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Darboux invariants for planar polynomial differential systems having an invariant conic

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Abstract. We characterize all the planar polynomial differential systems with a unique invariant algebraic curve given by a real conic and having a Darboux invariant.

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1. Introduction and statement of the main results

Real planar polynomial differential systems appear in many branches of applied mathematics, physics, and, in general, in applied sciences. For such differential systems, the existence of a first integral determines completely their phase portrait. The first integrals depending on the time, i.e., on the independent variable of the differential system, are called invariants. A special class of invariants is the Darboux invariants. As we shall see the invariants instead of determining the phase portrait of the system, we determine its $\alpha-$ and $\omega-$limit sets in the compactified polynomial differential system. That is, the Darboux invariants allow to describe the sets where all the orbits born or die.

In general, it is a very difficult problem to recognize when a given polynomial differential system in the plane has or not a first integral or a Darboux invariant. The goal of this paper is to classify all polynomial differential systems in the plane $\mathbb{R}^2$ having a Darboux invariant and a unique invariant algebraic curve given by a conic.

Let $\mathbb{K}[x, y]$ be the ring of the polynomials in the variables x and y with coefficients in $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We consider the *polynomial differential system* in $\mathbb{R}^2$ defined by

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, P and Q are relatively prime in $\mathbb{R}[x, y]$, and the dot denotes derivative with respect to the independent variable t usually called the *time*.

We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of system (1). We associate with system (1) the vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

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We say that the polynomial differential system (1) is *integrable* on an open subset $U \subset \mathbb{R}^2$ if there exists a nonlocally constant analytic function $H : U \to \mathbb{R}$, called a *first integral* of the system on U, which is constant on all solution curves $(x(t), y(t))$ of system (1) contained in U. Clearly, H is a first integral of system (1) if and only if $XH \equiv 0$ on U.

An *invariant* of system (1) on the open subset U of $\mathbb{R}^2$ is a nonlocally constant analytic function I in the variables x, y, and t such that I is constant on all solution curves $(x(t), y(t))$ of system (1) contained in U, i.e.,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial t} = 0.$$

Obviously, I is a first integral of system (1) depending on the time t.

Let $f \in \mathbb{C}[x, y]\backslash\{0\}$. The algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of system (1) if for some polynomial $K \in \mathbb{C}[x, y]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of invariant algebraic curve $f = 0$. Note that, when $K = 0$, then f is a polynomial first integral. We remark that in the definition of invariant algebraic curve $f = 0$, we always allow this curve to be complex, even in the case of a real polynomial system, due to the fact that sometimes for real polynomial systems, the existence of a real first integral can be forced by the existence of complex invariant algebraic curves. For more details on invariant algebraic curves see [8].

Let $g, h \in \mathbb{C}[x, y]\backslash\{0\}$ and assume that g and h are relatively prime in the ring $\mathbb{C}[x, y]$ or that $h = 1$. Then, the function $\exp(g/h)$ is called an *exponential factor* of system (1) if for some polynomial $L \in \mathbb{C}[x, y]$ of degree at most $m - 1$, we have that

$$X(\exp(g/h)) = L\exp(g/h).$$

We say that an invariant I is a *Darboux invariant* of the vector field X if it can be written as

$$I(x, y, t) = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^{st},$$

where $f_i = 0$ are invariant algebraic curves of X for $i = 1, \dots, p$; F_j are exponential factors of X for $j = 1, \dots, q$; $\lambda_i, \mu_j \in \mathbb{C}$, and $s \in \mathbb{R}\backslash\{0\}$.

The search of first integrals is a classic tool in order to describe the phase portraits of a 2-dimensional differential system. As usual, the *phase portrait* of a system is the decomposition of the domain of definition of this system as union of all its orbits. Every planar polynomial differential system in $\mathbb{R}^2$ can be analytically extended to infinity, in such a way that $\mathbb{R}^2$ is identified with the interior of a disc and its boundary $\mathbb{S}^1$ is identified with the infinity. This closed disc is called the *Poincaré disc*, for more details see Chapter 5 of [7]. The phase portrait of any planar polynomial differential system can be drawn on the Poincaré disc.

