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Lazer–Leach type conditions on periodic solutions of semilinear resonant Duffing equations with singularities

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Abstract. In this paper, we study the existence of positive periodic solutions of resonant Duffing equations with singularities. Some Lazer–Leach type conditions are given to ensure the existence of positive periodic solutions of singular resonant Duffing equations.

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1. Introduction

In this paper, we are concerned with periodic solutions of Duffing equations

$$x'' + g(x) = p(t), \tag{1.1}$$

where $g : (0, +\infty) \to \mathbf{R}$ is locally Lipschitz continuous and has a singularity at the origin, $p : \mathbf{R} \to \mathbf{R}$ is continuous and 2π periodic.

The periodic problem of differential equations with singularities has been widely studied with an increasing interest lately because of their background in applied sciences [1–15]. For example, the oscillation problem of a spherical thick shell made of an elastic material can be modeled by singular Duffing equations [3]. The focusing system of an electron beam immersed in a periodic magnetic field can be also modeled by this kind equations [5].

The opening work on the existence of periodic solutions of the second order differential equations with singularities was done by Lazer and Solimini [16], in which the equations

$$x'' - \frac{1}{x^\gamma} = p(t)$$

were studied. It was proved in [16] that if $\gamma \geq 1$, then this equation has at least one positive 2π-periodic solution if and only if

$$\int_0^{2\pi} p(t)\mathrm{d}t < 0.$$

From then on, the existence of periodic solutions of equations with the strong singularities ($\gamma \geq 1$) was widely studied.

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Del Pino et al. [4] studied the existence of periodic solutions of equations

$$x'' + g(t, x) = 0, \tag{1.2}$$

where $g : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$ is continuous and 2π-periodic in t. Assume that there exist positive constants η_1, η_2, δ and $\gamma \geq 1$ such that

$$\frac{\eta_1}{x^\gamma} \leq -g(t, x) \leq \frac{\eta_2}{x^\gamma}, \tag{1.3}$$

where $t \in [0, 2\pi], 0 < x < \delta$. Moreover, there exists an integer $n \geq 0$ such that, for $t \in [0, 2\pi]$,

$$\frac{n^2}{4} < \liminf_{x\to+\infty} \frac{g(t, x)}{x} \leq \limsup_{x\to+\infty} \frac{g(t, x)}{x} < \frac{(n+1)^2}{4}. \tag{1.4}$$

It was proved in [4] that Eq. (1.2) has at least one positive periodic solution under conditions (1.3) and (1.4). This result implies that if $\beta \geq 0$ and $\beta \neq \frac{n^2}{4}, n \in \mathbf{N}$ and $\gamma \geq 1$ then equation

$$x'' + \beta x - \frac{1}{x^\gamma} = p(t) \tag{1.5}$$

has at least one positive 2π-periodic solution. Meanwhile, they raised one open problem: What conditions should be imposed on $p(t)$ to ensure the existence of 2π-periodic solutions of Eq. (1.5) in the case that $\beta = \frac{n^2}{4}$ for $n \in \mathbf{N}$, which is usually called the resonant case. Wang and Ma [17] studied this problem. They considered more general equations as follows:

$$x'' + \frac{1}{4}n^2 x + g(x) = p(t), \tag{1.6}$$

where $n \geq 1$ is an integer. Assume that g satisfies the singular condition

$$\frac{\eta_1}{x^\gamma} \leq -g(x) \leq \frac{\eta_2}{x^\gamma}, \quad x \in (0, \delta), \tag{1.7}$$

where η_1, η_2, δ and $\gamma \geq 1$ are positive constants; moreover, g satisfies the limit condition

$$\lim_{x\to+\infty} g(x) = g(+\infty). \tag{1.8}$$

When (1.7) and (1.8) hold, it was proved in [17] that Eq. (1.6) has at least one 2π-periodic solution provided that the following condition is satisfied,

$$4g(+\infty) - \int_0^{2\pi} p(t) \left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t \neq 0, \quad \forall \theta \in \mathbf{R}.$$

One aim of this paper is to further study the existence of periodic solutions of Eq. (1.6). Assume that g satisfies

(h_1) $\lim_{x\to0^+} g(x) = -\infty$,

and the primitive function G of g satisfies

(h_2) $\lim_{x\to0^+} G(x) = +\infty$, $(G(x) = \int_1^x g(s)\mathrm{d}s)$.

Meanwhile, the following condition holds,

(h_3) $\lim_{x\to0^+} \frac{G(x)}{g(x)} = 0$.

Moreover, we assume that the condition as follows is satisfied,

(h_4) $g(x)$ is bounded for $x \in [1, +\infty)$ and G satisfies

$$\lim_{x\to+\infty} \frac{G(x)}{x} = G(+\infty).$$

Conditions $(h_i)(i=1,2,3)$ generalize condition (1.7). It is easy to check that if g satisfies (1.7), then conditions $(h_i)(i=1,2,3)$ hold. On the other hand, we can easily find functions g which satisfy conditions $(h_i)(i=1,2,3)$, but these functions g do not satisfy (1.7). For example, let us define

$$g(x)=\frac{\ln x}{x}, \quad x\in(0,+\infty).$$

By a direct calculation, we get

$$G(x)=\frac{1}{2}(\ln x)^2, \quad x\in(0,+\infty).$$

Therefore, we have

$$\lim_{x\to 0^+} g(x)=-\infty, \quad \lim_{x\to 0^+} G(x)=+\infty$$

and

$$\lim_{x\to 0^+}\frac{G(x)}{g(x)}=0.$$

Hence, $g(x)=\ln x/x$ satisfies conditions (h_i) $(i=1,2,3)$. However, it is not hard to check that (1.7) is not satisfied.

By using phase-plane analysis method and topological degree argument, we obtain the following result.

