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Optimal decay rates of the compressible fluid models of Korteweg type

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Abstract. We use a general energy method to prove the optimal time decay rates of the solutions to the compressible Navier–Stokes–Korteweg system in the whole space. In particular, the optimal decay rates of the higher-order spatial derivatives of solutions are obtained.

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1. Introduction

In this paper, we consider the compressible Navier–Stokes–Korteweg system for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u - (\mu + \lambda)\nabla \operatorname{div} u = \kappa \rho \nabla \Delta \rho. \tag{1.2}$$

This equation system governs the motions of the compressible isothermal viscous capillary fluids, where $\rho(t,x) and\, u(t,x)$ represent the density and velocity of the fluid, respectively, for all $(x,t) \in \mathbb{R}^3 \times (0,+\infty)$, and the smooth function $p = p(\rho)$ is the pressure satisfying $p'(\rho) > 0$ for $\rho > 0$. The constants μ, λ are the viscosity coefficients, and κ is the capillary coefficient, which satisfied the following condition:

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0, \quad \kappa > 0.$$

We supplement the system by the initial data

$$\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0. \tag{1.3}$$

The formulation of the theory of capillarity with diffuse interfaces was first introduced by Korteweg [15] and derived rigorously by Dunn and Serrin [2]. Recently, there is some mathematical theory concerning the compressible Navier–Stokes–Korteweg system. More precisely, Danchin and Desjardins [5] established the existence and uniqueness of solutions in the critical Besov spaces. Hattori and Li [12,13] proved the local existence and global existence of classical solutions in Sobolev space. Bresch et al. [1] and Haspot [9] considered the global existence of weak solutions for the compressible Navier–Stokes–Korteweg system. Kotschote [16] proved the local existence of the strong solutions. Wang and Tan [36] and Tan et al. [32] established the optimal decay rates of the global classical solutions and the optimal decay rates of the global strong solutions without external force, respectively. Li [23] proved the global existence and optimal decay rate of smooth solutions with potential external force. As is well known, for the compressible Navier–Stokes equations, i.e., $\kappa = 0$, many important progresses have been made on the existence and the convergence rates of solutions, see [3,4,6,7,10,11,16–19,22,24,25,30,33,37] and the reference therein

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for instance. Among them, when there is no force, Matsumura and Nishida obtained the optimal L^2 convergence rate for the compressible viscous and heat conductive fluid in $\mathbb{R}^3$ in [26]

$$\|(\rho-\bar\rho,u,\theta-\bar\theta)(t)\|_{L^2}\le C_0(1+t)^{-\frac{3}{4}},\quad t\ge 0. \tag{1.4}$$

If the small initial disturbance belongs to $H^3\cap L^1$, Ponce in [28] gave the optimal L^p convergence rate

$$\|\nabla^k(\rho-\bar\rho,u,\theta-\bar\theta)(t)\|_{L^2}\le C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{k}{2}},\quad t\ge 0, \tag{1.5}$$

for $2\le p\le\infty$ and $0\le k\le 2$. If the small initial disturbance belongs to $H^s\cap W^{s,1}$ with sufficient large integer s, the optimal L^p, $1\le p<2$ convergence rates were also obtained by using the Green function in [18,22,36]. When an external potential force is existing, the optimal L^p, $2\le p<6$ decay rate of the solutions and the optimal L^2 decay rate of the first-order derivatives were obtained in a series of papers [6,7,33]. When there is a self-consist electric potential force, i.e., for the compressible Navier–Stokes–Poisson equations, recently, H.-L. Li et al. proved the global existence and optimal decay rates of the solutions in [21]. It was showed that the rotating effect of electric field makes the momentum of the compressible Navier–Stokes–Poisson equations decay at a slower rate. Finally, the long-time decay rate of global solutions for half-space and exterior domain or the general external force were obtained, and we refer to the papers [4,5,17,19,20,30] for instance.

In this paper, by using a general energy method, we will obtain the optimal time decay rates of the solutions to the problem (1.1)–(1.3) when the initial data are small perturbations of given constant state $(\bar\rho,\bar\theta)$. The study is motivated by Guo and Wang [8], where the authors develop a pure energy method for proving the optimal time decay rates of the solutions for the compressible Navier–Stokes equations. Recently, the method has wide range of applications. Wang used the method to study the Vlasov–Poisson–Boltzmann system and the Navier–Stokes–Poisson equations, see [34,35]. Tan and Wang [31] considered the MHD equation by using this method. In our proof, the negative Sobolev norms are shown to be preserved along time evolution and enhance the decay rates. We use a family of scaled energy estimates with minimum derivative counts and interpolations among them without linear decay analysis.

Main results of this paper are stated as the following theorem.

Theorem 1.1. *Assume that $(\rho_0-\bar\rho)\in H^{N+1}, u\in H^N$ for an integer $N\ge 3$. Then, there exists a constant δ_0 such that if*

$$\|\rho_0-\bar\rho\|_{H^4}+\|u_0\|_{H^3}\le\delta_0, \tag{1.6}$$

then there exists a unique global solution (ρ,u) of the Cauchy problem (1.1)–(1.3) satisfying that for all $t\ge 0$,

$$\|(\rho-\bar\rho)(t)\|^2_{H^{N+1}}+\|u(t)\|^2_{H^N}+\int_0^t\|\nabla\rho(s)\|^2_{H^{N+1}}+\|\nabla u(s)\|^2_{H^N}ds\le C(\|\rho_0-\bar\rho\|^2_{H^{N+1}}+\|u_0\|^2_{H^N}). \tag{1.7}$$

If further, $(\rho_0-\bar\rho,u_0)\in\dot H^{-s}$ for some $s\in[0,\frac{3}{2})$, then

$$\|\Lambda^{-s}(\rho-\bar\rho)\|^2_{L^2}+\|\Lambda^{-s}u\|^2_{L^2}+\|\Lambda^{-s}\nabla(\rho-\bar\rho)\|^2_{L^2}\le C_0, \tag{1.8}$$

and

$$\|\nabla^l(\rho-\bar\rho)(t)\|^2_{H^{N-l+1}}+\|\nabla^l u(t)\|^2_{H^{N-l}}\le C_0(1+t)^{-(l+s)}\quad for\quad l=0,\ldots,N-1. \tag{1.9}$$

Remark 1.2. The constraint $s<3/2$ comes from applying Lemma 2.5 to estimate the nonlinear terms when doing the negative Sobolev estimates via Λ^{-s}, for when $s\ge 3/2$, the nonlinear estimates would not work.

Remark 1.3. Notice that we only assume that the lower-order Sobolev norm of initial data is small, while the higher-order Sobolev norm can be arbitrarily large.

From Sect. 3, we will use the pure energy estimates method to present the proof of Theorem 1.1. However, we will be not able to close the energy estimates at each $l-$th level as the heat equation. This is essentially caused by the "degenerate" dissipative structure of the nonlinear homogenous system of (1.1)–(1.3), when using the energy method. More precisely, the linear energy identity of the problem reads as: for $l = 0, \dots, N$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^l \varrho|^2 + |\nabla^l u|^2 + \kappa|\nabla^{l+1}\varrho|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \mu|\nabla\nabla^l u|^2 + (\mu+\lambda)|\mathrm{div}\nabla^l u|^2 \mathrm{d}x = 0. \tag{1.10}$$

The constraint of λ and μ implies that there exists a constant $\sigma_0 > 0$ such that

$$\int_{\mathbb{R}^3} \mu|\nabla\nabla^l u|^2 + (\mu+\lambda)|\mathrm{div}\nabla^l u|^2 \mathrm{d}x \geq \sigma_0 \|\nabla^{l+1} u\|_{L^2}^2. \tag{1.11}$$

Note that (1.10) and (1.11) only give the dissipative estimate for u. To rediscover the dissipative estimate for ϱ, we will use the linearized equations of (1.1)–(1.3) via constructing the interactive energy functional between u and $\nabla\varrho$ to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^l u \cdot \nabla\nabla^l \varrho \mathrm{d}x + C(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+2}\varrho\|_{L^2}^2) \lesssim \|\nabla^{l+1} u\|_{L^2}^2. \tag{1.12}$$

This implies that to get the dissipative estimate for ϱ, it requires us to do the energy estimates (1.10) at the l-th level. To get around this obstacle, the idea is to construct some energy functionals $\mathcal{E}_l^m(t)$, $1 \leq m \leq N$ and $0 \leq l \leq m-1$,

$$\mathcal{E}_l^m(t) \sim \sum_{l \leq k \leq m} \|[\nabla^k \varrho(t), \nabla^k \nabla\varrho(t), \nabla^k u(t)]\|_{L^2}^2, \tag{1.13}$$

which has a minimum derivative count l. We will then close the energy estimates at each l-th level in weak sense by deriving the Lyapunov-type inequality for these energy functionals in which the corresponding dissipation ($\mathcal{D}_l^m(t)$) can be related to the energy $\mathcal{E}_l^m(t)$ by the Sobolev interpolation. This can be easily established for the linear homogeneous problem along our analysis; however, for the nonlinear problem (2.1)–(2.3), it is much more complicated due to the nonlinear estimates. This is the second point of this paper that we will extensively and carefully use the Sobolev interpolation of the Gagliardo–Nirenberg inequality between high- and low-order spatial derivatives to bound the nonlinear terms by $\sqrt{\mathcal{E}_0^3(t)}\mathcal{D}_l^m(t)$ that can be absorbed. When deriving the negative Sobolev estimates, we need to restrict that $s < 3/2$ in order to estimate Λ^{-s} acting on the nonlinear terms by using the Hardy–Littlewood–Sobolev inequality, and also we need to separate the cases that $s \in (0, 1/2]$ and $s \in (1/2, 3/2)$. Once these estimates are obtained, Theorem 1.1 follows by the interpolation between negative and positive Sobolev norms.

The rest of this paper is devoted to prove Theorem 1.1. We briefly introduce the strategy of the proof. In Sect. 3, it suffices to derive (1.7). Then, the global existence will follow in a standard way as in [23] by the local existence, a priori estimates and the continuity argument. Finally, we use the energy estimates to deduce the Lyapunov-type energy inequality, then combining it with Lemma 4.1 to prove (1.8) and (1.9).

