Z. Angew. Math. Phys. 66 (2015), 677–687

0044-2275/15/030677-11
published online April 2, 2014
DOI 10.1007/s00033-014-0415-8

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global regularity of 2D generalized MHD equations with magnetic diffusion

Quansen Jiu and Jiefeng Zhao

Abstract. This paper is concerned with the global regularity of the 2D (two-dimensional) generalized magnetohydrodynamic equations with only magnetic diffusion $\Lambda^{2\beta}b$. It is proved that when $\beta > 1$ there exists a unique global regular solution for this equations. The obtained result improves the previous known ones which require that $\beta > \frac{3}{2}$. With help of Fourier analysis, Besov spaces and singular integral theory, some delicate estimates on the vorticity ω and the current j are established to prove our main result.

Mathematics Subject Classification. 35Q35 · 35Q60.

Keywords. MHD equations · Global existence · Uniqueness · Magnetic diffusion · Besov space.

1. Introduction

Consider the Cauchy problem of the following two-dimensional generalized magnetohydrodynamic (GMHD) equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - \nu \Lambda^{2\alpha} u, \\ b_t + u \cdot \nabla b = b \cdot \nabla u - \kappa \Lambda^{2\beta} b, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u\,(x,0) = u_0\,(x)\,, \quad b\,(x,0) = b_0\,(x) \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^2$ and $t > 0$, where $u = u\,(x,t)$ is the velocity, $b = b\,(x,t)$ is the magnetic, $p = p\,(x,t)$ is the pressure, and $u_0\,(x)\,, b_0\,(x)$ with $\mathrm{div} u_0\,(x) = \mathrm{div} b_0\,(x) = 0$ are the initial velocity and magnetic, respectively. Here $\nu, \kappa, \alpha, \beta \geq 0$ are nonnegative constants and Λ is defined by

$$\widehat{\Lambda f}(\xi) = |\xi|\,\widehat{f}(\xi),$$

where $\wedge$ denotes the Fourier transform. In the following sections, we will use the inverse Fourier transform $\vee$.

The global regularity of the 2D GMHD Eq. (1.1) has attracted a lot of attention and there have been extensively studies (see [2–8,12,14–19]). It follows from [15] that the problem (1.1) has a unique global regular solution if

$$\alpha \geqslant 1, \quad \beta > 0, \quad \alpha + \beta \geqslant 2.$$

Tran et al. [12] got a global regular solution under assumptions that

$$\alpha \geqslant 1/2,\ \beta \geqslant 1 \qquad \text{or} \qquad 0 \leqslant \alpha < 1/2,\ 2\alpha + \beta > 2 \qquad \text{or} \qquad \alpha \geqslant 2, \quad \beta = 0.$$

Recently, it was shown in [7] that if $0 \leqslant \alpha < 1/2,\ \beta \geqslant 1,\ 3\alpha + 2\beta > 3$, then the solution is globally regular. In particular, when $\alpha = 0, \beta > \frac{3}{2}$, the solution is globally regular. This was proved independently

The research is partially supported by National Natural Sciences Foundation of China (Nos. 11171229, 11231006) and Project of Beijing Chang Cheng Xue Zhe.

in [17,18]. Meanwhile, Fan et al. [5] used properties of the heat equation and presented a global regular solution when $0 < \alpha < \frac{1}{2}, \beta = 1$.

In this paper, we aim at getting the global regular solution of (1.1) when $\nu = 0, \kappa > 0$ and $\beta > 1$. For simplicity, we let $\kappa = 1$. That is, we consider

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u - \Lambda^{2\beta} b, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x). \end{cases} \tag{1.2}$$

Let $\omega = -\partial_2 u_1 + \partial_1 u_2$ and $j = -\partial_2 b_1 + \partial_1 b_2$ be the vorticity and the current, respectively. We will prove that $\omega, j \in L^2(0, T; L^\infty)$ and obtain the global regularity of the solution by the BKM type criterion in [2]. To this end, we will take advantage of the approaches used in [5] and [7] to deal with the higher regularity estimates of j. More precisely, using the equation satisfied by the current j, we will obtain the estimates of $\|\Lambda^r j\|_{L^2}^2 (t) + \int_0^t \|\Lambda^{\beta+r} j\|_{L^2}^2 \le C$ with $r = \beta - 1$. Using the singular integral representation of $\Lambda^\delta j$ with some $\delta > 0$, we will obtain the estimate $\|\nabla j\|_{L^2(0,T;L^\infty(\mathbb{R}^2))}$. Then, we get the estimates of $\|\omega\|_{L^2(0,T;L^\infty(\mathbb{R}^2))}$ using the particle trajectory method. In the last, we apply Besov space to prove that $(u, b) \in C([0, T]; H^\rho(\mathbb{R}^2))$ with $\rho > \max\{2, \beta\}$. It should be noted that after the paper is finished, at the almost same time, Cao et al. obtain the similar result independently using a different method (see [4]). In comparison with result obtained in [4], it is not required that $\|\nabla j_0\|_{L^\infty} < \infty$ in our result. Moreover, the proof is much more direct and concise here.

It should be remarked that the global existence and uniqueness of regular solution to (1.2) when $\beta = 1$ remain open in general. In this case, the global classical solution was constructed in [8] for bounded domain and [3] for exterior domain and half space under the assumption that the initial data is small, respectively. Also, when $\nu > 0, \kappa = 0$ and $\alpha = 1$, the global existence and uniqueness of regular solution to (1.1) remain unsolved except for recent results under assuming small perturbation around a steady state (see [9]).

The main result of this paper is stated as follows:

Theorem 1.1. *Let $\beta > 1$ and assume that $(u_0, b_0) \in H^\rho$ with $\rho > \max\{2, \beta\}$. Then for any $T > 0$, the Cauchy problem (1.2) has a unique regular solution*

$$(u, b) \in C([0, T]; H^\rho(\mathbb{R}^2)) \quad \textit{and} \quad b \in L^2([0, T]; H^{\rho+\beta}(\mathbb{R}^2)).$$

Remark 1.1. When $\alpha = 0, \beta > \frac{3}{2}, \rho > 2$, the global regularity has been obtained in [7,17] and [18].