Theorem 1.1. *Assume that conditions $(h_i)(i=1,2,3,4)$ hold. Then,* Eq. (1.6) *has at least one positive 2π-periodic solution provided that the following condition holds,*

$$4G(+\infty)-\int_0^{2\pi} p(t)\left|\sin\left(\theta+\frac{nt}{2}\right)\right| dt\neq 0, \quad \forall\theta\in\mathbf{R}. \tag{1.9}$$

Remark 1.1. Conditions (h_1) and (h_2) do not imply condition (h_3). When conditions (h_1) and (h_2) hold, we have

$$\limsup_{x\to 0^+}\frac{G(x)}{g(x)}=0.$$

In fact, let us define $f(x)=\ln G(x)$ for $x>0$ small enough. Since $\lim_{x\to 0^+} G(x)=+\infty$, we have $\lim_{x\to 0^+} f(x)=+\infty$. We claim $\liminf_{x\to 0^+} f'(x)=-\infty$. Otherwise, there exists $A>0$ such that

$$f'(x)\geq -A, \quad x\in(0,1),$$

which implies

$$f(x)\leq f(1)+A(1-x), \quad x\in(0,1).$$

This contradicts with the fact $\lim_{x\to 0^+} f(x)=+\infty$. Therefore, we get $\liminf_{x\to 0^+}\frac{g(x)}{G(x)}=-\infty$, which, together with conditions (h_1) and (h_2), implies $\limsup_{x\to 0^+}\frac{G(x)}{g(x)}=0$.

But, in general, we do not have

$$\liminf_{x\to 0^+}\frac{G(x)}{g(x)}=0.$$

For example, let us set

$$a_j=\frac{1}{j+1}-\frac{1}{(j+1)^3}, \quad b_j=\frac{1}{j+1}, \quad c_j=\frac{1}{j+1}+\frac{1}{(j+1)^3}, \quad (j=1,2,\ldots).$$

It is easy to check that $c_{j+1} < a_j$ and $b_j \ln b_j > -1, (j = 1, 2, \ldots)$. Define a function $g : (0, 1] \to \mathbf{R}$ as follows,

$$g(x) = \begin{cases} \dfrac{1 + a_j \ln b_j}{a_j(b_j - a_j)}(x - b_j) + \ln b_j, & x \in [a_j, b_j], j = 1, 2, \ldots, \\ \dfrac{1 + c_j \ln b_j}{c_j(b_j - c_j)}(x - b_j) + \ln b_j, & x \in [b_j, c_j], j = 1, 2, \ldots, \\ -\dfrac{1}{x}, & x \in (0, 1] \setminus \cup_{j=1}^{\infty} [a_j, c_j]. \end{cases}$$

Set

$$A_j = \left(a_j, -\frac{1}{a_j}\right), \quad B_j = (b_j, \ln b_j), \quad C_j = \left(c_j, -\frac{1}{c_j}\right).$$

Let us denote by S_j the area of the triangle $\Delta A_j B_j C_j$. Obviously, we have

$$S_j \leq \frac{2}{j^3} \left[\frac{(j+1)^3}{j^2 + 2j} - \ln(j+1) \right].$$

Since

$$\lim_{j \to \infty} \frac{1}{j} \left[\frac{(j+1)^3}{j^2 + 2j} - \ln(j+1) \right] = 1,$$

we know that the positive term series $\Sigma_{j=1}^{\infty} S_j$ is convergent. Set $S = \Sigma_{j=1}^{\infty} S_j$. Then, we have

$$-\ln b_j - S < G(b_j) < -\ln b_j, \quad (j = 1, 2, \ldots).$$

Consequently, we get

$$\lim_{j \to \infty} \frac{G(b_j)}{g(b_j)} = \lim_{j \to \infty} \frac{G(b_j)}{\ln b_j} = -1,$$

which implies

$$\liminf_{x \to 0^+} \frac{G(x)}{g(x)} \leq -1.$$

Therefore, conditions (h_1) and (h_2) cannot imply (h_3).

If the limit

(h_5) $\lim_{x \to +\infty} g(x) = g(+\infty)$

exists and is finite, then we can easily derive

$$\lim_{x \to +\infty} \frac{G(x)}{x} = g(+\infty).$$

Consequently, we obtain the following corollary.

Corollary 1.1. *Assume that conditions* $(h_i)(i = 1, 2, 3, 5)$ *hold. Then,* Eq. (1.6) *has at least one positive* 2π*-periodic solution provided that the following condition holds,*

$$4g(+\infty) - \int_0^{2\pi} p(t) \left| \sin\left(\theta + \frac{nt}{2}\right) \right| dt \neq 0, \quad \forall \theta \in \mathbf{R}. \tag{1.10}$$

Remark 1.2. Condition (1.9) or (1.10) can be compared with the well-known Lazer–Leach condition

$$2|g(+\infty) - g(-\infty)| - \int_0^{2\pi} p(t) \sin(\theta + nt) \mathrm{d}t \neq 0, \quad \theta \in \mathbf{R}, \tag{1.11}$$

where $g : \mathbf{R} \to \mathbf{R}$ is continuous, the limits $\lim_{x\to+\infty} g(x) = g(+\infty)$, $\lim_{x\to-\infty} g(x) = g(-\infty)$ exist and are finite. According to [18], if condition (1.11) is satisfied, then equation

$$x'' + n^2 x + g(x) = p(t)$$

has at least one 2π-periodic solution.

When $g(x)$ is bounded for sufficiently large $x > 0$ and condition (h_4) or (h_5) is not satisfied, we can also deal with the periodic solutions of Eq. (1.6). In this case, we introduce notations as follows,

$$\underline{g}(+\infty) = \liminf_{x\to+\infty} g(x), \quad \overline{g}(+\infty) = \limsup_{x\to+\infty} g(x).$$

We can prove the following theorem.

Theorem 1.2. *Assume that conditions $(h_i)(i = 1, 2, 3)$ hold and $g(x)$ is bounded for $x \in [1, +\infty)$. Then,* Eq. (1.6) *has at least one positive 2π-periodic solution provided that either*

$$4\underline{g}(+\infty) - \int_0^{2\pi} p(t) \left|\sin\left(\theta + \frac{nt}{2}\right)\right| dt > 0, \quad \forall\theta \in \mathbf{R}$$

or

$$4\overline{g}(+\infty) - \int_0^{2\pi} p(t) \left|\sin\left(\theta + \frac{nt}{2}\right)\right| dt < 0, \quad \forall\theta \in \mathbf{R}$$

holds.