Notation. In this paper, L^p, H^s denote the usual L^p and Sobolev spaces on $\mathbb{R}^3$, with norms $\|.\|_{L^p}$ and $\|.\|_{H^s}$, respectively. ∇^l with $l \in \mathbb{Z}^+$ stands for the usual spatial derivatives of order l. Λ^{-s} with $s > 0$ is defined in Definition 2.1 of Sect. 2. We use $\dot{H}^s$ denoting the homogeneous Sobolev spaces on $\mathbb{R}^3$ with norm $\|.\|_{\dot{H}^s}$ defined by (2.8). We use C to denote the constant depending only on the physical coefficient but may vary and C_0 to denote the constant depending additionally on the initial data.

2. Preliminaries

Before we present the energy estimates method, we should now recall the following useful Lemmas which we will use extensively.

First, we will review the Sobolev interpolation of the Gagliardo–Nirenberg inequality which we will use much often.

Lemma 2.1. *Let $0 \le s, \alpha \le l$, and then we have*

$$\|\nabla^\alpha f\|_{L^p} \le \|\nabla^s f\|_{L^q}^{1-\theta} \|\nabla^l f\|_{L^r}^{\theta}, \tag{2.1}$$

where α satisfies the following equation:

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{s}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta. \tag{2.2}$$

Proof. This is a special case of [27, pp. 125, Theorem].

Then, we shall recall the following estimate Lemmas to estimate the L^∞ norm of the spatial derivatives of h and f defined by (3.4) □

Lemma 2.2. *If $\|\varrho\|_{H^2} \le 1$, and $g(\varrho)$ is a smooth function of ϱ with bounded derivatives of any order, then for any integer $m \ge 1$, we have*

$$\|\nabla^m (g(\varrho))\|_{L^\infty} \le \|\nabla^m \varrho\|_{L^2}^{1/4} \|\nabla^{m+2} \varrho\|_{L^2}^{3/4}. \tag{2.3}$$

Proof. See [8, Lemma 3.1]. □

Lemma 2.3. (Commutator Estimate) *Let $m \ge 1$ be an integer, and then the commutator which is defined by the following*

$$[\nabla^m, f]g := \nabla^m (fg) - f\nabla^m g, \tag{2.4}$$

can be bounded by

$$\|[\nabla^m, f]g\|_{L^p} \le \|\nabla f\|_{L^{p_1}} \|\nabla^{m-1} g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{2.5}$$

where $p, p_2, p_3 \in (1, +\infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \tag{2.6}$$

Proof. See [14, Lemma 3.1]. □

Then, in order to establish the negative Sobolev estimates, we should review the following useful lemmas related to the negative Sobolev norms. But let us first introduce some necessary definitions.

Definition 2.1. The operator Λ^s, for $s \in \mathbb{R}$, is defined by

$$\Lambda^s g(x) = \int_{\mathbb{R}} |\xi|^s \hat{g}(\xi) e^{2\pi i x\cdot\xi} d\xi, \tag{2.7}$$

where $\hat{g}$ is the Fourier transform of g.

Definition 2.2. $\dot{H}^s$ is defined as the homogeneous Sobolev space of g, with the following norm:

$$\|g\|_{\dot{H}^s} := \|\Lambda^s g\|_{L^2} = \||\xi|^s \hat{g}\|_{L^2}. \tag{2.8}$$

The index s can be nonpositive real numbers. However, for convenience, we will change the index to be with $s \ge 0$, in this case. We will employ the following special Sobolev interpolation that related the negative index s:

Lemma 2.4. *Let $s \geq 0$ and $l \geq 0$, and then we have*

$$\|\nabla^l g\|_{L^2} \leq \|\nabla^{l+1} g\|_{L^2}^{1-\theta} \|g\|_{\dot{H}^{-s}}^{\theta}, \quad \text{where} \quad \theta = \frac{1}{l+s+1}. \tag{2.9}$$

Proof. By the Parseval theorem, the definition of (2.8) and Hölder's inequality, we have

$$\|\nabla^l g\|_{L^2} = \||\xi|^l \hat{g}\|_{L^2} \leq \||\xi|^{l+1}\hat{g}\|_{L^2}^{1-\theta} \||\xi|^{-s}\hat{g}\|_{L^2}^{\theta} = \|\nabla^{l+1} g\|_{L^2}^{1-\theta} \|g\|_{\dot{H}^{-s}}^{\theta}. \tag{2.10}$$

□

If $s \in (0, 3), \Lambda^{-s} g$ defined by (2.7) is the Riesz potential. The Hardy–Littlewood–Sobolev theorem implies the following L^p type inequality for the Riesz potential:

Lemma 2.5. *Let $0 < s < 3, 1 < p < q < \infty, 1/q + s/3 = 1/p$, and then*

$$\|\Lambda^{-s} g\|_{L^q} \leq \|g\|_{L^p}. \tag{2.11}$$

Proof. See [29, pp. 119, Theorem 1]. □

3. Energy estimates

To accomplish the energy estimates on the nonlinear problem of (1.1)–(1.3), we set $\varrho := \rho - 1$ and reformulate them as

$$\partial_t \varrho + \mathrm{div} u = -\mathrm{div}(\varrho u), \tag{3.1}$$

$$\partial_t u - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div} u + \nabla\varrho - \kappa\nabla\Delta\varrho = -(u\cdot\nabla)u - h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) - f(\varrho)\nabla\varrho, \tag{3.2}$$

$$(\varrho, u)|_{t=0} = (\varrho_0, u_0), \tag{3.3}$$

where the two nonlinear functions of ϱ are defined by

$$h(\varrho) := \frac{\varrho}{\varrho+1}, \quad \text{and} \quad f(\varrho) = \frac{p'(\varrho+1)}{\varrho+1} - 1. \tag{3.4}$$

And moreover, in order to derive the priori energy estimates for the equivalent problem (3.1)–(3.3), we assume a priori that for sufficiently small $\delta > 0$,

$$\sqrt{\mathcal{E}_0^3(t)} = \|\varrho(t)\|_{H^4} + \|u(t)\|_{H^3} \leq \delta. \tag{3.5}$$

This together with Sobolev's inequality implies in particular, and we obtain

$$\frac{1}{2} \leq \varrho + 1 \leq 2. \tag{3.6}$$

Hence, we immediately have

$$|h(\varrho)|, |f(\varrho)| \leq C|\varrho| \quad \text{and} \quad |h^{(k)}(\varrho)|, |f^{(k)}(\varrho)| \leq C, \quad \forall k \geq 1. \tag{3.7}$$

Next, we will start to exhibit the first type of energy estimates including the dissipation estimate for u.

Lemma 3.1. *If $\sqrt{\mathcal{E}_0^3(t)} \leq \delta$, then for $k = 0, 1, \ldots, N-1$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^k\varrho|^2 + |\nabla^k u|^2 + \kappa|\nabla^{k+1}\varrho|^2 dx + C\|\nabla^{k+1}u\|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2). \tag{3.8}$$

Proof. Applying ∇^k to (3.1), (3.2) and multiplying those two resulting identities by $\nabla^k\varrho$ and $\nabla^k u$, respectively, summing them up and then integrating the equation over $\mathbb{R}^3$ by parts, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\nabla^k\varrho|^2+|\nabla^k u|^2\mathrm{d}x+\int\limits_{\mathbb{R}^3}\mu|\nabla^{k+1}u|^2+(\mu+\lambda)|\nabla^k\mathrm{div}u|^2\mathrm{d}x+\int\limits_{\mathbb{R}^3}\kappa\nabla^k\mathrm{div}u\cdot\nabla^k\Delta\varrho\mathrm{d}x\\&=\int\limits_{\mathbb{R}^3}-\nabla^k(\mathrm{div}(\varrho u))\cdot\nabla^k\varrho-\nabla^k((u\cdot\nabla)u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)+f(\varrho)\nabla\varrho)\nabla^k u\mathrm{d}x.\end{aligned}\tag{3.9}$$

We can first estimate the left-hand side of (3.9). For the second term, we have

$$\int\limits_{\mathbb{R}^3}\mu|\nabla^{k+1}u|^2+(\mu+\lambda)|\nabla^k\mathrm{div}u|^2\mathrm{d}x\geq\sigma_0\|\nabla^{k+1}u\|_{L^2}^2,\tag{3.10}$$

since $\mu>0$ and $\lambda+\frac{2}{3}\mu\geq 0$. □

For the third term, by the continuity Eq. (3.1) and integrating over $\mathbb{R}^3$ by parts, we can obtain the following:

$$\begin{aligned}\int\limits_{\mathbb{R}^3}\kappa\nabla^k\mathrm{div}u\cdot\nabla^k\Delta\varrho\mathrm{d}x&=\int\limits_{\mathbb{R}^3}\kappa\nabla^k(-\partial_t\varrho-\mathrm{div}(\varrho u))\cdot\nabla^k\Delta\varrho\mathrm{d}x\\&=\int\limits_{\mathbb{R}^3}\kappa\nabla^{k+1}(\partial_t\varrho+\mathrm{div}(\varrho u))\cdot\nabla^{k+1}\varrho\mathrm{d}x\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}\kappa|\nabla^{k+1}\varrho|^2\mathrm{d}x+\int\limits_{\mathbb{R}^3}\kappa\nabla^{k+1}(\varrho\mathrm{div}u+u\cdot\nabla\varrho)\nabla^{k+1}\varrho\mathrm{d}x.\end{aligned}\tag{3.11}$$

In light of (3.10) and (3.11), we can rewrite (3.9) as:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\nabla^k\varrho|^2+|\nabla^k u|^2+\kappa|\nabla^{k+1}\varrho|^2\mathrm{d}x+\sigma_0\|\nabla^{k+1}u\|_{L^2}^2\\&\leq-\int\limits_{\mathbb{R}^3}\kappa\nabla^{k+1}(\varrho\mathrm{div}u+u\cdot\nabla\varrho)\nabla^{k+1}\varrho\mathrm{d}x+\int\limits_{\mathbb{R}^3}-\nabla^k(\mathrm{div}(\varrho u))\cdot\nabla^k\varrho\\&\quad-\nabla^k((u\cdot\nabla)u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)+f(\varrho)\nabla\varrho)\nabla^k u\mathrm{d}x.\end{aligned}\tag{3.12}$$

Then, we shall estimate each term in the right-hand side of (3.12). The key point is that we will carefully interpolate the spatial derivatives between the higher-order derivatives and the lower-order ones to bound these nonlinear terms by the right-hand side of (3.8). Firstly, we should consider one special situation, when $k=0$.