Let $\phi_p(t)$ be the solution of system (1) passing through the point $p \in \mathbb{R}^2$, defined on its maximal interval (α_p, ω_p) such that $\phi_p(0) = p$. If $\omega_p = \infty$, we define the set

$$\omega(p) = \{q \in \mathbb{R}^2 : \text{there exists } \{t_n\} \text{ with } t_n \to \infty \text{ and } \phi(t_n) \to q \text{ when } n \to \infty\}.$$

In the same way, if $\alpha_p = -\infty$, we define the set

$$\alpha(p) = \{q \in \mathbb{R}^2 : \text{there exists } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \phi(t_n) \to q \text{ when } n \to \infty\}.$$

The sets $\omega(p)$ and $\alpha(p)$ are called the ω-*limit set* (or simply ω-*limit*) and the α-*limit set* (or α-*limit*) of p, respectively.

The Darboux invariant provides information about the ω- and α-limit sets of all orbits of system (1). More precisely, there is the following result proved in [10], since its proof is short, we include it here for completeness.

Proposition 1. *Let $I(x,y,t) = f(x,y)e^{st}$ be a Darboux invariant of system (1). Let $p \in \mathbb{R}^2$ and $\phi_p(t)$ be the solution of system (1) with maximal interval (α_p, ω_p) such that $\phi_p(0) = p$.*

(a) *If $\omega_p = \infty$ then $\omega(p) \subset \{f(x,y) = 0\} \cup \mathbb{S}^1$,*
(b) *If $\alpha_p = -\infty$ then $\alpha(p) \subset \{f(x,y) = 0\} \cup \mathbb{S}^1$.*

Here $\mathbb{S}^1$ denotes the infinity of the Poincaré disc.

Proof. We prove statement (a), the proof of statement (b) is similar. Suppose $s > 0$ and let $\phi_p(t) = (x_p(t), y_p(t))$. Since $I(x,y,t)$ is an invariant $I(x_p(t), y_p(t), t) = a \in \mathbb{R}$ for all $t \in (\alpha_p, \omega_p)$. Then,

$$a = \lim_{t\to\infty} I(x_p(t), y_p(t), t) = \lim_{t\to\infty} f(x_p(t), y_p(t))e^{st}.$$

As $\lim_{t\to\infty} e^{st} = \infty$, we have that $\lim_{t\to\infty} f(x_p(t), y_p(t)) = 0$. So, by continuity and the definition of ω-limit set, it follows that $\omega(p) \subset \{f(x,y) = 0\}$, and for the α-limit set $\alpha(p) \in \mathbb{S}^1$. □

It is known that if a planar quadratic polynomial differential system has 3 invariant algebraic curves, then this system is Darboux integrable in the sense that it has a first integral which is a Darboux function, or it has an integrating factor given by a Darboux function; for more details see for instance, statement (v) of Theorem 8.7 of [7]. In this paper, we only study *polynomial differential systems of arbitrary degree having an invariant algebraic curve given by a real conic*, and our goal is to characterize which of these systems have a Darboux invariant. Note that, a real conic at most can produce two invariant algebraic curves, which can be real or complex curves.

The real conics in $\mathbb{R}^2$ (i.e., conics $f(x,y) = 0$ where $f(x,y)$ is a real quadratic polynomial) are classified as ellipses (E), complex ellipses (CE), hyperbolas (H), parabolas (P), two real straight lines intersecting at a point (LV), two real and parallel straight lines (RPL), two complex and parallel straight lines (CPL), one double real straight line (DL), and two complex straight lines intersecting at a real point p (p).

The characterization of all *quadratic systems*, i.e., systems (1) with $m = 2$, having two real or complex invariant straight lines taking into account their multiplicity was given in [2], and extensions to dimension 3 are given in [11].

Now, we do the characterization of all polynomial differential systems in $\mathbb{R}^2$ having an invariant conic and a Darboux invariant.

Theorem 2. (LV) *Every real polynomial differential system in $\mathbb{R}^2$ having a Darboux invariant and two real invariant straight lines intersecting at a point, after an affine change of coordinates, can be written as*

$$\dot{x} = xK_1(x,y), \quad \dot{y} = y(a + bK_1(x,y)), \tag{2}$$

where $K_1 \in \mathbb{R}[x,y], a, b \in \mathbb{R}$ and $a \neq 0$. Moreover its Darboux invariant is $I_1(x,y,t) = y/(x^b e^{at})$.

The polynomial differential systems in the plane having two real invariant straight lines intersecting at a point always can be written (as it is also showed in the proof of Theorem 2) into the form $\dot{x} = xg(x,y), \dot{y} = yh(x,y)$, which are the well-known Lotka-Volterra systems. For this reason, here we denote by LV this kind of systems.

