|\phi|_2})^{\frac{2N+8}{N}}}{2N+8}\int\limits_{\Omega}|\phi|^{\frac{2N+8}{N}} < 0 \end{aligned}$$

if $b \in (0, \frac{1}{\lambda_1}]$. Since $E_{a,b}(\phi_c) \to E_{0,b}(\phi_c)$ as $a \to 0^+$, we see that there exists $a^* > 0$ such that $e_{a,b}(c^2) \le E_{a,b}(\phi_c) < 0$ for all $0 < a < a^*$ and the lemma is proved. □

Lemma 2.3. *If $b \in (0, \frac{1}{\lambda_1}]$ and $a \in (0, a^*)$, where a^* is given in Lemma 2.2, then the function $c \mapsto e_{a,b}(c^2)$ is continuous on $(0, c^*)$.*

Proof. The proof follows from Lemma 2.1 (1) and is similar to that of Theorem 2.1 in [5], so we omit it. □

Lemma 2.4. *If $b \in (0, \frac{1}{\lambda_1}]$ and $a \in (0, a^*)$, then for any $0 < c < c_*$,*

$$e_{a,b}(c^2) < e_{a,b}(\alpha^2) + e_{a,b}(c^2 - \alpha^2),\ \forall\ 0 < \alpha < c.$$

Proof. If $b \in (0, \frac{1}{\lambda_1}]$ and $0 < a < a^*$, then $e_{a,b}(c^2) < 0$ for all $0 < c < c^*$. Let $\{u_n\} \subset S_c$ be a minimizing sequence for $e_{a,b}(c^2)$, then by Lemma 2.1, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and then by (V), there exists $k_1 > 0$ such that

$$\int\limits_{\mathbb{R}^N}|u_n|^{\frac{2N+8}{N}} \ge k_1. \tag{2.1}$$

Indeed, if $\int_{\mathbb{R}^N}|u_n|^{\frac{2N+8}{N}} \to 0$, then for any $\varepsilon > 0$, there exists $n_0 > 0$ such that $n > n_0$ implies that $\int_{\mathbb{R}^N}|u_n|^{\frac{2N+8}{N}} < \varepsilon$. By (V), there exists $R > 0$ such that $0 \le V(x) < \varepsilon$ for all $|x| \ge R$. Then, for $n > n_0$, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \int\limits_{\mathbb{R}^N}V(x)|u_n|^4 &= \int\limits_{B_R(0)}V(x)|u_n|^4 + \int\limits_{\mathbb{R}^N\setminus B_R(0)}V(x)|u_n|^4 \\ &\le V_0\int\limits_{B_R(0)}|u_n|^4 + \varepsilon\int\limits_{\mathbb{R}^N\setminus B_R(0)}|u_n|^4 \\ &\le V_0C_1|u_n|^4_{\frac{2N+8}{N}} + \varepsilon C_2 \\ &\le (V_0C_1 + C_2)\varepsilon, \end{aligned} \tag{2.2}$$

which implies that $\int_{\mathbb{R}^N}V(x)|u_n|^4 \to 0$ by the arbitrary of ε and $V(x) \ge 0$. Hence, by (1.4), we have

$$e_{a,b}(c^2) = \lim_{n\to+\infty}E_{a,b}(u_n) \ge \lim_{n\to+\infty} -\frac{1}{4}\int\limits_{\mathbb{R}^N}V(x)|u_n|^4 = 0,$$

which is a contradiction. Moreover, since $e_{a,b}(c^2) < 0$, for n large we have

$$b\left(\int\limits_{\mathbb{R}^N} |u_n|^2\right)^2 \le \int\limits_{\mathbb{R}^N} V(x)|u_n|^4 + \frac{2N}{N+4}\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}. \tag{2.3}$$