Let $k=0$, and by integrating by parts, we can get from (3.12) as

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\varrho|^2+|u|^2+\kappa|\nabla\varrho|^2\mathrm{d}x+\sigma_0\|\nabla u\|_{L^2}^2\\&\lesssim-\int\limits_{\mathbb{R}^3}\kappa\nabla(\varrho\mathrm{div}u+u\cdot\nabla\varrho)\nabla\varrho\mathrm{d}x\\&\quad-\int\limits_{\mathbb{R}^3}\mathrm{div}(\varrho u)\varrho+((u\cdot\nabla)u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)+f(\varrho)\nabla\varrho)u\mathrm{d}x\\&\lesssim\int\limits_{\mathbb{R}^3}\kappa(\varrho\mathrm{div}u+u\cdot\nabla\varrho)\nabla^2\varrho\mathrm{d}x-\int\limits_{\mathbb{R}^3}(\varrho\mathrm{div}u+u\cdot\nabla\varrho)\varrho+((u\cdot\nabla)u\\&\quad+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)+f(\varrho)\nabla\varrho)u\mathrm{d}x\\&:=I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned}\tag{3.13}$$

For the term I_1, by Höder's, Minkowshi's, Sobolev's and Young's inequality, we obtain

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^3} \kappa(\varrho \mathrm{div} u + u \cdot \nabla\varrho)\nabla^2\varrho \mathrm{d}x \lesssim (\|\varrho\|_{L^6}\|\nabla u\|_{L^2} + \|u\|_{L^6}\|\nabla\varrho\|_{L^2})\|\nabla^2\varrho\|_{L^3} \\ &\lesssim (\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla\varrho\|_{L^2})\|\nabla^2\varrho\|_{L^3} \lesssim (\|\nabla u\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2)\|\nabla^2\varrho\|_{H^1} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla u\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2). \end{aligned} \tag{3.14}$$

Similarly, we can bound I_2 and I_3 by

$$\begin{aligned} I_2 &:= -\int_{\mathbb{R}^3} (\varrho \mathrm{div} u + u \cdot \nabla\varrho)\varrho \mathrm{d}x \lesssim (\|\varrho\|_{L^3}\|\nabla u\|_{L^2} + \|u\|_{L^3}\|\nabla\varrho\|_{L^2})\|\varrho\|_{L^6} \\ &\lesssim (\|\varrho\|_{H^1}\|\nabla u\|_{L^2} + \|u\|_{H^1}\|\nabla\varrho\|_{L^2})\|\nabla\varrho\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla u\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2). \end{aligned} \tag{3.15}$$

$$I_3 := -\int_{\mathbb{R}^3} (u\cdot\nabla)u \cdot u \mathrm{d}x \lesssim \|u\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6} \lesssim \|u\|_{H^3}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla u\|_{L^2}^2. \tag{3.16}$$

For the term I_4, from the Eq. (3.7), we know that

$$\begin{aligned} I_4 &:= -\int_{\mathbb{R}^3} h(\varrho)(\mu\Delta u)u \mathrm{d}x = \int_{\mathbb{R}^3} \mu\nabla u \cdot \nabla(h(\varrho)u)\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \mu\nabla u \cdot (h'(\varrho)\nabla\varrho\cdot u + h(\varrho)\nabla u)\mathrm{d}x \\ &\lesssim \|\nabla u\|_{L^3}\|\nabla\varrho\|_{L^2}\|u\|_{L^6} + \|\nabla u\|_{L^3}\|h(\varrho)\|_{L^6}\|\nabla u\|_{L^2} \\ &\lesssim \|u\|_{H^3}\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2} + \|u\|_{H^3}\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \end{aligned} \tag{3.17}$$

Similarly, for the term I_5, we have

$$\begin{aligned} I_5 &:= -\int_{\mathbb{R}^3} h(\varrho)(\mu+\lambda)\nabla \mathrm{div} u \cdot u \mathrm{d}x = \int_{\mathbb{R}^3} (\mu+\lambda)\mathrm{div} u \cdot \nabla(h(\varrho)\cdot u)\mathrm{d}x \\ &= \int_{\mathbb{R}^3} (\mu+\lambda)\mathrm{div} u \cdot (h'(\varrho)\nabla\varrho\cdot u + h(\varrho)\nabla u)\mathrm{d}x \\ &\lesssim \|\nabla u\|_{L^3}(\|\nabla\varrho\|_{L^2}\|u\|_{L^6} + \|\varrho\|_{L^6}\|\nabla u\|_{L^2}) \\ &\lesssim \|u\|_{H^3}(\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2}) \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \end{aligned} \tag{3.18}$$

By Sobolev's inequality, we can bound the term I_6 as

$$\begin{aligned} I_6 &:= -\int_{\mathbb{R}^3} f(\varrho)\nabla\varrho u \mathrm{d}x \lesssim \|f(\varrho)\|_{L^3}\|\nabla\varrho\|_{L^2}\|u\|_{L^6} \\ &\lesssim \|\varrho\|_{H^4}\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \end{aligned} \tag{3.19}$$

From (3.13) to (3.19), we can get that for $k = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\varrho|^2 + |u|^2 + \kappa|\nabla\varrho|^2 \mathrm{d}x + C\|\nabla u\|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2), \tag{3.20}$$

which means that $k = 0$ satisfied the form in (3.8).

Next, we should estimate all the terms of the right-hand side of (3.12), when $k \geq 1$ and set

$$\begin{aligned} &-\int\limits_{\mathbb{R}^3} \kappa\nabla^{k+1}(\varrho\mathrm{div}u + u\cdot\nabla\varrho)\nabla^{k+1}\varrho\mathrm{d}x + \int\limits_{\mathbb{R}^3} -\nabla^k(\mathrm{div}(\varrho u))\cdot\nabla^k\varrho \\ &-\nabla^k[(u\cdot\nabla)u + h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) + f(\varrho)\nabla\varrho]\nabla^k u\mathrm{d}x \\ &:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{3.21}$$

First, for the term J_1, employing the Leibniz formula and by Hölder's inequality, we obtain the following:

$$\begin{aligned} J_1 &:= -\int\limits_{\mathbb{R}^3} \kappa\nabla^{k+1}(\varrho\mathrm{div}u)\nabla^{k+1}\varrho\mathrm{d}x \\ &= -\int\limits_{\mathbb{R}^3} \sum_{0\leq l\leq k+1} \kappa C_{k+1}^l \nabla^l\varrho\nabla^{k-l+2}u\nabla^{k+1}\varrho\mathrm{d}x \\ &\lesssim \sum_{0\leq l\leq k+1} \|\nabla^l\varrho\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}. \end{aligned} \tag{3.22}$$

To estimate the first factor above, we take the L^3-norm on the term with less number of derivatives. Hence, if $l \leq [\frac{k+1}{2}]$, together with the Sobolev interpolation of Lemma 2.1, we have

$$\begin{aligned} \|\nabla^l\varrho\nabla^{k-l+2}u\|_{L^2} &\lesssim \|\nabla^l\varrho\|_{L^3}\|\nabla^{k-l+2}u\|_{L^6} \\ &\lesssim \|\nabla^\alpha\varrho\|_{L^2}^{1-\frac{l-2}{k-1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l-2}{k-1}}\|\nabla^2 u\|_{L^2}^{\frac{l-2}{k-1}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l-2}{k-1}}, \end{aligned} \tag{3.23}$$

where α comes from the adjustment of the index between the energy and the dissipation, and it is defined by

$$\begin{aligned} &\frac{l}{3} - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right)\left(1 - \frac{l-2}{k-1}\right) + \left(\frac{k+1}{3} - \frac{1}{2}\right)\frac{l-2}{k-1} \\ &\Longrightarrow \alpha = 2 + \frac{k-1}{2(k-l+1)} \in [2,3), \quad \text{since} \quad 0 \leq l \leq \frac{k+1}{2}. \end{aligned} \tag{3.24}$$

Hence, by the definition of the energy $\mathcal{E}_0^3(t)$ and Young's inequality, we obtain that for $l \leq [\frac{k+1}{2}]$,

$$\|\nabla^l\varrho\nabla^{k-l+2}u\|_{L^2} \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}). \tag{3.25}$$

If $[\frac{k+1}{2}] + 1 \leq l \leq k$ (if $k < [\frac{k+1}{2}] + 1$, then it is nothing in this case, and hereafter, etc.), we have

$$\begin{aligned} &\|\nabla^l\varrho\nabla^{k-l+2}u\|_{L^2} \lesssim \|\nabla^l\varrho\|_{L^6}\|\nabla^{k-l+2}u\|_{L^3} \\ &\lesssim \|\nabla^2\varrho\|_{L^2}^{1-\frac{l-1}{k-1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l-1}{k-1}}\|\nabla^\alpha u\|_{L^2}^{\frac{l-1}{k-1}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l-1}{k-1}} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}), \end{aligned} \tag{3.26}$$

where α is defined by

$$\begin{aligned} &\frac{k-l+2}{3} - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right)\frac{l-1}{k-1} + \left(\frac{k+1}{3} - \frac{1}{2}\right)\left(1 - \frac{l-1}{k-1}\right) \\ &\qquad\Longrightarrow \alpha = 2 + \frac{k-1}{2(l-1)} \in [2,3), \quad \text{since} \quad \frac{k+1}{2} < l \leq k+1. \end{aligned} \tag{3.27}$$