Theorem 2 for the particular case of quadratic systems was proved in [1].

Theorem 3. (RPL) *Every real polynomial differential system in $\mathbb{R}^2$ having a Darboux invariant and two real and parallel invariant straight lines, after an affine change of coordinates and a rescaling of the time, can be written as*

$$\dot{x} = x^2 - 1, \quad \dot{y} = Q(x,y), \tag{3}$$

where $Q \in \mathbb{R}[x,y]$. Moreover its Darboux invariant is $I_2(x,y,t) = \sqrt{\dfrac{x+1}{x-1}}e^t$.

Theorem 4. (CPL) *Every real polynomial differential system in* $\mathbb{R}^2$ *having a Darboux invariant and two complex and parallel invariant straight lines, after an affine change of coordinates and a rescaling of the time, can be written as*

$$\dot{x} = x^2 + 1, \quad \dot{y} = Q(x, y), \tag{4}$$

where $Q \in \mathbb{R}[x, y]$. *Moreover its Darboux invariant is* $I_3(x, y, t) = e^{(t+\arctan(1/x))}$.

Theorem 5. (DL) *Every real polynomial differential system in* $\mathbb{R}^2$ *having a Darboux invariant and one double real invariant straight line, after an affine change of coordinates and a rescaling of the time, can be written as*

$$\dot{x} = x^2, \quad \dot{y} = Q(x, y), \tag{5}$$

where $Q \in \mathbb{R}[x, y]$. *Moreover its Darboux invariant is* $I_4(x, y, t) = e^{(t+1/x)}$.

See the definition of a double real invariant straight line or an invariant straight line of multiplicity 2 in Sect. 2.

Theorem 6. (p) *Every real polynomial differential system in* $\mathbb{R}^2$ *having a Darboux invariant and two complex invariant straight lines intersecting at a real point, after an affine change of coordinates, can be written either as*

$$\dot{x} = (x - ay)A(x, y) - by, \quad \dot{y} = (ax + y)A(x, y) + bx, \tag{6}$$

where $A \in \mathbb{R}[x, y], a, b \in \mathbb{R}$, *with* $b \neq 0$, *and its Darboux invariant is* $I_5(x, y, t) = (x^2 + y^2)^a e^{-2\arctan(y/x)+2bt}$, *or as*

$$\dot{x} = -yB(x, y) + cx, \quad \dot{y} = xB(x, y) + cy, \tag{7}$$

where $B \in \mathbb{R}[x, y], c \in \mathbb{R}\backslash\{0\}$, *and its Darboux invariant is* $I_6(x, y, t) = (x^2 + y^2)e^{-2ct}$.

Theorems 3, 4, 5, and 6 for the particular case of quadratic systems were proved in [10].

Theorem 7. (P) *Every real polynomial differential system in* $\mathbb{R}^2$ *having a Darboux invariant and an invariant parabola, after an affine change of coordinates, can be written as*

$$\dot{x} = (y - x^2)A(x, y) - C(x, y), \quad \dot{y} = (y - x^2)(b + 2xA(x, y)) - 2xC(x, y), \tag{8}$$

where $b \in \mathbb{R}\backslash\{0\}$ *and* $A, C \in \mathbb{R}[x, y]$. *Moreover its Darboux invariant is* $I_7(x, y, t) = e^t/(y - x^2)^{1/b}$.

Theorem 8. (E, CE, H) *Every real polynomial differential system in* $\mathbb{R}^2$ *having only one invariant algebraic curve given by either an ellipse, or a complex ellipse, or a hyperbola has no Darboux invariants.*

This paper is organized as follows. In Sect. 2, we present some results of the Darboux theory of integrability which we will use later on. In Sect. 3, we prove Theorems 2–8.

2. Darboux theory of integrability

In this section, we introduce some results on the Darboux theory of integrability which shall be used in the next section to prove Theorems 2, 3, 4, 5, 6, 7, and 8. This kind of integrability provides a link between the integrability of polynomial differential systems and their invariant algebraic curves. More details about this theory can be found in [4] and [7].

The following proposition is easy to prove.