Set $u_n^\theta := \theta u_n$ with $\theta > 1$, then $u_n^\theta \in S_{\theta c}$ and for n large,

$$\begin{aligned} E_{a,b}(u_n^\theta) - \theta^2 E_{a,b}(u_n) &= \frac{\theta^2}{4}\Big[b(\theta^2-1)\left(\int\limits_{\mathbb{R}^N} |\nabla u_n|^2\right)^2 - (\theta^2-1)\int\limits_{\mathbb{R}^N} V(x)|u_n|^4 \\ &\quad - \frac{2N(\theta^{\frac{8}{N}}-1)}{N+4}\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}\Big] \\ &:= \frac{\theta^2}{4} f_n(\theta), \end{aligned}$$

where $f_n : [1,+\infty) \to \mathbb{R}$ is given as

$$f_n(t) = b(t^2-1)\left(\int\limits_{\mathbb{R}^N} |\nabla u_n|^2\right)^2 - (t^2-1)\int\limits_{\mathbb{R}^N} V(x)|u_n|^4 - \frac{2N(t^{\frac{8}{N}}-1)}{N+4}\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}.$$

We easily see that $f_n(1) = 0$ and by (2.3) and (2.1),

$$\begin{aligned} f_n'(t)|_{t=1} &= 2b\left(\int\limits_{\mathbb{R}^N} |\nabla u_n|^2\right)^2 - 2\int\limits_{\mathbb{R}^N} V(x)|u_n|^4 - \frac{16}{N+4}\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}} \\ &\le -\frac{4(4-N)}{N+4}\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}} \le -\frac{4(4-N)}{N+4}k_1 < 0. \end{aligned} \tag{2.4}$$

Since

$$f_n''(t) = 2b\left(\int\limits_{\mathbb{R}^N} |\nabla u_n|^2\right)^2 - 2\int\limits_{\mathbb{R}^N} V(x)|u_n|^4 - \frac{16(8-N)}{N(N+4)}t^{\frac{8-2N}{N}}\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}},$$

it follows that $f_n''(t)$ is strictly decreasing on $[1,+\infty)$ and $f_n''(t) < f_n''(t)|_{t=1} < 0$ on $[1,+\infty)$ by (2.3). Hence, we get that

$$\begin{aligned} E_{a,b}(u_n^\theta) - \theta^2 E_{a,b}(u_n) &= \frac{\theta^2}{4} f_n(\theta) \\ &\le \frac{\theta^2}{4}[f_n(0) + f_n'(1)(\theta-1)] < -\frac{4-N}{N+4}k_1\theta^2(\theta-1) < 0, \end{aligned}$$

which implies that

$$e_{a,b}(\theta^2c^2) < \theta^2 e_{a,b}(c^2)$$

by letting $n \to +\infty$. Then, we easily conclude our result and the lemma is proved. $\square$

Proof of Theorem 1.2

Proof. For any $c \in (0, c^*)$, by Lemma 2.2, $e_{a,b}(c^2) < 0$. Let $\{u_n\} \subset S_c$ be a minimizing sequence for $e_{a,b}(c^2)$, then by Lemmas 2.1, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We may assume that for some $u \in H^1(\mathbb{R}^N)$,

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H^1(\mathbb{R}^N), \\ u_n \to u, & \text{in } L^q_{loc}(\mathbb{R}^N), \quad q \in [1, 2^*), \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{2.5}$$

Moreover, $u \not\equiv 0$. In fact, by contradiction, we just suppose that $u \equiv 0$, then by (V) and (2.5), similarly to (2.2), we have $\int_{\mathbb{R}^N} V(x)|u_n|^4 \to 0$, which implies that $e_{a,b}(c^2) \geq 0$. It is a contradiction. So $\alpha := |u|_2 \in (0, c]$.

