

## Limit cycles for $m$ -piecewise discontinuous polynomial Liénard differential equations

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**Abstract.** We provide lower bounds for the maximum number of limit cycles for the  $m$ -piecewise discontinuous polynomial differential equations  $\dot{x} = y + \text{sgn}(g_m(x, y))F(x)$ ,  $\dot{y} = -x$ , where the zero set of the function  $\text{sgn}(g_m(x, y))$  with  $m = 2, 4, 6, \dots$  is the product of  $m/2$  straight lines passing through the origin of coordinates dividing the plane into sectors of angle  $2\pi/m$ , and  $\text{sgn}(z)$  denotes the sign function.

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### 1. Introduction and statement of the main result

The discontinuous Liénard polynomial differential systems have many applications, see for instance the excellent paper of Makarenkov and Lamb [23]. As far as we know up to now, there are no papers studying their limit cycles. The main objective of this paper is to start this study.

Hilbert [14] in 1900 and in the second part of its 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree and also to study their distribution or configuration in the plane. Except for the Riemann hypothesis, the 16th problem seems to be the most elusive of Hilbert's problems. It has been one of the main problems in the qualitative theory of planar differential equations in the XX century. The contributions of Écalle [11] and Ilyashenko [15] proving that any polynomial differential system has finitely many limit cycles have been the best results in this area. But until now it is not proved the existence of a uniform upper bound. This problem remains open even for the quadratic polynomial differential systems. However, it is not difficult to see that any finite configuration of limit cycles is realizable for some polynomial differential system, see for details [21].

Thus, we have the finiteness of the number of limit cycles for every polynomial differential system of degree  $n$ , but we do not have uniform bounds for that number in the whole class of all polynomial differential systems of degree  $n$ . Following Smale [25], we consider an easier and special class of polynomial differential systems, the *Liénard polynomial differential systems*:

$$\begin{aligned}\dot{x} &= y + F(x), \\ \dot{y} &= -x,\end{aligned}\tag{1}$$

where  $F(x) = a_0 + a_1x + \dots + a_nx^n$ , and the dot denotes derivative with respect to the time  $t$ . For these systems, the existence of uniform bounds also remains unproved. But when the degree  $n$  of these systems

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is odd, Ilyashenko and Panov in [16] obtained a uniform upper bound for the number of limit cycles in a subclass of systems such that  $F$  is monic and its coefficients satisfy some estimations.

For the Liénard polynomial differential systems (1), Lins et al. [19] in 1977 conjectured that they have at most  $[(n-1)/2]$  limit cycles if  $F(x)$  is a polynomial of degree  $n$ . Here,  $[z]$  denotes the integer part function of  $z$ . Moreover, he provided how to construct Liénard polynomial differential systems of degree  $n$  with  $[(n-1)/2]$  limit cycles.

In 2007 Dumortier et al. [10] proved that for  $n = 7$ , there are 4 limit cycles when the conjecture stated at most 3. In fact as they comment in that paper, their arguments can be extended in order to show that for  $n \geq 7$  odd always there will be more limit cycles than the expected by the conjecture. Recently, De Maesschalck and Dumortier proved in [8] that the classical Liénard equation of degree  $n \geq 6$  can have  $[(n-1)/2] + 2$  limit cycles. In the last two papers, the discussions are based on singular perturbation theory, and the authors used relaxation oscillation solutions to study the number of limit cycles. Li and Llibre [18] proved in 2012 that the conjecture of Lins, de Melo and Pugh holds for the Liénard polynomial differential systems of degree 4. So, now only remains open the conjecture for degree 5.

The problem of determining the maximum number of limit cycles that a given differential system can have has become one of the main topics in the qualitative theory of differential systems. Our main concern is to bring this problem to a class of non-smooth dynamical systems. A good representative of this class is the mathematical model

$$\ddot{x} + x + f(x, \dot{x}) = \operatorname{sgn}(g(x, \dot{x}))h(x, \dot{x}), \quad (2)$$

which is commonly found in many applications such as control theory (see [4] and [9]), relay systems (see [3]), economy (see [13]), impact systems (see [6]), mechanical systems [2], nonlinear oscillations [17] among others. And of course in these areas the detection of limit cycles is of fundamental importance.

Thus, in this paper, we shall study the limit cycles of the  $m$ -piecewise discontinuous polynomial differential equations

$$\begin{aligned} \dot{x} &= y + \operatorname{sgn}(g_m(x, y))F(x), \\ \dot{y} &= -x, \end{aligned} \quad (3)$$

where the zero set of the function  $\operatorname{sgn}(g_m(x, y))$  with  $m = 2, 4, 6, \dots$  is the union of  $m/2$  different straight lines passing through the origin of coordinates dividing the plane into sectors of angle  $2\pi/m$ . Here,  $\operatorname{sgn}(z)$  denotes the sign function, i.e.,

$$\operatorname{sgn}(z) = \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0, \end{cases}$$

Note that the differential system (3) is a particular case of the differential systems (2), which in some sense generalize the class of Liénard differential systems to the non-smooth differential systems.

We also consider the case  $m = 0$  with  $g_0(x, y) = 1$ . Therefore, the differential equation (3) for  $m = 0$  coincides with the Liénard polynomial differential equation (1). For this reason, we shall call the  $m$ -piecewise discontinuous polynomial differential equations (3) for  $m = 2, 4, 6, \dots$  as the  *$m$ -piecewise discontinuous Liénard polynomial differential equation of degree  $n$*  if  $n$  is the degree of the polynomial  $F(x)$ .

Piecewise discontinuous differential equations or more general non-smooth differential equations derived from ordinary differential equations when the non-uniqueness of some solutions is allowed. The theory of non-smooth differential equations has been developed very fast in these recent years due to various facts: its mathematical beauty, its strong relation with other branches of science and the challenge in establishing reasonable and consistent definitions and conventions. It has become certainly one of the common frontiers between Mathematics, Physics and Engineering. Also appears in a natural way in control systems, impact in mechanical systems and in nonlinear oscillations, in particular in electrical circuits. We understand that non-smooth systems are driven by applications, and they play an intrinsic

role in a wide range of technological areas. See for more details on the non-smooth differential equations [26] and the references therein.

