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A Mathematical analysis of fluid motion in irreversible phase transitions

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Abstract. This article addresses the mathematical analysis of a model for the irreversible solidification process of certain materials by taking in consideration the effects of natural convection in molten regions. Such a model consists of a highly nonlinear system of partial differential equations coupled to a doubly nonlinear differential inclusion. The existence of weak–strong solutions for the system is proved, and certain mathematical effects of advection on the regularity of the solutions are discussed.

Mathematics Subject Classification. 35K55 · 35K59 · 35B65 · 35R35 · 35Q35 · 80A22.

Keywords. Irreversible phase transitions · Singular Stokes equations · Convection · Existence of solutions.

1. Introduction

The purpose of the present article is to investigate the existence of solutions to the following system

$$u_t - \Delta u + \nabla P + K(h(\omega))(u + \rho u_t) = \zeta\theta \text{ in } Q_{ml}, \tag{1}$$

$$\theta_t + \omega_t - \Delta\theta - \Delta_p\theta + u \cdot \nabla\theta = g(x,t) \text{ in } Q, \tag{2}$$

$$\omega_t + \alpha(\omega_t) - \Delta\omega - \Delta_q\omega + \kappa u \cdot \nabla\omega \ni \theta + f(\omega) \text{ in } Q, \tag{3}$$

$$\theta = \frac{\partial\omega}{\partial\nu} = 0, u = 0 \text{ on } \partial\Omega \times (0,T), \tag{4}$$

$$\nabla \cdot u = 0 \text{ in } Q_{ml}, \tag{5}$$

$$u + \rho u_t = 0 \text{ in } Q_s, \tag{6}$$

$$\theta(.,0) = \theta_0,\ \omega(.,0) = \omega_0,\ u(.,0) = u_0 \text{ in } \Omega, \tag{7}$$

where $0 < T < +\infty; \Omega \subset \mathbb{R}^N$, for $N = 2, 3$ or 4, is an open-bounded domain with a C^2-boundary $\partial\Omega, Q = \Omega \times (0,T)$, and ν represents the outer unit normal vector to the boundary $\partial\Omega$.

This is a model for the process of irreversible solidification that occurs to certain materials such as glue, organic matter or some types of polymers; in fact, once such materials change from liquid to solid, they cannot reverse the process. Although assuming slow flow of molten material, the present model also takes in consideration the effects of natural convection, by including suitable advection terms in the energy and phase equations.

A brief description of the model and the related literature will be given in Sect. 2. Here, we just observe that in this context, Eq. (1) is a modified Stokes system; the unknown u denotes the macroscopic velocity of the material, and P is the related hydrostatic pressure; the convection term is not present because we are assuming slow flow. Equation (2) is an energy balance equation; the unknown θ is related to the temperature; an advection term for the temperature is included. Equation (3) governs the material phases; ω is the so-called phase-field variable and indicates the physical phase of the material; an advection term for the phase may also be included (depending on the value of $\kappa \geq 0$, which is a given constant).

In Eq. (1), the term $\zeta\theta$ is the buoyancy force due to thermal differences given by the Boussinesq's approximation.

To explain the role of the last term in the left-hand side of Eq. (1), we first remark that the scalar function $h(.)$ is a given smooth function depending on the material being considered and relating the solid fraction and the phase-field variable: $h(\omega(x,t))$ gives the solid fraction at (x,t). In this way, it is natural to assume that h is a smooth real increasing function such that $h(z)=0$ when $z\leq 0$ and $h(z)=1$ when $z\geq 1$.

Hence, it is also natural to consider the following important a priori unknown space-time phase regions:

$$Q_s=\{(x,t)\in Q : h(\omega(x,t))=1\},\ \text{the solid region,}$$

and

$$Q_{ml}=\{(x,t)\in Q : 0\leq h(\omega(x,t))<1\},\ \text{the non-solid region.}$$

We stress that since the first equation (1) only holds in Q_{ml}, and this region depends on ω, system (1)–(7) is indeed a free-boundary problem.

The Carman–Kozeny type term $K(h(\omega))(u+\rho u_t)$, with ρ a given positive constant, brings a singularity to the velocity equation in the transition layers from non-solid to solid regions, since it is required that

$$\lim_{s\to 1^-} K(s)=+\infty.$$

This term describes the friction forces acting on the flow through the non-solid region and acts as a penalization in mushy regions.

The operators Δ_p and Δ_q are the p and q-Laplacian, respectively,

$$\Delta_p\theta=\operatorname{div}(|\nabla\theta|^{p-2}\nabla\theta)\ \text{ and }\ \Delta_q\omega=\operatorname{div}(|\nabla\omega|^{q-2}\nabla\omega),\ p,q>2.$$

In Eq. (3), $\alpha\subset\mathbb{R}^2$ is a maximal monotone operator with $\alpha(0)\ni 0$. A restriction on the domain of α, denoted $D(\alpha)$, furnishes the irreversibility of the phase transitions; in fact, by imposing that $D(\alpha)=[0,+\infty)$, it follows that ω_t is nonnegative for any possible solution, which guarantees that the transition occurs only on one direction (from liquid to solid, for instance).

Main contributions, mathematical difficulties and comparisons

The contributions of the present work are twofold.

First, regarding modeling aspects of the phenomenon, we observe that we are assuming slow flow of molten material, and thus, the convective inertial forces are small compared with viscous forces, which leads to a Stokes type Eq. (1). However, the transport effects of such flow on the energy and the phase are taken in consideration by the inclusion of suitable advection terms in both the temperature equation and the phase-field inclusion. Therefore, the present model is more realistic and extends a previously studied one in [9], where a two-dimensional version of (1)–(7) is considered for the case when advection in the phase-field equation is neglected. Moreover, regarding the spatial dimension, we are able to analyze the 2D, 3D and 4D cases. Further, we consider that flow through mushy regions is governed by a modified Darcy's law, which also takes into the account the effects of the acceleration of the fluid (see [34, p. 275] or [35, Sect. 4]). As we will explain in Sect. 2, the present Carman–Kozeny multiplier, namely $u+\rho u_t$, acts as a relaxation factor, allowing a smooth decay of the velocity once a portion of the material becomes solid, instead of forcing it to become immediately zero as in the case of the usual Carman–Kozeny term, which corresponds to the case $\rho=0$. The main mathematical advantage in the present case, corresponding to $\rho>0$, is that now we will be able to prove that u is more regular in time.

We remark that there are some works presenting mathematical analyses of solidification problems that take into account the effects of fluid motion. Without the intention of being complete, we cite [6,11,31], where reversible solidification with convection for alloys is investigated. Let us stress that the main reason to consider convection is that fluid motion interferes in the formation of the so-called dendrites, a microscopic phenomenon regarding the internal geometry of the material. The understanding of such phenomenon is a key issue to estimate a possible final pattern for the material. For further details, we also refer to [4] or [8], where the effects of natural convection are investigated by means of numerical simulations. We notice that the inclusion of convection in the model brings several new technical difficulties to an already hard problem.

Second, from the mathematical point of view, the analysis of (1)–(7) is more delicate than its previous versions in [6,9,11,31], mostly due that Eq. (3) includes a non-monotone perturbation to a doubly nonlinear differential inclusion. Actually, the techniques used to handle $\alpha(\omega_t)$ and its interactions with $-\Delta_q\omega$ might be undermined by the low regularity of the convective term $u \cdot \nabla\omega$. Further, one issue is that we cannot include extra hypotheses to the operator α (as extra coerciveness within certain Banach spaces to apply certain abstract theories, see [2]), since this may cause a loss of its physical role/interpretation. To overcome these difficulties, we introduce a different approximation from the usual one used to treat doubly nonlinear differential inclusions like (3). Loosely speaking, it is a consequence of the fact that we have to deal with the time derivative of (3) in order to control an appropriate norm for the approximations of $\alpha(.)$, namely $\alpha_\tau(.)$, where $\tau > 0$ is a regularization parameter. Usually, this norm is chosen to be the L^2-norm, but due to the singular character of (1), there only holds $(u \cdot \nabla\omega)_t \in L^s$, where $1 < s < 2$. Thus, we have to handle the low regularity of $u \cdot \nabla\omega$ by identifying the graph $\alpha(.) \subset \mathbb{R}^2$ as a maximal monotone operator in $L^{s'} \times L^s$, for $1 < s < 2$, instead of $L^2 \times L^2$. More precisely, it is usual to consider a linear perturbation of the inverse of α, namely, $\alpha_\tau(x) = (\gamma_\tau(x) + \tau x)^{-1}$ (for example, see [9,10,14]), where γ_τ is the Yosida's regularization for $\gamma = \alpha^{-1}$. Instead, we set $\alpha_\tau(x) = (\gamma_\tau(x) + \tau|x|^{s-2}x)^{-1}$, a nonlinear monotone perturbation of the inverse. By this choice, we include enough coercivity on the approximations, what allow us to pass to the limit in the approximate versions of (3) (see the proof of Theorem 3.2). To the best of our knowledge, this sort of approximation has not been considered in the literature and might be useful in other situations. Therefore, this approach provides a new tool that may be applied to other problems involving highly nonlinear terms.

Additionally, we investigate the effects of $u\cdot\nabla\omega$ regarding the regularity for solutions of (3). Indeed, by using tools of fractional regularity theory for p-Laplacian like equations (see [10,19]), we prove that there exists relevant gain of regularity for the solutions when $\kappa = 0$, which we briefly call "the non-advective case." This indicates a measure for the loss of regularity due to the presence of an advective term in the doubly nonlinear inclusion (3) and partially shows how the regularity of solutions might be improved in absence of non-monotone perturbations.

Plan of the paper

In Sect. 2, we give a brief description of the model. In Sect. 3, we introduce the basic notations and state our main results. A discrete version of the model is discussed in Sect. 4, whereas the associated time-dependent approximations and the derivation of their *a priori* estimates are done in Sect. 5. Section 6 is then devoted to the application of the previous estimates to obtain convergences of approximate solutions that are fundamental to our purposes. Section 7 is reserved to the proof of existence of solutions of system (1)–(7) (Theorem 3.2), and, finally, in Sect. 8, we explain how to improve the regularity of solutions in the case where there is no convection in the phase-field inclusion.

2. Brief description of the model

As we have remarked in the Introduction, system (1)–(7) may be seen as a model for solidification when the phase transition is irreversible and the flow in molten region is slow.

The interpretation of the system as a model for irreversibility is a consequence of the combination between a proper description for this sort of solidification and a rather standard approach to model the flow in the melt, which has been used both in the applied (computational) problems and in mathematical analyses.

Concerning the flow in the melt, following the main ideas of Ahmad [1], Beckermann et al. [4,5], Blanc et al. [6], Caginalp et al. [16], Voller et al. [34,35], among others, we coupled a diffuse interface model for phase transitions with a singular system for the fluid motion. In this approach, the flow is modeled as if the medium in the transition (mushy) layers was a type of porous media, with porosity related to the phase-field variable and decreasing to zero in solid regions. This brings up a singular equation for fluid motion, which can be a singular Navier–Stokes equation ([9,12,31]) or some modification of it, for instance, in which the effective fluid viscosity depends on the gradient of the velocity ([11]). In the present case, since we are considering slow flows, the approach leads to a singular Stokes equation.

This sort of fluid equations is referred as singular due to the inclusion of a Carman–Kozeny type term, which is related to friction forces due to flow through the non-solid region. It acts as a penalization due to a singularity at $s = 1$: $\lim_{s\to 1^-} K(s) = +\infty$. It is important to remark that, with the usual form of the Carman–Kozeny term, which corresponds to $\rho = 0$ and is reduced to $K(h(\omega))u$, it forces the velocity to satisfy $u = 0$ in the solid region Q_s; that is, any portion of the material that becomes solid must immediately come to full stop. This brings very serious mathematical difficulties in proving estimates for the time derivative of the velocity, u_t, in any region including parts of the transition layer from the non-solid region, Q_{ml}, to the solid one, Q_s; (see either [9] or [11], where local estimates in regions away from these transition layers are obtained). Differently, the model considered in the present work assumes that the flow through mushy regions is governed by a modified Darcy's law, which also takes into account effects of the acceleration of the fluid (see [34, p. 275] or [35, Sect. 4]). In this case, the present Carman–Kozeny multiplier, namely $u + \rho u_t$, with $\rho > 0$, forces the velocity to satisfy $u + \rho u_t = 0$ in Q_s, implying the exponential decay in time for the velocity after a portion of the material becomes solid instead of requiring it immediate full stop as in the case of the usual Carman–Koseny term ($\rho = 0$). As we will see, the present form of this term will allow us to obtain suitable estimates for u_t.

The present approach to the irreversibility is similar to the ones in Blanchard et al. [7], Frémond et. al [22], Bonfanti et al. [14], Bonetti [13], Colli et al. [18], Laurençot et al. [29], Luterotti et al. [30]. They deal with inclusions related to:

$$\omega_t + \alpha(\omega_t) + \beta(\omega) - \Delta\omega \ni \theta + f(\omega) \text{ in } Q,$$

where α and β are maximal monotone operators. A restriction on the domain of α then furnishes the irreversibility of the phase transition. Indeed, by imposing that the domain $D(\alpha) = [0, +\infty)$, the term ω_t is forced to be nonnegative, which guarantees that the transition occurs only on one direction (from liquid to solid, for instance). In Eq. (3), we have the same sort of maximal operator α.

Another aspect to be observed in the previously mentioned articles is that in most of them the phase-field variable ω has a direct physical meaning; it is exactly the solid fraction distribution in the material, which naturally requires that $0 \le \omega \le 1$. This condition is attained by imposing that the domain $D(\beta) = [0, 1]$, guaranteeing the proper physics requirement and also automatically a suitable L^∞-estimate for ω in space and time. By contrasting the last equation with (3), we observe that, besides having now the advective term $\kappa u \cdot \nabla\omega$ due to the flow transport, we have a term $-\Delta_q\omega$ in (3) instead of β. We do not have such β because our ω has not a direct physical meaning; in fact, as we previously explained, the solid fraction in our case is given by $h(\omega)$, and thus, we cannot require an estimate as $0 \le \omega \le 1$. However, as we will see, once q is chosen large enough, the term $-\Delta_q\omega$ will act at least in part as $\beta(\cdot)$, furnishing a L^∞-estimate for ω just in the space variable; moreover, such term in (3) will give information regarding $\nabla\omega$ that will be employed to control the advective terms. Of course, from the mathematical point of view, there is also a price to be paid: q-Laplacian like equations provide poor information regarding higher order regularity of their solutions.

As for the derivation of (3), we use the following standard approach: consider a free-energy functional depending on

$$\mathcal{E}(t) = \int_{\Omega} E(\omega(x,t), \nabla\omega(x,t), \theta(x,t))\mathrm{d}x$$

with the volumetric free-energy density $E = E(\omega, p_1, \ldots, p_N, \theta)$ where $p_i = \partial_{x_i}\omega$ for $i = 1, \ldots, N$. Then, the corresponding Allen–Cahn equation for the evolution of the phase-field ω is:

$$\omega_t + \nabla\cdot(\omega u) + \lambda\frac{\delta E}{\delta\omega} + \mathcal{F} = 0, \tag{8}$$

where λ is a (small) positive parameter, $\mathcal{F}$ is a source/drain term adequate to each specific situation and $\frac{\delta E}{\delta\omega}$ is the variational derivative of E, which is given by

$$\frac{\delta E}{\delta\omega} = \partial_\omega E - \sum_{i=1}^{N}\partial_{x_i}\partial_{p_i}E.$$

In the absence of the source/drain term, this equation guarantees that the associated free-energy $\mathcal{E}(t)$ decays along the time.

