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Twisted stacked central configurations for the spatial nine-body problem

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Abstract. In this article, we study the existence of the twisted stacked central configurations for the nine-body problem. More precisely, the position vectors x_1, x_2, x_3, x_4 and x_5 are at the vertices of a square pyramid Σ; the position vectors x_6, x_7, x_8 and x_9 are at the vertices of a square Π; the square (x_1, x_2, x_3, x_4) and the square (x_6, x_7, x_8, x_9) have twisted angle $\pi/4$.

Mathematical Subject Classification. 34C15 · 34C25.

Keywords. Nine-body problem · Twisted stacked central configuration · Newtonian law.

1. Introduction and main results

The Newtonian n-body problem [1–3] consists in the study of a system formed by n punctual bodies located at $x_1, x_2, \ldots, x_n$, $x_i \in R^d, d = 2, 3$ with positive masses $m_1, m_2, \ldots, m_n$ interacting among themselves by their mutual gravitational attraction according to Newtonian law:

$$m_i \ddot{x}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3}, \tag{1.1}$$

for $i = 1, 2, \ldots, n$. Here $r_{ij} = |x_i - x_j|$ is the Euclidean distance between x_i and x_j.

The space of configuration is defined by

$$X = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ for all } i \neq j\},$$

while the center of mass is given by

$$c = (m_1 x_1 + \cdots + m_n x_n)/M,$$

where $M = m_1 + \cdots + m_n$ is the total mass.

At a given instant $t = t_0$, the n bodies are in a **central configuration** [4,5] if there exists $\lambda \neq 0$ such that

$$-\lambda(x_i - c) = \sum_{j=1, j\neq i}^{n} \frac{m_j (x_j - x_i)}{r_{ij}^3}, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Two central configurations are said to be equivalent if one can be transformed to the other by a scalar multiplication and a rotation. So, we can study the classes of central configurations defined by the above equivalence relation.

There are several reasons why central configurations are of special importance in the study of the n-body problem, see [3–10] for details.

In 2005, for five-body problem, Hampton [11] provides a new family of planar central configurations called stacked central configurations, completed by Llibre and Mello [12]. Mello, Chaves, Fernandes and Garcia [13] consider the Stacked central configurations for the spatial six-body problem. Zhang and Zhou

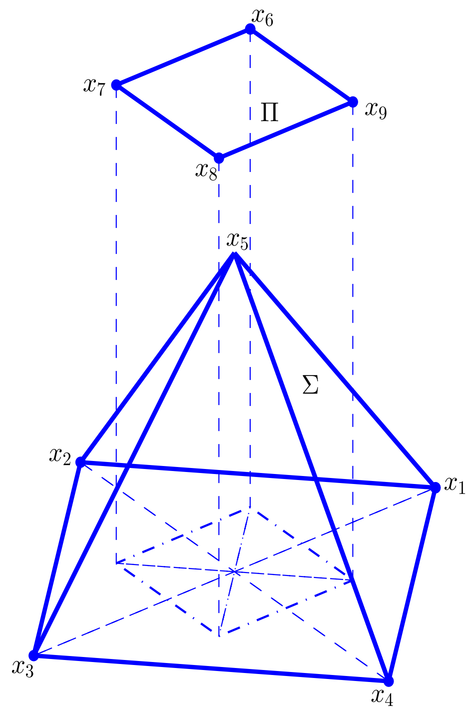

FIG. 1. Twisted stacked central configurations for the spatial nine-body problem

[14] showed the existence of double pyramidal central configurations of $N+2$-body problem. The authors [15–17] provided new examples of stacked central configurations for the spatial seven-body problem.

In this paper, we find new classes of stacked spatial central configurations for the nine-body problem which have five bodies at the vertices of a square pyramid and the other four bodies are located at the vertices of a square. More precisely, the spatial central configurations considered here satisfy (see Fig. 1): the position vectors x_1, x_2, x_3, x_4 and x_5 are at the vertices of a square pyramid Σ; the position vectors x_6, x_7, x_8 and x_9 are at the vertices of a square Π; the square (x_1, x_2, x_3, x_4) and the square(x_6, x_7, x_8, x_9) have twisted angle $\pi/4$.