Our main result on the limit cycles for  $m$ -piecewise discontinuous Liénard polynomial differential equations of degree  $n$  is the following theorem and conjecture. There and in the rest of the paper when we say that “the lower upper bound for the maximum number of limit cycles of a differential systems is  $M$ ” this means that this system can have  $M$  limit cycles if the right-hand side of the system is chosen conveniently.

**Theorem 1.** *Lower upper bounds  $L(m, n)$  for the maximum number of limit cycles of the  $m$ -piecewise discontinuous polynomial Liénard differential equations (3) of degree  $n$  are*

$$L(0, n) = \left\lceil \frac{n-1}{2} \right\rceil, \quad L(2, n) = \left\lceil \frac{n}{2} \right\rceil, \quad L(4, n) = \left\lceil \frac{n-1}{2} \right\rceil.$$

Theorem 1 is proved in Sect. 3 using the averaging theory. In the “Appendix,” we shall summarize the results on the averaging theory that we shall use in this paper. Of course the conjecture for  $L(0, n)$  was proved by Lins et al. [19], but here we shall present an easier and shorter proof using averaging theory.

**Conjecture.** *A lower upper bound  $L(m, n)$  for the maximum number of limit cycles of the  $m$ -piecewise discontinuous polynomial Liénard differential equations of degree  $n$  given by system (3) is*

$$L(m, n) = \left\lceil \frac{1}{2} \left( n - \frac{m-2}{2} \right) \right\rceil$$

if  $m = 6, 8, 10, \dots$

In Sect. 3, we shall provide some analytical results providing evidence that the conjecture must hold for the cases  $L(m, n)$  with  $m = 6, 8, 10, \dots$

We must mention that our proof of Theorem 1 will work also if the slopes of the  $m/2$  straight lines of  $g_m(x, y) = 0$  are  $\tan(\alpha + (2\pi j)/m)$  for an arbitrary  $\alpha$  instead of  $\tan((2\pi j)/m)$  for  $j = 1, 2, \dots, m$ .

## 2. Computations of $L(0, n)$ , $L(2, n)$ and $L(4, n)$

As we shall see the proof of Theorem 1 is reduced to prove the next Propositions 2, 3 and 4.

First, we shall provide the proof of  $L(0, n)$ . We recall the Descartes theorem about the number of zeros of a real polynomial. For a proof, see the pages 81–83 of the book [5], mainly the last lines of page 82 and the beginning of page 83. See also the Appendix A of [22] for checking that the maximum number of positive roots can be reached stated in the Descartes theorem can be reached. More details can be seen in “Appendix III”.

**Descartes theorem.** *Consider the real polynomial  $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \dots + a_{i_r}x^{i_r}$  with  $r > 1$ ,  $0 \leq i_1 < i_2 < \dots < i_r$  and the numbers  $a_{i_j}$  are not simultaneously zeros for  $j \in \{1, 2, \dots, r\}$ . When  $a_{i_j}a_{i_{j+1}} < 0$ , we say that  $a_{i_j}$  and  $a_{i_{j+1}}$  have a variation of sign. If the number of variations of signs is  $m$ , then  $p(x)$  has at most  $m$  positive real roots. Moreover, it is always possible to choose the coefficients of  $p(x)$  in such a way that  $p(x)$  has exactly  $r - 1$  positive real roots.*

**Proposition 2.** *The equality*

$$L(0, n) = \left\lceil \frac{n-1}{2} \right\rceil$$

holds.

*Proof.* For proving that  $[(n-1)/2]$  is a lower bound for the maximum number of limit cycles of the Liénard polynomial differential systems (1) of degree  $n$ , we shall prove that there are differential systems of the form

$$\begin{aligned}\dot{x} &= y + \varepsilon F(x), \\ \dot{y} &= -x,\end{aligned}\tag{4}$$

with  $F(x) = a_0 + a_1x + \dots + a_nx^n$  and  $a_n \neq 0$  having  $[(n-1)/2]$  limit cycles.

We consider the usual polar coordinates  $(r, \theta)$  such that  $x = r \cos \theta$  and  $y = r \sin \theta$ . The differential system (4) in polar coordinates becomes

$$\begin{aligned}\dot{r} &= \varepsilon \cos \theta F(r \cos \theta), \\ \dot{\theta} &= -1 - \varepsilon \frac{1}{r} \sin \theta F(r \cos \theta).\end{aligned}\tag{5}$$

Now taking as the new independent variable the variable  $\theta$ , system (5) can be written as

$$\frac{dr}{d\theta} = -\varepsilon \cos \theta F(r \cos \theta) + O(\varepsilon^2) = \varepsilon f(\theta, r) + \varepsilon^2 g(\theta, r, \varepsilon).\tag{6}$$

Since the differential equation (6) satisfies all the assumptions of Theorem 10 of the ‘‘Appendix,’’ we apply it to equation (6). Thus, using the notation of the ‘‘Appendix,’’ we have

$$\begin{aligned}f_0(r) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta, r) d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \cos \theta F(r \cos \theta) d\theta \\ &= -\frac{1}{2\pi} \sum_{i=0}^n a_i r^i \int_0^{2\pi} \cos^{i+1} \theta d\theta = -\frac{1}{2\pi} \sum_{j=0}^{[(n-1)/2]} a_{2j+1} r^{2j+1} \int_0^{2\pi} \cos^{2j+2} \theta d\theta.\end{aligned}$$