Proposition 9. *For the real polynomial differential system* (1), $f = 0$ *is an invariant algebraic curve with cofactor* K *if and only if* $\bar{f} = 0$ *is an invariant algebraic curve with cofactor* $\bar{K}$, *where* $\bar{f}$ *and* $\bar{K}$ *denote the conjugates of* f *and* K, *respectively.*

Observe that if among the invariant algebraic curves of system (1) a complex conjugate pair $f = 0$ and $\bar{f} = 0$ occurs, then the Darboux invariant has a real factor of the form $f^{\lambda}\bar{f}^{\bar{\lambda}}$, which is the multi-valued real function

$$[(\mathrm{Re}(f))^2 + (\mathrm{Im}(f))^2]^{\mathrm{Re}(\lambda)} \exp(-2\,\mathrm{Im}(\lambda) \arctan(\mathrm{Im}(f)/\,\mathrm{Re}(f))). \tag{9}$$

So, if system (1) is real, then the Darboux invariant is also real, independently of the fact of having complex invariant curves or complex exponential factors.

The next result is proved in Proposition 8.4 of [7].

Proposition 10. *Suppose $f \in \mathbb{C}[x, y]$ and let $f = f_1^{n_1} \ldots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x, y]$. Then for a polynomial differential system (1), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor K_{f_i}. Moreover $K_f = n_1K_{f_1} + \cdots + n_rK_{f_r}$.*

Concerning exponential factors, the following result holds (for a proof see Proposition 8.6 of [7]).

Proposition 11. *If $F = \exp(g/h)$ is an exponential factor for the polynomial differential system (1) and h is not a constant, then $h = 0$ is an invariant algebraic curve and g satisfies the equation $Xg = gK_h+hK_F$, where K_h and K_F are the cofactors of h and F, respectively.*

The next result and its proof can be found in [7] (see statement (vi) of Theorem 8.7), and it explains how to find Darboux invariants.

Proposition 12. *Suppose that a polynomial differential system (1) of degree m admits p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$, and q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$. There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{p}^{i=1} \lambda_i K_i + \sum_{q}^{j=1} \mu_j L_j = -s, \tag{10}$$

for some $s \in \mathbb{R}\backslash\{0\}$, if and only if the real (multi-valued) function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\mu_1} \ldots F_q^{\mu_q} e^{st}$$

is a Darboux invariant of system (1).

The next theorem, which is due to Christopher [3] and was rediscovered by Zholadek [12], an algebraic proof of it can be found in [6], shows that for the integrability of a polynomial differential system (1) of degree m, we do not need many algebraic solutions when these solutions are in generic position. Then, it is enough that the sum of their degrees can be $m + 1$.

Theorem 13. (Christopher–Zholadek Theorem) *Let $f_i = 0$ for $i = 1, \ldots, p$ be p irreducible algebraic curves in $\mathbb{C}^2$, and let $k = \sum_{i=1}^{q} \deg f_i$. We assume*

(i) *there are no points at which f_i and its first derivatives all vanish,*
(ii) *the highest order terms of f_i have no repeated factors,*
(iii) *no more than two curves meet at any point in the finite plane and are not tangent at these points,*
(iv) *no two curves have a common factor in their highest order terms,*

then any polynomial vector field X of degree m tangent to all $f_i = 0$ is of the form described bellow.

(*a*) *If $m > k - 1$ then*

$$X = Y\left(\prod_{i=1}^{p} f_i\right) + \sum_{i=1}^{p} h_i \left(\prod_{\substack{j=1,\\ j\neq i}}^{p} f_j\right) X_{f_i}, \tag{11}$$

where $X_{f_i} = (-\partial f_i/\partial y, \partial f_i/\partial x)$ is a Hamiltonian vector field, the h_i are polynomials of degree $\leq m-k+1$ and Y is a polynomial vector field of degree $\leq m - k$.

(*b*) *If* $m = k - 1$ *then*

$$X = \sum_{i=1}^{p} \alpha_i \left(\prod_{\substack{j=1,\\ j\neq i}}^{p} f_j \right) X_{f_i},$$

with $\alpha_i \in \mathbb{C}$. *In this case, a Darboux first integral exists.*

(*c*) *If* $m < k - 1$ *then* $X \equiv 0$.

It is known that if a planar polynomial differential system (1) has the invariant straight line $a+bx+cy = 0$, then $a + bx + cy$ divides the polynomial

$$R(x, y) = \begin{vmatrix} 1 & x & y \\ 0 & P & Q \\ 0 & PP_x + QP_y & PQ_x + QQ_y \end{vmatrix}.$$

Moreover, if $(a + bx + cy)^k|R(x, y)$ and $(a + bx + cy)^{k+1} \nmid R(x, y)$, then we say that the invariant straight line $a + bx + cy = 0$ has *multiplicity* k. For more details on the multiplicity of invariant straight lines see [5] and [9].