Remark 1.3. In [17], the multiplicity of periodic solutions of Eq. (1.1) was proved under condition (1.7) and some other conditions. Analogous results can also be proved when (1.7) is replaced with conditions $(h_i)(i = 1, 2, 3)$ by using the similar methods as in the present paper and [17].

Remark 1.4. In the following, for convenience and brevity, we move the singular point 0 to the point -1. In fact, we can take a transformation $x = u + 1$ to achieve this aim. We shall consider singular equations as follows:

$$x'' + \frac{1}{4}n^2 x + g(x) = p(t), \tag{1.6$'$}$$

where $g : (-1, +\infty) \to \mathbf{R}$ is continuous and has a singularity at $x = -1$. We now assume that the following conditions hold,

(h_1') $\lim_{x\to-1^+} g(x) = -\infty$,
(h_2') $\lim_{x\to-1^+} G(x) = +\infty$, $\quad (G(x) = \int_0^x g(s)\mathrm{d}s)$,
(h_3') $\lim_{x\to-1^+} \frac{G(x)}{g(x)} = 0$.

Next, we shall deal with the existence of periodic solutions of Eq. (1.6$'$) under conditions (h_i') $(i = 1, 2, 3)$ and (h_4) or (h_5).

Throughout this paper, we always use $\mathbf{R}$ and $\mathbf{N}$ to denote the real number set and the natural number set, respectively. For a multivariate function ζ, the notation $\zeta = o(1)$ (or $o(1/c)$) always means that, for $c \to \infty, \zeta \to 0$ (or $c \cdot \zeta \to 0$) holds uniformly with respect to other variables, whereas $\zeta = O(1)$ always means that ζ is bounded for c large enough.

2. Preliminary lemmas

Consider the equivalent equation of Eq. (1.6′),

$$x' = y, \quad y' = -\frac{1}{4}n^2 x - g(x) + p(t). \tag{2.1}$$

we shall perform some phase-plane analyses for Eq. (1.6′) when conditions $(h_i')(i = 1, 2, 3)$ and (h_4) or (h_5) hold. Let $(x(t), y(t)) = (x(t, x_0, y_0), y(t, x_0, y_0))$ be the solution of (2.1) through the initial point

$$x(0, x_0, y_0) = x_0, \quad y(0, x_0, y_0) = y_0.$$

Lemma 2.1. *Assume that conditions $(h_i')(i = 1, 2)$ hold and $g(x)$ is bounded for $x \in [0, +\infty)$. Then, every solution $(x(t), y(t))$ of system (2.1) exists uniquely on the whole t-axis.*

Proof. Define a potential function

$$V(x, y) = \frac{1}{2}y^2 + \frac{1}{8}n^2 x^2 + G(x).$$

Set

$$V(t) = \frac{1}{2}y^2(t) + \frac{1}{8}n^2 x^2(t) + G(x(t)).$$

Then, we have

$$V'(t) = y(t)y'(t) + \tfrac{1}{4}n^2 x(t)x'(t) + g(x(t))x'(t) = y(t)p(t) \le \tfrac{1}{2}y^2(t) + \tfrac{1}{2}p^2(t).$$

From (h_2') and the boundedness of g on the interval $(0, +\infty)$ we know that there exists a constant $M > 0$ such that

$$\frac{1}{8}n^2 x^2 + G(x) + M > 0, \quad x \in (-1, +\infty).$$

Hence,

$$V'(t) \le \frac{1}{2}y^2(t) + \frac{1}{8}n^2 x^2(t) + G(x(t)) + \frac{1}{2}p^2(t) + M \le V(t) + M',$$

where $M' = M + \frac{1}{2}M_0^2, M_0 = \max\{|p(t)| : t \in [0, 2\pi]\}$. Then, for any finite $T > 0$, we have

$$V(t) \le V(0)e^T + M'(e^T - 1), \quad t \in [0, T).$$

Therefore, there is no blow-up for $(x(t), y(t))$ in any finite interval $[0, T)$. Furthermore, $(x(t), y(t))$ exists on the interval $[0, +\infty)$. Similarly, we can prove that $(x(t), y(t))$ exists on the interval $(-\infty, 0]$. The uniqueness of the solution $(x(t), y(t))$ follows directly from the local Lipschitzian condition on g. □

To depict the position of orbit $(x(t), y(t))$ of Eq. (2.1), we introduce a function $\zeta : (-1, +\infty) \times \mathbf{R} \to \mathbf{R}$,

$$\zeta(x, y) = x^2 + y^2 + \frac{1}{(1 + x)^2}.$$

Lemma 2.2. ([3]) *Assume that conditions $(h_i')(i = 1, 2)$ hold and $g(x)$ is bounded for $x \in [0, +\infty)$. Then, for any $\varrho > 0$, there exists $\varrho_0 > 0$ sufficiently large such that, for $\zeta(x_0, y_0) \ge \varrho_0^2$,*

$$\zeta(x(t), y(t)) \ge \varrho^2, \quad t \in [0, 4\pi],$$

where $(x(t), y(t))$ is the solution of (2.1) through the initial point (x_0, y_0).