If

$$b_{2j+1} = -\frac{1}{2\pi} \int_0^{2\pi} \cos^{2j+2} \theta d\theta \neq 0$$

for  $j = 0, 1, \dots, [(n-1)/2]$ , it follows that

$$f_0(r) = \sum_{j=0}^{[(n-1)/2]} a_{2j+1} b_{2j+1} r^{2j+1}.$$

The monomials which appear in the polynomial  $f_0(r)$  are  $r, r^3, \dots, r^{2[(n-1)/2]+1}$ . Therefore, since in the coefficient of the monomial,  $r^{2j+1}$  appears  $a_{2j+1}$  that we can choose freely in such a way that the roots of this polynomial are 0 and  $\pm r_1, \pm r_2, \dots, \pm r_{[(n-1)/2]}$  with  $r_k > 0$  for  $k = 1, 2, \dots, [(n-1)/2]$  and  $r_1 < r_2 < \dots < r_{[(n-1)/2]}$ . Since all these roots are simple,  $f'(r_k) \neq 0$  for  $k = 1, 2, \dots, [(n-1)/2]$ . Hence, by Theorem 10, for  $\varepsilon$  sufficiently small the differential equation (6), and consequently the differential system (4) will have  $[(n-1)/2]$  limit cycles near the circles of radius  $r_k$  for  $k = 1, 2, \dots, [(n-1)/2]$ . Hence, the proposition is proved.  $\square$

**Proposition 3.** *The equality*

$$L(2, n) = \left\lfloor \frac{n}{2} \right\rfloor$$

*holds.*

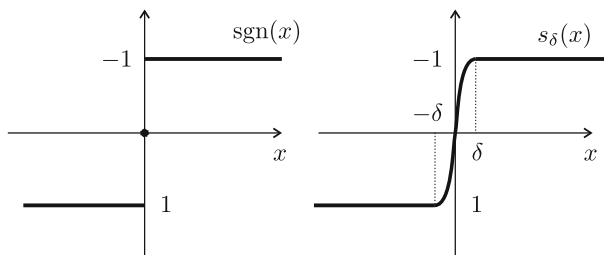


FIG. 1. The functions  $\text{sgn}(x)$  and  $s_\delta(x)$

*Proof.* For proving that  $[n/2]$  is a lower bound for the maximum number of limit cycles of the 2-piecewise discontinuous Liénard polynomial differential systems (1) of degree  $n$ , we shall prove that there are differential systems of the form

$$\begin{aligned} \dot{x} &= y + \varepsilon \text{sgn}(x)F(x), \\ \dot{y} &= -x, \end{aligned} \tag{7}$$

with  $F(x) = a_0 + a_1x + \dots + a_nx^n$  and  $a_n \neq 0$  having  $[n/2]$  limit cycles.

System (7) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= \varepsilon \text{sgn}(r \cos \theta) \cos \theta F(r \cos \theta), \\ \dot{\theta} &= -1 - \varepsilon \text{sgn}(r \cos \theta) \frac{1}{r} \sin \theta F(r \cos \theta). \end{aligned} \tag{8}$$

Instead of working with the discontinuous differential system (7), we shall work with the smooth differential system

$$\begin{aligned} \dot{x} &= y + \varepsilon s_\delta(x)F(x), \\ \dot{y} &= -x, \end{aligned} \tag{9}$$

where  $s_\delta(x)$  is the smooth function defined in Fig. 1, such that

$$\lim_{\delta \rightarrow 0} s_\delta(x) = \text{sgn}(x).$$

System (9) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= \varepsilon s_\delta(r \cos \theta) \cos \theta F(r \cos \theta), \\ \dot{\theta} &= -1 - \varepsilon s_\delta(r \cos \theta) \frac{1}{r} \sin \theta F(r \cos \theta). \end{aligned} \tag{10}$$

We must note that the Poincaré map of both systems (8) and (10) are smooth because in the first case it is composition of two smooth functions (one defined on  $x = 0$  by the flow in  $x > 0$ , and the other also defined on  $x = 0$  by the flow in  $x < 0$ ), and in the second, it is smooth by the general results on smooth ordinary differential equations. Clearly, the limit of the Poincaré map of system (9) when  $\delta \rightarrow 0$  tends to the Poincaré map of system (7). In fact, this convergence which is clear for the systems here studied has been proved in [20] for any discontinuous systems under convenient assumptions. On the other hand, if we do the Taylor expansion of the Poincaré map in the parameter  $\varepsilon$ , the averaged function  $f_0$  of the ‘‘Appendix’’ is the coefficient of  $\varepsilon$  in such expansion, and for more details, see for instance the section 3 of [7]. Therefore, if  $f_0^\delta(r)$  and  $f_0(r)$  denotes the averaged function of systems (10) and (8), respectively, then the limit of  $f_0^\delta(r)$  when  $\delta \rightarrow 0$  is the function  $f_0(r)$ . Hence, by Theorem 10 the simple zeros of the function  $f_0(r)$  provide limit cycles of the differential equation (8) and consequently of the discontinuous differential system (7). Now we shall compute the function  $f_0(r)$ .

Taking now  $\theta$  as the new independent variable system, Eq. (8) can be written as

$$\begin{aligned} \frac{dr}{d\theta} &= -\varepsilon \operatorname{sgn}(r \cos \theta) \cos \theta F(r \cos \theta) + O(\varepsilon^2) \\ &= \varepsilon f(\theta, r) + \varepsilon^2 g(\theta, r, \varepsilon). \end{aligned} \quad (11)$$

Since the differential equation (7) is the limit of systems satisfying all the assumptions of Theorem 10 of the ‘‘Appendix,’’ we apply it to Eq. (11). Thus, we have

$$\begin{aligned} f_0(r) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta, r) d\theta \\ &= \frac{1}{2\pi} \left( - \int_{-\pi/2}^{\pi/2} \cos \theta F(r \cos \theta) d\theta + \int_{\pi/2}^{3\pi/2} \cos \theta F(r \cos \theta) d\theta \right) \\ &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (-\cos \theta F(r \cos \theta) + \cos(\theta - \pi) F(r \cos(\theta - \pi))) d\theta \\ &= -\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (\cos \theta F(r \cos \theta) + \cos \theta F(-r \cos \theta)) d\theta \\ &= -\frac{1}{\pi} \sum_{j=0}^{[n/2]} a_{2j} r^{2j} \int_{-\pi/2}^{\pi/2} \cos^{2j+1} \theta d\theta. \end{aligned}$$