Next, we try to prove that $u \in S_c$. Just suppose that $\alpha < c$, then $u \in S_\alpha$. By (2.5), we have

$$|u_n|_2^2 = |u_n - u|_2^2 + |u|_2^2 + o_n(1), \tag{2.6}$$

where $o_n(1) \to 0$ as $n \to +\infty$. By the Brezis–Lieb Lemma and Lemma 2.3, we see that

$$e_{a,b}(c^2) = \lim_{n\to+\infty} E_{a,b}(u_n) \geq E_{a,b}(u) + \lim_{n\to+\infty} E_{a,b}(u_n - u) \geq e_{a,b}(\alpha^2) + e_{a,b}(c^2 - \alpha^2),$$

which contradicts Lemma 2.4. Then $|u|_2 = c$. So $u \in S_c$ and then by the Gagliardo–Nirenberg inequality (1.4), $u_n \to u$ in $L^p(\mathbb{R}^N)$, $p \in [2, 2^*)$. Hence, by (2.5),

$$e_{a,b}(c^2) \leq E_{a,b}(u) \leq \lim_{n\to+\infty} E_{a,b}(u_n) = e_{a,b}(c^2).$$

So u is a minimizer of $e_{a,b}(c^2)$ and then u is a constraint critical point of $E_{a,b}$ on S_c. Therefore, there exists $\lambda_c \in \mathbb{R}$ such that $E'_{a,b}(u) - \lambda_c u = 0$ in $H^{-1}(\mathbb{R}^N)$, i.e., (u, λ_c) is a couple of solution to the following equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u - V(x)u^3 - |u|^{\frac{8}{N}}u = \lambda_c u.$$

Moreover, the fact that $E_{a,b}(u) < 0$ shows that

$$\lambda_c c^2 = \langle E'_{a,b}(u), u\rangle = 4E_{a,b}(u) - a\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{4-N}{N+4}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}} < 0,$$

i.e., $\lambda_c < 0$. □

3. Proof of Theorem 1.3

In this section, we consider the case $c > c^*$. Let us first show that I possesses a mountain pass geometry on S_c.

Lemma 3.1. *For any $c > c^*$, there exists $K(c) > 0$ depending on c such that*

$$\gamma(c) = \inf_{h\in\Gamma(c)} \max_{\tau\in[0,1]} I(h(\tau)) > \max_{h\in\Gamma(c)} \max\{I(h(0)), I(h(1))\}, \tag{3.1}$$

where

$$\Gamma(c) = \{h \in C([0,1], S_c) |\ h(0) \in B_{K(c)}, I(h(1)) < 0\} \tag{3.2}$$

and $B_{K(c)} = \{u \in S_c |\ |\nabla u|_2^2 \leq K(c)\}$. Moreover, $K(c)$ is nonincreasing on $(c^*, +\infty)$ and $K(c) \to 0$ as $c \to +\infty$.

Proof. For any $k > 0$, set

$$B_k := \{u \in S_c \mid |\nabla u|_2^2 \le k\}.$$

For any $u \in S_c$, by (1.4), we have

$$I(u) \ge \frac{a}{2}|\nabla u|_2^2 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}} \frac{b}{4}|\nabla u|_2^4,$$

which implies that

$$I(u) \ge \frac{a}{4}|\nabla u|_2^2 \text{ for all } 0 < |\nabla u|_2^2 \le k_1(c) := \frac{a}{b}\left(\frac{c^*}{c}\right)^{\frac{8-2N}{N}}.$$

Since $|I(u)| \to 0$ as $|\nabla u|_2 \to 0^+$, there exist $0 < k_0(c) < k_1(c)$ such that

$$0 < \sup_{u \in B_{k_0(c)}} I(u) < \inf_{u \in \partial B_{k_1(c)}} I(u) \text{ and } I(u) > 0 \text{ for all } u \in B_{k_1(c)}. \tag{3.3}$$

Moreover, it easily sees that $k_1(c) \to 0$ as $c \to +\infty$. Then, without loss of generality, we may assume that $k_0(c)$ is nonincreasing on $(c^*, +\infty)$.