2. Preliminaries

Let us first consider the following equation

$$\begin{cases} v_t + \Lambda^{2\beta} v = f \\ v(x, 0) = v_0(x). \end{cases}$$

Similar to the heat equation, we can get

$$v(x,t) = \int_{\mathbb{R}^2} t^{-\frac{1}{\beta}} h\left(\frac{x-y}{t^{\frac{1}{2\beta}}}\right) v_0(y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}^2} (t-s)^{-\frac{1}{\beta}} h\left(\frac{x-y}{(t-s)^{\frac{1}{2\beta}}}\right) f(y,s)\, \mathrm{d}y \mathrm{d}s, \tag{2.3}$$

where $h(x) = \left(\mathrm{e}^{-|\cdot|^{2\beta}}\right)^\vee (x)$ and it has the similar properties as the heat kernel (see [10,13] and references therein). The following lemma will be needed to prove our main result.

Lemma 2.1. *Let l be a nonnegative integer and $\eta \geqslant 0$ be a nonnegative real number. Then*

$$\left\|\nabla^l h\right\|_{L^1} + \|\Lambda^\eta h\|_{L^1} \leqslant C. \tag{2.4}$$

Proof. First, we give the proof of the estimates of $\nabla^l h$.

$$\begin{aligned}
\left\| \nabla^l h\right\|_{L^1} &= C \sup_{|\gamma|=l} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \xi^\gamma \mathrm{e}^{-|\xi|^{2\beta}} \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \right| \mathrm{d}x \\
&= C \sup_{|\gamma|=l} \int_{|x|\leqslant 1} \left| \int_{\mathbb{R}^2} \xi^\gamma \mathrm{e}^{-|\xi|^{2\beta}} \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \right| \mathrm{d}x + C \int_{|x|\geqslant 1} \left| \int_{\mathbb{R}^2} \xi^\gamma \mathrm{e}^{-|\xi|^{2\beta}} \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \right| \mathrm{d}x \\
&\leqslant C + C \sup_{|\gamma|=l} \int_{|x|\geqslant 1} (1+|x|^2)^{-2} \left| \int_{\mathbb{R}^2} \xi^\gamma \mathrm{e}^{-|\xi|^{2\beta}} (1-\Delta_\xi)^2 \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \right| \mathrm{d}x \\
&\leqslant C + C \sup_{|\gamma|=l} \int_{|x|\geqslant 1} (1+|x|^2)^{-2} \left| \int_{\mathbb{R}^2} (1-\Delta_\xi)^2 (\xi^\gamma \mathrm{e}^{-|\xi|^{2\beta}}) \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \right| \mathrm{d}x \\
&\leqslant C.
\end{aligned}$$

Next, we start to estimate $\Lambda^\eta h$ and let $l > \eta$.

$$\begin{aligned}
\|\Lambda^\eta h\|_{L^1} &= \left\| \sum_{k\geqslant -1} \Delta_k \Lambda^\eta h \right\|_{L^1} \\
&\leqslant \|\Delta_{-1}\Lambda^\eta h\|_{L^1} + \sum_{k\geqslant 0} \|\Delta_k \Lambda^\eta h\|_{L^1} \\
&\leqslant C \|h\|_{L^1} + C \sum_{k\geqslant 0} 2^{k(-l+\eta)} \left\|\Delta_k \nabla^l h\right\|_{L^1} \\
&\leqslant C + C \sum_{k\geqslant 0} 2^{k(-l+\eta)} \left\|\nabla^l h\right\|_{L^1} \\
&\leqslant C,
\end{aligned}$$

where we used the nonhomogeneous Littlewood–Paley decompositions $\mathrm{Id} = \sum_k \Delta_k$ and Bernstein-type inequalities (see [1,11]). □

Denote $\omega = \nabla^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2$ the vorticity of the velocity fields and $j = \nabla^\perp \cdot b = -\partial_2 b_1 + \partial_1 b_2$ the current of the magnetic fields. Applying $\nabla^\perp \cdot$ on both sides of the Eq. (1.2), we obtain the following equations for ω and j:

$$\omega_t + u\cdot\nabla\omega = b\cdot\nabla j, \tag{2.5}$$

$$j_t + u\cdot\nabla j = b\cdot\nabla\omega + T(\nabla u, \nabla b) - \Lambda^{2\beta} j, \tag{2.6}$$

where

$$T(\nabla u, \nabla b) = 2\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2 (\partial_1 b_2 + \partial_2 b_1).$$

Lemma 2.2. *Let $u_0, b_0 \in H^1$. Then for any $T > 0$ and $0 < t < T$, we have*

$$\|\omega\|_{L^2}^2(t) + \|j\|_{L^2}^2(t) + \int_0^t \left\|\Lambda^\beta j\right\|_{L^2}^2 \mathrm{d}\tau \leqslant C(T). \tag{2.7}$$

The proof of Lemma 2.2 is referred to [12], and we omit it here.

The following lemma is from [7]. For completeness, we present the sketch of the proof in the following.