The proof of the next result can be found in [6].

Lemma 14. *Assume that the polynomial differential system* (1) *has an invariant algebraic curve* $f = 0$ *and that* f *satisfies condition (i) of Theorem* 13. *If* $(f_x, f_y) = 1$, *then system* (1) *has the following normal form*

$$\dot{x} = Af - Cf_y, \quad \dot{y} = Bf + Cf_x$$

where $A, B, C \in \mathbb{R}[x, y], f_x = \partial f/\partial x, f_y = \partial f/\partial y$. *Here* $(f_x, f_y) = 1$ *means that the greatest common divisor between* f_x *and* f_y *is* 1.

3. Proof of the theorems

Now, we will prove Theorems 2, 3, 4, 5, 6, 7, and 8, which were stated in Sect. 1.

Proof of Theorem 2. (LV) Suppose that the polynomial differential system (1) has two real invariant straight lines intersecting at a point. Then, after an affine change of coordinates, we can assume that $f_1 = x = 0$ and $f_2 = y = 0$ are the invariant straight lines of system (1) intersecting at the origin with cofactors K_1 and K_2, respectively. So, we have that

$$P\frac{\partial f_1}{\partial x} + Q\frac{\partial f_1}{\partial y} = K_1 f_1 \quad \Rightarrow \quad P = K_1 x.$$

Analogously,

$$P\frac{\partial f_2}{\partial x} + Q\frac{\partial f_2}{\partial y} = K_2 f_2 \quad \Rightarrow \quad Q = K_2 y.$$

Therefore, we can write system (1) as

$$\dot{x} = xK_1(x, y), \quad \dot{y} = yK_2(x, y), \tag{12}$$

where $K_1, K_2 \in \mathbb{R}[x, y]$.

Now, we suppose that system (12) has a Darboux invariant. By Proposition 12, there exist $\mu, \lambda \in \mathbb{C}$ not all zero such that from Eq. (10), we have $\mu K_1 + \lambda K_2 = -s$, where $s \in \mathbb{R}\backslash\{0\}$. In particular, without loss of generality we can consider $\mu, \lambda \in \mathbb{R}$ and $\lambda \neq 0$. Then,

$$K_2 = -\frac{s}{\lambda} - \frac{\mu}{\lambda}K_1.$$

Therefore, from system (12), we get the following system

$$\dot{x} = xK_1(x,y), \quad \dot{y} = -\frac{1}{\lambda} y(s + \mu K_1(x,y)).$$

So taking $a = -s/\lambda$ and $b = -\mu/\lambda$, we obtain system (2). Moreover, it follows directly from Proposition 12 that the Darboux invariant of system (2) is given by $I_1(x,y,t) = y/(x^b e^{at})$. □

Proof of Theorem 3. (RPL) Suppose that the polynomial differential system (1) has two real and parallel invariant straight lines. Then, after an affine change of coordinates, we can take $f_1 = x - 1 = 0$ and $f_2 = x + 1 = 0$ as the invariant parallel straight lines of system (1) with cofactors K_1 and K_2, respectively. So we have that

$$P\frac{\partial f_1}{\partial x} + Q\frac{\partial f_1}{\partial y} = K_1 f_1 \quad \Rightarrow \quad P = K_1(x-1).$$

Analogously,

$$P\frac{\partial f_2}{\partial x} + Q\frac{\partial f_2}{\partial y} = K_2 f_2 \quad \Rightarrow \quad P = K_2(x+1).$$

Then, $P(x,y) = (x^2 - 1)g(x,y)$, with $g \in \mathbb{R}[x,y]$. Therefore, we can write system (1) as

$$\dot{x} = (x^2 - 1)g(x,y), \quad \dot{y} = Q(x,y). \tag{13}$$

Now, we suppose that system (13) has a Darboux invariant. By Proposition 12, there exist $\mu, \lambda \in \mathbb{C}$ not all zero such that they satisfy (10). Since $K_1 = (x+1)g(x,y)$ and $K_2 = (x-1)g(x,y)$ are the cofactors of $f_1 = 0$ and $f_2 = 0$, respectively, then, from (10), we have

$$\mu(x+1)g(x,y) + \lambda(x-1)g(x,y) = -s \quad \Rightarrow \quad [\mu(x+1) + \lambda(x-1)]g(x,y) = -s. \tag{14}$$