We can assume

$$\begin{aligned} &x_1 = (1,0,0), \quad x_2 = (0,1,0), \quad x_3 = (-1,0,0), \quad x_4 = (0,-1,0), \quad x_5 = (0,0,h), \\ &x_6 = \left(\frac{\sqrt{2}}{2}x, \frac{\sqrt{2}}{2}x, y\right), \quad x_7 = \left(-\frac{\sqrt{2}}{2}x, \frac{\sqrt{2}}{2}x, y\right), \\ &x_8 = \left(-\frac{\sqrt{2}}{2}x, -\frac{\sqrt{2}}{2}x, y\right), \quad x_9 = \left(\frac{\sqrt{2}}{2}x, -\frac{\sqrt{2}}{2}x, y\right), \end{aligned} \tag{1.3}$$

where $x > 0$, $y \in \mathbb{R}$ and $y \neq 0$, the positive constant h satisfies the equation

$$\frac{2}{r_{15}^3} = \frac{1}{r_{12}^3} + \frac{1}{r_{13}^3},$$

(see [14]), that is, $h \approx 1.26276522$.

The main results of this paper are the following.

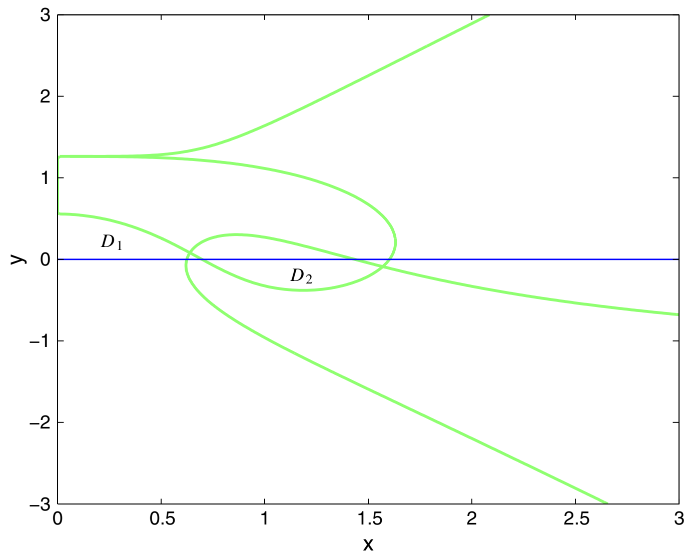

FIG. 2. The regions D_1 and D_2

Theorem 1.1. *Consider the spatial configurations according to Fig. 1, in order that the nine mass points are in a central configuration, the following statements are necessary:*

1. *The masses m_1, m_2, m_3 and m_4 must be equal;*
2. *The masses m_6, m_7, m_8 and m_9 must be equal.*

Theorem 1.2. *Let*

$$D_1 = \{(x, y)|a_{11} < 0, a_{12} > 0, a_{13} < 0, a_{21} > 0, a_{23} < 0\},$$
$$D_2 = \{(x, y)|a_{11} > 0, a_{12} < 0, a_{13} > 0, a_{21} < 0, a_{23} > 0\},$$

which can be seen in Fig. 2, where $a_{11}, a_{12}, a_{13}, a_{21}, a_{23}$ are defined in (2.11). Under the assumption (1.3) and the necessary statements in Theorem 1.1, there exist some points in $T^{-1}(0) \cap (D_1 \cup D_2)$ and accordingly the positive masses m_1, m_5, m_6, such that the nine bodies form a spatial central configuration (see Fig. 1), where T is defined in (2.8).