In light of (3.25) and (3.26), by Cauchy's inequality, we deduce from (3.22) that

$$J_1 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2). \tag{3.28}$$

Next, for the term J_2, we utilize the commutator notation (3.4) to rewrite it as

$$\begin{aligned} J_2 &:= -\int_{\mathbb{R}^3} \kappa \nabla^{k+1}(u\cdot\nabla\varrho)\nabla^{k+1}\varrho \mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \kappa(u\cdot\nabla\nabla^{k+1}\varrho + [\nabla^{k+1}, u]\nabla\varrho)\nabla^{k+1}\varrho \mathrm{d}x \\ &:= J_{21} + J_{22}. \end{aligned} \tag{3.29}$$

By integrating by part, by Sobolev's inequality, we have

$$\begin{aligned} J_{21} &:= -\int_{\mathbb{R}^3} \kappa u\cdot\nabla\nabla^{k+1}\varrho\cdot\nabla^{k+1}\varrho \mathrm{d}x = -\int_{\mathbb{R}^3} \kappa u\cdot\nabla\frac{|\nabla^{k+1}\varrho|^2}{2}\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \kappa \mathrm{div} u|\nabla^{k+1}\varrho|^2 \mathrm{d}x \\ &\lesssim \|\nabla u\|_{L^\infty}\|\nabla^{k+1}\varrho\|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}\varrho\|_{L^2}^2. \end{aligned} \tag{3.30}$$

We use the commutator estimate of Lemma 2.3 and Sobolev's inequality to bound

$$\begin{aligned} J_{22} &:= -\int_{\mathbb{R}^3} \kappa([\nabla^{k+1}, u]\nabla\varrho)\cdot\nabla^{k+1}\varrho \mathrm{d}x \lesssim \|[\nabla^{k+1}, u]\nabla\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \\ &\lesssim (\|\nabla u\|_{L^\infty}\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}\|\nabla\varrho\|_{L^\infty})\|\nabla^{k+1}\varrho\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2). \end{aligned} \tag{3.31}$$

In light of (3.30) and (3.31), we find

$$J_2 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2). \tag{3.32}$$

Now, we estimate the term J_3. By integrating by parts, by Hölder's, Minkowshi's and Sobolev's inequalities, we obtain

$$\begin{aligned} J_3 &:= \int_{\mathbb{R}^3} -\nabla^k(\mathrm{div}(\varrho u))\cdot\nabla^k\varrho \mathrm{d}x = \int_{\mathbb{R}^3} \nabla^{k-1}(\mathrm{div}(\varrho u))\cdot\nabla^{k+1}\varrho \mathrm{d}x \\ &\lesssim \|\nabla^k(\varrho u)\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} = \|\sum_{0\le l\le k} C_k^l\nabla^l\varrho\nabla^{k-l}u\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \\ &\lesssim \sum_{0\le l\le k}\|\nabla^l\varrho\nabla^{k-l}u\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}. \end{aligned} \tag{3.33}$$

If $l \le [\frac{k}{2}]$, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} \|\nabla^l\varrho\nabla^{k-l}u\|_{L^2}\| &\lesssim \|\nabla^l\varrho\|_{L^3}\|\nabla^{k-l}u\|_{L^6} \\ &\lesssim \|\nabla^\alpha\varrho\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l}{k}}\|\nabla u\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l}{k}}, \end{aligned} \tag{3.34}$$

where α is defined by

$$\begin{aligned} &\frac{l}{3}-\frac{1}{3} = \left(\frac{\alpha}{3}-\frac{1}{2}\right)\left(1-\frac{l}{k}\right) + \left(\frac{k+1}{3}-\frac{1}{2}\right)\frac{l}{k} \\ &\Longrightarrow \alpha = 2-\frac{k}{2(k-l)} \in \left[1, \frac{3}{2}\right], \quad \text{since} \quad 0\le l\le\frac{k}{2}. \end{aligned} \tag{3.35}$$

If $[\frac{k}{2}]+1 \le l \le k$, by Hölder's inequality and Lemma 2.1 again, we have

$$\begin{aligned} &\|\nabla^l\varrho\nabla^{k-l}u\|_{L^2}\| \lesssim \|\nabla^l\varrho\|_{L^6}\|\nabla^{k-l}u\|_{L^3} \\ &\lesssim \|\varrho\|_{L^2}^{1-\frac{l+1}{k+1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l+1}{k+1}}\|\nabla^\alpha u\|_{L^2}^{\frac{l+1}{k+1}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l+1}{k+1}}, \end{aligned} \tag{3.36}$$

where α is defined by

$$\frac{k-l}{3}-\frac{1}{3}=\left(\frac{\alpha}{3}-\frac{1}{2}\right)\frac{l+1}{k+1}+\left(\frac{k+1}{3}-\frac{1}{2}\right)\left(1-\frac{l+1}{k+1}\right)$$
$$\Longrightarrow \alpha=\frac{k+1}{2(l+1)}\in[\frac{1}{2},1),\quad \text{since}\quad \frac{k+1}{2}\leq l\leq k. \tag{3.37}$$

In light of (3.34) and (3.36), by Young's inequality, we deduce from (3.33) that

$$J_3\lesssim\sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2). \tag{3.38}$$

Next, we estimate the term J_4. By Hölder's and Sobolev's inequality, we obtain

$$\begin{aligned} J_4&:=\int_{\mathbb{R}^3}-\nabla^k((u\cdot\nabla)u)\cdot\nabla^k u\mathrm{d}x=\int_{\mathbb{R}^3}\nabla^{k-1}((u\cdot\nabla)u)\cdot\nabla^{k+1}u\mathrm{d}x\\ &=\sum_{0\leq l\leq k-1}C_{k-1}^l\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{k-l}u\cdot\nabla^{k+1}u\mathrm{d}x\\ &\lesssim\sum_{0\leq l\leq k-1}\|\nabla^l u\cdot\nabla^{k-l}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}. \end{aligned} \tag{3.39}$$

If $l\leq[\frac{k-1}{2}]$, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} &\|\nabla^l u\cdot\nabla^{k-l}u\|_{L^2}\lesssim\|\nabla^l u\|_{L^3}\|\nabla^{k-l}u\|_{L^6}\\ &\quad\lesssim\|\nabla^\alpha u\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}^{\frac{l}{k}}\|\nabla u\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l}{k}}\\ &\quad=\|\nabla^\alpha u\|_{L^2}^{1-\frac{l}{k}}\|\nabla u\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}\\ &\quad\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}u\|_{L^2}, \end{aligned} \tag{3.40}$$

where α is defined as (3.35), but

$$\alpha=2-\frac{k}{2(k-l)}\in(1,\frac{3}{2}],\quad \text{since}\quad 0\leq l\leq\frac{k-1}{2}. \tag{3.41}$$

If $[\frac{k-1}{2}]+1\leq l\leq k-1$, by Hölder's inequality and Lemma 2.1 again, we have

$$\begin{aligned} &\|\nabla^l u\cdot\nabla^{k-l}u\|_{L^2}\lesssim\|\nabla^l u\|_{L^6}\|\nabla^{k-l}u\|_{L^3}\\ &\lesssim\|\nabla u\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}u|_{L^2}^{\frac{l}{k}}\|\nabla^\alpha u\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l}{k}}\\ &=\|\nabla u\|_{L^2}^{1-\frac{l}{k}}\|\nabla^\alpha u\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}\\ &\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}u\|_{L^2}, \end{aligned} \tag{3.42}$$

where α is defined as (3.37), but

$$\alpha=\frac{k+1}{2(l+1)}\in\left(\frac{1}{2},1\right),\quad \text{since}\quad \frac{k}{2}\leq l\leq k-1. \tag{3.43}$$

In light of (3.40) and (3.42), we deduce from (3.39) that

$$J_4\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}u\|_{L^2}^2. \tag{3.44}$$

Next, we estimate the term J_5. We do the approximation to simplify the presentations by

$$J_5:=\int_{\mathbb{R}^3}-\nabla^k[h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)]\nabla^k u\mathrm{d}x\approx-\int_{\mathbb{R}^3}\nabla^k(h(\varrho)\nabla^2 u)\cdot\nabla^k u\mathrm{d}x. \tag{3.45}$$

Since $k \geq 1$, we can integrate by parts to have

$$\begin{aligned} J_5 &\approx \int_{\mathbb{R}^3} \nabla^{k-1}(h(\varrho)\nabla^2 u)\cdot\nabla^{k+1}u\mathrm{d}x \\ &\lesssim \sum_{0\leq l\leq k-1}\int_{\mathbb{R}^3}\nabla^l h(\varrho)\cdot\nabla^{k-l+1}u\cdot\nabla^{k+1}u\mathrm{d}x \\ &\lesssim \sum_{0\leq l\leq k-1}\|\nabla^l h(\varrho)\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}. \end{aligned} \tag{3.46}$$

In order to estimate the first term in (3.46), we shall discuss it in the following cases:
i) For $l=0$, since $|h(\varrho)|\leq C|\varrho|$, by Höder's and Sobolev's inequalities, we have

$$\begin{aligned} &\|h(\varrho)\cdot\nabla^{k+1}u\|_{L^2}\lesssim\|h(\varrho)\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2} \\ &\lesssim\|\varrho\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\lesssim\|\nabla\varrho\|_{L^2}\|\nabla^{k+1}u\|_{L^2} \\ &\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}u\|_{L^2}. \end{aligned} \tag{3.47}$$

ii) For $l=1$, since $|h^{(k)}(\varrho)|\leq C$, for any $k\geq 1$, we have

$$\begin{aligned} &\|\nabla h(\varrho)\cdot\nabla^k u\|_{L^2}\lesssim\|h'(\varrho)\cdot\nabla\varrho\cdot\nabla^k u\|_{L^2} \\ &\lesssim\|h'(\varrho)\cdot\nabla\varrho\|_{L^3}\|\nabla^k u\|_{L^6}\lesssim\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}u\|_{L^2} \\ &\lesssim\sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}u\|_{L^2}. \end{aligned} \tag{3.48}$$

iii) For $2\leq l\leq k-1$, by Lemma 2.2, we have

$$\begin{aligned} &\|\nabla^l h(\varrho)\|_{L^\infty}\lesssim\|\nabla^l\varrho\|_{L^2}^{1/4}\|\nabla^{l+2}\varrho\|_{L^2}^{3/4} \\ &\lesssim(\|\varrho\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l}{k+1}})^{1/4}(\|\varrho\|_{L^2}^{1-\frac{l+2}{k+1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l+2}{k+1}})^{3/4} \\ &\lesssim\|\varrho\|_{L^2}^{1-\frac{2l+3}{2(k+1)}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{2l+3}{2(k+1)}}. \end{aligned} \tag{3.49}$$