Next, we take the free-energy density of form

$$E(\omega, \nabla\omega, \theta) = W(\omega) - a\theta\omega + \frac{\nu}{2}|\nabla\omega|^2 + \frac{\nu_1}{q}|\nabla\omega|^q,$$

(this means that to the classical (total) free-energy, which has already the L^2-norm of $\nabla\omega$, its L^q-norm is also included) with a, ν and ν_1 positive constants, $W(\omega)$ a bulk functional suitable for the material being considered (for instance, a double-well potential) and $\mathcal{F} = \alpha(\omega_t)$, with $\alpha(\cdot)$ a maximal monotone operator with domain $D(\alpha) = [0, +\infty)$ as before (for instance, the extension of the subdifferential of the indicator function of the interval $[0, +\infty)$, $I_{[0,+\infty)}$). By taking for simplicity of exposition all the constants to be 1, recalling that we are assuming incompressible flows, and performing the required computations, from (8) we get exactly (3) with $f(\omega) = -W'(\omega)$ in the case $\kappa = 1$ (the inclusion appears since $\alpha(\cdot)$ may be a multivalued operator); the case $\kappa = 0$ is an approximation for the case that the time scale of the solidification process is faster than the time scale of the flow.

As for (2), it is obtained by approximations; our intention here is to clarify exactly which approximations were considered. We start with the simplified form of the internal energy balance:

$$e_t + \nabla\cdot(eu) + \nabla\cdot\mathbf{q} = g,$$

where the internal terms for heat production were neglected. Next, the internal energy is approximated by $e = C\theta + \ell\omega$, with C a positive constant related to the volumetric specific heat of the material and ℓ a positive constant related to the latent heat. As for the heat flux, we assume that is of form $\mathbf{q} = (k_1 + k_2|\nabla\theta|^{p-2})\nabla\theta$, with k_1 and k_2 positive constants; that is, we assume the heat diffusion coefficient is given by $k_1 + k_2|\nabla\theta|^{p-2}$ and thus increases as the temperature gradient increases. This sort of situation, although in a different physical context, was considered earlier; see for instance, Dimova and Meyer-Spasche [21]. Again for simplicity of exposition, by taking all the constants to be 1, recalling that we are assuming incompressible flows, and performing the required computations, we get Eq. (2).

Before finishing this section, let us mention that it is a delicate issue to decide how to take into consideration, both in physically sound and in mathematically feasible manner, the effects of fluid motion in an irreversible solidification process. In the present work, by analogy with [4,6,9,11,12,34,35], those effects are considered by coupling a modified Stokes system, (1), with a system which describes the phase transition, (2)–(3). In addition, its mathematical analysis is even more delicate, since we have to handle the interaction between a singular free-boundary problem with a doubly nonlinear differential inclusion. Since these two opposing situations bring several major mathematical difficulties to the analysis, it is common to neglect the effects of the material flow while investigating irreversible solidification; for instance, in [7,13,14,18,29,30] the macroscopical velocity of the material is supposed to be zero and thus the effects of fluid motion were not taken into account.

3. Notations and main results

First, we fix some basic notation. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega$ of class C^2. For $1 \leq p \leq \infty$ and $0 \leq s \leq \infty$, we denote by $W^{s,p}(\Omega)$ the standard Sobolev-Slobodeckii spaces. The so-called Nikolskii spaces are denoted by $\mathcal{N}^{\sigma,r}$, where $\sigma = 1 + \delta,\ 0 < \delta < 1,$ and $r \geq 2$. The reader is referred to [25, Chap. 1] for further details concerning Sobolev-Slobodeckii spaces and to [28, Sect. 8.2] for Nikolskii spaces.

In order to handle fluid equations, the following functional spaces of divergence-free vector fields are introduced:

$$\begin{aligned}\mathcal{V}(\Omega) &= \{\phi \in (C_0^\infty(\Omega))^N : \nabla \cdot \phi = 0 \text{ in } \Omega\},\\ H_r &= \overline{\mathcal{V}(\Omega)}^{(L^r)^N}, r > 1,\\ V &= \overline{\mathcal{V}(\Omega)}^{(W^{1,2})^N}.\end{aligned}$$

As usual, H_2 is denoted just by H.

For the sake of simplicity, we employ the same notations for scalars, vectors in $\mathbb{R}^N$ and matrices in $\mathbb{R}^{N^2}$. For instance, we denote by "." both the inner product in $\mathbb{R}^N$ and the tensor product in $\mathbb{R}^{N^2}$; the norms on $(W^{s,p}(\Omega))^N$ and $W^{s,p}(\Omega)$ are indicated by $\|.\|_{W^{s,p}}$; we write $u \in L^2(\Omega)$, even when u is a vector field, meaning that all of its components are in $L^2(\Omega)$, and so on. Naturally, the difference will be clear from the context. Also to simplify the notations, we omit the dependences of the spaces on Ω.

Next, we state our main hypotheses:

(H1) $\Omega \subset \mathbb{R}^N, N = 2, 3$ or 4, is an bounded domain with $\partial\Omega$ of class C^2;
(H2) $\alpha \subset \mathbb{R}^2$ is a maximal monotone operator such that $\alpha(0) \ni 0$; its domain is given by $D(\alpha) = [0, +\infty)$;
(H3) $f : \mathbb{R} \to \mathbb{R}, f$ is a Lipschitz continuous function with $f(0) = 0$;
(H4) $g : Q \to \mathbb{R},\ g$ belongs to $L^2(0, T; L^2)$;
(H5) $K : [0, 1) \to \mathbb{R},\ K \geq 0,\ K(0) = 0,\ K \in C^1([0, 1)),\ K'(x) \geq 0$ and $\lim_{x\to 1^-} K(x) = +\infty$;
(H6) $h : \mathbb{R} \to \mathbb{R}$ is a $C^1(\mathbb{R})$-increasing function such that $h(z) = 0$ when $z \leq 0$ and $h(z) = 1$ when $z \geq 1$ (and thus $0 \leq h(z) \leq 1,\ \forall z \in \mathbb{R}$);
(H7) $\rho > 0$ and $\kappa \geq 0$. In order to simplify the notation, without loosing any mathematical generality, from now on we fix $\rho = 1$;
(H8) Let

$$q > N \quad \text{and} \quad p > \max\left\{4, \frac{2q}{q-2}\right\};$$

we take the initial data such that

$$u_0 \in V,\ \theta_0 \in W_0^{1,p} \text{ and } \omega_0 \in W^{2,q}.$$

We remark that the previous conditions on p and q are justified by the coupling between Eqs. (2) and (3), and by phase-field/velocity or temperature/velocity interactions. Indeed, they guarantee the validity of certain key algebraic inequalities (see Lemma 4.1).

(**H9**) Consider the auxiliary maximal monotone graph defined by $\gamma = \alpha^{-1} \subset \mathbb{R}^2$, and a proper convex lower semicontinuous function $\Gamma : \mathbb{R} \to [0, +\infty]$, with $\Gamma(0) = 0$, such that $\partial\Gamma = \gamma$.
Assume, moreover, the following technical hypothesis on the initial data:

$$\Gamma(\theta_0 + f(\omega_0) + \Delta\omega_0 + \Delta_q\omega_0 - \kappa u_0 \cdot \nabla\omega_0) \in L^1.$$

The previous operators γ and Γ will play a important role in the derivation of certain key energy estimates (see Lemma 5.3).

The reader is refereed to [3,15,32] for detailed information on maximal monotone operators, subdifferentials and related topics.

Next, we define the concept of solutions we will consider in this work.

Definition 3.1. *A quadruple* $(u, \theta, \omega, \eta)$ *is a solution of* (1)–(7) *when*

- $u \in C([0,T]; H) \cap L^\infty(0,T; V) \cap W^{1,2}(0,T; H)$,
- $\theta \in C([0,T]; L^p) \cap L^\infty(0,T; W_0^{1,p}) \cap W^{1,2}(0,T; L^2)$,
- $\omega \in C([0,T]; C(\overline{\Omega})) \cap L^\infty(0,T; W^{1,q}) \cap W^{1,2}(0,T; W^{1,2}) \cap W^{1,\infty}(0,T; L^2)$,
- $\eta \in L^2(0,T; L^s)$, *for some* $1 < s \le 2$,

and they satisfy

$$\int_0^T \int_\Omega u_t \cdot \phi + \nabla u \cdot \nabla \phi + K(h(\omega))(u + u_t) \cdot \phi = \int_0^T \int_\Omega \zeta \theta \cdot \phi, \tag{9}$$

$$\theta_t + \omega_t - \Delta\theta - \Delta_p \theta + u \cdot \nabla\theta = g \ \textit{a.e. in } Q, \tag{10}$$

$$\eta + \omega_t - \Delta\omega - \Delta_q \omega + \kappa u \cdot \nabla\omega = \theta + f(\omega) \ \textit{a.e. in } Q, \tag{11}$$

$\forall \phi \in L^2(0,T; V)$ *with compact support in* $Q_{ml} = \{(x,t) \in Q : 0 \le h(\omega(x,t)) < 1\}$, *and also*

$$\eta \in \alpha(\omega_t) \subset L^2(0,T; L^s), \tag{12}$$

$$u + u_t = 0 \ \textit{a.e. in } Q_s = \{(x,t) \in Q : h(\omega(x,t)) = 1\}, \tag{13}$$

$$u(.,0) = u_0, \ \theta(.,0) = \theta_0, \ \omega(.,0) = \omega_0 \ \textit{a.e in } \ \Omega. \tag{14}$$

Remark: Observe that this type of solution is more regular in time than the ones considered in corresponding models with the usual Carman–Kozeny approximation (see the discussion in Sect. 2 and [9, Thm. 2.1], for instance). This is due to the present multiplier term attached to the singularity K, which allows us to obtain a suitable global estimate for u_t and not just local ones from the transition layers between non-solid and solid regions. On the other hand, notice that in the phase-field equation, we now deal with $L^2(0,T; L^s)$, for some $1 < s \le 2$, with s related to the non-monotone perturbation $\kappa u \cdot \nabla\omega$. Indeed, when $\kappa > 0$, we have $1 < s < 2$, since $(\kappa u \cdot \nabla\omega)_t$ belongs to $L^2(0,T; L^s)$, where $1 < s < 2$ when $N = 2$, $\frac{6}{5} \le s \le \min\{\frac{3}{2}, \frac{2q}{q+2}\}$ when $N = 3$, and $s = \frac{4}{3}$ when $N = 4$. In the case that $\kappa = 0$, we recover the case $s = 2$.

The main results of this paper concern the existence and regularity of solutions to system (1)–(7).

Theorem 3.2. *Suppose that*

$$1 < s < 2 \ \textit{if } N = 2, \ \textit{and } \frac{2N}{N+2} \le s \le \min\left\{\frac{N}{N-1}, \frac{2q}{q+2}\right\} \ \textit{if } N = 3, \ 4. \tag{15}$$

Under hypotheses **(H1)**–**(H9)**, *there exists a solution to* (1)–(7) *given by the quadruple* $(u, \theta, \omega, \eta)$.

It turns out that under an extra hypothesis, the solutions are indeed more regular. In fact, we expose a measure for the effect of convection in the phase-field inclusion on the solutions of (1)–(7).

Theorem 3.3. *Under hypotheses* **(H1)**–**(H9)**, *suppose additionally that* $\kappa = 0$ *and* $\partial\Omega \in C^3$. *Then there exists a solution to* (1)–(7) *satisfying*

$$\eta \in \alpha(\omega_t) \subset L^2(0,T; L^2) \tag{16}$$

and

$$\omega \in L^\infty(0,T; W^{2,2}) \cap L^\infty(0,T; \mathcal{N}^{1+2/q;q}). \tag{17}$$

Notice that besides improving the space L^s to L^2, we also obtain extra fractional regularity for ω due to the following continuous embedding of Niikolski spaces into Sobolev-Slobodeckii spaces (see, for instance, [26, Lem. 2.1]):

$$\mathcal{N}^{1+2/q,q} \hookrightarrow W^{1,q}.$$

This is not a surprise since it is well known that convection terms with low regularity coefficients have a weakening character regarding regularity for the solutions of general nonlinear partial differential equations.

Some technical remarks:

(i) As it will be clear in the next sections, we are led to work with the space $L^{s'} \times L^s$, with $1 < s < 2$, as the functional framework for the multivalued operator $\alpha(\cdot)$ due to the way we approximate it by using suitable regularizations α_τ. In fact, the usual alternative Hilbertian framework $W^{1,2} \times (W^{1,2})'$ cannot be used due to the presence of q-Laplacian term ($q \neq 2$ in our case). Another alternative would be the functional framework given by $W^{1,q} \times (W^{1,q})'$, $q > N$, together with a modified Yosida approximations for $\alpha(\cdot)$. However, in this setting, we still had to identify the corresponding limit terms as belonging to the multivalued operator $\alpha(\cdot)$, and for this, a strong estimate for the approximated solutions (namely, $\overline{\omega}'_\tau$ in $W^{1,q}$, cf. Lemma 5.2) would be required; but the nonlinearities of the problem prevent us to obtain such estimate. Anyway, even if a weaker functional framework approach could be used for the present problem, we emphasize that our approach provides better regularity for the elements of multivalued operator since our regularization guarantees that η belongs to a Lebesgue space.

(ii) The existence of local in time solutions in the case of homogeneous Neumann boundary conditions for the temperature can be handled with minor adaptations of the arguments presented in the following sections (see Remark 5.2). However, more serious technical difficulties appear when one tries to prove a corresponding global in time existence of solutions in this case; in fact, Poincaré inequality was strongly used in the arguments for global existence for homogeneous Dirichlet boundary conditions; since we were not able to circumvent its use in the Neumann case, we do not know whether the existing local solutions are global. As for non-homogeneous boundary conditions, technical difficulties appear even in the Dirichlet case since the standard change of variables usually used to reduce the problem to the homogeneous case cannot be easily applied to (2) due to its nonlinear pattern.

(iii) We do not know whether uniqueness of solutions hold for the present problem. The main difficulty in proving such a result appears when one tries to compare the flow velocities associated with two possible solutions with same initial conditions; since we have a free-boundary value problem, the corresponding fluid equations hold in different subsets of Q, and we cannot proceed as usual to get a equation for the difference of the velocities.

4. Preliminary results and an auxiliary problem

We begin with a technical lemma which will be used repeatedly along the text. As its proof follows by straightforward computations, we omit it.