If

$$b_{2j} = -\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos^{2j+1} \theta d\theta < 0$$

for  $j = 0, 1, \dots, [n/2]$ , it follows that

$$f_0(r) = \sum_{j=0}^{[n/2]} a_{2j} b_{2j} r^{2j}.$$

By Descartes theorem and choosing the coefficients  $a_{2j}$  conveniently, the polynomial  $f_0(r)$  has  $[n/2]$  positive roots  $r_k$  for  $k = 1, 2, \dots, [n/2]$ . Clearly the other roots of that polynomial of degree  $2[n/2]$  are  $-r_k$  for  $k = 1, 2, \dots, [n/2]$ . Therefore,  $f'(r_k) \neq 0$  for  $k = 1, 2, \dots, [n/2]$ . Hence, from the previous arguments, by Theorem 10 for  $\varepsilon$  sufficiently small the differential equation (11), and consequently the differential system (7) will have  $[n/2]$  limit cycles near the circles of radius  $r_k$  for  $k = 1, 2, \dots, [n/2]$ . Hence, the proposition is proved.  $\square$

**Proposition 4.** *The equality*

$$L(4, n) = \left\lceil \frac{n-1}{2} \right\rceil$$

holds.

*Proof.* For proving that  $[(n-1)/2]$  is a lower bound for the maximum number of limit cycles of the 4-piecewise discontinuous Liénard polynomial differential systems (1) of degree  $n$ , we shall prove that there are differential systems of the form

$$\begin{aligned} \dot{x} &= y + \varepsilon \operatorname{sgn}(x^2 - y^2)F(x), \\ \dot{y} &= -x, \end{aligned} \tag{12}$$

with  $F(x) = a_0 + a_1x + \dots + a_nx^n$  and  $a_n \neq 0$  having  $[(n - 1)/2]$  limit cycles.

System (12) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= \varepsilon \operatorname{sgn}(\cos(2\theta)) \cos \theta F(r \cos \theta), \\ \dot{\theta} &= -1 - \varepsilon \operatorname{sgn}(\cos(2\theta)) \frac{1}{r} \sin \theta F(r \cos \theta). \end{aligned} \tag{13}$$

Using similar arguments to the proof of Proposition 3, we shall see that the discontinuous differential system (13) is limit in  $\mathbb{R}^2 \setminus \{(0, 0)\}$  of smooth differential systems.

Taking  $\theta$  as the new independent variable system, Eq. (13) can be written as

$$\begin{aligned} \frac{dr}{d\theta} &= -\varepsilon \operatorname{sgn}(\cos(2\theta)) \cos \theta F(r \cos \theta) + O(\varepsilon^2) \\ &= \varepsilon f(\theta, r) + \varepsilon^2 g(\theta, r, \varepsilon). \end{aligned} \tag{14}$$

Since the differential equation (14) is the limit of systems satisfying all the assumptions of Theorem 10 of the ‘‘Appendix,’’ we shall apply directly Theorem 10 to system (14) for computing the averaged function  $f_0(r)$ . Thus, we have

$$f_0(r) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta, r) d\theta = \frac{1}{2\pi} (I_1 + I_2),$$

where

$$\begin{aligned} I_1 &= - \int_{-\pi/4}^{\pi/4} \cos \theta F(r \cos \theta) d\theta - \int_{3\pi/4}^{5\pi/4} \cos \theta F(r \cos \theta) d\theta, \\ I_2 &= \int_{\pi/4}^{3\pi/4} \cos \theta F(r \cos \theta) d\theta + \int_{5\pi/4}^{7\pi/4} \cos \theta F(r \cos \theta) d\theta. \end{aligned}$$

We have

$$\begin{aligned} I_1 &= - \int_{-\pi/4}^{\pi/4} \cos \theta F(r \cos \theta) d\theta + \int_{-\pi/4}^{\pi/4} \cos \theta F(-r \cos \theta) d\theta \\ &= \int_{-\pi/4}^{\pi/4} \cos \theta (-F(r \cos \theta) + F(-r \cos \theta)) d\theta \\ &= -2 \sum_{j=0}^{[(n-1)/2]} a_{2j+1} r^{2j+1} \int_{-\pi/4}^{\pi/4} \cos^{2j+2} \theta d\theta \end{aligned}$$

If

$$b_{2j+1} = -2 \int_{-\pi/4}^{\pi/4} \cos^{2j+2} \theta d\theta < 0$$

for  $j = 0, 1, \dots, [(n-1)/2]$ , it follows that

$$I_1 = \sum_{j=0}^{[(n-1)/2]} a_{2j+1} b_{2j+1} r^{2j+1}.$$

Similarly, we have

$$\begin{aligned} I_2 &= - \int_{\pi/4}^{3\pi/4} \cos \theta F(r \cos \theta) d\theta - \int_{\pi/4}^{3\pi/4} \cos \theta F(-r \cos \theta) d\theta \\ &= \int_{\pi/4}^{3\pi/4} \cos \theta (F(r \cos \theta) - F(-r \cos \theta)) d\theta \\ &= 2 \sum_{j=0}^{[(n-1)/2]} a_{2j+1} r^{2j+1} \int_{\pi/4}^{3\pi/4} \cos^{2j+2} \theta d\theta \end{aligned}$$

If

$$c_{2j+1} = 2 \int_{\pi/4}^{3\pi/4} \cos^{2j+2} \theta d\theta > 0$$

for  $j = 0, 1, \dots, [(n-1)/2]$ , it follows that

$$I_2 = \sum_{j=0}^{[(n-1)/2]} a_{2j+1} c_{2j+1} r^{2j+1}.$$

Consequently

$$2\pi f_0[r] = \sum_{j=0}^{[(n-1)/2]} a_{2j+1} (b_{2j+1} + c_{2j+1}) r^{2j+1}.$$