For any $t > 0$, set

$$Q^t(x) := \frac{ct^{\frac{N}{2}}Q(tx)}{|Q|_2},$$

where $Q := Q_{\frac{2N+8}{N}}$ is given in (1.5). Then, $Q^t \in S_c$, $|\nabla Q^t|_2 = tc|\nabla Q|_2$ and

$$I(Q^t) = \frac{ac^2}{2}t^2 - \left[\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}} - 1\right]\frac{bc^4}{4}t^4 \to -\infty \text{ as } t \to +\infty.$$

Hence, there exist $t_1 > 0$ small and $t_2 > 0$ large such that

$$|\nabla Q^{t_1}|_2^2 \le k_0(c),$$

$$|\nabla Q^{t_2}|_2^2 > k_1(c) \text{ and } I(Q^{t_2}) < 0.$$

Set $K(c) := k_0(c)$, then $K(c)$ is nonincreasing on $(c^*, +\infty)$ and $\lim_{c \to +\infty} K(c) = 0$. Let

$$h(\tau) := Q^{(1-\tau)t_1 + \tau t_2}, \ \tau \in [0, 1].$$

Then $h \in \Gamma(c)$, i.e., $\Gamma(c) \ne \varnothing$. For any $h \in \Gamma(c)$, $|\nabla h(0)|_2^2 \le K(c) = k_0(c) < k_1(c) < |\nabla h(1)|_2^2$. Then, there exists $\tau_0 \in (0, 1)$ such that $|\nabla h(\tau_0)|_2^2 = k_1(c)$, i.e., $h(\tau_0) \in \partial B_{k_1(c)}$. So by (3.3), we have

$$\max_{\tau \in [0,1]} I(h(\tau)) \ge I(h(\tau_0)) \ge \inf_{u \in \partial B_{k_1(c)}} I(u) > \sup_{u \in B_{k_0(c)}} I(u) \ge I(h(0)) > 0,$$

which implies that

$$\gamma(c) \ge \max_{h \in \Gamma(c)} \max\{I(h(0)), I(h(1))\}$$

since $h \in \Gamma(c)$ is arbitrary. □

Remark 3.2. *As it is clear from the proof of Lemma 3.1, we can assume that*

$$\sup_{u \in B_{K(c)}} I(u) < \frac{\gamma(c)}{2}.$$

In order to obtain a bounded $(PS)_{\gamma(c)}$ sequence for $I|_{S_c}$, we define a map $H(u,t): H^1(\mathbb{R}^N)\times\mathbb{R}\to H^1(\mathbb{R}^N)$ as

$$H(u,t)(x)=e^{\frac{N}{2}t}u(e^tx)$$

and define a C^1 functional $\widetilde{I}(u,t): H^1(\mathbb{R}^N)\times\mathbb{R}\to\mathbb{R}$ as

$$\widetilde{I}(u,t)=\frac{ae^{2t}}{2}\int\limits_{\mathbb{R}^N}|\nabla u|^2+\frac{be^{4t}}{4}\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\right)^2-\frac{Ne^{4t}}{2N+8}\int\limits_{\mathbb{R}^N}|u|^{\frac{2N+8}{N}},$$

i.e., $\widetilde{I}(u,t)=I(H(u,t))$. It easily sees that for all $t\in\mathbb{R}$, $H(u,t)\in S_c$ if $u\in S_c$.

Lemma 3.3. *For any $c>c^*$,*

$$\widetilde{\gamma}(c)=\inf_{h\in\widetilde{\Gamma}(c)}\max_{\tau\in[0,1]}\widetilde{I}(h(\tau))$$

is well defined with

$$\widetilde{\Gamma}(c)=\{h\in C([0,1],S_c\times\mathbb{R})|\ H(h(0))\in B_{K(c)},\widetilde{I}(h(1))<0\},$$

where $B_{K(c)}$ is given in Lemma 3.1. Moreover, $\widetilde{\gamma}(c)=\gamma(c)$.