Lemma 4.1. *Let $q > N$ and $p > \max\left\{4, \dfrac{2q}{q-2}\right\}$; then there exists r such that*

$$\max\left\{\frac{2p}{p-2}, \frac{2q}{q-2}\right\} \le r < \min\left\{p, \frac{2N}{N-2}\right\}. \tag{18}$$

(in particular, $r > 2$, $\dfrac{1}{2} + \dfrac{1}{p} + \dfrac{1}{r} \le 1$ and $\dfrac{1}{2} + \dfrac{1}{q} + \dfrac{1}{r} \le 1$) and

$$V \hookrightarrow\hookrightarrow H_r.$$

We stress that the compactness result of Lemma 4.1 will be only used in the proof of existence of solutions for the discretized problem. The restriction on r allows to apply the Hölder's inequality when estimating convective terms. Moreover, it comes from the interplay between the temperature and the velocity field (see proof of Lemma 5.1).

It is worth to notice that the convective term vanishes

$$\int_\Omega u \cdot \nabla\psi\psi = 0, \text{ when } \psi \in W^{1,2} \text{ and } u \in V, \tag{19}$$

which, in particular, implies that

$$\int_\Omega u \cdot \nabla\psi\phi = -\int_\Omega u \cdot \nabla\phi\psi, \text{ when } \psi,\ \phi \in W^{1,2} \text{ and } u \in V. \tag{20}$$

We now recall two algebraic inequalities that will be used frequently along the paper.

The first one is a key tool for proving the monotonicity of the p-Laplacian. Let $p \geq 2$. For all $a,\ b \in \mathbb{R}^N,\ N \in \mathbb{N}$, it holds

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b) \geq C|a-b|^p, \tag{21}$$

where C depends only upon p (see, [20, Lem. 4.4]). The other one is valid for any $k > 1$ and $a,\ b \in \mathbb{R}^N,\ N \in \mathbb{N}$,

$$|a|^{k-2}a \cdot (a-b) \geq \frac{|a|^k}{k} - \frac{|b|^k}{k}, \tag{22}$$

and follows from the Young's inequality.

4.1. An auxiliary problem: time discretization

Due to the nonlinear term regarding the time derivative of the phase-field unknown, namely $\alpha(\omega_t)$, it is natural to introduce a time-discrete version of (1)–(7).

Let $0 < \tau < 1$ and $M \in \mathbb{N}$ be such that

$$M\tau = T,$$

which by definition are the discretization parameters; the auxiliary discretized in time problem is based on the following:

Iterative scheme:

Given

$$u_{i-1} \in V, \quad \theta_{i-1} \in W_0^{1,p} \text{ and } \omega_{i-1} \in W^{1,q},$$

find

$$u_i \in V, \quad \theta_i \in W_0^{1,p} \text{ and } \omega_i \in W^{1,q}$$

such that

$$\frac{u_i - u_{i-1}}{\tau} - \Delta u_i + K_\tau(\omega_i)\left(u_i + \frac{u_i - u_{i-1}}{\tau}\right) = \zeta\theta_i \quad \text{in } V', \tag{23}$$

$$\frac{\theta_i - \theta_{i-1}}{\tau} + \frac{\omega_i - \omega_{i-1}}{\tau} - \Delta\theta_i - \Delta_p\theta_i + u_i \cdot \nabla\theta_i = g_i \quad \text{in } W^{-1,p'}, \tag{24}$$

$$\begin{aligned}&\frac{\omega_i - \omega_{i-1}}{\tau} + \alpha_\tau\left(\frac{\omega_i - \omega_{i-1}}{\tau}\right) - \Delta\omega_i - \Delta_q\omega_i + \kappa u_i \cdot \nabla\omega_i \\ &\quad = \theta_i + f(\omega_i) \quad \text{in } (W^{1,q})',\end{aligned} \tag{25}$$

$$u_i = 0, \quad \theta_i = \frac{\partial\omega_i}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{26}$$

The iteration starts with the initial conditions u_0, θ_0 and ω_0 given in **(H8)**. The terms g_i are defined by

$$g_i = \frac{1}{\tau} \int_{\tau(i-1)}^{\tau i} g(x,s)\ ds,$$

while

$$K_\tau(x) = K_{ext}(h(x) - \tau),\ x \in \mathbb{R}, \tag{27}$$

where K_{ext} is the extension of K defined on $(-\infty, 1)$ by letting $K_{ext}(x) = 0$ when $x < 0$.

The definition of α_τ, a special approximation for α, is more delicate.

Definition of α_τ:

Initially, we introduce γ_τ, the standard Yosida's regularization of $\gamma = \alpha^{-1}$ (see [3,15] for further information). Moreover, notice that

$$F_\tau : \mathbb{R} \to \mathbb{R},\ \text{where } F_\tau(x) = \gamma_\tau(x) + \tau|x|^{s-2}x, \quad 1 < s \leq 2,$$

is maximal monotone, differentiable a.e. and bijective. Indeed, γ_τ and $|I|^{s-2}I$ are both maximal monotone and continuous, so that F_τ is maximal monotone. Moreover, we readily check that F_τ is one to one, and since it is coercive, then F_τ is also surjective (see Barbu [3, Cor. 2.3]). In this way, we set

$$\alpha_\tau = (\gamma_\tau + \tau|I|^{s-2}I)^{-1}, \tag{28}$$

which is also bijective and differentiable a.e.. This choice will be justified when obtaining a priori estimates to system (23)–(26) (see Sect. 5).

We remark certain important properties for α_τ:

$$\alpha_\tau(0) = 0 \text{ and } \alpha_\tau(.) \text{ is maximal monotone,}$$

due to the very definition of γ, γ_τ and **(H2)**. Consequently, there holds

$$\alpha_\tau(x)x \geq 0.$$

Additionally, with a straightforward adaptation of [3, Prop. 2.3], we prove that α_τ is locally Lipschitz continuous. More precisely, it holds

$$|\alpha_\tau(x)| \leq C \frac{|x|^{\frac{1}{s-1}}}{\tau^{\frac{1}{s-1}}} \tag{29}$$

and

$$|\alpha_\tau(x) - \alpha_\tau(y)| \leq \frac{C}{\tau^{\frac{1}{s-1}}} |x-y| \big(|x|^{\frac{2-s}{s-1}} + |y|^{\frac{2-s}{s-1}}\big). \tag{30}$$

These two properties are only used in the proof of existence of solution for the discretized problem. For the reader's convenience, we give proofs for both inequalities.

Let $x_\tau = (\gamma_\tau + \tau|I|^{s-2}I)^{-1}x$ and $y_\tau = (\gamma_\tau + \tau|I|^{s-2}I)^{-1}y$ and observe that

$$x = \gamma_\tau(x_\tau) + \tau|x_\tau|^{s-2}x_\tau.$$

Thus, as $\gamma_\tau(x_\tau)x_\tau \geq 0$, we infer (29).

In addition, notice that

$$\gamma_\tau(x_\tau) - \gamma_\tau(y_\tau) + \tau\big(|x_\tau|^{s-2}x_\tau - |y_\tau|^{s-2}y_\tau\big) = x - y$$

Then, by multiplying the previous identity by $x_\tau - y_\tau$, we obtain that

$$\frac{|x_\tau - y_\tau|^2}{\big(|x_\tau| + |y_\tau|\big)^{2-s}} \leq \frac{C}{\tau}|x-y||x_\tau - y_\tau|,$$

since (see [17, p .3])

$$\big(|x_\tau|^{s-2}x_\tau - |y_\tau|^{s-2}y_\tau\big)(x_\tau - y_\tau) \geq C\frac{|x_\tau - y_\tau|^2}{\big(|x_\tau| + |y_\tau|\big)^{2-s}}.$$

In particular, as $\alpha_\tau(x) = x_\tau$ and $\alpha_\tau(y) = y_\tau$, by combining (29) and the last estimate we obtain (30).

We also observe that, by Moreau's proposition ([32, Prop. 1.8, p. 162]), for each τ there exists a proper lower semicontinuous function Γ_τ such that

$$\begin{aligned} &\Gamma_\tau : \mathbb{R} \to (0,\infty] \text{ is Fréchet differentiable,} \\ &\partial\Gamma_\tau = \gamma_\tau \ \text{ and } \ 0 \leq \Gamma_\tau(x) \leq \Gamma(x), \ \forall x \in \mathbb{R}. \end{aligned} \tag{31}$$

Since our purpose is to let $\tau \to 0$ in order to reobtain (1)–(7), we can fix an upper bound for $\tau > 0$; so in the following, we always assume that $\tau \leq 1$.

The next result proves the existence of solutions of the previously described iterative scheme. As the proof is a straightforward application of Leray–Schauder's fixed point theorem, we only sketch it.

Proposition 4.2. *Given $u_{i-1} \in V, \theta_{i-1} \in W_0^{1,p}$ and $\omega_{i-1} \in W^{1,q}$, for $\tau > 0$ sufficiently small, there exist $u_i \in V, \theta_i \in W_0^{1,p}$ and $\omega_i \in W^{1,q}$ satisfying (23)–(26).*

Proof. To apply the Leray–Schauder's fixed point theorem, we introduce the Banach space $\mathcal{B} = H_r \times L^p \times L^{\frac{2}{s-1}}$, where r is given by Lemma 4.1 and $1 < s \leq 2$. For every $0 \leq \lambda \leq 1$, we define the operator $T_\lambda : \mathcal{B} \to \mathcal{B}$ by

$$T_\lambda(u,\theta,\omega) = (\bar{u},\bar{\theta},\bar{\omega}),$$

where $(\bar{u},\bar{\theta},\bar{\omega})$ is the unique solution of

$$\frac{\bar{u}}{\tau} - \Delta\bar{u} + \lambda\left[K_\tau(\omega)\left(u + \frac{u - u_{i-1}}{\tau}\right)\right] = \lambda\zeta\theta + \frac{u_{i-1}}{\tau} \text{ in } V', \tag{32}$$

$$\frac{\bar{\theta}}{\tau} + \lambda\frac{\omega}{\tau} - \Delta\bar{\theta} - \Delta_p\bar{\theta} + u\cdot\nabla\bar{\theta} = \lambda\left[g_i + \frac{\theta_{i-1}}{\tau} + \frac{\omega_{i-1}}{\tau}\right] \text{ in } W^{-1,p'},$$

$$\frac{\bar{\omega}}{\tau} + \lambda\alpha_\tau\left(\frac{\omega - \omega_{i-1}}{\tau}\right) - \Delta\bar{\omega} - \Delta_q\bar{\omega} + \kappa u\cdot\nabla\bar{\omega} \tag{33}$$

$$= \lambda\left[\theta + f(\omega) + \frac{\omega_{i-1}}{\tau}\right] \text{ in } (W^{1,q})', \tag{34}$$

$$\bar{u} = 0, \ \bar{\theta} = \frac{\partial\bar{\omega}}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{35}$$

We first observe that T_λ is well defined for every $0 \leq \lambda \leq 1$. Indeed, clearly

- $\lambda\left[-K_\tau(\omega)\left(u + \dfrac{u - u_{i-1}}{\tau}\right) + \zeta\theta\right] + \dfrac{u_{i-1}}{\tau} \in L^2;$
- $\lambda\left[\dfrac{\omega_{i-1}}{\tau} - \dfrac{\omega}{\tau} + g_i + \dfrac{\theta_{i-1}}{\tau}\right] = \bar{g} \in L^2;$
- $\lambda\left[-\alpha_\tau\left(\dfrac{\omega - \omega_{i-1}}{\tau}\right) + \theta + f(\omega) + \dfrac{\omega_{i-1}}{\tau}\right] \in L^2.$

Notice that by [24, Thm. 5.1, p.80], there exists a unique $\bar{u} \in V$ satisfying (32). The existence of $\bar{\theta}$ follows by a standard iterative scheme (see [27] pp. 5–12). Indeed, given $\bar{\theta}^{n-1} \in W_0^{1,p}$, with $\bar{\theta}^0 = \theta_0$, consider $\bar{\theta}^n \in W_0^{1,p}$ such that

$$\frac{\bar{\theta}^n}{\tau} - \Delta\bar{\theta}^n - \Delta_p\bar{\theta}^n = \bar{g} - u\cdot\nabla\bar{\theta}^{n-1} \in L^2,$$

which exists by Prop.1 in [10]. Moreover, by taking $\bar{\theta}^n$ as a multiplier and by using Hölder's inequality, there follows that

$$\frac{1}{\tau}\int_\Omega |\bar{\theta}^n|^2 + \int_\Omega |\nabla\bar{\theta}^n|^p \leq C(\bar{g}, \|u\|_{H_r}^r) + \frac{1}{8}\int_\Omega |\nabla\bar{\theta}^{n-1}|^p.$$

Then, by interpolation,

$$\|\bar{\theta}^n\|_{W_0^{1,p}} \leq C(\bar{g}, u, \theta_0),$$

so that, there exists $\bar{\theta} \in W_0^{1,p}$, for which, up to subsequences, $\bar{\theta}^n \rightharpoonup \bar{\theta}$ in $W_0^{1,p}$.

In addition, the combination of Lemma 4.1 and identity (19) leads to

$$\left|\int_\Omega u\cdot\nabla\bar{\theta}^{n-1}\bar{\theta}^n - \int_\Omega u\cdot\nabla\bar{\theta}\ \bar{\theta}\right| \leq C(\bar{g}, \|u\|_{H_r}, \theta_0)\|\bar{\theta}^n - \bar{\theta}\|_{L^2}.$$

However, $-\Delta_p$ is maximal monotone in $W_0^{1,p}$, thus, by combining the so-called Minty's trick (see [3, Cor. 2.4]) and the last inequality, we see that $\bar{\theta}$ is the unique weak solution of (33).

We obtain a unique $\bar{\omega} \in W^{1,q}$ satisfying (34) in a analogous manner. Hence, remarking that $W^{1,q} \hookrightarrow L^{\frac{2}{s-1}}$, we see that T_λ is well defined for $0 \leq \lambda \leq 1$.

We check the hypotheses of Leray–Schauder's theorem in five steps.

Step 1. Let $\lambda = 0$. The existence and uniqueness of $u \in V, \theta \in W_0^{1,p}$ and $\omega \in W^{1,q}$ such that

$$T_0(u, \theta, \omega) = (u, \theta, \omega),$$

follows directly from the previous argument.