We claim that  $b_{2j+1} + c_{2j+1} < 0$ . This claim follows from the fact that if  $\theta \in (-\pi/4, \pi/4)$  then  $\cos \theta > 1/\sqrt{2}$ , while if  $\theta \in (\pi/4, 3\pi/4)$  then  $\cos \theta < 1/\sqrt{2}$ . Therefore, by Descartes theorem and choosing the coefficients  $a_{2j+1}$  conveniently, the polynomial  $f_0(r)$  has  $[(n-1)/2]$  positive roots  $r_k$  for  $k = 1, 2, \dots, [(n-1)/2]$ . Clearly the other roots of that polynomial of degree  $2[(n-1)/2] + 1$  are 0 and  $-r_k$  for  $k = 1, 2, \dots, [(n-1)/2]$ . Therefore,  $f'(r_k) \neq 0$  for  $k = 1, 2, \dots, [(n-1)/2]$ . Hence, by Theorem 10, for  $\varepsilon$  sufficiently small the differential equation (6), and consequently the differential system (4) will have  $[(n-1)/2]$  limit cycles near the circles of radius  $r_k$  for  $\square$

### 3. Some analytic results and numerical computations

For  $m = 2, 4, 6, \dots$ , let  $g_m(x, y)$  be the function which appears in system (3). The set of points  $(x, y)$  satisfying  $g_m(x, y) = 0$  divides the plane into  $m$  sectors. We can assume that the slopes of the  $m/2$  straight lines of  $g_m(x, y) = 0$  are  $\tan(\alpha + (2\pi j)/m)$  for  $j = 0, 1, \dots, m/2 - 1$ . Then, by the arguments of the proof of Propositions 3 and 4 for studying the limit cycles of the  $m$ -piecewise discontinuous Liénard polynomial differential equation of degree  $n$  via de averaging method, we must study the simple zeros of the averaged function



$$f_0(r) = -\frac{1}{2\pi} \sum_{j=0}^{m-1} \int_{\alpha+(2\pi j)/m}^{\alpha+(2\pi(j+1))/m} (-1)^j \cos \theta F(r \cos \theta) d\theta.$$

If as usual  $F(x) = a_0 + a_1x + \dots + a_nx^n$  with  $a_n \neq 0$ , then

$$\begin{aligned} f_0(r) &= -\frac{1}{2\pi} \sum_{i=0}^n a_i r^i \left( \sum_{j=0}^{m-1} \int_{\alpha+2\pi j/m}^{\alpha+2\pi(j+1)/m} (-1)^j \cos^{i+1} \theta d\theta \right) \\ &= \sum_{i=0}^n a_i d_i r^i. \end{aligned} \tag{15}$$

Consequently  $f_0(r)$  is always a polynomial of degree at most  $n$ . Note that we have taken  $(-1)^j$  in the expression of the function  $f_0(r)$ , but it could also be  $(-1)^{j+1}$ , this depends on the explicit expression of the function  $g_m(x, y)$ . But this change of sign in the function  $f_0(r)$  does not affect its zeros.

*Remark.* From (15), for studying the simple zeros of the polynomial  $f_0(r)$ , we must know if the constants  $d_i$  which depend on  $m$  are zero or not. The problem of solving the conjecture is reduced to know which  $d_i$  are zero for a given  $m$  and to apply the Descartes theorem.

We recall that a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is called *odd* if  $g(-r) = -g(r)$ , and it is called *even* if  $g(-r) = g(r)$ . If  $g$  is an odd polynomial, then  $g$  only has monomials of odd degree. If  $g$  is an even polynomial, then  $g$  only has monomials of even degree.

**Proposition 5.** *If  $m$  is multiple of 4, then  $d_i = 0$  for  $i = 0, 2, 4, \dots$ , and the polynomial  $f_0(r)$  is odd.*

*Proof.* Since  $m$  is multiple of 4, the signs  $(-1)^j$  and  $(-1)^{j+m/2}$  of  $\text{sgn}(g_m(x, y))$  in an open sector defined by  $g_m(x, y) = 0$  and in its symmetric with respect to the origin of coordinates are equal. But in one of these sectors,  $\cos^{i+1} \theta$  is positive and in the other, it is negative because  $i$  is even. So the addition of the two integrals of  $d_i$  over these two symmetric sectors is zero, and consequently  $d_i$  holds. Therefore, from (15) it follows that  $f_0(r)$  is an odd polynomial.  $\square$

**Proposition 6.** *If  $m$  is not a multiple of 4, then  $d_i = 0$  for  $i = 1, 3, 5, \dots$ , and the polynomial  $f_0(r)$  is even.*

*Proof.* If  $m$  is not a multiple of 4, the signs  $(-1)^j$  and  $(-1)^{j+m/2}$  in an open sector and in its symmetric with respect to the origin of coordinates are different. Since in these two sectors,  $\cos^{i+1} \theta$  is positive because  $i$  is odd; again the addition of the two integrals of  $d_i$  over these two symmetric sectors is zero, and consequently  $d_i$  holds. Therefore, from (15) it follows that  $f_0(r)$  is an even polynomial.  $\square$

Of course, the results of Propositions 5 and 6 agree with the expressions of  $f_0(r)$  obtained in the proofs of Propositions 3 and 4.

**Proposition 7.** *If  $m = 6$ , then*

$$f_0(r) = \sum_{j=1}^{\lfloor n/2 \rfloor} a_{2j} d_{2j} r^{2j}.$$

*Moreover, the maximum number of positive real roots of the polynomial  $f_0(r)$  is  $\lfloor (n - 2)/2 \rfloor$ .*

*Proof.* From Proposition 6 and (15), it follows that for  $m = 6$  we have

$$f_0(r) = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j} d_{2j} r^{2j}.$$

We must prove that  $d_0 = 0$ . We have that

$$\begin{aligned} d_0 &= \frac{1}{2\pi} \sum_{j=0}^5 \int_{\alpha+\pi j/3}^{\alpha+\pi(j+1)/3} (-1)^{j+1} \cos \theta d\theta \\ &= \frac{1}{2\pi} \int_{\alpha}^{\alpha+\pi/3} \sum_{j=0}^5 (-1)^{j+1} \cos \left( \theta - \frac{j\pi}{3} \right) d\theta \end{aligned}$$

An easy computation shows that

$$\sum_{j=0}^5 (-1)^{j+1} \cos \left( \theta - \frac{j\pi}{3} \right) = 0,$$

consequently  $d_0 = 0$ .