Hence, $g(x,y) = a \in \mathbb{R}\backslash\{0\}$, because otherwise $s = 0$, which is a contradiction with the fact that system (13) has a Darboux invariant. Then, we can write system (13) as

$$\dot{x} = a(x^2 - 1), \quad \dot{y} = Q(x,y).$$

We obtain system (3) doing the rescaling $T = at$, where T is the new time. So, we can take $g(x,y) = 1$ in Eq. (14), and then, we have that $\mu + \lambda = 0$ and $\mu - \lambda = -s$. Hence, $\lambda = -\mu = s/2$. Then, choosing $s = 1$, it follows directly from Proposition 12 that the Darboux invariant of system (3) is given by $I_2(x,y,t) = \sqrt{\frac{x+1}{x-1}}e^t$. □

Proof of Theorem 4. (CPL) Suppose that the polynomial differential system (1) has two complex and parallel invariant straight lines. Then, after an affine change of coordinates, we can take $f_1 = x - i = 0$ and $f_2 = x + i = 0$ as the invariant parallel straight lines of system (1) with cofactors K_1 and K_2, respectively. Using the same arguments of the proof of the previous theorem, we can assume $P(x,y) = (x^2 + 1)g(x,y)$, with $g \in \mathbb{R}[x,y]$. Therefore, we can write system (1) as

$$\dot{x} = (x^2 + 1)g(x,y), \quad \dot{y} = Q(x,y), \tag{15}$$

where $Q(x,y) \in \mathbb{R}[x,y]$.

Now, we suppose that system (15) has a Darboux invariant. By Proposition 12, there exist $\mu, \lambda \in \mathbb{C}$ not all zero such that they satisfy (10). Since $K_1 = (x+i)g(x,y)$ and $K_2 = \bar{K}_1$ are the cofactors of $f_1 = 0$ and $f_2 = 0$ respectively, then, from (10), we have

$$[\mu(x+i) + \lambda(x-i)]g(x,y) = -s. \tag{16}$$

As before we have $g(x,y) = a \in \mathbb{R}\backslash\{0\}$. Then, we can write system (15) as

$$\dot{x} = a(x^2 + 1), \quad \dot{y} = Q(x,y).$$

We obtain system (4) doing the rescaling $T = at$, where T is the new time.

Now, we consider $\mu = \mu_1 + i\mu_2$, with $\mu_1, \mu_2 \in \mathbb{R}$. From Eq. (16) note that $\lambda = \bar{\mu}$. So, we obtain $2a(\mu_1 x - \mu_2) = -s$, and consequently, $\mu_1 = 0$ and $\mu_2 = s/2$. It follows from Proposition 12 that the Darboux invariant of system (4) is given by $I_3(x, y, t) = (x - i)^{(is/2)}(x + i)^{(-is/2)}e^{st}$. Then, using (9), we can replace $f_1^{\mu}\bar{f}_1^{\bar{\mu}}$ and we obtain $I_3(x, y, t) = e^{(t+\arctan(1/x))}$. □

Proof of Theorem 5. (DL) Suppose that the polynomial differential system (1) has one double real invariant straight line. Then, after an affine change of coordinates, we can consider $f = x = 0$ as the double real invariant straight line of system (1). Given a small perturbation of system (1), the invariant straight line $x = 0$ bifurcates in two real and parallel straight lines $f_1 = x - \varepsilon = 0$ and $f_2 = x + \varepsilon = 0$. Other possibility is that $x = 0$ bifurcates in two real invariant straight lines intersecting at a point, but in this case, after doing some computations, we obtain that the polynomials P and Q of system (1) are not relatively prime. So, from the proof of Theorem 3 (RPL), we can write system (1) as

$$\dot{x} = (x^2 - \varepsilon^2)g(x, y), \quad \dot{y} = Q(x, y), \tag{17}$$

where $g \in \mathbb{R}[x, y]$. Then, making $\varepsilon \to 0$, we can write system (17) as

$$\dot{x} = x^2 g(x, y), \quad \dot{y} = Q(x, y), \tag{18}$$

and note that $f = x = 0$ is a double real invariant straight line of system (18).