From Lemma 2.2, we know that if $\zeta(x_0, y_0)$ is large enough, then $x^2(t) + y^2(t) > 0, t \in [0, 4\pi]$. Let us take a transformation as follows,

$$x = r\cos\theta, \quad y = \frac{n}{2}r\sin\theta.$$

Under this transformation, Eq. (2.1) becomes

$$\begin{cases} \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{n}{2} - \frac{2}{nr} g(r\cos\theta)\cos\theta + \frac{2}{nr} p(t)\cos\theta, \\ \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{2}{n} g(r\cos\theta)\sin\theta + \frac{2}{n} p(t)\sin\theta. \end{cases} \tag{2.2}$$

For simplicity, we denote by $(r(t), \theta(t)) = (r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ the solution of Eq. (2.2) satisfying the initial condition $(r(0), \theta(0)) = (r_0, \theta_0)$. We define the Poincaré map P as follows,

$$P : (r_0, \theta_0) \to (r_1, \theta_1) = (r(2\pi, r_0, \theta_0), \theta(2\pi, r_0, \theta_0)),$$

with $x_0 = r_0 \cos\theta_0 > -1, y_0 = r_0 \sin\theta_0$.

Lemma 2.3. *Assume that conditions $(h_i')(i = 1, 2)$ hold and $g(x)$ is bounded for $x \in [0, +\infty)$. Then, there exist $R_0 > 0$ and $\omega > 0$ such that, for $\zeta(x_0, y_0) \geq R_0^2$,*

$$\theta'(t) \leq -\omega < 0, \quad t \in [0, 4\pi].$$

Proof. From (h_1'), we know that there exist $\nu > 0$ and $-1 < \sigma < 0$ such that

$$\frac{g(x) - p(t)}{x} \geq \nu, \quad x \in (-1, \sigma), t \in \mathbf{R}.$$

Therefore, if $-1 < x(t) < \sigma, t \in [0, 4\pi]$, then

$$\theta'(t) \leq -\frac{n}{2} - \frac{2}{n}\nu\cos^2\theta \leq -\frac{n}{2}. \tag{2.3}$$

On the other hand, since $g(x)$ is bounded on the interval $[\sigma, +\infty)$, we know from Lemma 2.2 that there exists $R_0 > 0$ large enough such that, if $\zeta(x_0, y_0) \geq R_0^2$ and $x(t) \in [\sigma, +\infty), t \in [0, 4\pi]$, then

$$\frac{|g(x(t))| + |p(t)|}{r(t)} \leq \frac{n}{4}.$$

Consequently,

$$\theta'(t) \leq -\frac{n}{2} + \frac{|g(x(t))| + |p(t)|}{r(t)} |\cos\theta(t)| \leq -\frac{n}{4}. \tag{2.4}$$

From (2.3) and (2.4), we get the conclusion of Lemma 2.3. □

Lemma 2.4. *Assume that conditions $(h_i')(i = 1, 2)$ hold and $g(x)$ is bounded for $x \in [0, +\infty)$. Then, for $c \to +\infty$, the estimate*

$$r(t) = c + O(1)$$

holds uniformly with respect to $t \in [0, 4\pi]$ satisfying $\cos\theta(t) \geq 0$ and $(r_0, \theta_0) \in \mathbf{R}^+ \times \mathbf{S}^1$ satisfying $(r_0\cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c : \frac{1}{2}y^2 + F(x) = F(c), F(x) = \frac{1}{8}n^2x^2 + G(x)$.

Proof. Let $(x(t), y(t))$ be the solution of Eq. (2.1) through the initial point (x_0, y_0) with $x_0 = r_0\cos\theta_0, y_0 = \frac{n}{2} r_0 \sin\theta_0$. From the proof of Lemma 2.1, we know that there exists a positive constant M such that $F(x) + M > 0, x \in (-1, +\infty)$. Set

$$u(t) = \sqrt{y^2(t) + 2F(x(t)) + M}.$$

Then, we have

$$u'(t) = \frac{p(t)y(t)}{\sqrt{y^2(t) + 2F(x(t)) + M}}.$$

Therefore, for any $t, s \in [0, 4\pi]$, we get

$$|u(t) - u(s)| \leq D = \int_0^{4\pi} |p(t)| \mathrm{d}t.$$

Furthermore,

$$u(0) - D \le u(t) \le u(0) + D, \quad t \in [0, 4\pi],$$

which implies

$$\sqrt{2F(c) + M} - D \le u(t) \le \sqrt{2F(c) + M} + D, \quad t \in [0, 4\pi].$$

Since $F(x)$ is strictly increasing for $x > 0$ large enough, we know that, for c large enough, there exist two constants $a = a(c)$, $b = b(c)$ with $0 < a < c < b$ such that

$$\sqrt{2F(c) + M} - D = \sqrt{2F(a) + M}, \quad \sqrt{2F(c) + M} + D = \sqrt{2F(b) + M}.$$

Then, we get

$$\sqrt{2F(a) + M} \le u(t) \le \sqrt{2F(b) + M}, \quad t \in [0, 4\pi], \tag{2.5}$$

and

$$\sqrt{2F(b) + M} - \sqrt{2F(a) + M} = 2D. \tag{2.6}$$

According to (2.6), we know that there exists $\xi \in (a, b)$ such that

$$b - a = \frac{2D\sqrt{2F(\xi) + M}}{F'(\xi)} = \frac{4D\sqrt{n^2\xi^2 + 8G(\xi) + 4M}}{n^2\xi + 4g(\xi)}.$$

Since g is bounded on the interval $[0, +\infty)$, we know that there exists a positive constant ζ such that for $c > 0$ large enough,

$$|b - a| \le \zeta. \tag{2.7}$$

From (2.5), we get

$$2F(a) \le y^2(t) + 2F(x(t)) \le 2F(b), \quad t \in [0, 4\pi], \tag{2.8}$$

which yields $F(x(t)) \le F(b), t \in [0, 4\pi]$. Since $F(x)$ is increasing for $x > 0$ large enough, we know that, for $\cos\theta(t) \ge 0$ and c large enough,

$$0 \le x(t) = r(t)\cos\theta(t) \le b, \quad t \in [0, 4\pi].$$

It follows from (2.8) that

$$n^2a^2 + 8G(a) \le 4y^2(t) + n^2x^2(t) + 8G(x(t)) \le n^2b^2 + 8G(b), \quad t \in [0, 4\pi].$$