Lemma 2.3. *Let $r = \beta - 1$ and $k \geqslant \beta$. Let $u_0, b_0 \in H^k$. Then for any $T > 0$ and $0 < t < T$, we have*

$$\|\Lambda^r j\|_{L^2}^2 (t) + \int_0^t \left\|\Lambda^{\beta+r} j\right\|_{L^2}^2 d\tau \leqslant C(T). \tag{2.8}$$

Proof. Applying Λ^r on both sides of the Eq. (2.6), we obtain

$$(\Lambda^r j)_t + \Lambda^r(u \cdot \nabla j) = \Lambda^r(b \cdot \nabla \omega) + \Lambda^r(T(\nabla u, \nabla b)) - \Lambda^{2\beta+r} j. \tag{2.9}$$

Multiplying (2.9) by $\Lambda^r j$ and integrating with respect to x in $\mathbb{R}^2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^r j\|_{L^2}^2 + \left\|\Lambda^{\beta+r} j\right\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \Lambda^r(u \cdot \nabla j)\Lambda^r j \mathrm{d}x + \int_{\mathbb{R}^2} \Lambda^r(b \cdot \nabla \omega)\Lambda^r j \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} \Lambda^r T(\nabla u, \nabla b)\Lambda^r j \mathrm{d}x \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.10}$$

I_1 is estimated as follows:

$$\begin{aligned}|I_1| &= \left|\int_{\mathbb{R}^2} \Lambda^r(u \cdot \nabla j)\Lambda^r j \mathrm{d}x\right| \\ &= \left|\int_{\mathbb{R}^2} \Lambda^{-1}\nabla \cdot (uj)\Lambda^{\beta+r} j \mathrm{d}x\right| \\ &\leqslant \left\|\Lambda^{-1}\nabla \cdot (uj)\right\|_{L^2}\left\|\Lambda^{\beta+r} j\right\|_{L^2} \\ &\leqslant C\|uj\|_{L^2}\left\|\Lambda^{\beta+r} j\right\|_{L^2} \\ &\leqslant C\|u\|_{L^4}\|j\|_{L^4}\left\|\Lambda^{\beta+r} j\right\|_{L^2} \\ &\leqslant C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{1-\frac{1}{2\beta}}\left\|\Lambda^\beta j\right\|_{L^2}^{\frac{1}{2\beta}}\left\|\Lambda^{\beta+r} j\right\|_{L^2} \\ &\leqslant C\epsilon\left\|\Lambda^{\beta+r} j\right\|_{L^2}^2 + C(\epsilon)\|u\|_{L^2}\|\omega\|_{L^2}\|j\|_{L^2}^{2-\frac{1}{\beta}}\left\|\Lambda^\beta j\right\|_{L^2}^{\frac{1}{\beta}}\end{aligned}$$

where we have used the following Gagliardo–Nirenberg inequalities

$$\|u\|_{L^4} \leqslant C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}};$$
$$\|j\|_{L^4} \leqslant C\|j\|_{L^2}^{1-\frac{1}{2\beta}}\left\|\Lambda^\beta j\right\|_{L^2}^{\frac{1}{2\beta}}.$$

Similarly, we can deal with I_2 as follows:

$$\begin{aligned}|I_2| &= \left|\int_{\mathbb{R}^2} \Lambda^r(b \cdot \nabla \omega)\Lambda^r j \mathrm{d}x\right| \\ &= \left|\int_{\mathbb{R}^2} \Lambda^{-1}\nabla \cdot (b\omega)\Lambda^{\beta+r} j \mathrm{d}x\right| \\ &\leqslant \left\|\Lambda^{-1}\nabla \cdot (b\omega)\right\|_{L^2}\left\|\Lambda^{\beta+r} j\right\|_{L^2} \\ &\leqslant C\|b\omega\|_{L^2}\left\|\Lambda^{\beta+r} j\right\|_{L^2}\end{aligned}$$

$$\leqslant C\|b\|_{L^\infty}\|\omega\|_{L^2}\left\|\Lambda^{\beta+r}j\right\|_{L^2}$$
$$\leqslant C\|b\|_{L^2}^{\frac{\beta}{1+\beta}}\left\|\Lambda^{\beta+1}b\right\|_{L^2}^{\frac{1}{1+\beta}}\|\omega\|_{L^2}\left\|\Lambda^{\beta+r}j\right\|_{L^2}$$
$$\leqslant C\epsilon\left\|\Lambda^{\beta+r}j\right\|_{L^2}^2+C(\epsilon)\|b\|_{L^2}^{\frac{2\beta}{1+\beta}}\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{2}{1+\beta}}\|\omega\|_{L^2}^2,$$

where we have used the following Gagliardo–Nirenberg inequalities

$$\|b\|_{L^\infty}\leqslant C\|b\|_{L^2}^{\frac{\beta}{1+\beta}}\left\|\Lambda^{1+\beta}b\right\|_{L^2}^{\frac{1}{1+\beta}}.$$

Now, we give estimate of I_3.

$$\begin{aligned}
|I_3| &= \left|\int_{\mathbb{R}^2}\Lambda^r T(\nabla u,\nabla b)\,\Lambda^r j\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^2}T(\nabla u,\nabla b)\,\Lambda^{2r} j\,\mathrm{d}x\right| \\
&\leqslant C\|\nabla u\|_{L^2}\|\nabla b\|_{L^\infty}\left\|\Lambda^{2r}j\right\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\nabla b\|_{L^2}^{\frac{\beta+r-1}{\beta+r}}\left\|\Lambda^{\beta+r+1}b\right\|_{L^2}^{\frac{1}{\beta+r}}\|j\|_{L^2}^{\frac{\beta-r}{\beta+r}}\left\|\Lambda^{\beta+r}j\right\|_{L^2}^{\frac{2r}{\beta+r}} \\
&\leqslant C\|\nabla u\|_{L^2}\|j\|_{L^2}^{\frac{2\beta-1}{\beta+r}}\left\|\Lambda^{\beta+r}j\right\|_{L^2}^{\frac{2r+1}{\beta+r}} \\
&\leqslant C\epsilon\left\|\Lambda^{\beta+r}j\right\|_{L^2}^2+C(\epsilon)\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2(\beta+r)}{2\beta-1}},
\end{aligned}$$

where we have used the following Gagliardo–Nirenberg inequalities

$$\|\nabla b\|_{L^\infty}\leqslant C\|\nabla b\|_{L^2}^{\frac{\beta+r-1}{\beta+r}}\left\|\Lambda^{\beta+r+1}b\right\|_{L^2}^{\frac{1}{\beta+r}},$$
$$\left\|\Lambda^{2r}j\right\|_{L^2}\leqslant C\|j\|_{L^2}^{\frac{\beta-r}{\beta+r}}\left\|\Lambda^{\beta+r}j\right\|_{L^2}^{\frac{2r}{\beta+r}}.$$