Now the rest of the proof follows using Descartes theorem as at the end of the proof of Proposition 3.  $\square$

Note that for proving the conjecture for  $m = 6$ , from Proposition 7, we only need to show that  $d_{2j} \neq 0$  for  $j = 1, 2, 3, \dots$  and to apply the arguments of the end of the proof of Proposition 3.

With the help of the program mathematica, we obtain for  $m = 6$  and  $\alpha = \pi/2$  that

$$d_{2j} = \frac{1}{\pi} \left( 2B_{\frac{3}{4}} \left( j+1, \frac{1}{2} \right) - \frac{\sqrt{\pi} \Gamma(j+1)}{\Gamma(j+\frac{3}{2})} \right)$$

where  $B_{\frac{3}{4}}(a, b)$  is an incomplete beta function and  $\Gamma(z)$  is the Euler gamma function, for more details see [1]. Then, again we can verify that  $d_0 = 0$ , and we must compute  $d_{2j}$  for many  $j \in \{1, 2, 3, \dots\}$  and check that for those  $d_{2j} \neq 0$ . But we do not know how to prove that  $d_{2j} \neq 0$  for all  $j = 1, 2, 3, \dots$ . If this can be proved, then the conjecture is proved for  $m = 6$ .

**Proposition 8.** *If  $m = 8$ , then*

$$f_0(r) = \sum_{j=1}^{[(n-1)/2]} a_{2j+1} d_{2j+1} r^{2j+1}.$$

Moreover, the maximum number of positive real roots of the polynomial  $f_0(r)$  is  $[(n-3)/2]$ .

*Proof.* From Proposition 5 and (15), it follows that for  $m = 6$  we have

$$f_0(r) = \sum_{j=0}^{[(n-1)/2]} a_{2j+1} d_{2j+1} r^{2j+1}.$$

We must prove that  $d_1 = 0$ . We have that

$$\begin{aligned} d_1 &= \frac{1}{2\pi} \sum_{j=0}^7 \int_{\alpha+\pi j/4}^{\alpha+\pi(j+1)/4} (-1)^{j+1} \cos^2 \theta d\theta \\ &= \frac{1}{2\pi} \int_{\alpha}^{\alpha+\pi/4} \sum_{j=0}^7 (-1)^{j+1} \cos^2 \left( \theta - \frac{j\pi}{4} \right) d\theta \end{aligned}$$

An easy computation shows that

$$\sum_{j=0}^7 (-1)^{j+1} \cos^2 \left( \theta - \frac{j\pi}{4} \right) = 0,$$

consequently  $d_1 = 0$ .

Now the rest of the proof follows using Descartes theorem as in the end of the proof of Proposition 4. □

Again note that for proving the conjecture for  $m = 8$ , from Proposition 8, we only need to show that  $d_{2j+1} \neq 0$  for  $j = 1, 2, 3, \dots$  and to apply the arguments of the end of the proof of Proposition 4.

Again using the program mathematica, we obtain for  $m = 8$  and  $\alpha = \pi/8$  that

$$d_{2j+1} = \frac{1}{\pi\Gamma(j+2)} \left( \sqrt{\pi} \Gamma \left( j + \frac{3}{2} \right) + 2 \left( B_{\frac{1}{4}(2-\sqrt{2})} \left( j + \frac{3}{2}, \frac{1}{2} \right) - B_{\frac{1}{4}(2+\sqrt{2})} \left( j + \frac{3}{2}, \frac{1}{2} \right) \right) \Gamma(j+2) \right).$$

Then, we can verify that  $d_1 = 0$ , and we can compute  $d_{2j+1}$  for many  $j \in \{1, 2, 3, \dots\}$  and check that for those  $d_{2j+1} \neq 0$ . But again we do not know how to prove that  $d_{2j+1} \neq 0$  for all  $j = 1, 2, 3, \dots$ . If this can be proved, then the conjecture is proved for  $m = 8$ .

**Proposition 9.** *If  $m = 10$ , then*

$$f_0(r) = \sum_{j=2}^{[n/2]} a_{2j} d_{2j} r^{2j}.$$

Moreover, the maximum number of positive real roots of the polynomial  $f_0(r)$  is  $[(n - 4)/2]$ .

*Proof.* From Propositions 6 and (15), it follows that for  $m = 10$  we have

$$f_0(r) = \sum_{j=0}^{[n/2]} a_{2j} d_{2j} r^{2j}.$$

We must prove that  $d_0 = d_2 = 0$ . We have that

$$\begin{aligned} d_0 &= \frac{1}{2\pi} \sum_{j=0}^9 \int_{\alpha+\pi j/5}^{\alpha+\pi(j+1)/5} (-1)^{j+1} \cos \theta d\theta \\ &= \frac{1}{2\pi} \int_{\alpha}^{\alpha+\pi/5} \sum_{j=0}^9 (-1)^{j+1} \cos \left( \theta - \frac{j\pi}{5} \right) d\theta \end{aligned}$$

An easy computation shows that

$$\sum_{j=0}^9 (-1)^{j+1} \cos \left( \theta - \frac{j\pi}{5} \right) = 0,$$

consequently  $d_0 = 0$ .

On the other hand,

$$\begin{aligned} d_2 &= \frac{1}{2\pi} \sum_{j=0}^9 \int_{\alpha+\pi j/5}^{\alpha+\pi(j+1)/5} (-1)^{j+1} \cos^3 \theta d\theta \\ &= \frac{1}{2\pi} \int_{\alpha}^{\alpha+\pi/5} \sum_{j=0}^9 (-1)^{j+1} \cos^3 \left( \theta - \frac{j\pi}{5} \right) d\theta \end{aligned}$$

An easy computation shows that

$$\sum_{j=0}^9 (-1)^{j+1} \cos^3 \left( \theta - \frac{j\pi}{5} \right) = 0,$$

consequently  $d_2 = 0$ .