Therefore, we get

$$n^2a^2 + 8G(a) \le n^2r^2(t) + 8G(r(t)\cos\theta(t)) \le n^2b^2 + 8G(b), \quad t \in [0, 4\pi],$$

which implies

$$a^2 + \frac{8}{n^2}\left[G(a) - G(r(t)\cos\theta(t))\right] \le r^2(t) \le b^2 + \frac{8}{n^2}\left[G(b) - G(r(t)\cos\theta(t))\right], \quad t \in [0, 4\pi]. \tag{2.9}$$

As g is bounded on the interval $[0, +\infty)$, there exist two positive constants c_1, c_2 such that $|G(x)| \le c_1x + c_2$ for $x \in [0, +\infty)$. Hence, we get that, for $t \in [0, 4\pi]$ and $\cos\theta(t) \ge 0$,

$$|G(r(t)\cos\theta(t))| \le c_1b + c_2. \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$a^2 - \frac{8}{n^2}\left[(c_1(a + b) + 2c_2)\right] \le r^2(t) \le b^2 + \frac{16}{n^2}(c_1b + c_2), \quad t \in [0, 4\pi].$$

According to (2.7), we know that there exists a positive constant α such that, for $\cos\theta(t) \ge 0$ and c large enough,

$$c^2 - \alpha c \le r^2(t) \le c^2 + \alpha c, \quad t \in [0, 4\pi].$$

Consequently, we have that, for $c \to +\infty$,

$$r(t) = c + O(1)$$

holds uniformly for $t \in [0, 4\pi]$ satisfying $\cos\theta(t) \geq 0$ and $(r_0, \theta_0) \in \mathbf{R}^+ \times \mathbf{S}^1$ satisfying $(r_0\cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$. □

Lemma 2.5. *Assume that conditions* (h_i') $(i = 1, 2, 3)$ *hold and* $g(x)$ *is bounded for* $x \in [0, +\infty)$. *Let* $(x(t), y(t))$ *be a solution of system* (2.1) *with* $(x_0, y_0) \in \Gamma_c$ *(which is defined in* Lemma 2.4*) satisfying*

$$x(t_1) = 0, \quad x(t_2) = 0,$$

and

$$x(t) < 0, \quad t \in (t_1, t_2).$$

Then, for $c \to +\infty$, *the estimate*

$$t_2 - t_1 = \frac{4}{nc} + o\left(\frac{1}{c}\right)$$

holds uniformly with respect to $(x_0, y_0) \in \Gamma_c$.

Proof. At first, we know from [14] that, for $c \to +\infty$,

$$t_2 - t_1 = o(1)$$

holds uniformly for $(x_0, y_0) \in \Gamma_c$. Next, we shall give a more delicate estimate to $t_2 - t_1$. To this end, we put ourselves into the same situation as in the proof of Lemma 2.4. Set

$$f(x) = F'(x) = \frac{1}{4} n^2 x + g(x), \quad x \in (-1, +\infty).$$

Without loss of generality, we assume that

$$f(x) < 0, \quad x \in (-1, 0).$$

From conditions (h_1') and (h_4), we know that $F(x)$ is increasing for $x > 0$ large enough and $F(x)$ is decreasing for $|x + 1|$ small enough. On the other hand, it follows from conditions (h_2') and (h_4) that

$$\lim_{x \to +\infty} F(x) = +\infty, \quad \lim_{x \to -1^+} F(x) = +\infty.$$

Therefore, for any $c > 0$ sufficiently large, there exists a unique $-1 < d(c) < 0$ such that

$$F(d(c)) = F(c).$$

Consequently, there are two constants $-1 < d(b) < d(a) < 0$ satisfying

$$F(d(a)) = F(a), \quad F(d(b)) = F(b),$$

where a and b are given in the proof of Lemma 2.4. From (2.8), we have

$$2F(d(a)) \leq y^2(t) + 2F(x(t)) \leq 2F(d(b)), \quad t \in [0, 4\pi].$$

Hence,

$$2(F(a) - F(x(t))) \leq y^2(t) \leq 2(F(d(b)) - F(x(t))). \tag{2.11}$$

Let $t_a, t_* \in [t_1, t_2]$ satisfying $t_a < t_*, x(t_a) = d(a), y(t_a) \leq 0$ and $-1 < x(t_*) < 0, y(t_*) = 0$. In what follows, we shall estimate $t_a - t_1$ and $t_* - t_a$, respectively.

From (2.11), we know that, for $t \in [t_1, t_a]$,

$$\sqrt{2(F(a) - F(x(t)))} \leq -x'(t) \leq \sqrt{2(F(d(b)) - F(x(t)))},$$

which implies

$$\int_{d(a)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(d(b))-F(x))}} \le t_a - t_1 \le \int_{d(a)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(d(a))-F(x))}}. \tag{2.12}$$

Set

$$I_1 = \int_{d(a)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(d(a))-F(x))}} = \int_{d(a)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(a)-F(x))}},$$

and

$$I_2 = \int_{d(b)}^{d(a)} \frac{\mathrm{d}x}{\sqrt{2(F(d(b))-F(x))}} = \int_{d(b)}^{d(a)} \frac{\mathrm{d}x}{\sqrt{2(F(b)-F(x))}}.$$

Next, we shall estimate I_1 and I_2. Since $g(x)$ is locally Lipschitz continuous, we know that $g(x)$ is differentiable almost everywhere in the interval $(-1,0)$ and $g'(x)$ is Lebesgue integrable on any closed sub-interval of $(-1,0)$. From the expression of I_1, we have

$$F(a)I_1 = \frac{\sqrt{2}}{2} \int_{d(a)}^{0} \frac{F(a)\mathrm{d}x}{\sqrt{F(a)-F(x)}}.$$

It follows from [1] that

$$F(a)I_1 = \sqrt{2} \int_{d(a)}^{0} P(x)\sqrt{F(a)-F(x)}\mathrm{d}x,$$

where

$$P(x) = \frac{1}{2} + W'(x), \quad W(x) = \frac{F(x)}{F'(x)} = \frac{\frac{1}{8}n^2x^2 + G(x)}{\frac{1}{4}n^2x + g(x)}, \quad x \in (-1,0).$$