Proof. By the definition of $H(u,t)$, we see that $\Gamma(c)\times\{0\}\subset\widetilde{\Gamma}(c)$, then $\widetilde{\Gamma}(c)\neq\varnothing$ and $\widetilde{\gamma}(c)$ is well defined. Moreover, $H(\widetilde{\Gamma}(c))=\Gamma(c)$, which implies that $\widetilde{\gamma}(c)=\gamma(c)$. □

Denote $E:=H^1(\mathbb{R}^N)\times\mathbb{R}$. Let E be equipped with a norm defined by

$$\|(u,t)\|_E=(\|u\|^2+|s|^2)^{\frac{1}{2}},\quad\forall\ (u,t)\in E,$$

which is induced by the corresponding scalar product

$$\langle(u,t),(v,s)\rangle_E=\int\limits_{\mathbb{R}^N}(a\nabla u\nabla v+uv)+ts,\quad\forall\ (u,t),(v,s)\in E.$$

We recall in [24] that for any $c>0$, S_c is a submanifold of $H^1(\mathbb{R}^N)$ with codimension 1 and the tangent space at $u\in S_c$ is defined as $T_u=\{v\in H^1(\mathbb{R}^N)|\ \int_{\mathbb{R}^N}uv=0\}$. The norm of the derivative of $I|_{S(c)}$ is defined by

$$\|(I|_{S_c})'(u)\|_*=\sup_{v\in T_u,\|v\|=1}\langle I'(u),v\rangle.$$

Similarly, the tangent space at $(u,t)\in S_c\times\mathbb{R}$ is $\widetilde{T}_{(u,t)}=\{(v,s)\in E|\ \int_{\mathbb{R}^N}uv=0\}$. The norm of the derivative of $\widetilde{I}|_{S_c\times\mathbb{R}}$ is defined by

$$\|(\widetilde{I}|_{S_c\times\mathbb{R}})'(u,t)\|_*=\sup_{(v,s)\in\widetilde{T}_{(u,t)},\|(v,s)\|_E=1}\langle\widetilde{I}'(u,t),(v,s)\rangle.$$

Lemma 3.4. *For $c>c^*$, then there exists a sequence $\{v_n\}\subset S_c$ such that*

(1) $I(v_n)\to\gamma(c)$ *as* $n\to+\infty$;
(2) $\|(I|_{S_c})'(v_n)\|_*\to 0$ *as* $n\to+\infty$;
(3) $G(v_n)\to 0$, *where*

$$G(u)=a\int\limits_{\mathbb{R}^N}|\nabla u|^2+b\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\right)^2-\frac{2N}{N+4}\int\limits_{\mathbb{R}^N}|u|^{\frac{2N+8}{N}};\tag{3.4}$$

(4) $\{v_n\}$ *is bounded in* $H^1(\mathbb{R}^N)$.

Proof. The proof is similar to that of Lemma 2.4 in [10]. We give a detailed proof here for readers' convenience.

Let $\{h_n\} \subset \Gamma(c)$ satisfy $\max_{\tau\in[0,1]} I(h_n(\tau)) \leq \gamma(c) + \frac{1}{n}$. Set $\widetilde{h}_n(\tau) := (h_n(\tau), 0)$, then $\widetilde{h}_n \in \widetilde{\Gamma}(c)$ and

$$\max_{\tau\in[0,1]} \widetilde{I}(\widetilde{h}_n(\tau)) \leq \widetilde{\gamma}(c) + \frac{1}{n}$$

since $\widetilde{\gamma}(c) = \gamma(c)$. By Proposition 2.2 in [10], there exists a sequence $\{(u_n, t_n)\} \subset S_c \times \mathbb{R}$ satisfying

$$\widetilde{I}(u_n, t_n) \to \widetilde{\gamma}(c), \ \|(\widetilde{I}|_{S_c\times\mathbb{R}})'(u_n, t_n)\|_* \to 0 \tag{3.5}$$

and

$$\min_{\tau\in[0,1]} \|(u_n, t_n) - \widetilde{h}_n(\tau)\|_E \to 0.$$

Set $v_n := H(u_n, t_n)$, then $v_n \in S_c$. By Lemma 3.3 and (3.5), we have

$$I(v_n) \to \gamma(c). \tag{3.6}$$

For any $(w, s) \in H^1(\mathbb{R}^N) \times \mathbb{R}$, we have

$$\langle \widetilde{I}'(u_n, t_n), (w, s)\rangle = \langle I'(v_n), H(w, t_n)\rangle + G(v_n)s. \tag{3.7}$$