Step 2. We claim that T_λ is continuous in $\mathcal{B}$ for every $\lambda \in [0, 1]$ fixed. Suppose that $(u^n, \theta^n, \omega^n) \to (u, \theta, \omega)$ in $\mathcal{B}$. Let $(\bar{u}^n, \bar{\theta}^n, \bar{\omega}^n)$ and $(\bar{u}, \bar{\theta}, \bar{\omega})$ be such that

$$(\bar{u}^n, \bar{\theta}^n, \bar{\omega}^n) = T_\lambda(u^n, \theta^n, \omega^n) \ \text{ and } \ (\bar{u}, \bar{\theta}, \bar{\omega}) = T_\lambda(u, \theta, \omega).$$

We will take the difference between the systems associated with $(\bar{u}^n, \bar{\theta}^n, \bar{\omega}^n)$ and $(\bar{u}, \bar{\theta}, \bar{\omega})$. Indeed, observe that for u^n or u, we obtain different versions of (32), which are associated with $\bar{u}^n$ or $\bar{u}$. The difference between them provides another equation. Next, by multiplying this resulting equation by $\bar{u}^n - \bar{u}$ and integrating over Ω, one obtains

$$\begin{aligned}
&\int_\Omega \frac{1}{\tau}|\bar{u}^n - \bar{u}|^2 + |\nabla\bar{u}^n - \nabla\bar{u}|^2 \\
&\quad \leq \lambda\int_\Omega \left[K_\tau(\omega)\Big(u + \frac{u - u_{i-1}}{\tau}\Big) + \zeta(\theta_n - \theta)\right]\cdot(\bar{u}^n - \bar{u}) \\
&\quad -\lambda\int_\Omega K_\tau(\omega^n)\Big(u^n + \frac{u^n - u_{i-1}}{\tau}\Big)\cdot(\bar{u}^n - \bar{u}).
\end{aligned} \tag{36}$$

But observe that,

$$\begin{aligned}
&\int_\Omega \left[K_\tau(\omega)\Big(u + \frac{u - u_{i-1}}{\tau}\Big) - K_\tau(\omega^n)\Big(u^n + \frac{u^n - u_{i-1}}{\tau}\Big) + \zeta(\theta_n - \theta)\right]\cdot(\bar{u}^n - \bar{u}) \\
&\quad \leq \frac{1}{4\tau}\|\bar{u}^n - \bar{u}\|_H^2 + C_\tau\|\theta^n - \theta\|_{L^2}^2 + C_\tau\left\|K_\tau(\omega)\Big(u - u^n + \frac{u - u^n}{\tau}\Big)\right\|_{L^2}^2 \\
&\quad +C_\tau\left\|\big(K_\tau(\omega^n) - K_\tau(\omega)\big)\Big(u^n + \frac{u^n - u_{i-1}}{\tau}\Big)\right\|_{L^2}^2,
\end{aligned} \tag{37}$$

which follows by adding and subtracting

$$\int_\Omega K_\tau(\omega)\Big(u^n + \frac{u^n}{\tau}\Big)\cdot(\bar{u}^n - \bar{u}).$$

Moreover, since K_τ is Lipschitz continuous

$$\begin{aligned}
&\left\|\big(K_\tau(\omega^n) - K_\tau(\omega)\big)\Big(u^n + \frac{u^n - u_{i-1}}{\tau}\Big)\right\|_{L^2}^2 \\
&\quad\le C_\tau\left\|(\omega^n - \omega)\Big(u^n + \frac{u^n - u_{i-1}}{\tau}\Big)\right\|_{L^2}^2 \\
&\quad\le C_\tau\|\omega^n - \omega\|_{L^q}^2\left\|u^n + \frac{u^n - u_{i-1}}{\tau}\right\|_{H_r}^2 \le C_\tau\|\omega^n - \omega\|_{L^q}^2,
\end{aligned} \tag{38}$$

since $1/q + 1/r < 1/2$ and u^n is bounded in H_r.

Thus, recalling that K_τ is bounded, combining (36), (37) and (38) yields the following

$$\|\bar{u} - \bar{u}^n\|_V^2 \le C_\tau\big(\|u - u^n\|_{H_r}^2 + \|\theta^n - \theta\|_{L^2}^2 + \|\omega^n - \omega\|_{L^q}^2\big), \tag{39}$$

which goes to zero as $n \to +\infty$.

Next, we employ similar arguments to the other two equations. Consider the equation obtained by subtracting Eqs. (33) related to $\bar{\theta}_n$ and $\bar{\theta}$. By multiplying the result by $\bar{\theta}^n - \bar{\theta}$, using the algebraic inequality (21) and Hölder's inequality applied to the convective term, one obtains

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega |\bar{\theta}^n - \bar{\theta}|^2 + C\int_\Omega |\nabla\bar{\theta}^n - \nabla\bar{\theta}|^p \\
&\quad\le \frac{\lambda}{\tau}\int_\Omega |\omega^n - \omega||\bar{\theta}^n - \bar{\theta}| + \int_\Omega u^n\cdot\nabla\bar{\theta}^n(\bar{\theta}^n - \bar{\theta}) - u\cdot\nabla\bar{\theta}(\bar{\theta}^n - \bar{\theta}) \\
&\quad\le C_{\tau,\epsilon}\|\omega^n - \omega\|_{L^2}^2 + \frac{\epsilon}{2}\|\bar{\theta}^n - \bar{\theta}\|_{L^2}^2 + \int_\Omega |(u^n - u)\cdot\nabla(\bar{\theta}^n - \bar{\theta})\bar{\theta}| \\
&\quad\le C_{\tau,\epsilon}\|\omega^n - \omega\|_{L^2}^2 + \epsilon\|\nabla\bar{\theta}^n - \nabla\bar{\theta}\|_{L^p}^p + C_\epsilon\|u^n - u\|_{H_r}^{p'}\|\bar{\theta}\|_{L^2}^{p'},
\end{aligned}$$

where we used (19) and (20).

Thus, by choosing $\epsilon \in (0, \min\{\tau/2, C/2\})$, we get

$$\frac{1}{2\tau}\int_\Omega |\bar{\theta}^n - \bar{\theta}|^2 + \frac{C}{2}\int_\Omega |\nabla\bar{\theta}^n - \nabla\bar{\theta}|^p \le C_\tau\|\omega^n - \omega\|_{L^2}^2 + C\|u^n - u\|_{H_r}^{p'}\|\bar{\theta}\|_{L^2}^{p'}.$$

In particular, by interpolation, there follows

$$\|\bar{\theta}^n - \bar{\theta}\|_{W^{1,p}} \to 0 \qquad \text{as } n \to +\infty. \tag{40}$$

Next, consider the equation obtained after subtracting (34) for $\bar{\omega}$ from (34) for $\bar{\omega}^n$. By multiplying the result by $\bar{\omega}^n - \bar{\omega}$, integrating over Ω and using that f is Lipschitz, we are lead to

$$\begin{aligned}
&\frac{1}{\tau}\int\limits_{\Omega}|\bar{\omega}^n-\bar{\omega}|^2+C\int\limits_{\Omega}|\nabla\bar{\omega}^n-\nabla\bar{\omega}|^q\\
&\quad\le C(f,\tau)\int\limits_{\Omega}\Big(|\omega^n-\omega|+|\theta^n-\theta|+|\omega^n-\omega|\big(|\omega^n|^{\frac{2-s}{s-1}}+|\omega|^{\frac{2-s}{s-1}}\big)\Big)|\bar{\omega}^n-\bar{\omega}|\\
&\quad-\kappa\int\limits_{\Omega}u\cdot\nabla(\bar{\omega}^n-\bar{\omega})\bar{\omega}-u^n\cdot\nabla(\bar{\omega}^n-\bar{\omega})\bar{\omega}^n,
\end{aligned}$$

where (30) was employed.

By treating the convective terms as before and using again the Hölder's inequality, we arrive at

$$\begin{aligned}
&\frac{1}{2\tau}\|\bar{\omega}^n-\bar{\omega}\|_{L^2}^2+\Big(\frac{C}{2}-\epsilon\Big)\|\nabla\bar{\omega}^n-\nabla\bar{\omega}\|_{L^q}^q\\
&\quad\le C_\tau\Big(\|\omega^n-\omega\|^2_{L^{\frac{2}{s-1}}}+\|\theta^n-\theta\|_{L^2}^2+\|u^n-u\|_{H_r}^{q'}\|\bar{\omega}\|_{L^2}^{q'}\Big),
\end{aligned}\tag{41}$$

since

$$\|\omega^n\|_{L^{\frac{2}{s-1}}}+\|\omega\|_{L^{\frac{2}{s-1}}}\le C,\ \forall n\in\mathbb{N}.$$

Hence, from (39), (40) and (41), there follows that T_λ is continuous in $\mathcal{B}$.

Step 3. We will show that T_λ is a compact operator for any $\lambda\in[0,1]$ fixed. Since $V\times W_0^{1,p}\times W^{1,q}$ is compactly embedded in $\mathcal{B}$, is it enough to prove that $T_\lambda:\mathcal{B}\to V\times W_0^{1,p}\times W^{1,q}$ is bounded. This is achieved by performing standard energy estimates, so we omit the details.

Step 4. We claim that T_λ is uniformly continuous with respect to λ on bounded subsets of $\mathcal{B}$. Indeed, let $\mathcal{A}\subset\mathcal{B},\mathcal{A}$ bounded, $\lambda_l\in[0,1], l=1$ or 2 and $(u,\theta,\omega)\in\mathcal{A}$. Then, take $(\bar{u}^l,\bar{\theta}^l,\bar{\omega}^l)=T_{\lambda_l}(u,\theta,\omega)$ and fix $\bar{u}=\bar{u}^2-\bar{u}^1$, $\bar{\theta}=\bar{\theta}^2-\bar{\theta}^1$ and $\bar{\omega}=\bar{\omega}^2-\bar{\omega}^1$.

Now, consider (32) given by $l=1$ and 2 and take the difference between them. After that, by multiplying the result by $\bar{u}$ and integrating over Ω, we get

$$\begin{aligned}
\int\limits_{\Omega}\frac{1}{\tau}|\bar{u}|^2+|\nabla\bar{u}|^2&\le|\lambda_1-\lambda_2|\int\limits_{\Omega}\left(|K_\tau(\omega)|\left|u+\frac{u-u_{i-1}}{\tau}\right|+|\zeta\theta|\right)\\
&\le C(\mathcal{A},\tau)|\lambda_1-\lambda_2|.
\end{aligned}$$

In similar way, by using inequality (21) for both p and q, we obtain

$$\int\limits_{\Omega}\frac{1}{\tau}|\bar{\theta}|^2+C|\nabla\bar{\theta}|^p\le C(\mathcal{A},\tau)|\lambda_1-\lambda_2|,$$

$$\int\limits_{\Omega}\frac{1}{\tau}|\bar{\omega}|^2+C|\nabla\bar{\omega}|^q\le C(\mathcal{A},\tau)|\lambda_1-\lambda_2|.$$

Hence, from the previous estimates and interpolation, the claim is proved.

Step 5. Finally, we focus on the estimates for the set of fixed points of T_λ. In this way, set $(u,\theta,\omega)=T_\lambda(u,\theta,\omega)$ and consider the corresponding Eqs. (32)–(35), properly modified.

By multiplying (32) by u and integrating over Ω, there follows

$$\int\limits_{\Omega}\frac{1}{2\tau}|u|^2+|\nabla u|^2\le C_\tau+C\|\theta\|_{L^2}^2,\tag{42}$$

since **(H5)** and (27) imply that $\|K_\tau(.)\|_{L^\infty}\le C_\tau$ and $\int\limits_{\Omega}K_\tau(\omega)u^2\ge 0$.

Next, multiplying (33) by θ, (34) by $(\omega-\omega_{i-1})/\tau$, adding the result and integrating over Ω lead to

$$\int_\Omega \frac{1}{\tau}|\theta|^2+\frac{1}{2\tau^2}|\omega|^2+|\nabla\theta|^p+\frac{1}{4\tau}|\nabla\omega|^2+\frac{1}{q\tau}|\nabla\omega|^q$$
$$\leq C_\tau+C\int_\Omega|\omega|^2+|\theta|^2+\frac{C}{\tau}\int_\Omega|u|^2+|\omega|^2, \tag{43}$$

where we have used (19), algebraic inequality (22), that $\alpha_\tau(x)x\geq 0$, that $\omega_{i-1}\in L^\infty$, and the Hölder's inequality.

Hence, gathering (42) and (43), and by taking $\tau>0$ sufficiently small, there follows that

$$\|u\|_V^2+\|\theta\|_{L^p}^p+\|\omega\|_{L^2}^2+\|\nabla\omega\|_{L^q}^q\leq C,$$

from which we deduce that

$$\|u\|_{H_r}+\|\theta\|_{L^p}+\|\omega\|_{L^{\frac{2}{s-1}}}\leq C.$$

Therefore, by combining Steps 1–5, we can apply Leray–Schauder's theorem and conclude that there exists $(u,\theta,\omega)\in V\times W_0^{1,p}\times W^{1,q}$ such that $T_1(u,\theta,\omega)=(u,\theta,\omega)$.

By setting $u_i=u,\theta_i=\theta$ e $\omega_i=\omega$, the result follows. □

For the temperature and phase-field discretized equations, the next result is better:

Proposition 4.3. *Let (u_i,θ_i,ω_i) be given by Proposition 4.2. Then, Eqs. (24) and (25) hold almost everywhere in Ω.*

Proof. Observe that (25) may be written as

$$\int_\Omega(1+|\nabla\omega_i|^{q-2})\nabla\omega_i\cdot\nabla\xi=\int_\Omega F_i\xi,\ \forall\xi\in W^{1,q}$$

where

$$F_i=\theta_i+f(\omega_i)-\frac{\omega_i-\omega_{i-1}}{\tau}-\alpha_\tau\left(\frac{\omega_i-\omega_{i-1}}{\tau}\right)-\kappa u_i\cdot\nabla\omega_i.$$

Since $p,\ q>N$, by the Sobolev embedding, $W_0^{1,p},\ W^{1,q}\hookrightarrow L^\infty$. So that, by **(H3)**, (29), the Hölder's inequality and the choice of r (see Lemma 4.1), we have that $F_i\in L^\sigma$ for some $\sigma\geq 2$.

Therefore,

$$\operatorname{div}\big((1+|\nabla\omega_i|^{q-2})\nabla\omega_i\big)\in L^\sigma\subset L^{q'}.$$

Next, since $(1+|\nabla\omega_i|^{q-2})\nabla\omega_i\in L^{q'}$, by applying [23, Thm. III.2.2], we have

$$\int_\Omega(1+|\nabla\omega_i|^{q-2})\nabla\omega_i\cdot\nabla\xi=\int_\Omega-\Delta\omega_i\xi-\Delta_q\omega_i\xi,\forall\xi\in W^{1,q}.$$

For the proof that (24) is satisfied almost everywhere in Ω, we observe that

$$G_i=g_i-\frac{\theta_i-\theta_{i-1}}{\tau}-\frac{\omega_i-\omega_{i-1}}{\tau}-u_i\cdot\nabla\theta_i\in L^\sigma,\ \sigma\geq 2,$$

and proceed analogously as before. □

5. Time-dependent approximations and a priori estimates

After analyzing the previous time-discrete version of the problem, we move forward to the time-dependent case. This is done by introducing auxiliary functions and investigating the approximate versions of (1)–(7).

Let $\tau > 0$ be given by Proposition 4.2. We consider auxiliary functions u_τ and $\bar{u}_\tau$ defined by

$$\begin{aligned}&u_\tau : [0,T] \to V,\\&u_\tau(t) = u_i,\ (i-1)\tau \le t \le i\tau,\ i = 0,\dots,M,\\&\bar{u}_\tau : [0,T] \to V,\\&\bar{u}_\tau(t) = \left(\frac{u_i - u_{i-1}}{\tau}\right)(t-(i-1)\tau) + u_{i-1},\ (i-1)\tau \le t \le i\tau,\ i = 1,\dots,M.\end{aligned}$$

The time derivative of $\bar{u}_\tau$ is denoted by $\bar{u}'_\tau$.

Analogously, we define $\theta_\tau, \bar{\theta}_\tau, \omega_\tau, \bar{\omega}_\tau$, and g_τ.