Substituting estimates of I_1-I_3 into (2.10), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^r j\|_{L^2}^2+\left\|\Lambda^{\beta+r}j\right\|_{L^2}^2 &\leqslant C\epsilon\left\|\Lambda^{\beta+r}j\right\|_{L^2}^2+C(\epsilon)\|u\|_{L^2}\|\omega\|_{L^2}\|j\|_{L^2}^{2-\frac{1}{\beta}}\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{1}{\beta}} \\
&\quad +C(\epsilon)\|b\|_{L^2}^{\frac{2\beta}{1+\beta}}\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{2}{1+\beta}}\|\omega\|_{L^2}^2+C(\epsilon)\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2(\beta+r)}{2\beta-1}}.
\end{aligned}$$

Choosing $\epsilon=\frac{1}{2C}$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^r j\|_{L^2}^2+\left\|\Lambda^{\beta+r}j\right\|_{L^2}^2 &\leqslant C(\epsilon)\|u\|_{L^2}\|\omega\|_{L^2}\|j\|_{L^2}^{2-\frac{1}{\beta}}\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{1}{\beta}}+C(\epsilon)\|b\|_{L^2}^{\frac{2\beta}{1+\beta}}\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{2}{1+\beta}}\|\omega\|_{L^2}^2 \\
&\quad +C(\epsilon)\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2(\beta+r)}{2\beta-1}}. \qquad (2.11)
\end{aligned}$$

By assumptions of the lemma, we have $\beta>1,\ r=\beta-1>0$, and hence $\frac{1}{\beta}\leqslant 1, \frac{2}{1+\beta}\leqslant 2$. Thus, due to Lemma 2.2, we have

$$\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{1}{\beta}},\left\|\Lambda^{\beta}j\right\|_{L^2}^{\frac{2}{1+\beta}}\in L^1(0,T).$$

Using the Gronwall's inequality in (2.11), we obtain

$$\|\Lambda^r j\|_{L^2}^2(t)+\int_0^t\left\|\Lambda^{\beta+r}j\right\|_{L^2}^2\mathrm{d}\tau\leqslant C(u_0,b_0,T).$$

The proof of the lemma is complete. □

3. The proof of Theorem 1.1

In this section, we will prove our main result, Theorem 1.1. The proof is based on the local wellposedness and a priori estimates in $C([0,T];H^\rho(\mathbb{R}^2))$. It is divided into two parts. In the first part, we will prove that $u\in L^\infty([0,T];H^\rho(\mathbb{R}^2))$ and $b\in L^\infty([0,T];H^\rho(\mathbb{R}^2))\cap L^2([0,T];H^{\rho+\beta}(\mathbb{R}^2))$ for any $T>0$. In the second part, we prove that $(u,b)\in C([0,T];H^\rho(\mathbb{R}^2))$.

Part 1. $(u,b)\in L^\infty([0,T];H^\rho(\mathbb{R}^2))$ and $b\in L^2([0,T];H^{\rho+\beta}(\mathbb{R}^2))$ for any $T>0$

In this part, our aim is to utilize the BKM type criterion to deduce the global regularity of the solution (see [2]). Namely, we will obtain the a priori estimates of ω and j in $L^1([0,T];L^\infty(\mathbb{R}^2))$. There are three steps as follows:

Step 1. $\omega\in L^\infty(0,T;L^p(\mathbb{R}^2)), j\in L^p(0,T;\mathbb{R}^2)$ for any $2<p<\infty$.

The second equation in (1.1) can be rewritten as

$$b_t+\Lambda^{2\beta}b=\sum_{i=1}^{2}\partial_i(b_iu-u_ib)$$

Due to (2.3), we have

$$b(x,t)=\int\limits_{\mathbb{R}^2}t^{-\frac{1}{\beta}}h\left(\frac{x-y}{t^{\frac{1}{2\beta}}}\right)b_0(y)\mathrm{d}y+\int\limits_0^t\int\limits_{\mathbb{R}^2}(t-s)^{-\frac{1}{\beta}}h\left(\frac{x-y}{(t-s)^{\frac{1}{2\beta}}}\right)\sum_{i=1}^{2}\partial_i(b_iu-u_ib)(y,s)\,\mathrm{d}y\mathrm{d}s. \tag{3.12}$$

It follows from Lemmas 2.2 and 2.3, respectively, that $u\in L^\infty(0,T;L^p(\mathbb{R}^2))$ for any $2\leqslant p<\infty$ and $b\in L^\infty(0,T;L^\infty(\mathbb{R}^2))$ by the Gagliardo–Nirenberg inequalities. Thanks to Lemma 2.1, we can get

$$\begin{aligned}
\|\nabla b\|_{L^p(0,T;\mathbb{R}^2)}&\leqslant C(T)\|\nabla b_0\|_{L^p(\mathbb{R}^2)}+C\|bu\|_{L^p(0,T;\mathbb{R}^2)}\int\limits_0^T\left\|t^{-\frac{2}{\beta}}(\nabla^2h)\left(\frac{\cdot}{t^{\frac{1}{2\beta}}}\right)\right\|_{L^1(\mathbb{R}^2)}\mathrm{d}t\\
&\leqslant C(T)\\
\|\nabla^2 b\|_{L^2(0,t;L^p(\mathbb{R}^2))}&\leqslant C\|\nabla b_0\|_{L^p(\mathbb{R}^2)}\left(\int\limits_0^t\left\|\tau^{-\frac{3}{2\beta}}(\nabla h)\left(\frac{\cdot}{\tau^{\frac{1}{2\beta}}}\right)\right\|^2_{L^1(\mathbb{R}^2)}\mathrm{d}\tau\right)^{\frac{1}{2}}\\
&\quad+C\|b\cdot\nabla u-u\cdot\nabla b\|_{L^2(0,t;L^p(\mathbb{R}^2))}\int\limits_0^t\left\|\tau^{-\frac{2}{\beta}}(\nabla^2h)\left(\frac{\cdot}{\tau^{\frac{1}{2\beta}}}\right)\right\|_{L^1(\mathbb{R}^2)}\mathrm{d}\tau\\
&\leqslant C(T)\|\nabla b_0\|_{L^p(\mathbb{R}^2)}+C(T)\|b\cdot\nabla u-u\cdot\nabla b\|_{L^2(0,t;L^p(\mathbb{R}^2))}.
\end{aligned}\tag{3.13}$$

for any $2\leq p<\infty$ and $t\in(0,T)$.