2. Proof of Theorem 1.1

From Eq. (1.2), it is easy to obtain

$$\lambda(x_i - x_j) = (m_i + m_j)d_{ij}(x_i - x_j) + \sum_{k \neq i,j} [d_{ik}(x_i - x_k) - d_{jk}(x_j - x_k)] \tag{2.1}$$

where $d_{ij} = 1/r_{ij}^3$. Taking the wedge product of Eq. (2.1) with the vector $x_i - x_j$ we get [7,18]

$$f_{ij} = \sum_{k \neq i,j} m_k(d_{ik} - d_{jk})\Delta_{ijk} = 0,$$

where $\Delta_{ijk} = (x_i - x_j) \wedge (x_i - x_k)$. The above equations, in the particular case of a planar central configuration, are known as the Laura–Andoyer equations, and the bi-vector Δ_{ijk} is simply twice the oriented area of the triangle (q_i, q_j, q_k). Taking the wedge product of Eq. (2.1) with $(x_i - x_j) \wedge (x_j - x_l)$, we get (see equation (6), p. 295 of [7] and the references therein)

$$f_{ijk} = \sum_{l=1, l\neq i,j,k}^{n} m_l (d_{il} - d_{jl}) \Delta_{ijkl} = 0, \tag{2.2}$$

for $1 \leq i < j \leq n$, $k = 1, \dots, n, k \neq i, j$. Here, $\Delta_{ijkl} = (x_i - x_j) \wedge (x_i - x_k) \cdot (x_i - x_l)$. Thus, Δ_{ijkl} gives six times the signed volume of the tetrahedron formed by the bodies with positions x_i, x_j, x_k and x_l; Eq. (2.2) is a system of $n(n-1)(n-2)/2$ equations, which are called Dziobek–Laura–Andoyer equations ([16]). For the nine-body problem, the system of Eq. (2.2) provides 252 equations. According to Fig. 1, our class of configurations with nine bodies must satisfy

$$\begin{aligned}
&r_{12} = r_{23} = r_{34} = r_{14} = \sqrt{2}, r_{13} = r_{24} = 2,\\
&r_{67} = r_{78} = r_{89} = r_{69} = \sqrt{2}x, r_{68} = r_{79} = 2x,\\
&r_{16} = r_{19} = r_{26} = r_{27} = r_{37} = r_{38} = r_{48} = r_{49} = \sqrt{x^2 - \sqrt{2}x + 1 + y^2},\\
&r_{17} = r_{18} = r_{28} = r_{29} = r_{36} = r_{39} = r_{46} = r_{47} = \sqrt{x^2 + \sqrt{2}x + 1 + y^2},\\
&r_{15} = r_{25} = r_{35} = r_{45} = \sqrt{1 + h^2},\\
&r_{56} = r_{57} = r_{58} = r_{59} = \sqrt{x^2 + (y - h)^2}.
\end{aligned} \tag{2.3}$$

Due to the assumption (1.3) and the definition of Δ_{ijkl}, we have several symmetries in the signed volumes.

By using the symmetries and the properties of Δ_{ijkl}, we obtain the following results.

Lemma 2.1. *In order to have a spatial central configuration according to Fig. 1, a necessary condition is that the masses m_1, m_2, m_3 and m_4 must be equal.*

Proof. It is sufficient to consider the equations $f_{678} = 0$, $f_{681} = 0$ and $f_{786} = 0$.

$$\begin{aligned}
f_{678} &= (m_1 - m_3)(d_{16} - d_{17})\Delta_{6,781} = 0,\\
f_{681} &= (m_2 - m_3)(d_{26} - d_{28})\Delta_{6,812} = 0,\\
f_{786} &= (m_2 - m_4)(d_{27} - d_{28})\Delta_{7,862} = 0.
\end{aligned}$$

For our class of central configurations, we have $d_{16} - d_{17} = d_{26} - d_{28} = d_{27} - d_{28} \neq 0$, $\Delta_{6,781} = \Delta_{7,862} \neq 0$ and $\Delta_{6,812} \neq 0$. So, the above equations hold if and only if $m_1 = m_2 = m_3 = m_4$. So the statement 1 of Theorem 1.1 is proved. □

Lemma 2.2. *If the configuration according to Fig. 1 is a central configuration, a necessary condition is that the masses m_6, m_7, m_8 and m_9 must be equal.*

By the symmetries of the configurations studied here, Lemma 2.2 can be easily obtained. The proof of Theorem 1.1 is completed.