Hence,

$$\sqrt{F(a)}I_1 = \sqrt{2} \int_{d(a)}^{0} P(x)\sqrt{1-\frac{F(x)}{F(a)}}\mathrm{d}x.$$

Obviously, we have $W(0)=0$. According to conditions (h_3'), we get

$$\lim_{x\to -1^+} W(x) = 0.$$

By using Lebesgue dominated convergent theorem and the fact $\lim_{a\to+\infty} d(a) = -1$, we get

$$\lim_{a\to+\infty} \sqrt{F(a)}I_1 = \lim_{a\to+\infty} \sqrt{2} \int_{d(a)}^{0} P(x)\sqrt{1-\frac{F(x)}{F(a)}}\mathrm{d}x = \sqrt{2}\int_{-1}^{0} P(x)\mathrm{d}x = \frac{\sqrt{2}}{2}.$$

Therefore,

$$I_1 = \sqrt{\frac{1}{2F(a)}} + o\left(\frac{1}{\sqrt{F(a)}}\right), \quad a \to +\infty.$$

Since $F(a) = \frac{1}{8}n^2a^2 + G(a)$, we obtain

$$I_1 = \frac{2}{na} + o\left(\frac{1}{a}\right), \quad a \to +\infty.$$

Recalling $a < c < b$ and $|b - a| \le l$, we know

$$I_1 = \frac{2}{nc} + o\left(\frac{1}{c}\right), \quad c \to +\infty. \tag{2.13}$$

We now estimate I_2. From the expression of I_2, we have

$$F(b)I_2 = \frac{\sqrt{2}}{2}\int_{d(b)}^{d(a)} \frac{F(b)\mathrm{d}x}{\sqrt{F(b) - F(x)}}.$$

From [1], we get

$$F(b)I_2 = \sqrt{2}\int_{d(b)}^{d(a)} P(x)\sqrt{F(b) - F(x)}\mathrm{d}x - \sqrt{2(F(b) - F(a))}\frac{F(d(a))}{2f(d(a))}.$$

Thus, we have

$$\sqrt{F(b)}I_2 = \sqrt{2}\int_{d(b)}^{d(a)} P(x)\sqrt{1 - \frac{F(x)}{F(b)}}\mathrm{d}x - \sqrt{2\left(1 - \frac{F(a)}{F(b)}\right)}\frac{F(d(a))}{2f(d(a))}.$$

Since $a = a(c) < c < b = b(c), |b - a| \le l$ and $\lim_{c\to+\infty} a(c) = +\infty, \lim_{c\to+\infty} b(c) = +\infty$ and $\lim_{a\to+\infty} d(a) = -1, \lim_{b\to+\infty} d(b) = -1$, we infer from condition (h_3') that

$$\lim_{c\to+\infty} \frac{F(d(a))}{f(d(a))} = \lim_{c\to+\infty} \frac{n^2d^2(a) + 8G(d(a))}{2n^2d(a) + 8g(d(a))} = 0. \tag{2.14}$$

From the boundedness of g on the interval $[0, +\infty)$, we obtain

$$\lim_{c\to+\infty} \frac{F(a)}{F(b)} = \lim_{c\to+\infty} \frac{n^2a^2 + 8G(a)}{n^2b^2 + 8G(b)} = 1. \tag{2.15}$$

Hence, we know from (2.14) and (2.15) that

$$\lim_{c\to+\infty} \sqrt{F(b)}I_2 = \lim_{c\to+\infty} \sqrt{2}\int_{d(b)}^{d(a)} P(x)\sqrt{1 - \frac{F(x)}{F(b)}}\mathrm{d}x - \lim_{c\to+\infty} \sqrt{2\left(1 - \frac{F(a)}{F(b)}\right)}\frac{F(d(a))}{2f(d(a))} = 0.$$

Furthermore, we have that, for $c \to +\infty$,

$$I_2 = o\left(\frac{1}{\sqrt{F(b)}}\right) = o\left(\frac{1}{c}\right). \tag{2.16}$$

On the other hand, we know from (2.13) that, for $c \to +\infty$,

$$\int_{d(b)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(d(b)) - F(x))}} = \int_{d(b)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(b) - F(x))}} = \frac{2}{nc} + o\left(\frac{1}{c}\right), \quad c \to +\infty,$$

which, together with (2.16), implies that, for $c \to +\infty$,

$$\int_{d(a)}^{0} \frac{\mathrm{d}x}{\sqrt{2(F(d(b)) - F(x))}} = \frac{2}{nc} + o\left(\frac{1}{c}\right), \quad c \to +\infty. \tag{2.17}$$

Combining (2.12), (2.13) with (2.17), we get that, for $c \to +\infty$, the estimate

$$t_a - t_1 = \frac{2}{nc} + o\left(\frac{1}{c}\right) \tag{2.18}$$

holds uniformly with respect to $(x_0, y_0) \in \Gamma_c$.

In what follows, we estimate $t_* - t_a$. Since $x'(t_*) = y(t_*) = 0$, we have that, for $t \in (t_a, t_*)$,

$$\int_{t}^{t_*} x''(s)x'(s)\mathrm{d}s = -\int_{t}^{t_*} f(x(s))x'(s)\mathrm{d}s + \int_{t}^{t_*} p(s)x'(s)\mathrm{d}s,$$

which implies

$$\begin{aligned} {x'}^2(t) &= 2(F(x(t_*)) - F(x(t))) - 2\int_{t}^{t_*} p(s)x'(s)\mathrm{d}s \\ &\geq 2(F(x(t_*)) - F(x(t))) + 2||p||_\infty (x(t_*) - x(t)), \end{aligned}$$

where $||p||_\infty = max\{|p(t)| : t \in [0, 2\pi]\}$. Therefore, for $t \in (t_a, t_*)$,

$$-x'(t) \geq \sqrt{2(F(x(t_*)) - F(x(t))) + 2||p||_\infty (x(t_*) - x(t))}.$$

Furthermore,

$$-\frac{x'(t)}{\sqrt{2(F(x(t_*)) - F(x(t))) + 2||p||_\infty (x(t_*) - x(t))}} \geq 1.$$