By Proposition 4.3, system (23)–(25) can be written as

$$\int_\Omega \bar{u}'_\tau \cdot \phi + \nabla u_\tau \cdot \nabla \phi + K_\tau(\omega_\tau)(u_\tau + \bar{u}'_\tau)\cdot \phi = \int_\Omega \zeta\theta_\tau \cdot \phi \quad \forall \phi \in V, \tag{44}$$

$$\bar{\theta}'_\tau + \bar{\omega}'_\tau - \Delta\theta_\tau - \Delta_p \theta_\tau + u_\tau \cdot \nabla\theta_\tau = g_\tau, \quad \text{a.e. in } Q, \tag{45}$$

$$\bar{\omega}'_\tau + \alpha_\tau(\bar{\omega}'_\tau) - \Delta\omega_\tau - \Delta_q\omega_\tau + \kappa u_\tau \cdot \nabla\omega_\tau = \theta_\tau + f(\omega_\tau) \quad \text{a.e. in } Q. \tag{46}$$

Moreover,

$$\begin{aligned}\nabla\cdot u_\tau = \nabla \cdot \bar{u}_\tau = 0 \text{ in } Q,\\ u_\tau(.,0) = \bar{u}_\tau(.,0) = u_0 \text{ a.e. in } \Omega,\\ \theta_\tau(.,0) = \bar{\theta}_\tau(.,0) = \theta_0 \text{ in } \Omega,\\ \omega_\tau(.,0) = \bar{\omega}_\tau(.,0) = \omega_0 \text{ in } \Omega.\end{aligned} \tag{47}$$

The main goal in this section is to obtain estimates, independent of $\tau > 0$, for the unknowns under consideration. We begin with the following lemma.

Lemma 5.1. *There exists* $C = C(\Omega, T, p, q, f, g, u_0, \theta_0, \omega_0) > 0$, *not depending on* $\tau > 0$, *such that*

$$\|\bar{u}'_\tau\|_{L^2(0,T;H)} + \|u_\tau\|_{L^\infty(0,T;V)} \le C, \tag{48}$$

$$\|\bar{\theta}'_\tau\|_{L^2(0,T;L^2)} + \|\theta_\tau\|_{L^\infty(0,T;W_0^{1,p})} \le C, \tag{49}$$

$$\|\bar{\omega}'_\tau\|_{L^2(0,T;L^2)} + \|\omega_\tau\|_{L^\infty(0,T;W^{1,q})} \le C, \tag{50}$$

$$\int_0^T \int_\Omega K_\tau(\omega_\tau)|u_\tau + \bar{u}'_\tau|^2 \le C. \tag{51}$$

Proof. By multiplying (24) by θ_i, (25) $+\omega_i$ by $(\omega_i - \omega_{i-1})/\tau$, integrating over Ω and adding the resulting equations, we obtain

$$\begin{aligned}&\int_\Omega \frac{|\theta_i|^2}{2\tau} - \frac{|\theta_{i-1}|^2}{2\tau} + |\nabla\theta_i|^p + \frac{|\omega_i|^2}{2\tau} - \frac{|\omega_{i-1}|^2}{2\tau} + \left|\frac{\omega_i - \omega_{i-1}}{2\tau}\right|^2\\&\quad + \frac{1}{\tau}\int_\Omega \frac{|\nabla\omega_i|^2}{2} - \frac{|\nabla\omega_{i-1}|^2}{2} + \frac{|\nabla\omega_i|^q}{q} - \frac{|\nabla\omega_{i-1}|^q}{q}\\&\quad \le C\int_\Omega |\theta_i|^2 + |g_i|^2 + |\omega_i|^2 - \kappa u_i \cdot \nabla\omega_i \frac{\omega_i - \omega_{i-1}}{\tau},\end{aligned} \tag{52}$$

where we employed (19), algebraic inequality (22), the fact that $\alpha_\tau(x)x \geq 0$ and **(H3)**.

By the choice of r done in Lemma 4.1, we can apply the Hölder's inequality in the convective term; so we have

$$\int_\Omega u_i \cdot \nabla\omega_i \frac{\omega_i - \omega_{i-1}}{\tau} \leq C\Big(\int_\Omega |u_i|^r\Big)^{1/r}\Big(\int_\Omega |\nabla\omega_i|^q\Big)^{1/q}\Big(\int_\Omega \Big|\frac{\omega_i-\omega_{i-1}}{\tau}\Big|^2\Big)^{1/2}$$
$$\leq C\Big(\int_\Omega |u_i|^r + |\nabla\omega_i|^q\Big) + \frac{1}{2}\int_\Omega \Big|\frac{\omega_i-\omega_{i-1}}{2\tau}\Big|^2 + C. \tag{53}$$

Then, plugging (53) into (52) leads to

$$\int_\Omega \frac{|\theta_i|^2}{\tau} - \frac{|\theta_{i-1}|^2}{\tau} + |\nabla\theta_i|^p + \frac{|\omega_i|^2}{\tau} - \frac{|\omega_{i-1}|^2}{\tau} + \Big|\frac{\omega_i-\omega_{i-1}}{2\tau}\Big|^2$$
$$+ \frac{1}{\tau}\int_\Omega |\nabla\omega_i|^2 - |\nabla\omega_{i-1}|^2 + \frac{2}{q}|\nabla\omega_i|^q - \frac{2}{q}|\nabla\omega_{i-1}|^q$$
$$\leq C\left(1 + \int_\Omega |\theta_i|^2 + |g_i|^2 + |\omega_i|^2 + |u_i|^r + |\nabla\omega_i|^q\right).$$

Now, recall that M is a discretization parameter such that $M\tau = T$. So, by multiplying the previous inequality by τ and adding from $i = 1$ to $m \leq M$, we obtain

$$\int_\Omega |\theta_m|^2 + |\omega_m|^2 + |\nabla\omega_m|^q + \sum_{i=1}^m \tau|\nabla\theta_i|^p + \sum_{i=1}^m \tau\Big|\frac{\omega_i-\omega_{i-1}}{\tau}\Big|^2$$
$$\leq C\left(1 + \int_\Omega \sum_{i=1}^m \tau\big(|\theta_i|^2 + |g_i|^2 + |\omega_i|^2 + |u_i|^r + |\nabla\omega_i|^q\big)\right). \tag{54}$$

In this way, by the definition of θ_τ, ω_τ and u_τ, one finds

$$\|\theta_\tau(t)\|_{L^2}^2 + \|\omega_\tau(t)\|_{L^2}^2 + \|\nabla\omega_\tau(t)\|_{L^q}^q$$
$$\leq C\Big(1 + \|\theta_\tau\|_{L^2(0,t;L^2)}^2 + \|\omega_\tau\|_{L^2(0,t;L^2)}^2 + \|\nabla\omega_\tau\|_{L^q(0,t;L^q)}^q + \|u_\tau\|_{L^r(0,t;L^r)}^r\Big),$$

for $0 < \tau < 1/2$ and a.e. t in $[0, T]$. Thus, by Gronwall's lemma,

$$\|\theta_\tau(t)\|_{L^2}^2 + \|\omega_\tau(t)\|_{L^2}^2 + \|\nabla\omega_\tau(t)\|_{L^q}^q \leq C\Big(1 + \|u_\tau\|_{L^r(0,t;L^r)}^r\Big), \tag{55}$$

where $C = C(T, q, f, g, u_0, \theta_0, \omega_0) > 0$.

From (54) and (55), we have that

$$\|\nabla\theta_\tau\|_{L^p(0,t;L^p)}^p + \|\bar{\omega}_\tau'\|_{L^2(0,t;L^2)}^2 \leq C\Big(1 + \|u_\tau\|_{L^r(0,t;L^r)}^r\Big). \tag{56}$$

Next, we multiply (23) by $(u_i + (u_i - u_{i-1})/\tau)$, integrate over Ω, and use algebraic inequality (22), to arrive at

$$\int_\Omega \frac{|u_i|^2}{2\tau} - \frac{|u_{i-1}|^2}{2\tau} + \Big|\frac{u_i-u_{i-1}}{2\tau}\Big|^2 + |\nabla u_i|^2 + \int_\Omega \frac{|\nabla u_i|^2}{2\tau} - \frac{|\nabla u_{i-1}|^2}{2\tau}$$
$$+ \int_\Omega K_\tau(\omega_i)\Big|u_i + \frac{u_i-u_{i-1}}{\tau}\Big|^2 \leq C\int_\Omega |\theta_i|^2 + |u_i|^2.$$

Once again, multiplying the last equation by τ and adding from $i = 1$ to $m \leq M$ lead to

$$\int_\Omega |u_m|^2 + |\nabla u_m|^2 + \sum_{i=1}^m \tau \left|\frac{u_i - u_{i-1}}{\tau}\right|^2 + \sum_{i=1}^m \tau |\nabla u_i|^2$$
$$+ \sum_{i=1}^m \tau \int_\Omega K_\tau(\omega_i)\left|u_i + \frac{u_i - u_{i-1}}{\tau}\right|^2 \leq C\Big(1 + \int_\Omega \sum_{i=1}^m \tau\big(|u_i|^2 + |\theta_i|^2\big)\Big).$$

By proceeding in the same way as was done to obtain (55) and (56), we get

$$\|u_\tau(t)\|^2_{L^2} + \|\nabla u_\tau(t)\|^2_{L^2} + \|\bar{u}'_\tau(t)\|^2_{L^2(0,t;L^2)} + \int_0^t \int_\Omega K_\tau(\omega_\tau)|u_\tau + \bar{u}'_\tau|^2$$
$$\leq C\Big(1 + \|\theta_\tau\|^2_{L^2(0,t;L^2)}\Big). \tag{57}$$

Finally, we multiply (24) by $(\theta_i - \theta_{i-1})/\tau$ and integrate over Ω. Recalling the choice of r in Lemma 4.1 and proceeding similarly as before, we obtain

$$\int_\Omega \left|\frac{\theta_i - \theta_{i-1}}{2\tau}\right|^2 + \frac{|\nabla\theta_i|^2}{2\tau} - \frac{|\nabla\theta_{i-1}|^2}{2\tau} + \frac{|\nabla\theta_i|^p}{p\tau} - \frac{|\nabla\theta_{i-1}|^p}{p\tau}$$
$$\leq C\left(1 + \int_\Omega |g_i|^2 + |u_i|^r + |\nabla\theta_i|^p + \left|\frac{\omega_i - \omega_{i-1}}{\tau}\right|^2\right).$$

By multiplying by $\tau > 0$, adding from $i = 1$ to $m \leq M$, and using estimate (56), we get

$$\|\bar{\theta}'_\tau\|^2_{L^2(0,t;L^2)} + \|\nabla\theta_\tau(t)\|^2_{L^2} + \|\nabla\theta_\tau(t)\|^p_{L^p} \leq C\big(1 + \|u_\tau\|^r_{L^r(0,t;L^r)}\big), \tag{58}$$

a.e. t in $[0, T]$. In particular, we have

$$\|\theta_\tau(t)\|^2_{L^2} + \|\nabla\theta_\tau(t)\|^p_{L^p} \leq C\big(1 + \|\nabla u_\tau\|^r_{L^\infty(0,t;L^2)}\big),$$

which combined with (57) leads to

$$\|\theta_\tau(t)\|^2_{L^2} + \|\nabla\theta_\tau(t)\|^p_{L^2} \leq \|\theta_\tau(t)\|^2_{L^2} + C\|\nabla\theta_\tau(t)\|^p_{L^p} \leq C\big(1 + \|\theta_\tau\|^r_{L^2(0,t;L^2)}\big). \tag{59}$$

Then, by applying the Poincaré's inequality and recalling that $r < p$ (see (18)), we obtain

$$\|\nabla\theta_\tau(t)\|^2_{L^2} \leq C\big(1 + \|\nabla\theta_\tau\|^{\frac{2r}{p}}_{L^2(0,t;L^2)}\big) \leq C\big(1 + \|\nabla\theta_\tau\|^2_{L^2(0,t;L^2)}\big).$$

In this fashion, by Gronwall's lemma, we prove that there holds, for a.e. t in $[0, T]$,

$$\|\nabla\theta_\tau(t)\|^2_{L^2} \leq C,$$

where $C = C(T, p, q, f, g, u_0, \theta_0, \omega_0) > 0$. Now, by Poincaré's inequality we have

$$\|\theta_\tau\|^2_{L^2(0,t;L^2)} \leq C.$$

Thus, (48) and (51) follow from (57) and the previous estimate. Further, (49) and (50) are a consequence of (48), (55), (56) and (58). □

Remark 5.2. *Let us observe that all the previous results, except Lemma 5.1, are still valid if we assume homogeneous Neumann boundary conditions for the temperature. However, in this case we can obtain a local version of Lemma 5.1 by means of simple adaptations of the arguments. Indeed, firstly observe that by* (59) *we get that*

$$\|\theta_\tau(t)\|^2_{L^2} \leq C\big(1 + \|\theta_\tau\|^r_{L^2(0,t;L^2)}\big).$$

Next, since $r > 2$, the Gronwall's Lemma implies the existence of a certain T^*, such that $0 < T^* \leq T$, and

$$\|\theta_\tau(t)\|_{L^2} \leq C \ a.e \ in \ [0, T^*]$$

where $C = C(T^*, T, p, q, f, g, u_0, \theta_0, \omega_0) > 0$.

This estimate and the argument presented in Lemma 5.1 then provide local versions of (48)–(51), which combined with the subsequent results of this paper, imply the existence of a local solution to (1)–(7) for homogeneous Neumann boundary conditions for the temperature θ.

Next, we obtain estimates for the time derivative of ω_τ, which will be important when controlling approximations of $\alpha(\omega_t)$.

Lemma 5.3. *There exists $C = C(\Omega, T, p, q, f, g, u_0, \theta_0, \omega_0) > 0$, not depending on $\tau > 0$, such that*

$$\|\bar{\omega}_\tau'\|_{L^\infty(0,T;L^2)} + \|\nabla\bar{\omega}_\tau'\|_{L^2(0,T;L^2)} \leq C, \tag{60}$$

$$\sum_{i=1}^{M} \int_\Omega \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^q}{\tau} \leq C. \tag{61}$$

Proof. Let us fix

$$\xi_i = \alpha_\tau\Big(\frac{\omega_i - \omega_{i-1}}{\tau}\Big), \text{ for } i \geq 1. \tag{62}$$

Then, given $i \geq 2$, from Eq. (25) associated with i, we subtract the same Eq. (25) but now associated with $i - 1$. After that, we multiply the result by $(\omega_i - \omega_{i-1})/\tau$ and integrate over Ω, to obtain

$$\begin{aligned}
&\int_\Omega \Big(\frac{\omega_i - \omega_{i-1}}{\tau} + \xi_i - \frac{\omega_{i-1} - \omega_{i-2}}{\tau} - \xi_{i-1}\Big)\frac{\omega_i - \omega_{i-1}}{\tau} + \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^2}{\tau} \\
&\quad + C\int_\Omega \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^q}{\tau} + \kappa\int_\Omega u_i \cdot \nabla\omega_i\frac{\omega_i - \omega_{i-1}}{\tau} - u_{i-1} \cdot \nabla\omega_{i-1}\frac{\omega_i - \omega_{i-1}}{\tau} \\
&\quad \leq C\int_\Omega \frac{|\theta_i - \theta_{i-1}|^2}{\tau} + \frac{|\omega_i - \omega_{i-1}|^2}{\tau},
\end{aligned} \tag{63}$$

where we employed **(H3)** and algebraic inequality (21).