Now the rest of the proof follows using Descartes theorem as in the end of the proof of Proposition 3.  $\square$

Note that for proving the conjecture for  $m = 6$ , from Proposition 7, we only need to show that  $d_{2j} \neq 0$  for  $j = 1, 2, 3, \dots$  and to apply the arguments of the end of the proof of Proposition 3.

With the program mathematica, we obtain for  $m = 10$  and  $\alpha = \pi/2$  that

$$d_{2j} = \frac{1}{\pi} \left[ 2B_{\frac{1}{8}(5-\sqrt{5})} \left( j+1, \frac{1}{2} \right) - 2B_{\frac{1}{8}(5+\sqrt{5})} \left( j+1, \frac{1}{2} \right) + \frac{\sqrt{\pi}\Gamma(j+1)}{\Gamma(j+\frac{3}{2})} \right].$$

Then, again we can verify that  $d_0 = d_2 = 0$ , and we can compute  $d_{2j}$  for many  $j \in \{2, 3, \dots\}$  and check that for those  $d_{2j} \neq 0$ . But we do not know how to prove that  $d_{2j} \neq 0$  for all  $j = 2, 3, \dots$ . If this can be proved, then the conjecture is proved for  $m = 10$  and so on.

## Appendix I: Averaging theory of first order

We first briefly recall the basic elements of averaging theory that we shall need in this paper. Roughly speaking, the method gives a quantitative relation between the solutions of a non-autonomous periodic differential system and the solutions of its averaged differential system, which is autonomous. The following theorem provides a first order approximation for periodic solutions of the original system (for a proof, see for example [12, 24, 27]).

We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon f(t, \mathbf{x}) + \varepsilon^2 g(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (16)$$

and

$$\dot{\mathbf{y}} = \varepsilon f_0(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \quad (17)$$

with  $f, \mathbf{y}$  and  $\mathbf{x}_0$  in some open subset  $\Omega$  of  $\mathbb{R}^n$ ,  $t \in [0, \infty)$ ,  $\varepsilon \in (0, \varepsilon_0]$ . Here,  $\varepsilon$  is a small parameter. We assume that  $f$  and  $g$  are periodic of period  $T$  in the variable  $t$ , and we set

$$f_0(\mathbf{y}) = \frac{1}{T} \int_0^T f(t, \mathbf{y}) dt.$$

**Theorem 10.** *Assume that  $f, D_{\mathbf{x}}f, D_{\mathbf{x}\mathbf{x}}f$  and  $D_{\mathbf{x}}g$  are continuous and bounded by a constant independent of  $\varepsilon$  in  $[0, \infty) \times \Omega \times (0, \varepsilon_0]$  and that  $\mathbf{y}(t) \in \Omega$  for  $t \in [0, 1/\varepsilon]$ . Then the following statements hold.*

(1) *For  $t \in [0, 1/\varepsilon]$ , we have  $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$  as  $\varepsilon \rightarrow 0$ .*

- (2) If  $p \neq 0$  is a singular point of system (17) and  $\det D_{\mathbf{y}}F(p) \neq 0$ , then there exists a periodic solution  $\phi(t, \varepsilon)$  of period  $T$  for system  $\dot{\mathbf{x}} = \varepsilon f(t, \mathbf{x}) + \varepsilon^2 g(t, \mathbf{x}, \varepsilon)$  which is close to  $p$ , more precisely  $\phi(0, \varepsilon) - p = O(\varepsilon)$  as  $\varepsilon \rightarrow 0$ .
- (3) The stability of the periodic solution  $\phi(t, \varepsilon)$  is given by the stability of the singular point.

We have used the notation  $D_{\mathbf{x}}f$  for all the first derivatives of  $f$  and  $D_{\mathbf{xx}}f$  for all the second derivatives of  $f$ .

### Appendix II: An example in control systems

The constants in the following model can be chosen in such a way that it fits in the universe of systems treated in this paper.

Consider the second-order equation

$$\ddot{x} + a\dot{x} + bx = \varepsilon\alpha x$$

with  $a, b$  arbitrary constants,  $\varepsilon$  a real parameter, and  $\alpha$  satisfying  $|\alpha| \leq 1$ .

Let  $Z_{\alpha}$  be the vector field represented by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -bx - ay + \varepsilon\alpha x. \end{aligned}$$

Note that when  $\varepsilon = 0$ ,  $a^2 - 4b < 0$  and  $a \neq 0$ , then the system becomes a linear vector field with complex eigenvalues (when  $a = 0$  the system becomes a center).

Let  $v = v(x, y)$  be a smooth real function defined in the plane and we want to find  $\alpha$  that minimizes the derivative of  $v$  along the orbits of  $Z_{\alpha}$ . So,

$$\dot{v} = yv_x - (bx + ay)v_y + \varepsilon\alpha xv_y,$$

and  $\min\{\dot{v}\}$  is attained by setting

$$\alpha_0 = \text{sgn}\{x.v_y\}$$

Choosing  $v = (x^2 + xy + y^2)/2$ , we consider  $Z_{\alpha_0}$  with  $\alpha_0 = \text{sgn}\{x(x+y)\}$ . So this system can be represented by the following differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -bx - ay + \varepsilon x, \end{aligned}$$

if  $x(x+y) > 0$  and

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -bx - ay - \varepsilon x, \end{aligned}$$

if  $x(x+y) < 0$ .

Assume  $Z_{\alpha_0}$  satisfying the Filippov rules on the straight lines  $L_1 : \{x = 0\}$  and  $L_2 : \{x + y = 0\}$ , which divide the plane into four regions I, II, III and IV.

If  $(1 - b - a)^2 > \varepsilon^2$ , then the orbits of the system spiral the singularity 0. Observe that on the discontinuity set one finds just sewing regions.

### Appendix III: Descartes Theorem

Consider the polynomial

$$p(x) = b_{i_1}x^{i_1} + b_{i_2}x^{i_2} + \dots + b_{i_r}x^{i_r}.$$

where  $0 \leq i_1 < i_2 < \dots < i_r$ ,  $r > 1$  and  $b_{i_j}$  are not zeros simultaneously for  $j \in \{1, 2, \dots, r\}$ .