Let $(w, s) = (0, 1) \in \widetilde{T}_{(u_n,t_n)}$, it follows from (3.5) that

$$G(v_n) \to 0. \tag{3.8}$$

For any $w \in T_{v_n}$, if we take $s = 0$ in (3.7), then we have

$$\langle I'(v_n), w\rangle = \langle \widetilde{I}'(u_n, t_n), (H(w, -t_n), 0)\rangle. \tag{3.9}$$

Moreover,

$$w \in T_{v_n} \Longleftrightarrow (H(w, -t_n), 0) \in \widetilde{T}_{(u_n,t_n)}.$$

Then, by (3.9) and (3.5), we see that to prove that $\|(I|_{S_c})'(v_n)\|_* \to 0$ is equivalent to show that $\{(H(w, -t_n), 0)\}$ is uniformly bounded in E for n large, which is indeed ensured by the fact that

$$|t_n| \leq \min_{\tau\in[0,1]} \|(u_n, t_n) - \widetilde{h}_n(\tau)\|_E \leq 1 \text{ for } n \text{ large.}$$

Since

$$I(v_n) - \frac{1}{4}G(v_n) = \frac{a}{4}\int_{\mathbb{R}^N} |\nabla v_n|^2,$$

it follows from (3.6) and (3.8) that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

In order to prove Theorem 1.3, we need the following two crucial lemmas.

Lemma 3.5. *The function $c \mapsto \gamma(c)$ is nonincreasing on $(c^*, +\infty)$.*

Proof. For any $c^* < c_1 < c_2 < +\infty$, it is enough to show that $\gamma(c_2) \leq \gamma(c_1)$.

Let $\{h_n\} \subset \Gamma(c_1)$ be a sequence of paths such that

$$\max_{\tau\in[0,1]} I(h_n(\tau)) \leq \gamma(c_1) + \frac{1}{n}. \tag{3.10}$$

Take $u \in B_{K(c_2)}$, i.e., $u \in S_{c_2}$ and $|\nabla u|_2^2 \leq K(c_2)$. By Lemma 3.1, $K(c_2) \leq K(c_1)$. Let $\gamma_n(\tau) \in C([0, \frac{1}{2}], B_{K(c_1)})$ satisfy that $\gamma_n(0) = \frac{c_1}{c_2}u$ and $\gamma_n(\frac{1}{2}) = h_n(0)$. Then, we define a new path $\widetilde{h}_n : [0, 1] \to S_{c_1}$ as follows:

$$\widetilde{h}_n(\tau) = \begin{cases} \gamma_n(\tau), & \tau \in [0, \frac{1}{2}], \\ h_n(2\tau - 1), & \tau \in [\frac{1}{2}, 1]. \end{cases} \tag{3.11}$$

It is easy to see that $\widetilde{h}_n \in \Gamma(c_1)$. Moreover, we conclude from Remark 3.2 that

$$\max_{\tau\in[0,1]} I(\widetilde{h}_n(\tau)) = \max_{\tau\in[\frac{1}{2},1]} I(\widetilde{h}_n(\tau)) = \max_{\tau\in[0,1]} I(h_n(\tau)). \tag{3.12}$$

Set

$$g_n(\tau) := \left(\frac{c_2}{c_1}\right)^{1-\frac{N}{2}} \widetilde{h}_n\left(\frac{c_1}{c_2}\tau\right).$$

Then, $|g_n|_2 = c_2$, i.e., $g_n \in S_{c_2}$ and for all $\tau \in [0,1]$,

$$\int_{\mathbb{R}^N} |\nabla g_n(\tau)|^2 = \int_{\mathbb{R}^N} |\nabla \widetilde{h}_n(\tau)|^2, \quad \int_{\mathbb{R}^N} |g_n(\tau)|^{\frac{2N+8}{N}} = \left(\frac{c_2}{c_1}\right)^{\frac{8-2N}{N}} \int_{\mathbb{R}^N} |\widetilde{h}_n(\tau)|^{\frac{2N+8}{N}}. \tag{3.13}$$