To estimate the convective terms in (63), we first rewrite them by adding and subtracting $u_{i-1} \cdot \nabla\omega_i(\omega_i - \omega_{i-1})/\tau$, and using property (19). Then, by the Hölder's inequality, we obtain

$$\begin{aligned}
&\int_\Omega u_i \cdot \nabla\omega_i\frac{\omega_i - \omega_{i-1}}{\tau} - u_{i-1} \cdot \nabla\omega_{i-1}\frac{\omega_i - \omega_{i-1}}{\tau} \\
&\quad = \int_\Omega (u_i - u_{i-1}) \cdot \nabla\omega_i\frac{\omega_i - \omega_{i-1}}{\tau} \\
&\quad \leq \frac{C}{\tau}\Big(\int_\Omega |u_i - u_{i-1}|^2\Big)^{1/2}\Big(\int_\Omega |\nabla\omega_i|^q\Big)^{1/q}\Big(\int_\Omega |\omega_i - \omega_{i-1}|^r\Big)^{1/r}.
\end{aligned}$$

Next, observe that from (50), $\|\nabla\omega_i\|_{L^q} \leq C$ for every i, with $C > 0$ independent on $\tau > 0$. Hence, by using the embedding $W^{1,2} \hookrightarrow L^r$ and the Young's inequality, we arrive at

$$\int_\Omega u_i \cdot \nabla\omega_i \frac{\omega_i - \omega_{i-1}}{\tau} - u_{i-1} \cdot \nabla\omega_{i-1} \frac{\omega_i - \omega_{i-1}}{\tau}$$
$$\le \int_\Omega C \frac{|u_i - u_{i-1}|^2}{\tau} + \frac{|\omega_i - \omega_{i-1}|^2}{2\tau} + \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^2}{2\tau}. \tag{64}$$

Moreover, by recalling the definitions of α_τ in (28) and Γ_τ in (31), direct calculations lead to

$$\gamma_\tau(\xi_i) + \tau|\xi_i|^{s-2}\xi_i = \frac{\omega_i - \omega_{i-1}}{\tau} \tag{65}$$

and

$$\int_\Omega \Gamma_\tau(\xi_i) - \Gamma_\tau(\xi_{i-1}) \le \int_\Omega \gamma_\tau(\xi_i)(\xi_i - \xi_{i-1}). \tag{66}$$

Thus, by using (65) in (66) yields

$$\int_\Omega \Gamma_\tau(\xi_i) - \Gamma_\tau(\xi_{i-1}) + \tau|\xi_i|^{s-2}\xi_i(\xi_i - \xi_{i-1}) \le \int_\Omega \left(\xi_i - \xi_{i-1}\right)\frac{\omega_i - \omega_{i-1}}{\tau},$$

and algebraic inequality (22) gives

$$\int_\Omega \Gamma_\tau(\xi_i) - \Gamma_\tau(\xi_{i-1}) + \frac{\tau}{s}|\xi_i|^s - \frac{\tau}{s}|\xi_{i-1}|^s \le \int_\Omega \left(\xi_i - \xi_{i-1}\right)\frac{\omega_i - \omega_{i-1}}{\tau}. \tag{67}$$

Let us remark that the previous inequality is valid for any $i \ge 1$.

Therefore, gathering (64) and (67) in (63), we end up with

$$\int_\Omega \Gamma_\tau(\xi_i) - \Gamma_\tau(\xi_{i-1}) + \frac{\tau}{s}|\xi_i|^s - \frac{\tau}{s}|\xi_{i-1}|^s + \frac{|\omega_i - \omega_{i-1}|^2}{\tau^2} - \frac{|\omega_{i-1} - \omega_{i-2}|^2}{\tau^2}$$
$$+ \int_\Omega \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^2}{\tau} + \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^q}{\tau}$$
$$\le C \int_\Omega \frac{|u_i - u_{i-1}|^2}{\tau} + \frac{|\theta_i - \theta_{i-1}|^2}{\tau} + \frac{|\omega_i - \omega_{i-1}|^2}{\tau}.$$

So, by adding from $i = 2$ to $m \le M$ and using (48)–(50), we get

$$\int_\Omega \Gamma_\tau(\xi_m) + \frac{\tau}{s}|\xi_m|^s + \left|\frac{\omega_m - \omega_{m-1}}{\tau}\right|^2$$
$$+ \sum_{i=1}^m \int_\Omega \tau\left|\frac{\nabla\omega_i - \nabla\omega_{i-1}}{\tau}\right|^2 + \frac{|\nabla\omega_i - \nabla\omega_{i-1}|^q}{\tau}$$
$$\le \ C\Bigg(1 + \int_\Omega \Gamma_\tau(\xi_1) + \frac{\tau}{s}|\xi_1|^s + \left|\frac{\omega_1 - \omega_0}{\tau}\right|^2$$
$$+ \frac{|\nabla\omega_1 - \nabla\omega_0|}{\tau} + \frac{|\nabla\omega_1 - \nabla\omega_0|^q}{\tau}\Bigg). \tag{68}$$

Now, we have to estimate the right-hand side of (68). First recall that in (62) we have set ξ_i for $i \ge 1$, so that we need to choose properly ξ_0. Indeed, fix

$$\xi_0 = \theta_0 + f(\omega_0) + \Delta\omega_0 + \Delta_q\omega_0 - \kappa u_0 \cdot \nabla\omega_0 \in L^s \tag{69}$$

by hypothesis **(H8)**, (see [10, Rmk. 1]).

By multiplying (25) for $i = 1$ by $(\omega_1 - \omega_0)/\tau$ and integrating over Ω, we obtain

$$\int_\Omega \left(\frac{\omega_1 - \omega_0}{\tau} + \alpha_\tau\Big(\frac{\omega_1 - \omega_0}{\tau}\Big)\right)\frac{\omega_1 - \omega_0}{\tau} + (1 + |\nabla\omega_1|^{q-2})\nabla\omega_1 \cdot \frac{\nabla\omega_1 - \nabla\omega_0}{\tau}$$
$$+\kappa\int_\Omega u_1 \cdot \nabla\omega_1 \frac{\omega_1 - \omega_0}{\tau} = \int_\Omega (\theta_1 + f(\omega_1))\frac{\omega_1 - \omega_0}{\tau}.$$

Thence, by adding and subtracting $\int_\Omega \xi_0(\omega_1 - \omega_0)/\tau$ (which is well defined since $\omega_1 - \omega_0 \in L^\infty$), using (19), algebraic inequality (21) and arranging terms, we deduce

$$\int_\Omega (\xi_1 - \xi_0)\cdot\frac{\omega_1 - \omega_0}{\tau} + \left|\frac{\omega_1 - \omega_0}{\tau}\right|^2 + \frac{|\nabla\omega_1 - \nabla\omega_0|^2}{\tau} + \frac{|\nabla\omega_1 - \nabla\omega_0|^q}{\tau}$$
$$\le \int_\Omega \frac{|\theta_1 - \theta_0|^2}{\tau} + C\frac{|\omega_1 - \omega_0|^2}{\tau} - \kappa(u_1 - u_0)\cdot\nabla\omega_0\frac{\omega_1 - \omega_0}{\tau}. \tag{70}$$

To estimate the convective term, notice that as $q > N$ and $\omega_0 \in W^{2,q}$, we have that $\nabla\omega_0 \in L^\infty$. Thus, by using Hölder's and Young's inequalities, there follows

$$\int_\Omega (u_1 - u_0)\cdot\nabla\omega_0\frac{\omega_1 - \omega_0}{\tau} \le C\int_\Omega \frac{|u_1 - u_0|^2}{\tau} + \frac{|\omega_1 - \omega_0|^2}{\tau}. \tag{71}$$

To obtain a lower bound for the first term appearing in (70), we use (67) for $i = 1$. Thus, plugging (71) into (70) yields

$$\int_\Omega \Gamma_\tau(\xi_1) + \frac{\tau}{s}|\xi_1|^s + \left|\frac{\omega_1 - \omega_0}{\tau}\right|^2 + \frac{|\nabla\omega_1 - \nabla\omega_0|^2}{\tau} + \frac{|\nabla\omega_1 - \nabla\omega_0|^q}{\tau}$$
$$\le C\Big(\int_\Omega \frac{|\theta_1 - \theta_0|^2}{\tau} + \frac{|\omega_1 - \omega_0|^2}{\tau} + \frac{|u_1 - u_0|^2}{\tau} + \Gamma_\tau(\xi_0) + \frac{\tau}{s}|\xi_0|^s\Big)$$
$$\le C\Big(1 + \int_\Omega \Gamma(\xi_0) + \frac{\tau}{s}|\xi_0|^s\Big), \tag{72}$$

where we have used (31) and estimates (48)–(50). Notice that the last expression is finite due to **(H9)** and (69). Therefore, by combining (68) and (72), we prove (60) and (61). □

The last results of this section concern estimates that will be used while proving the identification of the element of $\alpha(\omega_t)$.

It is worth to observe at this point that convection in the phase-field equation plays a key role. When such convection is omitted, it is possible to obtain a rather regular solution; with the convection term, the solutions are weaker, but this weakening is not severe since such solutions remain in Lebesgue spaces.

Lemma 5.4. *Suppose that* (15) *holds. Then, there exists $C > 0$, not depending on $\tau > 0$, such that*

$$\|\alpha_\tau(\bar\omega'_\tau)\|_{L^\infty(0,T;L^s)} \le C, \tag{73}$$
$$\|\Delta\omega_\tau + \Delta_q\omega_\tau\|_{L^\infty(0,T;L^s)} \le C. \tag{74}$$

Proof. We proceed analogously as in the proof of Lemma 5.3; we let $i \ge 2$ and again subtract the Eqs. (25) associated with i and $i - 1$; but this time we set

$$\hat{\xi}_i = \alpha_\tau\left(\frac{\omega_i - \omega_{i-1}}{\tau}\right) + \frac{\omega_i - \omega_{i-1}}{\tau}$$

and take $|\hat{\xi}_i|^{s-2}\hat{\xi}_i$ as multiplier.

By using algebraic inequality (22), we get

$$\begin{aligned}\int_\Omega \frac{|\hat{\xi}_i|^s}{s} - \frac{|\hat{\xi}_{i-1}|^s}{s} + \Big((1+|\nabla\omega_i|^{q-2})\nabla\omega_i - (1+|\nabla\omega_{i-1}|^{q-2})\nabla\omega_{i-1}\Big)\cdot\nabla(|\hat{\xi}_i|^{s-2}\hat{\xi}_i)\\ + \kappa\int_\Omega \big(u_i\cdot\nabla\omega_i - u_{i-1}\cdot\nabla\omega_{i-1}\big)|\hat{\xi}_i|^{s-2}\hat{\xi}_i\\ \le \int_\Omega \Big(\big(\theta_i - \theta_{i-1}\big) + \big(f(\omega_i) - f(\omega_{i-1})\big)\Big)|\hat{\xi}_i|^{s-2}\hat{\xi}_i. \end{aligned} \tag{75}$$

For the sake of clarity, we analyze (75) term by term.

To begin, we focus on delicate estimates on convective terms. Observe that by the choice of s in (15),

$$\frac{1}{q} + \frac{1}{2} + \frac{s-1}{s} \le 1.$$

Moreover, since $s \le \dfrac{N}{N-1}$, there exists $\sigma > 2$ such that

$$\frac{1}{\sigma} + \frac{1}{2} + \frac{s-1}{s} \le 1 \text{ and } V \hookrightarrow L^\sigma.$$

Thenceforth, by Hölder's inequality,

$$\begin{aligned}&\int_\Omega \Big|u_i\cdot\nabla\omega_i - u_{i-1}\cdot\nabla\omega_{i-1}\Big||\hat{\xi}_i|^{s-1}\\ &= \tau\int_\Omega \left|\frac{u_i - u_{i-1}}{\tau}\cdot\nabla\omega_i + u_{i-1}\cdot\frac{\nabla\omega_i - \nabla\omega_{i-1}}{\tau}\right||\hat{\xi}_i|^{s-1}\\ &\le C\tau\left(\left\|\frac{u_i - u_{i-1}}{\tau}\right\|_{L^2}\|\nabla\omega_i\|_{L^q}\|\hat{\xi}_i\|_{L^s}^{s-1} + \|u_{i-1}\|_{L^\sigma}\left\|\frac{\nabla\omega_i - \nabla\omega_{i-1}}{\tau}\right\|_{L^2}\|\hat{\xi}_i\|_{L^s}^{s-1}\right).\end{aligned}$$

Thus, by using estimates (48) and (50), and Young's inequality, there follows

$$\begin{aligned}&\left|\int_\Omega \Big(u_i\cdot\nabla\omega_i - u_{i-1}\cdot\nabla\omega_{i-1}\Big)|\hat{\xi}_i|^{s-2}\hat{\xi}_i\right|\\ &\quad\le \tau C\left(\left\|\frac{u_i - u_{i-1}}{\tau}\right\|_{L^2}\|\hat{\xi}_i\|_{L^s}^{s-1} + \left\|\frac{\nabla\omega_i - \nabla\omega_{i-1}}{\tau}\right\|_{L^2}\|\hat{\xi}_i\|_{L^s}^{s-1}\right)\\ &\quad\le \tau C\left(1 + \left\|\frac{u_i - u_{i-1}}{\tau}\right\|_{L^2}^2 + \left\|\frac{\nabla\omega_i - \nabla\omega_{i-1}}{\tau}\right\|_{L^2}^2 + \|\hat{\xi}_i\|_{L^s}^s\right).\end{aligned} \tag{76}$$

Next, we turn to dissipative terms. Indeed, notice that

$$\begin{aligned}&\int_\Omega \Big((1+|\nabla\omega_i|^{q-2})\nabla\omega_i - (1+|\nabla\omega_{i-1}|^{q-2})\nabla\omega_{i-1}\Big)\cdot\nabla(|\hat{\xi}_i|^{s-2}\hat{\xi}_i)\\ &\quad= (s-1)\int_\Omega |\hat{\xi}_i|^{s-2}\Big((1+|\nabla\omega_i|^{q-2})\nabla\omega_i - (1+|\nabla\omega_{i-1}|^{q-2})\nabla\omega_{i-1}\Big)\cdot\nabla\hat{\xi}_i.\end{aligned}$$

However,

$$\nabla\hat{\xi}_i = \Big(1+\alpha_\tau'(\bar{\omega}_\tau')\Big)\frac{\nabla\omega_i-\nabla\omega_{i-1}}{\tau}$$

and, by algebraic inequality (21)

$$\Big((1+|\nabla\omega_i|^{q-2})\nabla\omega_i-(1+|\nabla\omega_{i-1}|^{q-2})\nabla\omega_{i-1}\Big)\cdot\Big(\frac{\nabla\omega_i-\nabla\omega_{i-1}}{\tau}\Big)\geq 0.$$

Then, as $\alpha_\tau'(.)\geq 0$ a.e., there follows that

$$\int\limits_\Omega\Big((1+|\nabla\omega_i|^{q-2})\nabla\omega_i-(1+|\nabla\omega_{i-1}|^{q-2})\nabla\omega_{i-1}\Big)\cdot\nabla(|\hat{\xi}_i|^{s-2}\hat{\xi}_i)\geq 0. \tag{77}$$

Finally, we investigate the terms on the right-hand side of (75). Recalling that $f(.)$ is Lipschitz continuous by **(H3)**, we get

$$\begin{aligned}
&\int\limits_\Omega\Big|\big(\theta_i-\theta_{i-1}\big)+\big(f(\omega_i)-f(\omega_{i-1})\big)\Big||\hat{\xi}_i|^{s-1}\\
&\quad\leq C\tau\int\limits_\Omega\Big|\frac{\theta_i-\theta_{i-1}}{\tau}\Big||\hat{\xi}_i|^{s-1}+\Big|\frac{\omega_i-\omega_{i-1}}{\tau}\Big||\hat{\xi}_i|^{s-1}\\
&\quad\leq C\tau\left(1+\left\|\frac{\theta_i-\theta_{i-1}}{\tau}\right\|_{L^2}^2+\left\|\frac{\omega_i-\omega_{i-1}}{\tau}\right\|_{L^2}^2+\|\hat{\xi}_i\|_{L^s}^s\right),
\end{aligned}\tag{78}$$

where we have used the Hölder's inequality.