From the above inequality, and by Lemma 2.1 and Young's inequality, we have

$$\begin{aligned} &\|\nabla^l h(\varrho)\nabla^{k-l+1}u\|_{L^2}\lesssim\|\nabla^l h(\varrho)\|_{L^\infty}\|\nabla^{k-l+1}u\|_{L^2} \\ &\lesssim\|\varrho\|_{L^2}^{1-\frac{2l+3}{2(k+1)}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{2l+3}{2(k+1)}}\|\nabla^{k-l+1}u\|_{L^2} \\ &\lesssim\|\varrho\|_{L^2}^{1-\frac{2l+3}{2(k+1)}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{2l+3}{2(k+1)}}\|\nabla^\alpha u\|_{L^2}^{\frac{2l+3}{2(k+1)}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{2l+3}{2(k+1)}} \\ &\lesssim\sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}+\|\nabla^{k+1}u\|_{L^2}), \end{aligned} \tag{3.50}$$

where α is defined by

$$\begin{aligned} &\frac{k-l+1}{3}-\frac{1}{2}=\left(\frac{\alpha}{3}-\frac{1}{2}\right)\frac{2l+3}{2(k+1)}+\left(\frac{k+1}{3}-\frac{1}{2}\right)\left(1-\frac{2l+3}{2(k+1)}\right) \\ &\Longrightarrow\alpha=\frac{3(k+1)}{2l+3)}<3,\quad\text{since}\quad 2\leq l\leq k-1. \end{aligned} \tag{3.51}$$

In light of (3.47), (3.48) and (3.50), we deduce from (3.46) that

$$J_5\lesssim\sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2). \tag{3.52}$$

Finally, we can estimate the last term J_6 in the same way as J_5. Since $k \geq 1$, we can integrate by parts to have

$$\begin{aligned} J_6 &:= -\int_{\mathbb{R}^3} \nabla^k (f(\varrho)\nabla\varrho)\nabla^k u \mathrm{d}x = \int_{\mathbb{R}^3} \nabla^{k-1}(f(\varrho)\nabla\varrho)\nabla^{k+1}u \mathrm{d}x \\ &= \sum_{0\leq l\leq k-1} C_{k-1}^l \int_{\mathbb{R}^3} \nabla^l f(\varrho)\cdot\nabla^{k-l}\varrho\cdot\nabla^{k+1}u \mathrm{d}x \\ &\lesssim \sum_{0\leq l\leq k-1} \|\nabla^l f(\varrho)\cdot\nabla^{k-l}\varrho\|_{L^2}\|\nabla^{k+1}u\|_{L^2}. \end{aligned} \tag{3.53}$$

In order to estimate the first term above, we also need to discuss in the following cases:

i) For $l=0$, since $|f(\varrho)|\leq C|\varrho|$, by Höder's and Sobolev's inequalities, we have

$$\begin{aligned} &\|f(\varrho)\cdot\nabla^k\varrho\|_{L^2} \lesssim \|f(\varrho)\|_{L^3}\|\nabla^k\varrho\|_{L^6} \\ &\lesssim \|\varrho\|_{L^3}\|\nabla^k\varrho\|_{L^6} \lesssim \|\nabla\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}\varrho\|_{L^2}. \end{aligned} \tag{3.54}$$

ii) For $1\leq l\leq k-1$, by Lemma 2.2, we have

$$\begin{aligned} &\|\nabla^l f(\varrho)\cdot\nabla^{k-l}\varrho\|_{L^2} \lesssim \|\nabla^l f(\varrho)\|_{L^\infty}\|\nabla^{k-l}\varrho\|_{L^2} \\ &\lesssim \|\nabla^l\varrho\|_{L^2}^{1/4}\|\nabla^{l+2}\varrho\|_{L^2}^{3/4}\|\nabla^{k-l}\varrho\|_{L^2} \\ &\lesssim (\|\varrho\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l}{k+1}})^{1/4}(\|\varrho\|_{L^2}^{1-\frac{l+2}{k+1}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l+2}{k+1}})^{3/4}\|\nabla^{k-l}\varrho\|_{L^2} \\ &\lesssim \|\varrho\|_{L^2}^{1-\frac{2l+3}{2(k+1)}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{2l+3}{2(k+1)}}\|\nabla^\alpha\varrho\|_{L^2}^{\frac{2l+3}{2(k+1)}}\|\nabla^{k+1}\varrho\|_{L^2}^{1-\frac{2l+3}{2(k+1)}} \\ &\lesssim \sqrt{\mathcal{E}_0^3}\|\nabla^{k+1}\varrho\|_{L^2}, \end{aligned} \tag{3.55}$$

where α is defined by

$$\begin{aligned} &\frac{k-l}{3}-\frac{1}{2}=\left(\frac{\alpha}{3}-\frac{1}{2}\right)\frac{2l+3}{2(k+1)}+\left(\frac{k+1}{3}-\frac{1}{2}\right)\left(1-\frac{2l+3}{2(k+1)}\right) \\ &\Longrightarrow \alpha=\frac{k+1}{2l+3}<3, \quad \text{since} \quad 1\leq l\leq k-1. \end{aligned} \tag{3.56}$$

In light of (3.54) and (3.55), we deduce from (3.53) that

$$J_6 \lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2). \tag{3.57}$$

Plugging the estimates for $I_1 \sim I_6$ and $J_1 \sim J_6$, i.e., (3.20),(3.28),(3.32),(3.38),(3.44),(3.52),(3.57) into (3.12), we get (3.8).

For the next lemma, we will give the dissipative estimate for ϱ, via constructing the interactive energy function between u and $\nabla\varrho$.

Lemma 3.2. *If* $\sqrt{\mathcal{E}_0^3}\leq\delta$, *then for* $k=0,1,\ldots,N$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k\varrho dx + C(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+2}\varrho\|_{L^2}^2)\lesssim\|\nabla^{k+1}u\|_{L^2}^2. \tag{3.58}$$

Proof. Applying ∇^k to (3.2) and then multiplying the resulting equation by $\nabla\nabla^k\varrho$ under L^2 inner product, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla\nabla^k\varrho|^2 \mathrm{d}x - \kappa \int_{\mathbb{R}^3} \nabla^k\nabla\Delta\varrho\cdot\nabla\nabla^k\varrho \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \nabla^k\partial_t u\cdot\nabla\nabla^k\varrho\mathrm{d}x + \int_{\mathbb{R}^3} \mu\nabla^k\Delta u\cdot\nabla\nabla^k\varrho + (\mu+\lambda)\nabla^k\nabla\mathrm{div}u\cdot\nabla\nabla^k\varrho\mathrm{d}x \\
&\quad -\int_{\mathbb{R}^3} \nabla^k(u\cdot\nabla u + h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) + f(\varrho)\nabla\varrho)\cdot\nabla\nabla^k\varrho\mathrm{d}x \\
&\lesssim -\int_{\mathbb{R}^3} \nabla^k\partial_t u\cdot\nabla\nabla^k\varrho\mathrm{d}x - \int_{\mathbb{R}^3} \mu\nabla^{k+1}u\cdot\nabla^{k+2}\varrho + (\mu+\lambda)\nabla^{k+1}u\cdot\nabla^{k+2}\varrho\mathrm{d}x \\
&\quad +\int_{\mathbb{R}^3} \nabla^{k-1}(u\cdot\nabla u + h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) + f(\varrho)\nabla\varrho)\cdot\nabla^{k+2}\varrho\mathrm{d}x \\
&\lesssim -\int_{\mathbb{R}^3} \nabla^k\partial_t u\cdot\nabla\nabla^k\varrho\mathrm{d}x + C\|\nabla^{k+1}u\|_{L^2}\|\nabla^{k+2}\varrho\|_{L^2} + \|\nabla^{k-1}(u\cdot\nabla u \\
&\quad + h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) + f(\varrho)\nabla\varrho)\|_{L^2}\|\nabla^{k+2}\varrho\|_{L^2}.
\end{aligned} \tag{3.59}$$

Notice that the first term in the right-hand side of (3.59) involves the time derivative; thus, by the continuity Eq. (3.1) and integrating by parts for both the $t-$ and $x-$ variables, we can get that

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla^k\partial_t u\cdot\nabla\nabla^k\varrho\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k\varrho\mathrm{d}x + \int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k\partial_t\varrho\mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k\varrho\mathrm{d}x - \int_{\mathbb{R}^3} \nabla^k\mathrm{div}u\cdot\nabla^k\partial_t\varrho\mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k\varrho\mathrm{d}x + \int_{\mathbb{R}^3} \nabla^k\mathrm{div}u\cdot\nabla^k(\mathrm{div}u + \mathrm{div}(\varrho u))\mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k\varrho\mathrm{d}x + \|\nabla^k\mathrm{div}u\|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla^k\mathrm{div}u\cdot\nabla^k\mathrm{div}(\varrho u)\mathrm{d}x.
\end{aligned} \tag{3.60}$$

By applying Hölder's inequality, Leibniz formula and Minkowshi's inequality to the last term of the right side of (3.60), we have

$$\int_{\mathbb{R}^3} \nabla^k\mathrm{div}u\cdot\nabla^k\mathrm{div}(\varrho u)\mathrm{d}x \lesssim \|\nabla^{k+1}(\varrho u)\|_{L^2}\|\nabla^{k+1}u\|_{L^2} \lesssim \sum_{0\le l\le k+1} \|\nabla^l\varrho\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}. \tag{3.61}$$