Multiplying (2.5) by $|\omega|^{p-2}\omega(p>2)$, and integrating with respect to x, we get

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^p_{L^p}&\leqslant\int\limits_{\mathbb{R}^2}|b|\,|\nabla j|\,|\omega|^{p-1}\,\mathrm{d}x,\\
&\leqslant\|b\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|^{p-1}_{L^p}
\end{aligned}$$

Thus, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2_{L^p}\leqslant\|b\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|_{L^p}$$

and

$$\begin{aligned}\|\omega\|_{L^p}^2 &\leqslant C\|\omega(x,0)\|_{L^p}^2 + C\int_0^t (\|\nabla j\|_{L^p}^2 + \|\omega\|_{L^p}^2)\mathrm{d}s\\ &\leqslant C + C\int_0^t (\left\|\nabla^2 b\right\|_{L^p}^2 + \|\omega\|_{L^p}^2)\mathrm{d}s\\ &\overset{(3.13)}{\leqslant} C + C\int_0^t (\|b\cdot\nabla u - u\cdot\nabla b\|_{L^p}^2 + \|\omega\|_{L^p}^2)\mathrm{d}s\\ &\leqslant C + C\int_0^t (\|\nabla b\|_{L^p}^2\|u\|_{L^\infty}^2 + \|\omega\|_{L^p}^2)\mathrm{d}s\\ &\leqslant C + C\int_0^t (1+\|\nabla b\|_{L^p}^2)\|\omega\|_{L^p}^2\,\mathrm{d}s.\end{aligned}$$

This, combining with the Gronwall's inequality, leads to $\omega \in L^\infty(0,T;L^p(\mathbb{R}^2))$ for any $2 < p < \infty$.

Step 2. $\nabla j \in L^2(0,T;L^\infty(\mathbb{R}^2))$.

Similar to [7], we apply $\Lambda^\delta(0 < \delta < \min\{2\beta-2, \rho-2\})$ on both sides of (2.6) to obtain

$$(\Lambda^\delta j)_t + \Lambda^{2\beta}\Lambda^\delta j = -\Lambda^\delta(u\cdot\nabla j) + \Lambda^\delta(b\cdot\nabla\omega) + \Lambda^\delta(T(\nabla u,\nabla b)). \tag{3.14}$$

Thanks to Lemma 2.2 and Step 1, we have that $uj, b\omega,$ and $T(\nabla u,\nabla b) \in L^p(0,T;\mathbb{R}^2)$ for any $2 < p < \infty$. In the same way as in Step 1, we have

$$\begin{aligned}\Lambda^\delta j(x,t) &= \int_{\mathbb{R}^2} t^{-\frac{1}{\beta}} h\left(\frac{x-y}{t^{\frac{1}{2\beta}}}\right)\Lambda^\delta j_0(y)\mathrm{d}y\\ &\quad + \int_0^t\int_{\mathbb{R}^2}(t-s)^{-\frac{1}{\beta}} h\left(\frac{x-y}{(t-s)^{\frac{1}{2\beta}}}\right)\left(-\Lambda^\delta(u\cdot\nabla j) + \Lambda^\delta(b\cdot\nabla\omega)\right)(y,s)\,\mathrm{d}y\mathrm{d}s\\ &\quad + \int_0^t\int_{\mathbb{R}^2}(t-s)^{-\frac{1}{\beta}} h\left(\frac{x-y}{(t-s)^{\frac{1}{2\beta}}}\right)\Lambda^\delta(T(\nabla u,\nabla b))(y,s)\,\mathrm{d}y\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\|\nabla\Lambda^\delta j\|_{L^2(0,T;L^p(\mathbb{R}^2))} &\leqslant C\|\Lambda^\delta j_0\|_{L^p(\mathbb{R}^2)}\left(\int_0^T \left\|t^{-\frac{3}{2\beta}}(\nabla h)\left(\frac{\cdot}{t^{\frac{1}{2\beta}}}\right)\right\|_{L^1(\mathbb{R}^2)}^2 \mathrm{d}t\right)^{\frac{1}{2}}\\ &\quad + C\|uj\|_{L^2(0,T;L^p(\mathbb{R}^2))}\int_0^T \left\|t^{-\frac{4+\delta}{2\beta}}(\Lambda^\delta\nabla^2 h)\left(\frac{\cdot}{t^{\frac{1}{2\beta}}}\right)\right\|_{L^1(\mathbb{R}^2)}\mathrm{d}t\\ &\quad + C\|bw\|_{L^2(0,T;l^p(\mathbb{R}^2))}\int_0^T \left\|t^{-\frac{4+\delta}{2\beta}}(\Lambda^\delta\nabla^2 h)\left(\frac{\cdot}{t^{\frac{1}{2\beta}}}\right)\right\|_{L^1(\mathbb{R}^2)}\mathrm{d}t\end{aligned}$$

$$+C\|T(\nabla u,\nabla b)\|_{L^2(0,T;L^p(\mathbb{R}^2))}\int_0^T\left\|t^{-\frac{3+\delta}{2\beta}}(\Lambda^\delta\nabla h)\left(\frac{\cdot}{t^{\frac{1}{2\beta}}}\right)\right\|_{L^1(\mathbb{R}^2)}\mathrm{d}t$$
$$\leqslant C(T)$$

for any $2<p<\infty$. So we can choose δ small and p large enough such that $\nabla j\in L^2(0,T;L^\infty(\mathbb{R}^2))$ and $\|\Lambda^\delta j_0\|_{L^p}\leqslant C\|j_0\|_{H^\rho}$.

Step 3. $\omega\in L^\infty(0,T;L^\infty)$.