Due to Lemmas 2.1 and 2.2, henceforth, we restrict the set of admissible masses to

$$m_1 = m_2 = m_3 = m_4 = \alpha, \quad m_6 = m_7 = m_8 = m_9 = \beta.$$

In order to study the given twisted stacked central configurations, it is sufficient to study the following 4 equations:

$$f_{152} = \beta((d_{16} - d_{56})(\Delta_{1,526} + \Delta_{1,529}) + (d_{17} - d_{57})(\Delta_{1,527} + \Delta_{1,528})) = 0, \tag{2.4}$$

$$f_{164} = \alpha(d_{12} + d_{13} - d_{16} - d_{17})\Delta_{1,642} + m_5(d_{15} - d_{56})\Delta_{1,645} + \beta(d_{17} - d_{16})\Delta_{1,647} = 0, \tag{2.5}$$

$$f_{165} = \alpha((d_{12} + d_{13} - d_{16} - d_{17})\Delta_{1,653} + (d_{16} - d_{17})\Delta_{1,654}) + \beta((d_{17} - d_{16})\Delta_{1,657} + (d_{16} + d_{17} - d_{67} - d_{68})\Delta_{1,658}) = 0, \tag{2.6}$$

$$f_{561} = \alpha((d_{15} - d_{16})\Delta_{5,612} + (d_{15} - d_{17})\Delta_{5,613} + (d_{15} - d_{17})\Delta_{5,614}) + \beta((d_{56} - d_{67})\Delta_{5,617} + (d_{56} - d_{68})\Delta_{5,618} + (d_{56} - d_{67})\Delta_{5,619}) = 0. \tag{2.7}$$

If we write

$$f_{152} = \beta T = \beta((d_{16} - d_{56})(\Delta_{1,526} + \Delta_{1,529}) + (d_{17} - d_{57})(\Delta_{1,527} + \Delta_{1,528})) = 0,$$

it follows that

$$T = (d_{16} - d_{56})(\Delta_{1,526} + \Delta_{1,529}) + (d_{17} - d_{57})(\Delta_{1,527} + \Delta_{1,528}) = 0. \tag{2.8}$$

So in the following, we restrict our central configurations in the set $T^{-1}(0)$.

Lemma 2.3. *According to our assumptions and in the set $T^{-1}(0)$, the system of Eq. (2.2) is satisfied if the Eqs. (2.5) and (2.6) are satisfied.*

Proof. Under the assumptions (1.3), we have

$$T = (d_{16} - d_{56})(2y + \sqrt{2}hx - 2h) + (d_{17} - d_{56})(2y - \sqrt{2}hx - 2h) = 0,$$

that is,

$$4(y - h)d_{56} = 2(y - h)(d_{16} + d_{17}) + \sqrt{2}hx(d_{16} - d_{17}). \tag{2.9}$$

Substituting Eq. (2.9) into Eq. (2.7), we obtain the equation $-f_{165} = 0$.
Hence, in the set $T^{-1}(0)$, $f_{165} = 0$ implies $f_{561} = 0$. This completes the proof. □

From Lemma 2.3, in order to study central configurations according to Fig. 1 in the set $T^{-1}(0)$, it is sufficient to study the following two equations:

$$f_{164} = 0, \quad f_{165} = 0. \tag{2.10}$$

Denote by $A = (a_{ij})$ the matrix of the coefficients of the homogeneous linear system in the variables α, m_5, β defined by Eq. (2.10). Thus,