As a result, we get

$$\int_{x_*}^{d(a)} \frac{\mathrm{d}x}{\sqrt{2(F(x_*) - F(x)) + 2||p||_\infty (x_* - x)}} \geq t_* - t_a, \tag{2.19}$$

where $x_* = x(t_*)$. Since $\lim_{c\to+\infty} d(a) = -1$, $\lim_{c\to+\infty} d(b) = -1$, and $d(b) \leq x_* \leq d(a)$, we have $\lim_{c\to+\infty}(d(a) - x_*) = 0$. Meanwhile, there exists $a \leq x^* \leq b$ such that $x_* = d(x^*)$ and $F(x^*) = F(x_*)$. Using a similar method as estimating I_2, we can prove

$$\int_{x_*}^{d(a)} \frac{\mathrm{d}x}{\sqrt{2(F(x_*) - F(x)) + 2||p||_\infty (x_* - x)}} = o\left(\frac{1}{c}\right), \quad c \to +\infty,$$

which, together with (2.19), yields

$$t_* - t_a = o\left(\frac{1}{c}\right), \quad c \to +\infty. \tag{2.20}$$

From (2.18) and (2.20), we know that, for $c \to +\infty$,

$$t_* - t_1 = \frac{2}{nc} + o\left(\frac{1}{c}\right)$$

holds uniformly with respect to $(x_0, y_0) \in \Gamma_c$. Similarly, we have

$$t_2 - t_* = \frac{2}{nc} + o\left(\frac{1}{c}\right), \quad c \to +\infty.$$

Consequently, we get

$$t_2 - t_1 = \frac{4}{nc} + o\left(\frac{1}{c}\right), \quad c \to +\infty$$

holds uniformly with respect to $(x_0, y_0) \in \Gamma_c$. The proof is complete. □

3. Proof of Theorem 1.1

In this section, we shall use basic lemmas in Sect. 2 to prove Theorem 1.1.

Proposition 3.1. *Assume that conditions* $(h_i')(i = 1, 2, 3)$ *and* (h_4) *hold. Then,* Eq. (1.6′) *has at least one positive* 2π*-periodic solution provided that the following condition holds,*

$$4G(+\infty) - \int_0^{2\pi} p(t) \left|\sin\left(\theta + \frac{nt}{2}\right)\right| dt \neq n^2, \quad \forall \theta \in \mathbf{R}.$$

Proof. From Lemma 2.3, we know that $\theta'(t) < 0, t \in [0, 4\pi]$ for $\zeta(x_0, y_0)$ large enough. Then, the inverse of $\theta = \theta(t), t \in [0, 4\pi]$ exists for $\zeta(x_0, y_0)$ large enough. Let $t = t(\theta)$ be the inverse of $\theta = \theta(t)$. We denote by $\tau_n(r_0, \theta_0)$ the required time for the solution $(r(t), \theta(t))$ of (2.2) to complete n turns around the origin. In what follows, we shall estimate $\tau_n(r_0, \theta_0)$.

It follows from Lemma 2.4 that, for $c \to +\infty$, the estimate

$$\frac{1}{r(t)} = \frac{1}{c} + o\left(\frac{1}{c}\right) \tag{3.1}$$

holds uniformly for $t \in [0, 4\pi]$ satisfying $\cos\theta(t) \geq 0$ and (r_0, θ_0) with $(r_0 \cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$.

From the first equality of (2.2), we know that, for $\zeta(x_0, y_0)$ large enough and $t \in [0, 4\pi]$,

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{2}{n} \frac{1}{1 + \frac{4}{n^2 r} g(r\cos\theta)\cos\theta - \frac{4}{n^2 r} p(t)\cos\theta}.$$

According to condition (h_4), we know that g is bounded on the interval $[0, +\infty)$. Hence, we have that, for $c \to +\infty$, the estimate

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{2}{n} + \frac{8}{n^3 c} g(r\cos\theta)\cos\theta - \frac{8}{n^3 c} p(t)\cos\theta + o\left(\frac{1}{c}\right).$$

holds uniformly for $t \in [0, 4\pi]$ satisfying $\cos\theta(t) \geq 0$ and (r_0, θ_0) with $(r_0 \cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$.

Without loss of generality, we assume that $\theta_0 \in [-\frac{3\pi}{2}, \frac{\pi}{2}]$. We first deal with the case $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Set

$$J_0 = \int_{-\frac{\pi}{2}}^{\theta_0} \left[\frac{2}{n} - \frac{8}{n^3 c} g(r\cos\theta)\cos\theta + \frac{8}{n^3 c} p(t)\cos\theta\right] \mathrm{d}\theta,$$

and for $i = 1, \ldots, n-1$,

$$J_i = \int_{-2i\pi - \frac{\pi}{2}}^{-2i\pi + \frac{\pi}{2}} \left[\frac{2}{n} - \frac{8}{n^3 c} g(r\cos\theta)\cos\theta + \frac{8}{n^3 c} p(t)\cos\theta\right] \mathrm{d}\theta,$$

and

$$J_n = \int_{-2n\pi + \theta_0}^{-2n\pi + \frac{\pi}{2}} \left[\frac{2}{n} - \frac{8}{n^3 c} g(r\cos\theta)\cos\theta + \frac{8}{n^3 c} p(t)\cos\theta\right] \mathrm{d}\theta.$$

In what follows, we shall estimate $J_i, (i = 0, \ldots, n)$,, respectively. We now estimate J_0. Obviously,

$$J_0 = \frac{\pi}{n} + \frac{2\theta_0}{n} - \frac{8}{n^3 c} \int_{-\frac{\pi}{2}}^{\theta_0} [g(r\cos\theta)\cos\theta - p(t)\cos\theta]\,\mathrm{d}\theta.$$