Because of the estimates of the step 2, and the following equation

$$\omega_t+u\cdot\nabla\omega=b\cdot\nabla j,$$

we can prove that $\omega\in L^\infty(0,T;L^\infty)$ by using the particle trajectory method. By taking advantage of the BKM type criterion for global regularity (see [2]), we have $(u,b)\in L^\infty([0,T];H^\rho(\mathbb{R}^2))$ and $b\in L^2([0,T];H^{\rho+\beta}(\mathbb{R}^2))$. The proof of the first part of Theorem 1.1 is finished.

Part 2. $(u,b)\in C([0,T];H^\rho(\mathbb{R}^2))$

In Part 1, we have obtained $(u,b)\in L^\infty([0,T];H^\rho(\mathbb{R}^2))$. Now, we prove that $(u,b)\in C([0,T];H^\rho(\mathbb{R}^2))$, to this end, we will apply the Besov spaces and the notations are referred as in [11]. We will prove that $\sum_{q\geqslant-1}2^{2\rho q}(\|\Delta_q u\|^2_{L^\infty([0,T);L^2(\mathbb{R}^2))}+\|\Delta_q b\|^2_{L^\infty([0,T);L^2(\mathbb{R}^2))})\leqslant C$, where we use the nonhomogeneous Littlewood–Paley decompositions $Id=\sum_k\Delta_k$ (see [1,11]). And we also use the following denotation $S_q u=\sum_{l\leqslant q-1}\Delta_l u$.

Applying Δ_q to Eq. (1.2), we get

$$\begin{aligned}(\Delta_q u)_t+S_{q+1}u\cdot\nabla\Delta_q u=&-\nabla\Delta_q p+S_{q+1}b\cdot\nabla\Delta_q b+(S_{q+1}u\cdot\nabla\Delta_q u-\Delta_q(u\cdot\nabla u))\\&-(S_{q+1}b\cdot\nabla\Delta_q b-\Delta_q(b\cdot\nabla b))\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}(\Delta_q b)_t+S_{q+1}u\cdot\nabla\Delta_q b=&(S_{q+1}u\cdot\nabla\Delta_q b-\Delta_q(u\cdot\nabla b))-(S_{q+1}b\cdot\nabla\Delta_q u-\Delta_q(b\cdot\nabla u))\\&+S_{q+1}b\cdot\nabla\Delta_q b-\Lambda^{2\beta}\Delta_q b.\end{aligned}\tag{3.16}$$

Denote

$$\begin{aligned}R_{1q}&=S_{q+1}u\cdot\nabla\Delta_q u-\Delta_q(u\cdot\nabla u),\\R_{2q}&=-(S_{q+1}b\cdot\nabla\Delta_q b-\Delta_q(b\cdot\nabla b)),\\R_{3q}&=S_{q+1}u\cdot\nabla\Delta_q b-\Delta_q(u\cdot\nabla b),\\R_{4q}&=-(S_{q+1}b\cdot\nabla\Delta_q u-\Delta_q(b\cdot\nabla u)).\end{aligned}$$

Then, we have (see Lemma 2.5 in [11])

$$\begin{aligned}\|R_{1q}\|_{L^2}&\leqslant Cc_q2^{-\rho q}\|u\|^2_{H^\rho},\\\|R_{2q}\|_{L^2}&\leqslant Cc_q2^{-\rho q}\|b\|^2_{H^\rho},\\\|R_{3q}\|_{L^2}&\leqslant Cc_q2^{-\rho q}\|u\|_{H^\rho}\|b\|_{H^\rho},\\\|R_{4q}\|_{L^2}&\leqslant Cc_q2^{-\rho q}\|b\|_{H^\rho}\|u\|_{H^\rho}.\end{aligned}\tag{3.17}$$

Multiplying by $\Delta_q u$ and $\Delta_q b$ on both sides of (3.15) and (3.16), respectively, integrating with respect to x, and summing up, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta_q u\|^2_{L^2}+\|\Delta_q b\|^2_{L^2}\right)\leqslant&\int_{\mathbb{R}^2}-\nabla\Delta_q p\cdot\Delta_q u\mathrm{d}x+\int_{\mathbb{R}^2}((R_{1q}+R_{2q})\Delta_q u+(R_{3q}+R_{4q})\Delta_q b)\mathrm{d}x\\&+\int_{\mathbb{R}^2}((S_{q+1}b\cdot\nabla\Delta_q b)\Delta_q u+(S_{q+1}b\cdot\nabla\Delta_q b)\Delta_q b)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leqslant (\|R_{1q}\|_{L^2} + \|R_{2q}\|_{L^2})\|\Delta_q u\|_{L^2} + (\|R_{3q}\|_{L^2} + \|R_{4q}\|_{L^2})\|\Delta_q b\|_{L^2}\\
&\leqslant Cc_q 2^{-\rho q}\left(\|u\|_{H^\rho}^2 + \|b\|_{H^\rho}^2\right)\|\Delta_q u\|_{L^2}\\
&\quad + Cc_q 2^{-\rho q}\|u\|_{H^\rho}\|b\|_{H^\rho}\|\Delta_q b\|_{L^2},
\end{aligned}
$$

where we have used $\nabla\cdot u = \nabla\cdot b = 0$, (3.17) and $\|c_q\|_{l^2} \leqslant 1$. Due to the first part of Theorem 1.1, we have

$$
\|u\|_{L^\infty([0,T);H^\rho(\mathbb{R}^2))} \leqslant C, \qquad \|b\|_{L^\infty([0,T);H^\rho(\mathbb{R}^2))} \leqslant C.
$$

Thus, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta_q u\|_{L^2}^2 + \|\Delta_q b\|_{L^2}^2\right) \leqslant Cc_q 2^{-\rho q}\left(\|\Delta_q u\|_{L^2} + \|\Delta_q b\|_{L^2}\right)
$$