$$\begin{aligned}
a_{11} &= (d_{12} + d_{13} - d_{16} - d_{17})\Delta_{1,642} \\
&= 2y\left(\frac{1}{2\sqrt{2}} + \frac{1}{8} - \frac{1}{(x^2 - \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} - \frac{1}{(x^2 + \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}}\right), \\
a_{12} &= (d_{15} - d_{56})\Delta_{1,645} = (y-h)\left(\frac{1}{(1+h^2)^{\frac{3}{2}}} - \frac{1}{(x^2 + (y-h)^2)^{\frac{3}{2}}}\right), \\
a_{13} &= (d_{17} - d_{16})\Delta_{1,647} = \sqrt{2}xy\left(\frac{1}{(x^2 + \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} - \frac{1}{(x^2 - \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}}\right), \\
a_{21} &= (d_{12} + d_{13} - d_{16} - d_{17})\Delta_{1,653} + (d_{16} - d_{17})\Delta_{1,654} \\
&= \sqrt{2}hx\left(\frac{1}{8} + \frac{1}{2\sqrt{2}} - \frac{1}{(x^2 - \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} - \frac{1}{(x^2 + \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}}\right) \\
&\quad + (y-h)\left(\frac{1}{(x^2 + \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} - \frac{1}{(x^2 - \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}}\right), \\
a_{22} &= 0, \\
a_{23} &= (d_{17} - d_{16})\Delta_{1,657} + (d_{16} + d_{17} - d_{67} - d_{68})\Delta_{1,658} \\
&= hx^2\left(\frac{1}{(x^2 + \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} - \frac{1}{(x^2 - \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}}\right) \\
&\quad - \sqrt{2}x(y-h)\left(\frac{1}{(x^2 + \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} + \frac{1}{(x^2 - \sqrt{2}x + 1 + y^2)^{\frac{3}{2}}} - \frac{1}{8x^3} - \frac{1}{2\sqrt{2}x^3}\right). \qquad (2.11)
\end{aligned}$$

Let $x = \begin{pmatrix} \alpha \\ m_5 \\ \beta \end{pmatrix}$. Then, in order to get the spatial central configuration as in Fig. 1, we need to find positive solution α, m_5, β of the following system:

$$Ax = 0. \qquad (2.12)$$

2.1. The existence of spatial central configurations

In order to prove the existence of positive solutions of (2.12) in the set $T^{-1}(0)$, it is sufficient to prove that the entries in each row of A change the signs. So, if the entries of some row of A have the same signs, there are no admissible masses such that the bodies are in a central configuration according to Fig. 1.

Proof of Theorem 1.2. Since the rank of matrix A is two in the set $T^{-1}(0)$, there are nontrivial solutions of (2.12) in the set $T^{-1}(0)$.

Firstly, we prove the existence of spatial central configurations for some points in the set D_1 (see Fig. 2). In order to prove the existence of positive solutions of (2.12) in the set $T^{-1}(0)$, the entries a_{21}, a_{23} of the second line in the matrix A should have opposite signs. Thus, we consider the following set D_1, where D_1 is surrounded by curves $x = 0, y = 0, a_{21} = 0$ and $a_{23} = 0$.

In the set D_1, the entries of the matrix A have the following signs: $a_{21} > 0, a_{23} < 0$ (see Figs. 3, 4); $a_{11} < 0, a_{12} > 0$ for $D_1 \subset E_1$, where E_1 is surrounded by curves $x = 0, y = 0, a_{11} = 0, a_{12} = 0$, and in which $y > 0$ (see Figs. 5, 6, 7); $a_{13} < 0$ for $d_{17} - d_{16} < 0$ for all $x > 0$. In short, the signs of the entries of the matrix A restricted to the set D_1 are the following:

$$A = \begin{pmatrix} - & + & - \\ + & 0 & - \end{pmatrix}.$$

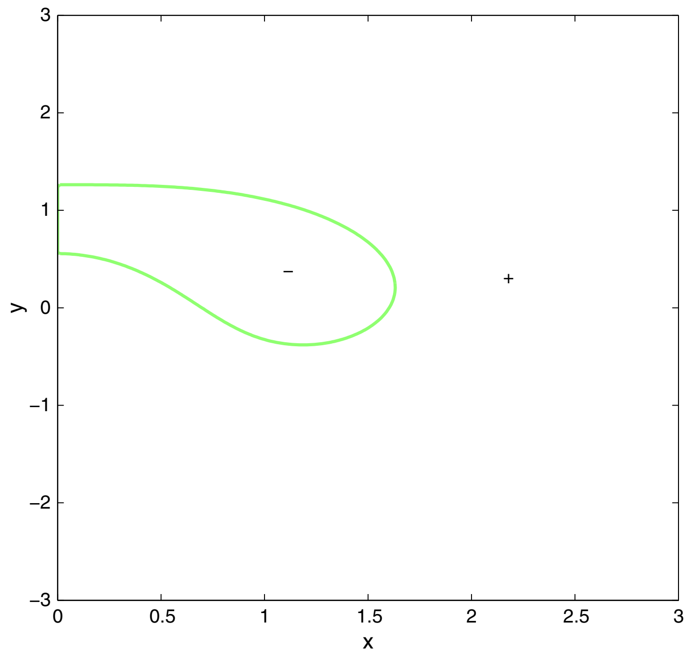

FIG. 3. The curve $a_{21} = 0$

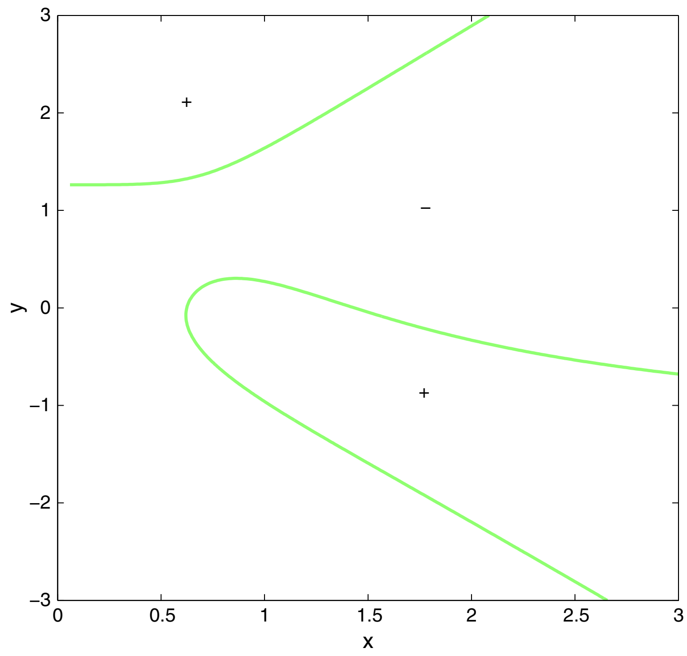

FIG. 4. The curve $a_{23} = 0$

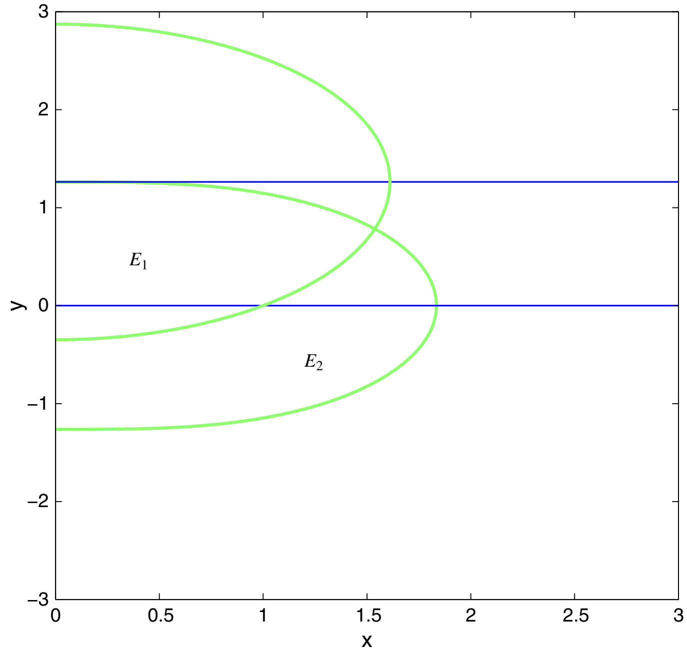

FIG. 5. The regions E_1 and E_2

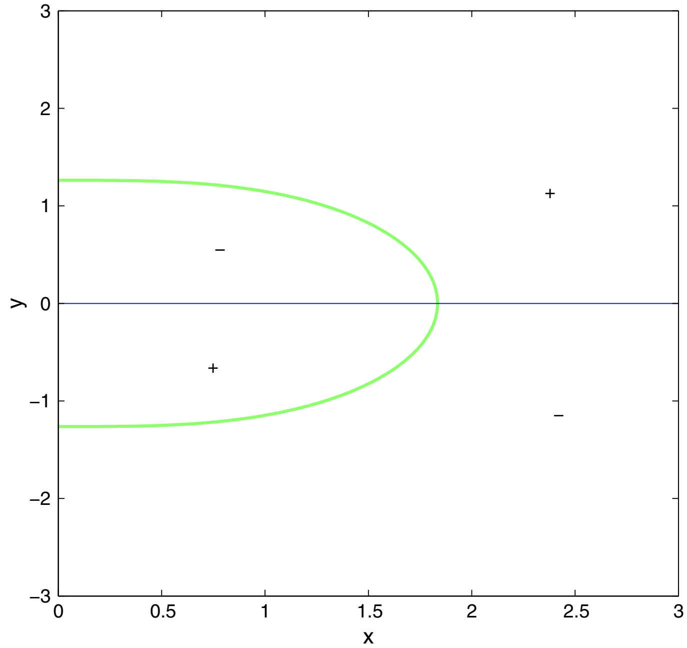

FIG. 6. The curve $a_{11} = 0$

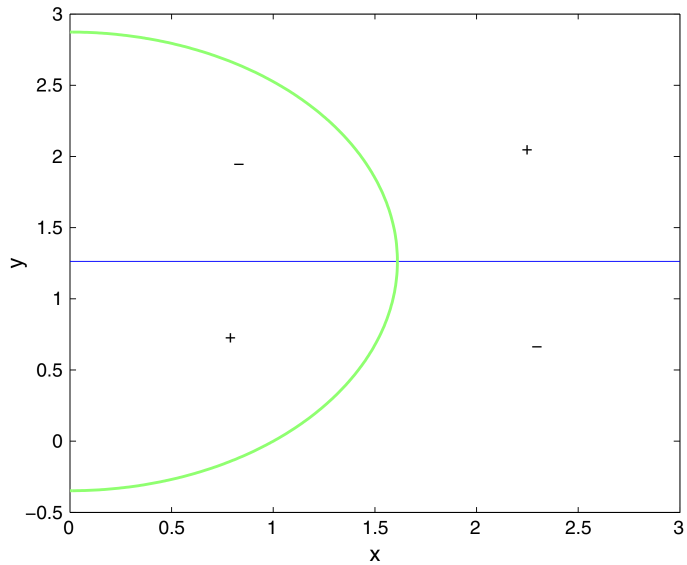

FIG. 7. The curve $a_{12} = 0$

In the rest of the proof, we show that the set $T^{-1}(0)$ has intersection with the set D_1. We consider the subset of D_1 :

$$L_1 = \{(x, y) : x = 0.3, \quad 0 < y < 0.44917949\}.$$

Obviously, L_1 is a segment with endpoints

$$P_1 = (0.3, 0), \quad P_2 = (0.3, 0.44917949).$$

(See Fig. 8). Evaluating the function T at these points, we have

$$T(P_1) = -2.99580811 < 0, \quad T(P_2) = 2.67822008 > 0.$$

Thus, there exists a point $P_0(x_0, y_0) \in L_1$, such that $T(P_0) = 0$. So, at the point P_0, we have nontrivial positive solutions of (2.12). By continuity of the entries of the matrix A, an interval I_1 containing x_0 such that for each $x \in I_1$, there exists y with $(x, y) \in T^{-1}(0) \cap D_1$, we have positive solutions of (2.12).

Secondly, we prove the existence of spatial central configurations according to Fig. 1 for some points in the set D_2 (see Fig. 2). In order to prove the existence of positive solutions of (2.12) in the set $T^{-1}(0)$, the entries a_{21}, a_{23} of the second line in the matrix A should have opposite signs. Thus, we consider the following set D_2, where D_2 is surrounded by curves $y = 0, a_{21} = 0, a_{23} = 0$, and in which $y < 0$.