Note that, $f = x = 0$ has cofactor $K = xg(x, y)$, and by Proposition 11, we have that $F = e^{1/x}$ is an exponential factor of system (18) with cofactor $L = -g(x, y)$. Suppose that system (18) has a Darboux invariant. So, by Proposition 12, there exist $\mu, \lambda \in \mathbb{C}$ not all zero such that

$$\mu K + \lambda L = -s \quad \Rightarrow \quad (\mu x - \lambda)g(x, y) = -s, \tag{19}$$

where $s \in \mathbb{R}\backslash\{0\}$. Using the same arguments of the proof of the previous theorems, we can take $g(x, y) = a \in \mathbb{R}\backslash\{0\}$. Then, we can write system (18) as

$$\dot{x} = ax^2, \quad \dot{y} = Q(x, y).$$

We obtain system (5) doing the rescaling $T = at$, where T is the new time. So, we can take $g(x, y) = 1$ in Eq. (19). Then, $\mu = 0$ and $\lambda = s$. Hence, it follows directly from Proposition 12 that the Darboux invariant of system (5) is given by $I_4(x, y, t) = e^{(t+1/x)}$. □

Proof of Theorem 6. (p) Suppose that the polynomial differential system (1) has two complex invariant straight lines intersecting at a real point. Then, after an affine change of coordinates, we can take $f_1 = x + iy = 0$ and $f_2 = x - iy = 0$ as the invariant straight lines of system (1) with cofactors K_1 and K_2, respectively. We denote the vector field of degree m associated with system (1) by $X = (P, Q)$. According to Theorem 13, we can write

$$X = f_1 f_2 Y + h_1 f_2 X_{f_1} + h_2 f_1 X_{f_2},$$

where $X_{f_i} = (-\partial f_i/\partial y, \partial f_i/\partial x)$ is a Hamiltonian vector field, h_1 and h_2 are polynomials of degree $\leq m - 1$, and $Y = (P_1, Q_1)$ is a polynomial vector field of degree $\leq m - 2$. So $(P, Q) = f_1 f_2(P_1, Q_1) + h_1 f_2(-i, 1) + h_2 f_1(i, 1)$ and consequently

$$P = (x^2 + y^2)P_1 - (h_1 + h_2)y + i(h_2 - h_1)x,$$
$$Q = (x^2 + y^2)Q_1 + (h_1 + h_2)x + i(h_2 - h_1)y.$$

Let $h_1(x, y) = m_1(x, y) + in_1(x, y)$ and $h_2(x, y) = m_2(x, y) + in_2(x, y)$, where $m_1, m_2, n_1, n_2 \in \mathbb{R}[x, y]$. Then, we have that

$$P = (x^2 + y^2)P_1 - y((m_1 + m_2) + i(n_1 + n_2)) + x((n_1 - n_2) + (m_2 - m_1)i),$$
$$Q = (x^2 + y^2)Q_1 + x((m_1 + m_2) + i(n_1 + n_2)) + y((n_1 - n_2) + (m_2 - m_1)i).$$

So $n_1 = -n_2$ and $m_1 = m_2$. Therefore, we can write system (1) as

$$\begin{aligned}\dot{x} &= (x^2+y^2)P_1(x,y) - 2ym_1(x,y) + 2xn_1(x,y),\\ \dot{y} &= (x^2+y^2)Q_1(x,y) + 2xm_1(x,y) + 2yn_1(x,y).\end{aligned} \tag{20}$$

Now, we consider $m_1(x,y) = M(x,y) + m$ and $n_1(x,y) = N(x,y) + n$, with $M(0,0) = N(0,0) = 0$. We can rewrite system (20) into the form

$$\begin{aligned}\dot{x} &= x\hat{A}(x,y) - y\hat{B}(x,y) + 2(nx - my),\\ \dot{y} &= y\hat{A}(x,y) + x\hat{B}(x,y) + 2(ny + mx),\end{aligned} \tag{21}$$

where $\hat{A}(x,y) = xP_1(x,y) + yQ_1(x,y) + 2N(x,y)$ and $\hat{B}(x,y) = xQ_1(x,y) - yP_1(x,y) + 2M(x,y)$. We suppose that system (21) has a Darboux invariant. By Proposition 12, there exist $\mu, \lambda \in \mathbb{C}$ not all zero such that they satisfy (10). We can show that $K_1 = (x - iy)(P_1 + iQ_1) + 2i(M + m) + 2(N + n)$ and $K_2 = \bar{K}_1$ are the cofactors of $f_1 = 0$ and $f_2 = 0$, respectively. Then, from Eq. (10), we have

$$(\mu+\lambda)(xP_1 + yQ_1 + 2(N+n)) + (\mu-\lambda)i(xQ_1 - yP_1 + 2(M+m)) = -s,$$

i.e.

$$(\mu+\lambda)(\hat{A} + 2n)) + (\mu-\lambda)i(\hat{B} + 2m)) = -s, \tag{22}$$

So $\lambda = \bar{\mu}$. We take $\mu = \mu_1 + i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$. Then, from Eq. (22), we obtain

$$2\mu_1(\hat{A} + 2n) - 2\mu_2(\hat{B} + 2m) = -s,$$

which provides the following linear system

$$\begin{aligned}\mu_1\hat{A}(x,y) - \mu_2\hat{B}(x,y) &= 0,\\ \mu_1 n - \mu_2 m &= -\frac{s}{4},\end{aligned} \tag{23}$$

in the variables μ_1 and μ_2.