Hence, by (3.11) and (3.13), we have $g_n(0) \in B_{K(c_2)}$ and $I(g_n(1)) < I(\widetilde{h}_n(1)) < 0$, so $g_n \in \Gamma(c_2)$. There exists a sequence $\{\tau_n\} \subset (0,1)$ such that $I(g_n(\tau_n)) = \max_{\tau\in[0,1]} I(g_n(\tau))$, then by (3.10)–(3.13), we have

$$\begin{aligned}\gamma(c_2) \le \max_{\tau\in[0,1]} I(g_n(\tau)) &= I(g_n(\tau_n)) \\ &< I(\widetilde{h}_n(\tau_n)) \le \max_{\tau\in[0,1]} I(\widetilde{h}_n(\tau)) \\ &\le \max_{\tau\in[0,1]} I(h_n(\tau)) \le \gamma(c_1) + \frac{1}{n},\end{aligned}$$

which implies the conclusion by letting $n \to +\infty$. □

Lemma 3.6. *For any $u \in H^1(\mathbb{R}^N)$ with $G(u) < 0$, where G is given in (3.4), then*

1. *there exists a unique $0 < t_0 < 1$ such that $G(u^{t_0}) = 0$, where $u^t(x) = t^{\frac{N}{2}} u(tx)$.*
2. *there exists $t_1 > t_0$ such that $I(u^{t_1}) < 0$.*
3. *Moreover, $I(u^{t_0}) = \max_{t\in[0,1]} I(u^{tt_1}) = \max_{t>0} I(u^t)$.*

Proof. Since $G(u) < 0$ implies that $u \not\equiv 0$, $G(u^t) > 0$ if $t > 0$ is small. Then, it must have a unique $t_0 \in (0,1)$ such that $G(u^{t_0}) = 0$. Indeed, the uniqueness is ensured by the fact that $G(u^{tt_0}) < t^4 G(u^{t_0}) = 0$ for all $t > 1$.

By $G(u) < 0$, we have

$$b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 < \frac{2N}{N+4} \int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}.$$

Then,

$$I(u^t) = \frac{at^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{t^4}{4}\left[b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{2N}{N+4} \int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}\right] \to -\infty$$

as $t \to +\infty$. Hence, there exists $t_1 > 0$ large such that $I(u^{t_1}) < 0$. Since $I(u^t) > 0$ for $t > 0$ small and $G(v^t) = t\frac{\partial}{\partial t} I(v^t)$, we see that $t_1 > t_0$ and $I(u^{t_0}) = \max_{t\in[0,1]} I(u^{tt_1})$. Moreover, we see that $I(u^t) < 0$ for all $t > t_1$, which shows that $I(u^{t_0}) = \max_{t>0} I(u^t)$. □

Proof of Theorem 1.3

Proof. For any $c > c^*$, by Lemma 3.4, there exists a bounded sequence $\{v_n\} \subset S_c$ such that

$$I(v_n) \to \gamma(c), \ \|(I|_{S_c})'(v_n)\|_* \to 0 \text{ and } G(v_n) \to 0. \tag{3.14}$$

Since $\gamma(c) > 0$, by the vanishing lemma, we may assume that, up to a subsequence and up to translations,

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^N) \tag{3.15}$$

for some $v \not\equiv 0$. Moreover, there exists $A \in \mathbb{R}\backslash\{0\}$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \to A^2. \tag{3.16}$$

By (3.14), there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$I'(v_n) - \lambda_n v_n \to 0 \text{ in } H^{-1}(\mathbb{R}^N). \tag{3.17}$$

Then, by (3.16), we have

$$\lambda_n = \frac{\langle I'(v_n), v_n\rangle}{c^2} \to \lambda_c := -\frac{4-N}{2Nc^2}(aA^2 + bA^4) < 0. \tag{3.18}$$