In the set D_2, the entries of the matrix A have the following signs: $a_{21} < 0, a_{23} > 0$ (see Figs. 3, 4); $a_{11} > 0, a_{12} < 0$ for $D_2 \subset E_2$, where E_2 is surrounded by curves $x = 0, y = 0, a_{11} = 0, a_{12} = 0$, and in which $y < 0$ (see Figs. 5, 6, 7); $a_{13} > 0$ for $d_{17} - d_{16}$ is negative for all $x > 0$. In short, the signs of the entries of the matrix A restricted to the set D_2 are the following:

$$A = \begin{pmatrix} + & - & + \\ - & 0 & + \end{pmatrix}.$$

In the rest, the proof we show that the set $T^{-1}(0)$ has intersection with the set D_2. We consider the subset of D_2 :

$$L_2 = \{(x, y) : x = 1.3, -0.36022766 < y < 0\}.$$

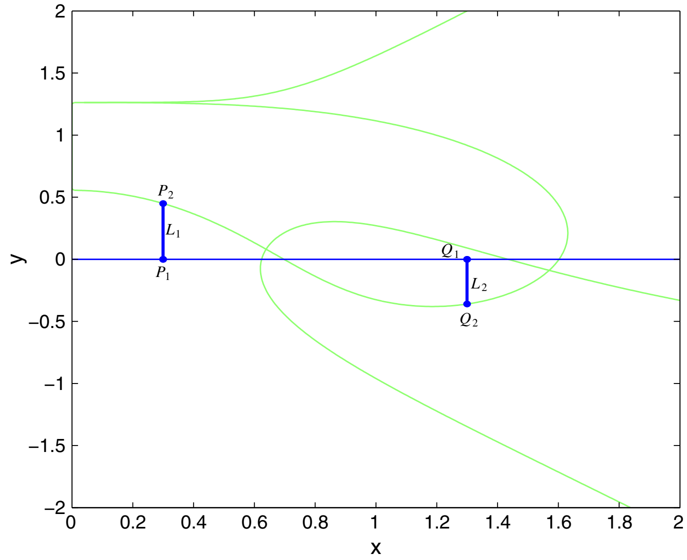

FIG. 8. Segment L_1 with endpoints P_1, P_2 and segment L_2 with endpoints Q_1, Q_2

Obviously, L_2 is a segment with endpoints

$$Q_1 = (1.3, 0), \quad Q_2 = (1.3, -0.36022766),$$

(See Fig. 8). Evaluating the function T at these points, we have

$$T(Q_1) = 0.08596624 > 0, \quad T(Q_2) = -0.78277013 < 0.$$

Thus, there exists a point $Q_0 = (x_0', y_0') \in L_2$, such that $T(Q_0) = 0$. So, at the point Q_0, we have nontrivial positive solutions of (2.12). By continuity of the entries of the matrix A, an interval I_2 containing x_0' such that for each $x \in I_2$, there exists y with $(x, y) \in T^{-1}(0) \cap D_2$, we have positive solutions of (2.12).

Thus, the proof of Theorem 1.2 is completed. □

Remarks. In order to give some information about the values of the masses and positions at $P_0(x_0, y_0) \in L_1$, we consider

$$\alpha = m_1 = m_2 = m_3 = m_4 = 1.$$

From numerical evaluations with eight decimal round-off coordinates, we have

$$x_0 = 0.3, \quad y_0 = 0.26887870.$$

Thus,

$$m_5 = 1.32808529, \quad \beta = m_6 = m_7 = m_8 = m_9 = 0.03177623.$$

In order to give some information about the values of the masses and positions at $Q_0(x_0', y_0') \in L_2$, we consider

$$\alpha = m_1 = m_2 = m_3 = m_4 = 1.$$

From numerical evaluations with eight decimal round-off coordinates, we have

$$x_0' = 1.3, \quad y_0' = -0.02823851.$$

Thus,

$$m_5 = 1.88854183, \quad \beta = m_6 = m_7 = m_8 = m_9 = 2.24996397.$$

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