If $0 \le l \le [\frac{k+1}{2}]$, by Lemma 2.1, we have

$$\begin{aligned}
&\|\nabla^l\varrho\nabla^{k-l+1}u\|_{L^2} \lesssim \|\nabla^l\varrho\|_{L^3}\|\nabla^{k-l+1}u\|_{L^6} \\
&\lesssim \|\nabla^\alpha\varrho\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{l-1}{k}}\|\nabla u\|_{L^2}^{\frac{l-1}{k}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l-1}{k}} \\
&\lesssim \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}),
\end{aligned} \tag{3.62}$$

where α is defined by

$$\begin{aligned}
&\frac{l}{3} - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right)\times\left(1 - \frac{l-1}{k}\right) + \left(\frac{k+1}{3} - \frac{1}{2}\right)\times\frac{l-1}{k} \\
&\Longrightarrow \alpha = 1 + \frac{k}{2(k-l+1)} \in [\frac{5}{4}, 2) \quad \text{since} \quad 0 \le l \le \frac{k+1}{2}.
\end{aligned} \tag{3.63}$$

While for $l > [\frac{k+1}{2}] + 1$ (then $k - l + 1 \leq [\frac{k+1}{2}]$), in the same way above, we have

$$\begin{aligned} \|\nabla^l \varrho \nabla^{k-l+1} u\|_{L^2} &\lesssim \|\nabla^l \varrho\|_{L^6} \|\nabla^{k-l+1} u\|_{L^3} \\ &\lesssim \|\nabla \varrho\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} \varrho\|_{L^2}^{\frac{l}{k}} \|\nabla^{\alpha} u\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l}{k}} \\ &\lesssim \sqrt{\mathcal{E}_0^3} (\|\nabla^{k+1} \varrho\|_{L^2} + \|\nabla^{k+1} u\|_{L^2}), \end{aligned} \tag{3.64}$$

where α is defined by

$$\begin{aligned} &\frac{k - l + 1}{3} - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right) \times \frac{l}{k} + \left(\frac{k+1}{3} - \frac{1}{2}\right) \times \left(1 - \frac{l}{k}\right) \\ &\Longrightarrow \alpha = 1 + \frac{k}{2l} \in [\frac{5}{4}, 2) \quad \text{since} \quad \frac{k+1}{2} < l \leq k + 1. \end{aligned} \tag{3.65}$$

Thus, in light of (3.62) and (3.64), we can deduce from (3.61) that

$$\int_{\mathbb{R}^3} \nabla^k \mathrm{div} u \cdot \nabla^k \mathrm{div}(\varrho u) dx \lesssim \sqrt{\mathcal{E}_0^3} (\|\nabla^{k+1} \varrho\|_{L^2} + \|\nabla^{k+1} u\|_{L^2}). \tag{3.66}$$

Plugging the above inequality into (3.60), we can obtain that

$$-\int_{\mathbb{R}^3} \nabla^k \partial_t u \cdot \nabla \nabla^k \mathrm{d}x \leq -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \nabla^k u \cdot \nabla \nabla^k \varrho \mathrm{d}x + C\|\nabla^{k+1} u\|_{L^2}^2 + C\sqrt{\mathcal{E}_0^3} \|\nabla^{k+1} \varrho\|_{L^2}^2. \tag{3.67}$$

□

Next, for the last term of the right-hand side of (3.59), notice that it has been already proved in the proof of Lemma 3.1 that

$$\begin{aligned} &\|\nabla^{k-1}(u \cdot \nabla u + h(\varrho)(\mu \Delta u + (\mu + \lambda) \nabla \mathrm{div} u) + f(\varrho) \nabla \varrho)\|_{L^2} \\ &\lesssim \sqrt{\mathcal{E}_0^3} (\|\nabla^{k+1} \varrho\|_{L^2} + \|\nabla^{k+1} u\|_{L^2}). \end{aligned} \tag{3.68}$$

As to the left-hand side of (3.59), by integrating by parts, the second term of it can be rewritten as

$$-\kappa \int_{\mathbb{R}^3} \nabla^k \nabla \Delta \varrho \cdot \nabla \nabla^k \varrho \mathrm{d}x = \kappa \int_{\mathbb{R}^3} \nabla^{k+2} \varrho \cdot \nabla^{k+2} \varrho \mathrm{d}x = \kappa \|\nabla^{k+2} \varrho\|_{L^2}^2. \tag{3.69}$$

Consequently, by (3.67)–(3.69), together with Cauchy's inequality, since $\sqrt{\mathcal{E}_0^3} \leq \delta$ is small, we can then deduce (3.58) from (3.59).

4. Negative Sobolev estimates

In this section, we will derive the evolution of the negative Sobolev norms of the solution to the problem (3.1)–(3.3). In order to estimate the nonlinear terms, we need to restrict ourselves to that $s \in (0, \frac{3}{2})$.

Lemma 4.1. *If $\sqrt{\mathcal{E}_0^3} \leq \delta$, then for $s \in (0, \frac{1}{2}]$, we have*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} |\Lambda^{-s} \varrho|^2 + |\Lambda^{-s} u|^2 + \kappa |\Lambda^{-s} \nabla \varrho|^2 dx + C\|\nabla \Lambda^{-s} u\|_{L^2}^2 \\ &\lesssim (\|\varrho\|_{H^3}^2 + \|\nabla u\|_{H^1}^2)(\|\Lambda^{-s} \varrho\|_{L^2} + \|\Lambda^{-s} u\|_{L^2} + \kappa \|\Lambda^{-s} \nabla \varrho\|_{L^2}), \end{aligned} \tag{4.1}$$

and for $s \in (\frac{1}{2}, \frac{3}{2})$, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} |\Lambda^{-s} \varrho|^2 + |\Lambda^{-s} u|^2 + \kappa |\Lambda^{-s} \nabla \varrho|^2 dx + C\|\nabla \Lambda^{-s} u\|_{L^2}^2 \\ &\lesssim \|(\varrho, u)\|_{L^2}^{s-\frac{1}{2}} (\|\varrho\|_{H^2} + \|\nabla u\|_{H^1})^{\frac{5}{2}-s} (\|\Lambda^{-s} \varrho\|_{L^2} + \|\Lambda^{-s} u\|_{L^2} + \kappa \|\Lambda^{-s} \nabla \varrho\|_{L^2}). \end{aligned} \tag{4.2}$$

Proof. Applying Λ^{-s} to (3.1) and (3.2), and multiplying those two resulting identities by $\Lambda^{-s}\varrho and \Lambda^{-s}u$, respectively, summing them up and then integrating this equation over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\Lambda^{-s}\varrho|^2+|\Lambda^{-s}u|^2\mathrm{d}x+\int_{\mathbb{R}^3}\mu|\nabla\Lambda^{-s}u|^2+(\mu+\lambda)|\mathrm{div}\Lambda^{-s}u|^2\mathrm{d}x\\
&=\kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\Delta\varrho\cdot\Lambda^{-s}u\mathrm{d}x+\int_{\mathbb{R}^3}\Lambda^{-s}(-\varrho\mathrm{div}u-u\cdot\nabla\varrho)\Lambda^{-s}\varrho\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)+f(\varrho)\nabla\varrho)\cdot\Lambda^{-s}u\mathrm{d}x\\
&:=W_1+W_2+W_3+W_4+W_5++W_6.
\end{aligned}\tag{4.3}$$

□

Firstly, we should bound the left-hand side of (4.3). For the second term, we have

$$\int_{\mathbb{R}^3}\mu|\nabla\Lambda^{-s}u|^2+(\mu+\lambda)|\mathrm{div}\Lambda^{-s}u|^2\mathrm{d}x\geq\sigma_0\|\nabla\Lambda^{-s}u\|_{L^2}^2.\tag{4.4}$$

In order to estimate the nonlinear terms in the right-hand side of (4.3), we shall use the estimate (2.11) of Riesz potential in Lemma 2.5. This forces us to require that $s\in(0,\frac{3}{2})$. If $s\in(0,\frac{1}{2}]$, then $1/2+s/3<1$ and $3/s\geq 6$. For the first term, by the continuity Eq. (3.1) and by integrating by parts, we get

$$\begin{aligned}
W_1&:=\kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\Delta\varrho\cdot\Lambda^{-s}u\mathrm{d}x=\kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\varrho\cdot\Lambda^{-s}\nabla(\mathrm{div}u)\mathrm{d}x\\
&=\kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\varrho\cdot\Lambda^{-s}\nabla(-\partial_t\varrho-\mathrm{div}(\varrho u))\mathrm{d}x\\
&=-\kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\varrho\cdot\Lambda^{-s}\nabla\partial_t\varrho\mathrm{d}x-\kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\varrho\cdot\Lambda^{-s}\nabla\mathrm{div}(\varrho u)\mathrm{d}x\\
&\leq-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\kappa|\Lambda^{-s}\nabla\varrho|^2\mathrm{d}x+\kappa\|\Lambda^{-s}\nabla\varrho\|_{L^2}\|\Lambda^{-s}\nabla\mathrm{div}(\varrho u)\|_{L^2}.
\end{aligned}\tag{4.5}$$

In order to estimate the term W_1, by employing the Leibniz formula and Minkowshi's inequality, we should first estimate that

$$\begin{aligned}
&\|\Lambda^{-s}\nabla\mathrm{div}(\varrho u)\|_{L^2}\lesssim\|\Lambda^{-s}(\nabla^2(\varrho u))\|_{L^2}\\
&\lesssim\|\Lambda^{-s}(\varrho\nabla^2u)\|_{L^2}+\|\Lambda^{-s}(\nabla\varrho\cdot\nabla u)\|_{L^2}+\|\Lambda^{-s}(\nabla^2\varrho\cdot u)\|_{L^2}\\
&:=S_1+S_2+S_3.
\end{aligned}\tag{4.6}$$