Thus, combining (75)–(78) leads to

$$\begin{aligned}
&\int\limits_\Omega\frac{|\hat{\xi}_i|^s}{s}-\frac{|\hat{\xi}_{i-1}|^s}{s}\\
&\quad\leq\tau C\left(1+\left\|\frac{u_i-u_{i-1}}{\tau}\right\|_{L^2}^2+\left\|\frac{\theta_i-\theta_{i-1}}{\tau}\right\|_{L^2}^2+\left\|\frac{\omega_i-\omega_{i-1}}{\tau}\right\|_{W^{1,2}}^2+\|\hat{\xi}_i\|_{L^s}^s\right),
\end{aligned}$$

where $i\geq 2$.

By summing from $i=2$ to $m\leq M$ and using the estimates (48), (49) and (60), we end up with

$$\int\limits_\Omega|\hat{\xi}_m|^s\leq C+\int\limits_\Omega|\hat{\xi}_1|^s+C\sum_{i=1}^m\tau\int\limits_\Omega|\hat{\xi}_i|^s. \tag{79}$$

To obtain a bound for the L^s norm of $\hat{\xi}_i$ by using Gronwall's lemma, we must look for an estimate to $\hat{\xi}_1$. To this end, consider (25) for $i=1$ and rewrite it in the form

$$\begin{aligned}
&\hat{\xi}_1-\Delta\omega_1+\Delta\omega_0-\Delta_q\omega_1+\Delta_q\omega_0+\kappa u_1\cdot\nabla\omega_1-\kappa u_0\cdot\nabla\omega_0\\
&\quad=\theta_1-\theta_0+f(\omega_1)-f(\omega_0)+\xi_0,
\end{aligned}\tag{80}$$

recalling that

$$\xi_0=\theta_0+f(\omega_0)+\Delta\omega_0+\Delta_q\omega_0-\kappa u_0\cdot\nabla\omega_0\in L^s\ \text{(see (69))}.$$

Observe that (76), (77) and (78) remain true for $i=1$. Hence, by multiplying (80) by $|\hat{\xi}_1|^{s-2}\hat{\xi}_1$ and by integrating over Ω, there follows that

$$\int\limits_\Omega|\hat{\xi}_1|^s\leq\tau C\left(1+\left\|\frac{u_1-u_0}{\tau}\right\|_{L^2}^2+\left\|\frac{\theta_1-\theta_0}{\tau}\right\|_{L^2}^2+\left\|\frac{\nabla\omega_1-\nabla\omega_0}{\tau}\right\|_{W^{1,2}}^2+\|\hat{\xi}_1\|_{L^s}^s\right).$$

Estimates (48), (49) and (60), give us

$$\int_\Omega |\hat{\xi}_1|^s \leq C + \tau C \|\hat{\xi}_1\|_{L^s}^s.$$

Thus, for $0 < \tau < 1/2C$, we end up with

$$\int_\Omega |\hat{\xi}_1|^s \leq C. \tag{81}$$

We are now ready to prove (73). Indeed, plugging (81) into (79) yields

$$\|\bar{\omega}_\tau' + \alpha_\tau(\bar{\omega}_\tau')\|_{L^s}^s \leq C\left(1 + \int_0^t \|\bar{\omega}_\tau' + \alpha_\tau(\bar{\omega}_\tau')\|_{L^s}^s\right),$$

which by Gronwall's lemma implies

$$\|\bar{\omega}_\tau' + \alpha_\tau(\bar{\omega}_\tau')\|_{L^\infty(0,T;L^s)} \leq C.$$

From this last result and (60), we conclude that (73).

Finally, (74) follows from Eq. (46) and estimate (73). □

6. Convergences

From the estimates established in the previous section, we obtain certain convergences for the approximate functions and vector fields, which allow us to prove the existence of solutions for (1)–(7), according to Definition 3.1.

We start with weak and weak-star convergences, once they are straightforward consequences of Lemmas 5.1, 5.3 and 5.4.

Proposition 6.1. *There exist u, θ, ω and Ψ such that, up to subsequences, as $\tau \to 0$*

$$u_\tau \overset{*}{\rightharpoonup} u \quad in \quad L^\infty(0,T;V), \tag{82}$$

$$\theta_\tau \overset{*}{\rightharpoonup} \theta \quad in \quad L^\infty(0,T;W_0^{1,p}), \tag{83}$$

$$\omega_\tau \overset{*}{\rightharpoonup} \omega \quad in \quad L^\infty(0,T;W^{1,q}), \tag{84}$$

$$\bar{u}_\tau' \rightharpoonup u_t \quad in \quad L^2(0,T;H), \tag{85}$$

$$\bar{\theta}_\tau' \rightharpoonup \theta_t \quad in \quad L^2(0,T;L^2), \tag{86}$$

$$\bar{\omega}_\tau' \overset{*}{\rightharpoonup} \omega_t \quad in \quad L^\infty(0,T;L^2), \tag{87}$$

$$\bar{\omega}_\tau' \rightharpoonup \omega_t \quad in \quad L^2(0,T;W^{1,2}), \tag{88}$$

$$-\Delta_p\theta_\tau \overset{*}{\rightharpoonup} \Psi \quad in \quad L^\infty(0,T;W^{-1,p'}). \tag{89}$$

Moreover, if (15) *holds, there exist η and Φ such that, up to subsequences, as $\tau \to 0$*

$$\alpha_\tau(\bar{\omega}_\tau') \overset{*}{\rightharpoonup} \eta \ in \ L^\infty(0,T;L^s), \tag{90}$$

$$-\Delta\omega_\tau - \Delta_q\omega_\tau \overset{*}{\rightharpoonup} \Phi \ in \ L^\infty(0,T;L^s). \tag{91}$$

Some strong convergences are required to pass to the limit in the approximate problem. We compile them in the following proposition.

Proposition 6.2. *There holds that, up to subsequences, as* $\tau \to 0$

$$u_\tau \to u \text{ in } L^\infty(0,T;H), \tag{92}$$
$$\bar{u}_\tau \to u \text{ in } C([0,T];H), \tag{93}$$
$$\theta_\tau \to \theta \text{ in } L^\infty(0,T;L^2) \tag{94}$$
$$\bar{\theta}_\tau \to \theta \text{ in } C([0,T];L^2), \tag{95}$$
$$\omega_\tau \to \omega \text{ in } L^\infty(0,T;C(\overline{\Omega})) \cap L^\infty(0,T;W^{1-\epsilon,q}), \tag{96}$$
$$\bar{\omega}_\tau \to \omega \text{ in } C([0,T];C(\overline{\Omega})) \cap C([0,T];W^{1-\epsilon,q}). \tag{97}$$

Proof. Since we have obtained estimates regarding different kinds of approximations for the unknowns (step functions and linear approximations), we must combine them in order to obtain further information on the approximations. Indeed, we claim that, as $\tau \to 0$,

$$u_\tau - \bar{u}_\tau \to 0 \text{ in } L^\infty(0,T;H), \tag{98}$$
$$\theta_\tau - \bar{\theta}_\tau \to 0 \text{ in } L^\infty(0,T;L^2), \tag{99}$$
$$\omega_\tau - \bar{\omega}_\tau \to 0 \text{ in } L^\infty(0,T;W^{1,q}). \tag{100}$$

To prove (98), notice that $\|u_\tau - \bar{u}_\tau\|_H \leq \|u_i - u_{i-1}\|_H$ and observe that $u_i - u_{i-1} = \int_{(i-1)\tau}^{i\tau} \frac{u_i - u_{i-1}}{\tau} ds$, then use estimate (48). For (99), we proceed analogously by using (49). The proof of (100) is a little different by the lack of estimates for $\bar{\omega}'_\tau$ in $W^{1,q}$. However, this is handled by using estimates (60) and (61) (for details see [10, Prop. 3]).

Since (96) is a direct consequence of (97) and (100), we concentrate now in (97). Notice that the definitions of $\bar{\omega}_\tau$ and ω_τ combined with (50) lead to

$$\|\bar{\omega}_\tau\|_{L^\infty(0,T;W^{1,q})} \leq C,$$

with C independent on $\tau > 0$.

However, as $q > N$, there exists $\epsilon > 0$, sufficiently small, such that

$$1 - \epsilon - N/q > 0.$$

In this way, due to [25, Thm 1.4.3.2 and 1.4.4.1], there holds

$$W^{1,q} \hookrightarrow\hookrightarrow W^{1-\epsilon,q} \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^2.$$

So, since $\bar{\omega}'_\tau$ is bounded in $L^2(0,T;L^2)$ by (50), we can apply [33, Cor. 4] and conclude

$$\bar{\omega}_\tau \to \omega \text{ in } C([0,T];W^{1-\epsilon,q}) \cap C([0,T];C(\bar{\Omega})).$$

Similar arguments prove (92)–(95). □

7. Proof of Theorem 3.2: existence of solutions

In this section, the essential tools developed in the previous sections will be employed for proving Theorem 3.2.

Let us remark that the regularity established in Definition 3.1 for u, θ, ω and η follows from Propositions 6.1 and 6.2. We prove that Eqs. (9)–(11) are satisfied by passing to the limit in the approximate system (44)–(46).

We split the proof in five steps.

Step 1. Let us consider Eq. (44). Let $\phi \in L^2(0,T;V)$ be such that *supp* $\phi \subset\subset Q_{ml} = \{(x,t) : 0 \leq h(\omega(x,t)) < 1\}$, which is an open set because $\omega \in C(\overline{Q})$.

Then, by (85), (82), and (94) one obtains that, as $\tau \to 0$,

$$\int_0^T \int_\Omega \bar{u}'_\tau \cdot \phi \to \int_0^T \int_\Omega u_t \cdot \phi,$$
$$\int_0^T \int_\Omega \nabla u_\tau \cdot \nabla \phi \to \int_0^T \int_\Omega \nabla u \cdot \nabla \phi, \tag{101}$$
$$\int_0^T \int_\Omega \zeta\theta_\tau \cdot \phi \to \int_0^T \int_\Omega \zeta\theta \cdot \phi.$$

We claim that, as $\tau \to 0$,

$$K_\tau(h(\omega_\tau)) = K_{ext}(h(\omega_\tau) - \tau) \to \; K(h(\omega)) \;\text{ uniformly a.e. in } supp\ \phi. \tag{102}$$

Indeed, on the one hand, observe that by (96) $\omega_\tau \to \omega$ uniformly a.e. in $(0, T) \times \overline{\Omega}$. Then, by **(H6)**, as $\tau \to 0$,

$$h(\omega_\tau) \to h(\omega) \text{ uniformly a.e. in } supp\ \phi.$$

On the other hand, fix $\delta > 0$ such that $h(\omega(x, t)) \leq 1-2\delta$, for every $(x, t) \in supp\ \phi$; so there exists $\tau_1 > 0$ such that for every $0 < \tau \leq \tau_1$,

$$h(\omega_\tau(x, t)) \leq 1 - \delta \text{ a.e. in } supp\ \phi$$

and then, for any $\tau > 0$ such that $0 < \tau \leq \tau_1$

$$h(\omega_\tau(x, t)) - \tau \in [-\tau_1, 1 - \delta] \text{ a.e. in } supp\ \phi.$$

Finally, as by construction K_{ext} is uniformly continuous in $[-\tau_1, 1 - \delta]$, the claim is proved.

Hence, from (92), (85) and (102), we find that

$$\int_0^T \int_\Omega K_\tau(h(\omega_\tau))(u_\tau + \bar{u}'_\tau) \cdot \phi \to \int_0^T \int_\Omega K(h(\omega))(u + u_t) \cdot \phi. \tag{103}$$

Thus, by taking $\tau \to 0$ in Eq. (44), from (101) and (103), we conclude that Eq. (9) is satisfied.

Step 2. We now consider Eq. (45). Let $\psi \in L^p(0, T; W_0^{1,p})$. As $\tau \to 0$, by (83), (86) and (87), we have

$$\int_0^T \int_\Omega (\bar{\theta}'_\tau + \bar{\omega}'_\tau)\psi \to \int_0^T \int_\Omega (\theta_t + \omega_t)\psi,$$
$$\int_0^T \int_\Omega \nabla\theta_\tau \cdot \nabla\psi \to \int_0^T \int_\Omega \nabla\theta \cdot \nabla\psi. \tag{104}$$

Moreover, it is straightforward to check that, as $\tau \to 0$

$$\int_0^T \int_\Omega g_\tau \psi \to \int_0^T \int_\Omega g\psi. \tag{105}$$

Thus, by (83) and (92) there follows that

$$\int_0^T \int_\Omega u_\tau \cdot \nabla\theta_\tau \psi \to \int_0^T \int_\Omega u \cdot \nabla\theta\psi. \tag{106}$$

Futher, we focus on the identification of the p-Laplacian. Indeed, notice that by (89), there exists $\Psi \in L^{p'}(0,T;W^{-1,p'})$ such that, as $\tau \to 0$,

$$\int_0^T \langle -\Delta_p \theta_\tau, \psi\rangle = \int_0^T\int_\Omega |\nabla\theta_\tau|^{p-2}\nabla\theta_\tau \cdot \nabla\psi \to \int_0^T \langle \Psi, \psi\rangle\,, \tag{107}$$

for all $\psi \in L^p(0,T;W^{1,p})$, where $\langle\ ,\ \rangle$ denotes the duality product between $W_0^{1,p}$ and $W^{-1,p'}$.