To complete the proof, it is enough to show that

$$\int_{\mathbb{R}^N} |\nabla v|^2 = A^2 \tag{3.19}$$

holds. Indeed, if (3.19) holds, then

$$|\nabla v_n - \nabla v|_2 \to 0, \tag{3.20}$$

hence by the Gagliardo–Nirenberg inequality (1.4) and the boundedness of $\{v_n\}$,

$$v_n \to v \text{ in } L^p(\mathbb{R}^N), \ \forall\, 2 < p < 2^*. \tag{3.21}$$

Then, by (3.15), (3.17) and (3.20), we have

$$\langle I'(v_n) - \lambda_n v_n - I'(v) - \lambda_c v, v_n - v\rangle \to 0,$$

which and (3.18)–(3.21) show that $v_n \to v$ in $L^2(\mathbb{R}^N)$, i.e., $v_n \to v$ in $H^1(\mathbb{R}^N)$. So, $v \in S_c$ and (v, λ_c) is a couple of solution of (1.1) with $I(v) = \gamma(c)$.

We next prove (3.19) by contradiction. We just suppose that $\int_{\mathbb{R}^N} |\nabla v|^2 < A^2$. Then, $\alpha := |v|_2 \in (0, c)$. By (3.14)–(3.18), we see that v is a nontrivial critical point of the following local functional

$$I_A(v) = \frac{a + bA^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{N}{2N+8} \int_{\mathbb{R}^N} |v|^{\frac{2N+8}{N}}.$$

Then, v satisfies the following Pohozaev identity

$$P_A(v) := \frac{(N-2)(a + bA^2)}{2} \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{N^2}{2N+8} \int_{\mathbb{R}^N} |v|^{\frac{2N+8}{N}} = 0,$$

hence $G_A(v) := \frac{N}{2}\langle I'_A(v), v\rangle - P_A(v) = 0$, i.e.,

$$(a + bA^2) \int_{\mathbb{R}^N} |\nabla v|^2 - \frac{2N}{N+4} \int_{\mathbb{R}^N} |v|^{\frac{2N+8}{N}} = 0.$$

So $G(v) < 0$. By Lemma 3.6, there exists a unique $t_0 \in (0,1)$ and $t_1 > t_0$ such that $G(v^{t_0}) = 0$ and $I(v^{t_1}) < 0$, where $v^t(x) := t^{\frac{N}{2}} v(tx)$. Then, $v^{tt_1} \in \Gamma(\alpha)$. Moreover, by Lemma 3.6, $I(v^{t_0}) = \max_{t\in[0,1]} I(v^{tt_1})$. Then, by (3.14) and (3.15), we have

$$\begin{aligned}
\gamma(\alpha) &\leq \max_{t\in[0,1]} I(v^{tt_1}) = I(v^{t_0}) = I(v^{t_0}) - \frac{1}{4}G(v^{t_0}) \\
&= \frac{at_0^2}{4}\int_{\mathbb{R}^N} |\nabla v|^2 \\
&< \frac{a}{4}\int_{\mathbb{R}^N} |\nabla v|^2 \\
&\leq \liminf_{n\to+\infty}\left(\frac{a}{4}\int_{\mathbb{R}^N} |\nabla v_n|^2\right) = \liminf_{n\to+\infty}\left[I(v_n) - \frac{1}{4}G(v_n)\right] = \gamma(c),
\end{aligned}$$

which contradicts Lemma 3.5. Then, we have completed the proof of the theorem. □

4. Proof of Theorem 1.4

Proof of Theorem 1.4

Proof. The idea of the proof comes from [11].

(1) For any $c > 0$ and $u \in S_c$, then $I_2(u) \geq -\frac{c^2}{2}$. Set $u^t(x) := t^{\frac{N}{2}} u(tx)$, $t > 0$. Then, $u^t \in S_c$ and

$$I_{2,c^2} \leq I_2(u^t) = \frac{at^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \to -\frac{c^2}{2}$$

as $t \to 0^+$. So $I_{2,c^2} = -\frac{c^2}{2}$.

Just suppose that there exists $u \in S_c$ such that $I_2(u) = I_{2,c^2} = -\frac{c^2}{2}$, then it follows that

$$0 < \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 = 0,$$

which is a contradiction.

□

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