If $s\in(0,\frac{1}{2}]$, by the Lemmas 2.5 and 2.1, we have

$$\begin{aligned}
&S_1:=\|\Lambda^{-s}(\varrho\nabla^2u)\|_{L^2}\lesssim\|\varrho\nabla^2u\|_{L^{\frac{1}{1/2+s/3}}}\lesssim\|\varrho\|_{L^{\frac{3}{s}}}\|\nabla^2u\|_{L^2}\\
&\quad\lesssim\|\nabla\varrho\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2\varrho\|_{L^2}^{\frac{1}{2}-s}\|\nabla^2u\|_{L^2}\\
&\quad\lesssim\|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2.
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
&S_2:=\|\Lambda^{-s}(\nabla\varrho\cdot\nabla u)\|_{L^2}\lesssim\|\nabla\varrho\cdot\nabla u\|_{L^{\frac{1}{1/2+s/3}}}\lesssim\|\nabla\varrho\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\\
&\lesssim\|\nabla^2\varrho\|_{L^2}^{\frac{1}{2}+s}\|\nabla^3\varrho\|_{L^2}^{\frac{1}{2}-s}\|\nabla u\|_{L^2}\\
&\lesssim\|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2.
\end{aligned}\tag{4.8}$$

$$\begin{aligned} S_3 &:= \|\Lambda^{-s}(\nabla^2\varrho\cdot u)\|_{L^2} \lesssim \|\nabla^2\varrho\cdot u\|_{L^{\frac{1}{1/2+s/3}}} \lesssim \|\nabla^2\varrho\|_{L^2}\|u\|_{L^{\frac{3}{s}}} \\ &\lesssim \|\nabla^2\varrho\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}-s} \\ &\lesssim \|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2. \end{aligned} \tag{4.9}$$

From (4.6) to (4.9), we can get that for $s\in(0,\frac{1}{2}]$,

$$\|\Lambda^{-s}\nabla\mathrm{div}(\varrho u)\|_{L^2} \lesssim \|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2. \tag{4.10}$$

And this together with (4.5), we can obtain that for $s\in[0,\frac{1}{2}]$,

$$W_1 := \kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\Delta\varrho\cdot\Lambda^{-s}u\mathrm{d}x \lesssim -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\kappa|\Lambda^{-s}\nabla\varrho|^2\mathrm{d}x+\kappa(\|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2)\|\Lambda^{-s}\nabla\varrho\|_{L^2}. \tag{4.11}$$

For the second term, we can apply Lemmas 2.5 and 2.1, Hölder's as well as Young's inequalities to have

$$\begin{aligned} W_2 &:= -\int_{\mathbb{R}^3}\Lambda^{-s}(\varrho\mathrm{div}u)\Lambda^{-s}\varrho\mathrm{d}x \lesssim \|\Lambda^{-s}(\varrho\mathrm{div}u)\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|\varrho\mathrm{div}u\|_{L^{\frac{1}{1/2+s/3}}}\|\Lambda^{-s}\varrho\|_{L^2} \lesssim \|\varrho\|_{L^{3/s}}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim (\|\nabla\varrho\|_{H^1}^2+\|\nabla u\|_{L^2}^2)\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.12}$$

Similarly, we can bound the remaining terms by

$$\begin{aligned} W_3 &:= -\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla\varrho)\Lambda^{-s}\varrho\mathrm{d}x \lesssim \|\Lambda^{-s}(u\cdot\nabla\varrho)\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|\nabla u\|_{L^2}^{1/2+s}\|\nabla^2 u\|_{L^2}^{1/2-s}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim (\|\nabla u\|_{H^1}^2+\|\nabla\varrho\|_{L^2}^2)\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.13}$$

$$\begin{aligned} W_4 &:= -\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla u)\Lambda^{-s}u\mathrm{d}x \lesssim \|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\nabla u\|_{L^2}^{1/2+s}\|\nabla^2 u\|_{L^2}^{1/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\nabla u\|_{H^1}\|\nabla u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\nabla u\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2}. \end{aligned} \tag{4.14}$$

$$\begin{aligned} W_5 &:= -\int_{\mathbb{R}^3}\Lambda^{-s}(h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u))\Lambda^{-s}u\mathrm{d}x \\ &\lesssim \|\Lambda^{-s}(h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u))\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\Lambda^{-s}(h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u))\|_{L^{\frac{1}{1/2+s/3}}}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\varrho\|_{L^{3/s}}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim (\|\nabla\varrho\|_{H^1}^2+\|\nabla^2 u\|_{L^2}^2)\|\Lambda^{-s}u\|_{L^2}. \end{aligned} \tag{4.15}$$

$$\begin{aligned} W_6 &:= -\int_{\mathbb{R}^3} \Lambda^{-s}(f(\varrho)\nabla\varrho)\cdot\Lambda^{-s}u\mathrm{d}x \lesssim \|\Lambda^{-s}(f(\varrho)\nabla\varrho)\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\varrho\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|\nabla\varrho\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2}. \end{aligned} \tag{4.16}$$

Now, if $s\in(1/2,3/2)$, we shall estimate each term on the right-hand side of (4.3) in a different way. Since $s\in(1/2,3/2)$, we have that $1/2+s/3<1$ and $2<3/s<6$. For the first term W_1, we should estimate each term in (4.6). Then, by Lemmas 2.5 and 2.1, we obtain

$$\begin{aligned} S_1 &:= \|\Lambda^{-s}(\varrho\nabla^2 u)\|_{L^2} \lesssim \|\varrho\nabla^2 u\|_{L^{\frac{1}{1/2+s/3}}} \lesssim \|\varrho\|_{L^{\frac{3}{s}}}\|\nabla^2 u\|_{L^2} \\ &\lesssim \|\varrho\|_{L^2}^{s-\frac12}\|\nabla\varrho\|_{L^2}^{\frac32-s}\|\nabla^2 u\|_{L^2} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac12}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac52-s}. \end{aligned} \tag{4.17}$$

$$\begin{aligned} S_2 &:= \|\Lambda^{-s}(\nabla\varrho\cdot\nabla u)\|_{L^2} \lesssim \|\nabla\varrho\cdot\nabla u\|_{L^{\frac{1}{1/2+s/3}}} \lesssim \|\nabla\varrho\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2} \\ &\lesssim \|\varrho\|_{L^2}^{\frac{s}{2}-\frac14}\|\nabla^2\varrho\|_{L^2}^{\frac54-\frac{s}{2}}\|u\|_{L^2}^{\frac{s}{2}-\frac14}\|\nabla^\alpha u\|_{L^2}^{\frac54-\frac{s}{2}} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac12}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac52-s}, \end{aligned} \tag{4.18}$$

where α is defined by

$$\begin{aligned} &\frac13-\frac12 = -\frac12\left(\frac{s}{2}-\frac14\right)+\left(\frac{\alpha}{3}-\frac12\right)\left(\frac54-\frac{s}{2}\right) \\ &\Longrightarrow \alpha=\frac{4}{5-2s}\in(1,2), \quad \text{since} \quad s\in\left(\frac12,\frac32\right). \end{aligned} \tag{4.19}$$

$$\begin{aligned} S_3 &:= \|\Lambda^{-s}(\nabla^2\varrho\cdot u)\|_{L^2} \lesssim \|\nabla^2\varrho\cdot u\|_{L^{\frac{1}{1/2+s/3}}} \lesssim \|\nabla^2\varrho\|_{L^2}\|u\|_{L^{\frac{3}{s}}} \\ &\lesssim \|\nabla^2\varrho\|_{L^2}\|u\|_{L^2}^{s-\frac12}\|\nabla u\|_{L^2}^{\frac32-s} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac12}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac52-s}. \end{aligned} \tag{4.20}$$

From (4.6), (4.17), (4.18) and (4.20), we can obtain that for $s\in(\frac12,\frac32)$

$$\|\Lambda^{-s}\nabla\mathrm{div}(\varrho u)\|_{L^2} \lesssim \|(\varrho,u)\|_{L^2}^{s-\frac12}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac52-s}. \tag{4.21}$$

This together with (4.5), we obtain that for $s\in(\frac12,\frac32)$

$$W_1 := \kappa\int_{\mathbb{R}^3}\Lambda^{-s}\nabla\Delta\varrho\cdot\Lambda^{-s}u\mathrm{d}x \lesssim -\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\kappa|\Lambda^{-s}\nabla\varrho|^2\mathrm{d}x+\kappa\|(\varrho,u)\|_{L^2}^{s-\frac12}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac52-s}\|\Lambda^{-s}\nabla\varrho\|_{L^2}. \tag{4.22}$$

For the second term, by Lemmas 2.5 and 2.1 again, we obtain that

$$\begin{aligned} W_2 &:= -\int_{\mathbb{R}^3}\Lambda^{-s}(\varrho\mathrm{div}u)\Lambda^{-s}\varrho\mathrm{d}x \lesssim \|\Lambda^{-s}(\varrho\mathrm{div}u)\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|\varrho\mathrm{div}u\|_{L^{\frac{1}{1/2+s/3}}}\|\Lambda^{-s}\varrho\|_{L^2} \lesssim \|\varrho\|_{L^{3/s}}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac12}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac52-s}\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.23}$$

Similarly, we can bound the remaining terms by

$$\begin{aligned} W_3 := -\int_{\mathbb{R}^3} \Lambda^{-s}(u\cdot\nabla\varrho)\Lambda^{-s}\varrho \mathrm{d}x &\lesssim \|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac{1}{2}}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac{5}{2}-s}\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.24}$$

$$\begin{aligned} W_4 := -\int_{\mathbb{R}^3} \Lambda^{-s}(u\cdot\nabla u)\Lambda^{-s}u \mathrm{d}x &\lesssim \|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{5}{2}-s}\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.25}$$

$$\begin{aligned} W_5 &:= -\int_{\mathbb{R}^3} \Lambda^{-s}(h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla \mathrm{div} u))\Lambda^{-s}u \mathrm{d}x \\ &\lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac{1}{2}}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1})^{\frac{5}{2}-s}\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.26}$$

$$\begin{aligned} W_6 := -\int_{\mathbb{R}^3} \Lambda^{-s}(f(\varrho)\nabla\varrho)\cdot\Lambda^{-s}u \mathrm{d}x &\lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \\ &\lesssim \|(\varrho,u)\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{5}{2}-s}\|\Lambda^{-s}\varrho\|_{L^2}. \end{aligned} \tag{4.27}$$

Consequently, in light of the estimates of W_1, W_2, W_3, W_4, W_5, W_6 both in the case of $s\in(0,1/2]$ and $s\in(1/2,3/2)$, together with (4.4), we can deduce (4.1) and (4.2) from (4.3) seperately.

5. Proof of Theorem 1.1

In this section, with all the energy estimates that we have derived in the previous two sections, we are able to prove Theorem 1.1 with the assistance of the Sobolev interpolation.