Additionally, we have that

$$\int_0^T\int_\Omega (\bar\theta'_\tau + \bar\omega'_\tau)\theta_\tau \to \int_0^T\int_\Omega (\theta_t + \omega_t)\theta \text{ by (94), (86) and (87),}$$

$$\int_0^T\int_\Omega g_\tau\theta_\tau \to \int_0^T\int_\Omega g\theta \text{ by (94) ,}$$

$$\limsup_{\tau\to 0} -\int_0^T\int_\Omega |\nabla\theta_\tau|^2 \le -\int_0^T\int_\Omega |\nabla\theta|^2 \text{ by (83).}$$

Thus, by multiplying Eq. (45) by θ_τ and using the previous convergences, we arrive at

$$\limsup_{\tau\to 0}\int_0^T \langle -\Delta_p\theta_\tau, \theta_\tau\rangle \le \int_0^T \langle \Psi, \theta\rangle\,.$$

Hence, recalling that $-\Delta_p$ is an maximal monotone operator on $L^p(0,T;W_0^{1,p}) \times L^{p'}(0,T;W^{-1,p'})$, by [3, Cor. 2.4], we have that

$$\Psi = -\Delta_p\theta. \tag{108}$$

From (104)–(108), we can pass to the limit in Eq. (45), leading to

$$\int_0^T\int_\Omega (\theta_t + \omega_t)\psi + (1 + |\nabla\theta|^{p-2})\nabla\theta\cdot\nabla\psi + u\cdot\nabla\theta\psi = \int_0^T\int_\Omega g\psi,$$

for any $\psi \in L^p(0,T;W^{1,p})$. Now, by an argument similar to that used in the proof of Proposition 4.3, there follows that

$$\theta_t + \omega_t - \Delta\theta - \Delta_p\theta + u\cdot\nabla\theta = g \text{ a.e. in } Q$$

and (10) holds.

Step 3. At this step, we consider Eq. (46). Initially, observe that given $\xi \in L^q(0,T;W^{1,q})$, we obtain the following convergences, as $\tau \to 0$,

$$\int_0^T\int_\Omega (\alpha_\tau(\bar\omega'_\tau) + \bar\omega'_\tau)\xi \to \int_0^T\int_\Omega (\eta + \omega_t)\xi \text{ by (87) and (90),}$$

$$\int_0^T\int_\Omega \nabla\omega_\tau\cdot\nabla\xi + \kappa u_\tau\cdot\nabla\omega_\tau\xi \to \int_0^T\int_\Omega \nabla\omega\cdot\nabla\xi + \kappa u\cdot\nabla\omega\xi \text{ by (92) and (96),}$$

$$\int_0^T\int_\Omega (\theta_\tau + f(\omega_\tau))\xi \to \int_0^T\int_\Omega (\theta + f(\omega))\xi \text{ by (94) and (96),}$$

where we have used that $f(.)$ is Lipschitz continuous by **(H3)**.

For the identification of the q-Laplacian, notice that by proceeding as it was done to obtain (107)–(108), we prove the convergence

$$-\Delta_q\omega_\tau \overset{*}{\rightharpoonup} -\Delta_q\omega \text{ em } L^\infty(0,T;(W^{1,q})'),$$

and in particular, we have that

$$\int_0^T\int_\Omega |\nabla\omega_\tau|^{q-2}\nabla\omega_\tau\cdot\nabla\xi \to \int_0^T\langle -\Delta_q\omega,\xi\rangle,$$

where now $\langle\ ,\rangle$ represents the duality product between $W^{1,q}$ and $(W^{1,q})'$.

The previous convergences allow us to pass to the limit in (46) and obtain

$$\int_0^T\int_\Omega \eta\xi+\omega_t\xi+(1+|\nabla\omega|^{q-2})\nabla\omega\cdot\nabla\xi+\kappa u\cdot\nabla\omega\xi=\int_0^T\int_\Omega(\theta+f(\omega))\xi,$$

for any $\xi\in L^q(0,T;W^{1,q})$. Once again, by a straightforward adaptation of Proposition 4.3, we infer that

$$\eta+\omega_t-\Delta\omega-\Delta_q\omega+\kappa u\cdot\nabla\omega=\theta+f(\omega)\ \text{ a.e. in } Q, \tag{109}$$

and (11) holds.

Step 4. We identify η by employing the so-called Minty's trick.

Notice that

$$\limsup_{\tau\to 0} -\int_0^T\int_\Omega |\bar\omega_\tau'|^2 \le -\int_0^T\int_\Omega \omega_t^2 \text{ by (87)}. \tag{110}$$

Moreover, from (88), (92), (94) and (96)

$$\int_0^T\int_\Omega u_\tau\cdot\nabla\omega_\tau\bar\omega_\tau' = -\int_0^T\int_\Omega u_\tau\cdot\nabla\bar\omega_\tau'\omega_\tau \to \int_0^T\int_\Omega u\cdot\nabla\omega\omega_t,$$

$$\int_0^T\int_\Omega (f(\omega_\tau)+\theta_\tau)\bar\omega_\tau' \to \int_0^T\int_\Omega (f(\omega)+\theta)\omega_t. \tag{111}$$

By using algebraic inequality (22), we have that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla\omega_\tau|^{q-2}\nabla\omega_\tau\cdot\nabla\bar\omega_\tau' &= \sum_{i=1}^M \tau\int_\Omega |\nabla\omega_i|^{q-2}\nabla\omega_i\cdot\frac{\nabla\omega_i-\nabla\omega_{i-1}}{\tau}\\ &\ge \int_\Omega\frac{|\nabla\omega_M|^q}{q}-\int_\Omega\frac{|\nabla\omega_0|^q}{q}\\ &= \frac{\|\nabla\bar\omega_\tau(T)\|_{L^q}^q}{q}-\frac{\|\nabla\omega_0\|_{L^q}^q}{q}.\end{aligned}$$

From (84) and (97), $\bar\omega_\tau(T)\rightharpoonup\omega(T)$ in $W^{1,q}$ as $\tau\to 0$, then we find that

$$\liminf_{\tau\to 0}\int_0^T\int_\Omega |\nabla\omega_\tau|^{q-2}\nabla\omega_\tau\cdot\nabla\bar\omega_\tau' \ge \frac{\|\nabla\omega(T)\|_{L^q}^q}{q}-\frac{\|\nabla\omega_0\|_{L^q}^q}{q}. \tag{112}$$

Similarly, we have that

$$\liminf_{\tau\to 0}\int_0^T\int_\Omega \nabla\omega_\tau\cdot\nabla\bar{\omega}'_\tau \geq \frac{\|\nabla\omega(T)\|^2_{L^2}}{2}-\frac{\|\nabla\omega_0\|^2_{L^2}}{2}. \tag{113}$$

Thus, by taking $\xi=\bar{\omega}'_\tau$ in Eq. (46), integrating in time, and combining (110)-(113), there follows

$$\begin{aligned}\limsup_{\tau\to 0}&\int_0^T\int_\Omega \alpha_\tau(\bar{\omega}'_\tau)\bar{\omega}'_\tau\\ &\leq \int_0^T\int_\Omega (f(\omega)+\theta)\omega_t-\omega_t^2-\kappa u\cdot\nabla\omega\omega_t\\ &\quad -\frac{\|\nabla\omega(T)\|^2_{L^2}}{2}+\frac{\|\nabla\omega_0\|^2_{L^2}}{2}-\frac{\|\nabla\omega(T)\|^q_{L^q}}{q}+\frac{\|\nabla\omega_0\|^q_{L^q}}{q}.\end{aligned} \tag{114}$$

We claim that $\Upsilon(\omega(.))=\int_\Omega \frac{|\nabla\omega(.)|^2}{2}+\frac{|\nabla\omega(.)|^q}{q}:L^{s'}\to\mathbb{R}\cup\{+\infty\}$ is absolutely continuous in $[0,T]$ and that it holds

$$-\int_0^T\int_\Omega (\Delta\omega+\Delta_q\omega)\omega_t=\frac{\|\nabla\omega(T)\|^2_{L^2}}{2}-\frac{\|\nabla\omega_0\|^2_{L^2}}{2}+\frac{\|\nabla\omega(T)\|^q_{L^q}}{q}-\frac{\|\nabla\omega_0\|^q_{L^q}}{q}. \tag{115}$$

Indeed, this follows by a direct adaptation of [18, Prop. 4.2] when $\omega_t\in L^2(0,T;L^{s'})$ and $-(\Delta\omega+\Delta_q\omega)\in L^2(0,T;L^s)$. Observe that s is such that $W^{1,2}\hookrightarrow L^{s'}$ (see (15)).

Thus, by multiplying (109) by ω_t, after an integration over Q, and using (115), we arrive at

$$\begin{aligned}\int_0^T\int_\Omega \eta\omega_t=&\int_0^T\int_\Omega (f(\omega)+\theta)\omega_t-\omega_t^2-\kappa u\cdot\nabla\omega\omega_t\\ &-\frac{\|\nabla\omega(T)\|^2_{L^2}}{2}+\frac{\|\nabla\omega_0\|^2_{L^2}}{2}-\frac{\|\nabla\omega(T)\|^q_{L^q}}{q}+\frac{\|\nabla\omega_0\|^q_{L^q}}{q}.\end{aligned} \tag{116}$$

Hence, by comparing (114) and (116), we conclude that

$$\limsup_{\tau\to 0}\int_0^T\int_\Omega \alpha_\tau(\bar{\omega}'_\tau)\bar{\omega}'_\tau\leq\int_0^T\int_\Omega \eta\omega_t. \tag{117}$$

Next, recall that (see (28))

$$\alpha_\tau=(\gamma_\tau+\tau|I|^{s-2}I)^{-1},$$

and fix $\eta_\tau=\alpha_\tau(\bar{\omega}'_\tau)$, so that

$$\gamma_\tau(\eta_\tau)+\tau|\eta_\tau|^{s-2}\eta_\tau=\bar{\omega}'_\tau.$$

Now, as $\gamma_\tau(\eta_\tau)\eta_\tau+\tau|\eta_\tau|^s=\eta_\tau\bar{\omega}'_\tau$, from estimative (73) there follows that

$$\limsup_{\tau\to 0}\int_0^T\int_\Omega \gamma_\tau(\eta_\tau)\eta_\tau\leq\int_0^T\int_\Omega \eta\omega_t.$$

By using again estimate (73), we obtain that

$$\tau|\eta_\tau|^{s-2}\eta_\tau \to 0 \text{ in } L^2(0,T;L^{s'}).$$

Therefore, as $W^{1,2} \hookrightarrow L^{s'}$ (see (15)), the combination of (88) and (90) entails that

$$\gamma_\tau(\eta_\tau) \rightharpoonup \omega_t \text{ in } L^2(0,T;L^{s'}) \ \text{ and } \ \eta_\tau \rightharpoonup \eta \text{ in } L^2(0,T;L^s), \tag{118}$$

Hence, by [3, Cor. 2.4], we conclude that

$$\omega_t \in \gamma(\eta) \ \text{ or equivalently } \ \eta \in \alpha(\omega_t);$$

thus (12) is also satisfied.

Step 5. At this final step, we prove that both Eq. (13) and the initial conditions (14) hold. Indeed, as $\omega_\tau \to \omega$ uniformly a.e. in Q,

$$h(\omega_\tau) \to 1 \text{ uniformly a.e. in } Q_s, \text{ by } \mathbf{(H6)}.$$

From **(H5)** and by the definition of K_τ, there follows that

$$K_\tau(h(\omega_\tau)) \to +\infty \text{ uniformly a.e. in } Q_s.$$

By combining the last convergence with the estimate (51), we infer $\|u_\tau + \bar{u}'_\tau\|_{L^2(Q_s)} \to 0$, as $\tau \to 0$. Hence, convergences (92) and (85) furnish

$$u + u_t = 0 \text{ a.e. in } Q_s,$$

so that (13) is satisfied.

Finally, the initial data (14) follow directly from (47) and convergences (93), (95) and (97). The proof of Theorem 3.2 is then complete. □

8. Improved regularity for the non-advective case

In this section, we assume that $\kappa = 0$, and thus, there is no convective term in the phase-field inclusion. We use the same notations as in the proof of Theorem 3.2.

Let us remark that, in this case, we can take $s = 2$ in the definition of α_τ (cf. (28)) and can follow all the arguments in previous sections, which are in fact simplified. Moreover, we obtain better estimates and convergences for the approximations.

We start by improving estimates (90) and (91) in the following lemma.

Lemma 8.1. *There exists $C > 0$, not depending on $\tau > 0$, such that*

$$\|\alpha_\tau(\bar{\omega}'_\tau)\|_{L^\infty(0,T;L^2)} \le C, \tag{119}$$

$$\|\Delta\omega_\tau + \Delta_q\omega_\tau\|_{L^\infty(0,T;L^2)} \le C. \tag{120}$$

Proof. The argument for proving both (119) and (120) is a direct adaptation of the proof of Lemma 5.4. Indeed, since $\kappa = 0$, we just need to consider $\hat{\xi}_i$ as the multiplier and proceed analogously. □

It turns out that it is possible to obtain extra fractional regularity for the solutions of (1)–(7). This is due to the application of certain tools regarding regularity theory and p-Laplacian operators, which will require $\partial\Omega \in C^3$ (cf. [10, Lem. 1]). To the best of our knowledge, it is unknown whether this regularity can be obtained for the full model when $\kappa \neq 0$. The reader is referred to [10] for further details.

Lemma 8.2. *There exists $C > 0$, not depending on $\tau > 0$, such that*

$$\|\omega_\tau\|_{L^\infty(0,T;\mathcal{N}^{1+2/q,q})} + \|\omega_\tau\|_{L^\infty(0,T;W^{2,2})} \leq C. \tag{121}$$

Proof. Since $\kappa = 0$, Eq. (46) becomes

$$\bar{\omega}'_\tau + \alpha_\tau(\bar{\omega}'_\tau) - \Delta\omega_\tau - \Delta_q\omega_\tau = \theta_\tau + f(\omega_\tau) \text{ a.e. in } Q.$$

Thus, by [10, Lem. 7], we have the estimate

$$\begin{aligned} &\|\omega_\tau\|^q_{L^\infty(0,T;\mathcal{N}^{1+2/q,q})} + \|\omega_\tau\|^2_{L^\infty(0,T;W^{2,2})} \\ &\quad \leq C\big(\|f(\omega_\tau) + \theta_\tau\|^2_{W^{1,2}(0,T;L^2)} + \|\omega_0\|^2_{W^{2,2}} + \|\omega_0\|^{2p}_{W^{1,2q}} + \|\omega_0\|^p_{W^{2,q}}\big). \end{aligned}$$

By **(H3)** and **(H8)**, combined with (49) and (50), we conclude the proof. □

Proof of Theorem 3.3.

Essentially, we use the extra information given in the last two lemmas in order to improve the regularity of both ω and η.

Indeed, it is clear that (17) is consequence of estimate (121). The identification of η follows by the Minty's trick as in Step 4 of the proof of Theorem 3.2, but now from (119)–(120), we can take $s = s' = 2$. In this way, (118) becomes

$$\gamma_\tau(\eta_\tau) \rightharpoonup \omega_t \text{ in } L^2(0,T;L^2) \text{ and } \eta_\tau \rightharpoonup \eta \text{ in } L^2(0,T;L^2)$$

which combined with (117), gives (16). The proof of Theorem 3.3 is finished.

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