Our concern is to prove the following statement:

**Statement:** *It is always possible to choose the coefficients  $b_{i_1}, \dots, b_{i_r}$  in such a way that  $p(x)$  has exactly  $r - 1$  positive real roots.*

First, we reproduce the Appendix A of [22].

## Appendix A

Let  $A$  be a set and let  $f_1, f_2, \dots, f_n : A \rightarrow \mathbb{R}$ . We say that  $f_1, \dots, f_n$  are *linearly independent* functions if and only if there holds

$$\forall x \in A \sum_{i=1}^n \alpha_i f_i(x) = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

**Proposition 11.** *For  $n \geq 2$  if  $f_1, \dots, f_n : A \rightarrow \mathbb{R}$  are linearly independent, then there exist  $a_1, \dots, a_{n-1} \in A$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  with*

$$\sum_{k=1}^n \alpha_k^2 \neq 0.$$

such that for every  $i \in \{1, \dots, n - 1\}$

$$\sum_{k=1}^n \alpha_k f_k(a_i) = 0.$$

We observe that, for our purposes, only the case when  $n > 1$  is considered.

**Lemma 12.** *There exist  $a_1, \dots, a_n$  such that  $n$  vectors*

$$\begin{pmatrix} f_1(a_1) \\ f_1(a_2) \\ \vdots \\ f_1(a_n) \end{pmatrix} \begin{pmatrix} f_2(a_1) \\ f_2(a_2) \\ \vdots \\ f_2(a_n) \end{pmatrix} \cdots \begin{pmatrix} f_n(a_1) \\ f_n(a_2) \\ \vdots \\ f_n(a_n) \end{pmatrix}$$

are linearly independent.

*Proof.* By induction hypothesis for  $n = 1$  (or  $n = 2$ ), it is trivially true. Let us assume that Lemma 12 is true for  $n - 1$  and suppose that it is not true for  $n$ . That would mean that for every  $a \in A$ , there exist  $\alpha_1(a), \dots, \alpha_n(a)$  not all equal to zero such that

$$\alpha_1(a) \begin{pmatrix} f_1(a_1) \\ f_1(a_2) \\ \vdots \\ f_1(a_{n-1}) \\ f_1(a) \end{pmatrix} + \alpha_2(a) \begin{pmatrix} f_2(a_1) \\ f_2(a_2) \\ \vdots \\ f_2(a_{n-1}) \\ f_2(a) \end{pmatrix} + \dots + \alpha_n(a) \begin{pmatrix} f_n(a_1) \\ f_n(a_2) \\ \vdots \\ f_n(a_{n-1}) \\ f_n(a) \end{pmatrix} = 0.$$

By induction hypothesis,  $\alpha_n(a) \neq 0$  and we have two possibilities:

- (i) There exists  $i \in \{1, \dots, n - 1\}$  such that  $f_n(a_i) \neq 0$ . In this case,  $\alpha_k(a)/\alpha_n(a)$  do not depend on  $a$  for  $k = 1, 2, \dots, n - 1$  (induction hypothesis). But then for every  $a$ ,  $f_n(a) = \sum_{k=1}^{n-1} \alpha_k(a)/\alpha_n(a) f_k(a)$  contradicting independence of  $f_1, \dots, f_n$ .
- (ii) For every  $i \in \{1, \dots, n - 1\}$ ,  $f_n(a_i) = 0$ . In this case, by induction hypothesis  $\alpha_k(a) \equiv 0$  for  $k = 1, 2, \dots, n - 1$ , and therefore,  $f_n(a) \equiv 0$ —contradiction.  $\square$

Now we prove the Proposition 11.

*Proof.* Taking  $a_1, \dots, a_n$  from Lemma 12, then the matrix

$$\mathcal{A} = \begin{bmatrix} f_1(a_1) & f_2(a_1) & \dots & f_n(a_1) \\ f_1(a_2) & f_2(a_2) & \dots & f_n(a_2) \\ \vdots & \vdots & \dots & \vdots \\ f_1(a_n) & f_2(a_n) & \dots & f_n(a_n) \end{bmatrix}$$

is invertible; therefore, the equation  $\mathcal{A} \cdot \vec{\alpha} = [0, 0, \dots, 0, 1]^T$  has a solution  $\vec{\alpha}$ . This means in particular there exists  $\alpha_1, \dots, \alpha_n$  such that  $[f_1(a_i), f_2(a_i), \dots, f_n(a_i)] \cdot [\alpha_1, \dots, \alpha_n]^T = 0$  for  $i = 1, 2, \dots, n-1$ .  $\square$

We claim that, "it is always possible to choose the coefficients of  $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} \dots a_{i_r}x^{i_r}$  in such a way that  $p(x)$  has exactly  $r-1$  positive real roots, where  $0 \leq i_1 \leq i_2 \leq \dots \leq i_r$  and  $a_{i_j} \neq 0$  for  $j \in \{1, 2, \dots, r\}$ ."

*Proof of the claim.* Using the notation of the statement of Proposition 1, we take  $n = r$ ,  $A = \mathbb{R}$ , and

$$f_1(x) = x^{i_1}, f_2(x) = x^{i_2}, \dots, f_r(x) = x^{i_r}.$$

Since  $0 \leq i_1 < i_2 < \dots < i_r$ , the functions  $f_1, \dots, f_r$  are linearly independent. So we are in the assumptions of Proposition 1, it follows that there exist  $a_1, \dots, a_{r-1} \in \mathbb{R}$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  such that for every  $i \in \{1, \dots, n-1\}$

$$\sum_{k=1}^n \alpha_k f_k(a_i) = 0.$$

That is, the  $a_i$  for  $i = 1, \dots, r-1$  are zeros of the function

$$\sum_{k=1}^r \alpha_k f_k(x) = 0.$$

Therefore, taking  $\alpha_k = b_{i_k}$ , the statement is proved.  $\square$

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