Then, we obtain

$$
\|\Delta_q u\|_{L^\infty([0,T);L^2(\mathbb{R}^2))}^2 + \|\Delta_q b\|_{L^\infty([0,T);L^2(\mathbb{R}^2))}^2 \leqslant \|\Delta_q u_0\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta_q b_0\|_{L^2(\mathbb{R}^2)}^2 + C(c_q 2^{-\rho q})^2 \tag{3.18}
$$

Multiplying (3.18) by $2^{2\rho q}$ and summing up over q, we get

$$
\begin{aligned}
\sum_{q\geqslant -1} 2^{2\rho q}\left(\|\Delta_q u\|_{L^\infty([0,T);L^2(\mathbb{R}^2))}^2 + \|\Delta_q b\|_{L^\infty([0,T);L^2(\mathbb{R}^2))}^2\right) &\leqslant \sum_{q\geqslant 1} 2^{2\rho q}\left(\|\Delta_q u_0\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta_q b_0\|_{L^2(\mathbb{R}^2)}^2\right)\\
&\quad + C\sum_{q\geqslant 1} c_q^2 \leqslant C(\|u_0\|_{H^\rho(\mathbb{R}^2)}^2 + \|b_0\|_{H^\rho(\mathbb{R}^2)}^2) + C\\
&\leqslant C.
\end{aligned}
$$

Therefore, for any $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$
\left(\sum_{q\geqslant N(\epsilon)} 2^{2\rho q}\left(\|\Delta_q u\|_{L^\infty([0,T);L^2(\mathbb{R}^2))}^2 + \|\Delta_q b\|_{L^\infty([0,T);L^2(\mathbb{R}^2))}^2\right)\right)^{\frac{1}{2}} \leqslant \epsilon. \tag{3.19}
$$

Thanks to the first part of Theorem 1.1, we can easily get the following estimates.

$$
\begin{aligned}
\|u\cdot\nabla u\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))} &\leqslant C\|u\|_{L^\infty([0,T);L^\infty(\mathbb{R}^2))}\|u\|_{L^\infty([0,T);H^\rho(\mathbb{R}^2))}\\
&\quad + C\|\nabla u\|_{L^\infty([0,T);L^\infty(\mathbb{R}^2))}\|u\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))}\\
&\leqslant C\\
\|b\cdot\nabla b\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))} &\leqslant C\|b\|_{L^\infty([0,T);L^\infty(\mathbb{R}^2))}\|b\|_{L^\infty([0,T);H^\rho(\mathbb{R}^2))}\\
&\quad + C\|\nabla b\|_{L^\infty([0,T);L^\infty(\mathbb{R}^2))}\|b\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))}\\
&\leqslant C\\
\|\nabla p\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))} &\leqslant C(\|u\cdot\nabla u\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))} + \|b\cdot\nabla b\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))})\\
&\leqslant C\\
\|u\cdot\nabla b\|_{L^2([0,T);H^{\rho-\beta}(\mathbb{R}^2))} &\leqslant C\|u\|_{L^4([0,T);L^\infty(\mathbb{R}^2))}\|b\|_{L^4([0,T);H^{\rho+1-\beta}(\mathbb{R}^2))}\\
&\quad + C\|\nabla b\|_{L^4([0,T);L^\infty(\mathbb{R}^2))}\|u\|_{L^\infty([0,T);H^{\rho-\beta}(\mathbb{R}^2))}\\
&\leqslant C\\
\|b\cdot\nabla u\|_{L^2([0,T);H^{\rho-\beta}(\mathbb{R}^2))} &\leqslant C\|b\|_{L^4([0,T);L^\infty(\mathbb{R}^2))}\|u\|_{L^4([0,T);H^{\rho+1-\beta}(\mathbb{R}^2))}\\
&\quad + C\|\nabla u\|_{L^4([0,T);L^\infty(\mathbb{R}^2))}\|b\|_{L^\infty([0,T);H^{\rho-\beta}(\mathbb{R}^2))}\\
&\leqslant C\\
\left\|\Lambda^{2\beta} b\right\|_{L^2([0,T);H^{\rho-\beta}(\mathbb{R}^2))} &\leqslant \|b\|_{L^2([0,T);H^{\rho+\beta}(\mathbb{R}^2))} \leqslant C.
\end{aligned}
$$

Combining the above estimates with Eq. (1.2), we get

$$\|u_t\|_{L^\infty([0,T);H^{\rho-1}(\mathbb{R}^2))} \leqslant C, \quad \|b_t\|_{L^2([0,T);H^{\rho-\beta}(\mathbb{R}^2))} \leqslant C. \tag{3.20}$$

Therefore, for any $t_1, t_2 \in [0,T)$ and $t_2 > t_1$,

$$\begin{aligned}
&\left(\sum_{-1\leqslant q\leqslant N(\epsilon)} 2^{2\rho q}\left(\|\Delta_q u(t_2)-\Delta_q u(t_1)\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_q b(t_2)-\Delta_q b(t_1)\|^2_{L^2(\mathbb{R}^2)}\right)\right)^{\frac12}\\
&= \left(\sum_{-1\leqslant q\leqslant N(\epsilon)} 2^{2\rho q}\left(\left\|\int_{t_1}^{t_2}\partial_t\Delta_q u(t)\mathrm{d}t\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\int_{t_1}^{t_2}\partial_t\Delta_q b(t)\mathrm{d}t\right\|^2_{L^2(\mathbb{R}^2)}\right)\right)^{\frac12}\\
&\leqslant (t_2-t_1)^{\frac12}\left(\sum_{-1\leqslant q\leqslant N(\epsilon)} 2^{2\rho q}\left(\int_{t_1}^{t_2}\|\partial_t\Delta_q u(t)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}t + \int_{t_1}^{t_2}\|\partial_t\Delta_q b(t)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}t\right)\right)^{\frac12}\\
&\leqslant C2^{N(\epsilon)}(t_2-t_1)^{\frac12}\left(\sum_{-1\leqslant q\leqslant N(\epsilon)} 2^{2(\rho-1)q}\int_{t_1}^{t_2}\|\partial_t\Delta_q u(t)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}t\right)^{\frac12}\\
&\quad + C2^{N(\epsilon)\beta}(t_2-t_1)^{\frac12}\left(\sum_{-1\leqslant q\leqslant N(\epsilon)} 2^{2(\rho-\beta)q}\int_{t_1}^{t_2}\|\partial_t\Delta_q b(t)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}t\right)^{\frac12}\\
&\leqslant C2^{N(\epsilon)}(t_2-t_1)^{\frac12}\left(\int_{t_1}^{t_2}\|\partial_t u(t)\|^2_{H^{\rho-1}(\mathbb{R}^2)}\,\mathrm{d}t\right)^{\frac12} + C2^{N(\epsilon)\beta}(t_2-t_1)^{\frac12}\left(\int_{t_1}^{t_2}\|\partial_t b(t)\|^2_{H^{\rho-\beta}(\mathbb{R}^2)}\,\mathrm{d}t\right)^{\frac12}\\
&\overset{(3.20)}{\leqslant} C2^{N(\epsilon)\beta}(t_2-t_1)^{\frac12}.
\end{aligned} \tag{3.21}$$