In order to prove (1.7), we need to close the energy estimates at each $l-$th level in weak sense. For convenience, we should first rewrite (3.8) of Lemma 3.1 as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^k u\|_{L^2}^2)+C\|\nabla^{k+1}u\|_{L^2}^2 \le \sqrt{\mathcal{E}_0^3}(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2). \tag{5.1}$$

Then, let $N\ge 3$ and $0\le l\le m-1$ with $1\le m\le N$. Summing up the estimates (5.1) for from $k=l$ to m, since we assume the priori $\sqrt{\mathcal{E}_0^3}\le\delta$ is sufficiently small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{l\le k\le m+1}\|\nabla^k\varrho\|_{L^2}^2+\sum_{l\le k\le m}\|\nabla^k u\|_{L^2}^2)+C_1\sum_{l+1\le k\le m+1}\|\nabla^k u\|_{L^2}^2 \lesssim C_2\delta\sum_{l+1\le k\le m+1}\|\nabla^k\varrho\|_{L^2}^2. \tag{5.2}$$

Then summing up the estimates (3.58) of Lemma 3.2 for from $k=l$ to m, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\le k\le m}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k\varrho \mathrm{d}x + C_3\sum_{l+1\le m+2}\|\nabla^k\varrho\|_{L^2}^2 \lesssim C_4\sum_{l+1\le k\le m+1}\|\nabla^k u\|_{L^2}^2. \tag{5.3}$$

Multiplying (5.3) by $2C_2\delta/C_3$, adding the resulting inequality with (5.2), since $\delta>0$ is small enough, we can deduce that there exists a constant $C_5>0$ such that for $0\le l\le m$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{l\le k\le m+1}\|\nabla^k\varrho\|_{L^2}^2+\sum_{l\le k\le m}\|\nabla^k u\|_{L^2}^2+\frac{2C_2\delta}{C_3}\sum_{l\le k\le m}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k\varrho \mathrm{d}x\right\} \\ &+C_5\left\{\sum_{l+1\le k\le m+2}\|\nabla^k\varrho\|_{L^2}^2+\sum_{l+1\le k\le m+1}\|\nabla^k u\|_{L^2}^2\right\}\lesssim 0. \end{aligned} \tag{5.4}$$

We define

$$\mathcal{E}_l^m(t) := C_5^{-1}\left(\sum_{l\le k\le m+1}\|\nabla^k\varrho\|_{L^2}^2+\sum_{l\le k\le m}\|\nabla^k u\|_{L^2}^2+\frac{2C_2\delta}{C_3}\sum_{l\le k\le m}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k\varrho\mathrm{d}x\right). \quad (5.5)$$

Since δ is so small that $\mathcal{E}_l^m(t)$ can be equivalent to $\|\nabla^l\varrho\|_{H^{m+1-l}}^2+\|\nabla^l u\|_{H^{m-l}}^2$, then we may reformulate (5.4) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l^m(t)+\|\nabla^{l+1}\varrho\|_{H^{m+1-l}}^2+\|\nabla^{l+1}u\|_{H^{m-l}}^2\le 0, \quad (5.6)$$

where $0\le l\le m-1$.

Now, let $l=0$ and $m=3$ in (5.6), and then integrating the equation directly in time, we get

$$\|\varrho(t)\|_{H^4}^2+\|u(t)\|_{H^3}^2\lesssim\mathcal{E}_0^3(t)\le\mathcal{E}_0^3(0)\lesssim\|\varrho_0\|_{H^4}^2+\|u_0\|_{H^3}^2. \quad (5.7)$$

By a standard continuity argument, this closes the priori estimates (3.5) if at the initial time, we assume that $\|\varrho_0\|_{H^4}^2+\|u_0\|_{H^3}^2\le\delta_0$ is sufficiently small. This in turn allows us to take $l=0$ and $m=N$ in (5.6) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0^N(t)+\|\nabla\varrho\|_{H^{N+1}}^2+\|\nabla u\|_{H^N}^2\le 0. \quad (5.8)$$

Then, by integrating it directly in time, we can obtain (1.7).

Next, we turn to prove (1.8) and (1.9) of Theorem 1.1. However, we are not able to prove them for all $s\in[0,3/2)$ at one time, because, in Sect. 4, we have proved that the negative Sobolev estimates of the problem (3.1)–(3.3) have different bounded forms in the case $s\in(0,\frac{1}{2}]$ and $s\in(\frac{1}{2},\frac{3}{2})$. As a result, we have to present the proof separately.

Firstly, let us prove them for $s\in[0,1/2]$.

To begin, we define the notation

$$\mathcal{E}_{-s}(t):=\|\Lambda^{-s}\varrho(t)\|_{L^2}^2+\|\Lambda^{-s}u(t)\|_{L^2}^2+\kappa\|\Lambda^{-s}\nabla\varrho(t)\|_{L^2}^2. \quad (5.9)$$

With this notation, (4.1) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{-s}(t)+C\|\nabla\Lambda^{-s}u\|_{L^2}^2\lesssim(\|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2)\sqrt{\mathcal{E}_{-s}(t)}. \quad (5.10)$$

Then, integrating (5.10) in time, and by the bound (1.7), we obtain that for $s\in(0,1/2]$,

$$\begin{aligned}\mathcal{E}_{-s}(t)&\lesssim\mathcal{E}_{-s}(0)+C\int_0^t(\|\varrho\|_{H^3}^2+\|\nabla u\|_{H^1}^2)\sqrt{\mathcal{E}_{-s}(\tau)}\mathrm{d}\tau\\&\lesssim C_0\left(1+\sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)}\right).\end{aligned} \quad (5.11)$$

This implies (1.8) for $s\in[0,1/2]$, that is,

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2+\|\Lambda^{-s}u(t)\|_{L^2}^2+\kappa\|\Lambda^{-s}\nabla\varrho(t)\|_{L^2}^2\lesssim C_0. \quad (5.12)$$

If $l=0,1,\dots,N-1$, by Lemma 2.4, we have

$$\begin{aligned}&\|\nabla^l f\|_{L^2}\le\|\nabla^{l+1}f\|_{L^2}^{1-\theta}\|\Lambda^{-s}f\|_{L^2}^{\theta},\quad \textit{and}\quad \theta=1/(l+1+s)\\&\Longrightarrow\|\nabla^{l+1}f\|_{L^2}\ge C\|\Lambda^{-s}f\|_{L^2}^{-\frac{1}{l+s}}\|\nabla^l f\|_{L^2}^{1+\frac{1}{l+s}}.\end{aligned} \quad (5.13)$$

By this fact and (5.12), we may find

$$\|\nabla^{l+1}\varrho\|_{L^2}^2+\|\nabla^{l+1}\nabla\varrho\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2\ge C_0(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^l\nabla\varrho\|_{L^2}^2+\|\nabla^l u\|_{L^2}^2)^{1+\frac{1}{l+s}}. \quad (5.14)$$

This implies that for $l=0,1,\dots,N-1$, we have

$$\|\nabla^{l+1}\varrho\|_{H^{N-l+1}}^2+\|\nabla^{l+1}u\|_{H^{N-l}}^2\ge C_0(\|\nabla^l\varrho\|_{H^{N-l+1}}^2+\|\nabla^l u\|_{H^{N-l}}^2)^{1+\frac{1}{l+s}}. \quad (5.15)$$

This together with (5.6) in the case of $m = N$, we can obtain the following time differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l^N(t) + C_0(\mathcal{E}_l^N(t))^{1+\frac{1}{l+s}} \leq 0, \quad \text{for} \quad l = 0, 1, \ldots, N-1. \tag{5.16}$$

Solving this inequality directly, we can get

$$\mathcal{E}_l^N(t) \leq C_0(1+t)^{-(l+s)}, \quad \text{for} \quad l = 0, 1, \ldots, N-1. \tag{5.17}$$

This implies (1.9) that for $s \in [0, 1/2]$,

$$\|\nabla^l \varrho\|_{H^{N-l+1}}^2 + \|\nabla^l u\|_{H^{N-l}}^2 \leq C_0(1+t)^{-(l+s)}, \quad \text{for} \quad l = 0, 1, \ldots, N-1. \tag{5.18}$$

Now, we can present the proof of (1.8) and (1.9) in the case of $s \in (1/2, 3/2)$. Although the arguments for the case $s \in [0, 1/2]$ cannot be applied to this case directly, we can deduce them from what we have proved for (1.8) and (1.9) with $s = 1/2$, since we have ϱ_0, $u_0 \in \dot{H}^{-1/2}$ (since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$). Then, we have the following decay result:

$$\|\nabla^l \varrho(t)\|_{H^{N-l+1}}^2 + \|\nabla^l u(t)\|_{H^{N-l}}^2 \leq C_0(1+t)^{-(l+1/2)}, \quad \text{for} \quad l = 0, 1, \ldots, N-1. \tag{5.19}$$

Hence, by (5.19), we deduce from (4.2) that for $s \in (1/2, 3/2)$,

$$\begin{aligned}\mathcal{E}_{-s}(t) &\lesssim \mathcal{E}_{-s}(0) + C\int_0^t \|(\varrho, u)\|_{L^2}^{s-1/2}(\|\varrho\|_{H^2} + \|\nabla u\|_{H^1})^{5/2-s}\sqrt{\mathcal{E}_{-s}(\tau)}\mathrm{d}\tau \\ &\lesssim C_0 + C_0\int_0^t (1+\tau)^{-(7/4-s/2)}\mathrm{d}\tau \sup_{0\leq\tau\leq t}\sqrt{\mathcal{E}_{-s}(\tau)} \\ &\lesssim C_0(1 + \sup_{0\leq\tau\leq t}\sqrt{\mathcal{E}_{-s}(\tau)}).\end{aligned} \tag{5.20}$$

This implies (1.8) for $s \in (1/2, 3/2)$, that is,

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \kappa\|\Lambda^{-s}\nabla\varrho(t)\|_{L^2}^2 \lesssim C_0. \tag{5.21}$$

Now that we have proved (5.21), we may repeat the arguments leading to (1.9) for $s \in [0, 1/2]$ to prove that they hold also for $s \in (1/2, 3/2)$.

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