We first consider $\mu_2 \neq 0$. Then, from Eq. (23), we have that $\hat{B}(x,y) = (\mu_1/\mu_2)\hat{A}(x,y)$ and $m = s/(4\mu_2) + n\mu_1/\mu_2$. We obtain system (6) replacing $\hat{B}$ and m in system (21) and taking $A(x,y) = \hat{A}(x,y) + 2n, a = \mu_1/\mu_2$ and $b = s/(2\mu_2)$. Moreover, from Proposition 12 and Eq. (9), it follows that the Darboux invariant of system (6) is $I_5(x,y,t) = (x^2+y^2)^a e^{-2\arctan(y/x)+2bt}$.

Now, we consider $\mu_2 = 0$. Obviously, $\mu_1 \neq 0$. So, from Eq. (23), we have that $\hat{A}(x,y) = 0$ and $n = -s/(4\mu_1)$. We obtain system (7) replacing $\hat{A}$ and n in system (21) and taking $B(x,y) = \hat{B}(x,y) + 2m$ and $c = -s/(2\mu_1)$. Moreover, from Proposition 12, it follows that Darboux invariant of system (7) is $I_6(x,y,t) = (x^2+y^2)e^{-2ct}$. □

Proof of Theorem 7. (P) Suppose that the polynomial differential system (1) has an invariant parabola. Then, after an affine change of coordinates, we can assume that $f = y - x^2 = 0$ is the invariant parabola of system (1) with cofactor K. Using Lemma 14, we can write system (1) as

$$\dot{x} = (y - x^2)A(x,y) - C(x,y), \quad \dot{y} = (y - x^2)B(x,y) - 2xC(x,y), \tag{24}$$

where $A, B, C \in \mathbb{R}[x,y]$.

Now, we suppose that system (24) has a Darboux invariant. Then, from Proposition 12, there exists $\mu \in \mathbb{R}\backslash\{0\}$ such that $\mu K = -s$, where $s \in \mathbb{R}\backslash\{0\}$ and $K = B(x,y) - 2xA(x,y)$. Let $B(x,y) = b + \hat{B}(x,y)$, with $\hat{B}(0,0) = 0$. So $b = -s/\mu \neq 0$ and $\hat{B}(x,y) = 2xA(x,y)$. Replacing $B(x,y)$ in system (24), we obtain system (8). We get from Proposition 12 and choosing $s = 1$ that the Darboux invariant of system (8) is $I_7(x,y,t) = e^t/(y - x^2)^{1/b}$. □

Proof of Theorem 8. (E, CE, H) Suppose that the polynomial differential system (1) has an invariant ellipse. Then, after an affine change of coordinates, we can assume that $f = x^2 + y^2 - 1 = 0$ is the invariant ellipse of system (1) with cofactor K. Using Lemma 14, we can write system (1) as

$$\dot{x} = (x^2 + y^2 - 1)A(x, y) - 2yC(x, y), \quad \dot{y} = (x^2 + y^2 - 1)B(x, y) + 2xC(x, y), \tag{25}$$

where $A, B, C \in \mathbb{R}[x, y]$.

It is easy to show that the cofactor of $f = 0$ is $K = 2(Ax + By)$. Then, Eq. (10) of Proposition 12 becomes $\mu K = -s$, but since the polynomial μK has no independent term, the equation never holds. Therefore, by Proposition 12, it follows that system (25) does not have Darboux invariants.

Note that, if the polynomial differential system (1) has either an invariant complex ellipse or an invariant hyperbola, then, after an affine change of coordinates, we can consider $x^2 + y^2 + 1 = 0$ and $x^2 - y^2 - 1 = 0$ as the invariant complex ellipse and invariant hyperbola, respectively. Analogously to the case of invariant ellipse, we can prove that differential polynomial systems in $\mathbb{R}^2$ with these invariant conics do not have Darboux invariants. □

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