Thus,

$$\begin{aligned}
&\|u(t_2)-u(t_1)\|_{H^\rho(\mathbb{R}^2)} + \|b(t_2)-b(t_1)\|_{H^\rho(\mathbb{R}^2)}\\
&\quad\leqslant C\left(\sum_{q\geqslant -1} 2^{2\rho q}\left(\|\Delta_q u(t_2)-\Delta_q u(t_1)\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_q b(t_2)-\Delta_q b(t_1)\|^2_{L^2(\mathbb{R}^2)}\right)\right)^{\frac12}\\
&\quad\leqslant C\left(\sum_{-1\leqslant q\leqslant N(\epsilon)} 2^{2\rho q}\left(\|\Delta_q u(t_2)-\Delta_q u(t_1)\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_q b(t_2)-\Delta_q b(t_1)\|^2_{L^2(\mathbb{R}^2)}\right)\right)^{\frac12}\\
&\qquad + C\left(\sum_{q\geqslant N(\epsilon)} 2^{2\rho q}\left(\|\Delta_q u(t_2)-\Delta_q u(t_1)\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_q b(t_2)-\Delta_q b(t_1)\|^2_{L^2(\mathbb{R}^2)}\right)\right)^{\frac12}\\
&\quad\overset{(3.19)(3.21)}{\leqslant} C2^{N(\epsilon)\beta}(t_2-t_1)^{\frac12} + C\epsilon.
\end{aligned}$$

The proof of Theorem 1.1 is complete.

Acknowledgments

The authors would like to thank Huan Yu for her valuable discussions.

References

1. Bahouri, H., Chemin, J.Y., Danchine, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Springer, Berlin (2011)
2. Caflisch, R.E., Klapper, I., Steele, G.: Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. Commun. Math. Phys. **184**, 443–455 (1997)
3. Casella, E., Secchi, P., Trebeschi, P.: Global clssical solutions for MHD system. J. Math. Fluid Mech. **5**, 70–91 (2003)
4. Cao, C., Wu, J., Yuan, B.: The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion. SIAM J. Math. Anal. **46**, 588–602 (2014)
5. Fan, J., Nakamura, G., Zhou, Y.: Global Cauchy problem of 2D generalized MHD equations, preprint
6. Jiu, Q., Niu, D.: Mathematical results related to a two-dimensional magneto-hydrodynamic equations. Acta Math. Sci. Ser. B English. Ed. **26**, 744–756 (2006)
7. Jiu, Q., Zhao, J.: A remark on global regularity of 2D generalized magnetohydrodynamic equations. J. Math. Anal. Appl. **412**, 478–484 (2014)
8. Kozono, H.: Weak and classical solutions of the two-dimensional magnetohydrodynamic equations. Tohoku Math. J. (2) **41**, 471–488 (1989)
9. Lin, F.H., Xu, L., Zhang, P.: Global small solution of 2-D incompressible MHD system, arXiv:1302.5877v2, 2013
10. Miao, C., Yuan, B., Zhang, B.: Well-posedness of the Cauchy problem for the fractional power dissipative equations. Nonlinear Anal. **68**, 461–484 (2008)
11. Miao, C., Wu, J., Zhang, Z.: Littlewood–Paley theory and its applications in partial differential equations of fluid dynamics. Science Press, Beijing (2012)
12. Tran, C.V., Yu, X., Zhai, Z.: On global regularity of 2D generalized magnetodydrodynamics equations. J. Differ. Equ. **254**, 4194–4216 (2013)
13. Wu, J.: Dissipative quasi-geostrophic equations with L^p data. Electron J. Differ. Equ. **2001**, 1–13 (2001)
14. Wu, J.: Generalized MHD equations. J. Differ. Equ. **195**(2), 284–312 (2003)
15. Wu, J.: Global regularity for a class of generalized magnetohydrodynamic equations. J. Math. Fluid Mech. **13**, 295–305 (2011)
16. Yamazaki, K.: On the global regularity of two-dimension generalized magnetohydrodynamics system, arXiv:1306.2842, 2013
17. Yamazaki, K.: Remarks on the global regularity of two-dimensional magnetohydrodynamics system with zero dissipation. Nonlinear Anal. **94**, 194–205 (2014)
18. Yuan, B., Bai, L.: Remarks on global regularity of 2D generalized MHD equations. J. Math. Anal. Appl. **413**, 633–640 (2014)
19. Zhou, Y., Fan, J.: A regularity criterion for the 2D MHD system with zero magnetic diffusivity. J. Math. Anal. Appl. **378**, 169–172 (2011)

Quansen Jiu
School of Mathematical Sciences and
Beijing Center of Mathematics and Information Sciences
Capital Normal University
Beijing 100048
People's Republic of China
e-mail: qsjiumath@gmail.com

Jiefeng Zhao
School of Mathematical Sciences
Capital Normal University
Beijing 100048
People's Republic of China
e-mail: zhaojiefeng001@163.com

(Received: January 5, 2014; revised: March 13, 2014)