From Lemma 5.1 in Appendix, we know that, for $c \to +\infty$, the estimate

$$\int_{-\frac{\pi}{2}}^{\theta_0} g(r\cos\theta)\cos\theta \mathrm{d}\theta = (1 + \sin\theta_0)G(+\infty) + o(1) \tag{3.2}$$

holds uniformly for (r_0, θ_0) satisfying $(r_0\cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$. When $\theta(t) \in [-\frac{\pi}{2}, \theta_0]$, we have

$$t(\theta) = \frac{2}{n}(\theta_0 - \theta) + o(1). \tag{3.3}$$

Since p is uniformly continuous on $[0, 2\pi]$ and $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we get from (3.3) that

$$\int_{-\frac{\pi}{2}}^{\theta_0} p(t)\cos\theta \mathrm{d}\theta = \frac{n}{2} \int_0^{\frac{2\theta_0+\pi}{n}} p(\tau) \left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau + o(1).$$

Therefore,

$$J_0 = \frac{\pi}{n} + \frac{2\theta_0}{n} + \frac{4}{n^3 c}\left[-2(1+\sin\theta_0)G(+\infty) + n \int_0^{\frac{2\theta_0+\pi}{n}} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau\right] + o\left(\frac{1}{c}\right).$$

We next estimate $J_i, (i = 1, \ldots, n-1)$. Obviously, we have

$$J_i = \frac{2\pi}{n} - \frac{8}{n^3 c} \int_{-2i\pi-\frac{\pi}{2}}^{-2i\pi+\frac{\pi}{2}} [g(r\cos\theta)\cos\theta - p(t)\cos\theta]\,\mathrm{d}\theta.$$

Similarly, we know from Lemma 5.1 in Appendix that, for $c \to +\infty$, the estimate

$$\int_{-2i\pi-\frac{\pi}{2}}^{-2i\pi+\frac{\pi}{2}} g(r\cos\theta)\cos\theta \mathrm{d}\theta = 2G(+\infty) + o(1) \tag{3.4}$$

holds uniformly for (r_0, θ_0) satisfying $(r_0\cos\theta_0, \frac{n}{2} r_0\sin\theta_0) \in \Gamma_c$. When $\theta(t) \in [-2i\pi - \frac{\pi}{2}, -2i\pi + \frac{\pi}{2}], (i = 1, \ldots, n-1)$, we have

$$t(\theta) = \frac{2}{n}(\theta_0 - \theta) - \frac{2i\pi}{n} + o(1).$$

Then, we obtain

$$\int_{-2i\pi-\frac{\pi}{2}}^{-2i\pi+\frac{\pi}{2}} p(t)\cos\theta \mathrm{d}\theta = \frac{n}{2} \int_{\frac{2\theta_0+(2i-1)\pi}{n}}^{\frac{2\theta_0+(2i+1)\pi}{n}} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau + o(1).$$

Hence,

$$J_i = \frac{2\pi}{n} + \frac{4}{n^3 c}\left[-4G(+\infty) + n \int_{\frac{2\theta_0+(2i-1)\pi}{n}}^{\frac{2\theta_0+(2i+1)\pi}{n}} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau\right] + o\left(\frac{1}{c}\right).$$

We finally estimate J_n. It is easy to see

$$J_n = \frac{\pi - 2\theta_0}{n} - \frac{8}{n^3 c} \int_{-2n\pi+\theta_0}^{-2n\pi+\frac{\pi}{2}} [g(r\cos\theta)\cos\theta - p(t)\cos\theta]\,\mathrm{d}\theta.$$

We can also get

$$\int_{-2n\pi+\theta_0}^{-2n\pi+\frac{\pi}{2}} g(r\cos\theta)\cos\theta\mathrm{d}\theta = (1-\sin\theta_0)G(+\infty) + o(1). \tag{3.5}$$

When $\theta(t) \in [-2n\pi+\theta_0, -2n\pi+\frac{\pi}{2}]$, we have

$$t(\theta) = \frac{2}{n}(\theta_0 - \theta) - 2\pi + o(1).$$

Hence,

$$\int_{-2n\pi+\theta_0}^{-2n\pi+\frac{\pi}{2}} p(t)\cos\theta\mathrm{d}\theta = \frac{n}{2} \int_{\frac{2\theta_0+(2n-1)\pi}{n}}^{2\pi} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right|\mathrm{d}\tau + o(1).$$

Thus, we get

$$J_n = \frac{\pi}{n} - \frac{2\theta_0}{n} + \frac{4}{n^3 c}\left[-2(1-\sin\theta_0)G(+\infty) + n\int_{\frac{2\theta_0+(2n-1)\pi}{n}}^{2\pi} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right|\mathrm{d}\tau\right] + o\left(\frac{1}{c}\right).$$

Consequently, we obtain

$$\sum_{i=0}^{n} J_i = 2\pi + \frac{4}{n^2 c}\left[-4G(+\infty) + \int_0^{2\pi} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right|\mathrm{d}\tau\right] + o\left(\frac{1}{c}\right).$$

On the other hand, we know from Lemma 2.5 that the time Δt needed for the solution $(x(t), y(t))$ to pass through the region $\{(x,y) : -1 < x \le 0, -\infty < y < +\infty\}$ once satisfies

$$\Delta t = \frac{4}{nc} + o\left(\frac{1}{c}\right).$$

Hence, we get that, for $c \to +\infty$,

$$\tau_n(r_0,\theta_0) = 2\pi + \frac{4}{n^2 c}\left[n^2 - 4G(+\infty) + \int_0^{2\pi} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right|\mathrm{d}\tau\right] + o\left(\frac{1}{c}\right). \tag{3.6}$$

In case, when $\theta_0 \in [-\frac{3\pi}{2}, -\frac{\pi}{2}]$, we can prove by using (3.6) and the same method in (Lemma [17,19]) that, for $c \to +\infty$,

$$\tau_n(r_0,\theta_0) = 2\pi + \frac{4}{n^2 c}\left[n^2 - 4G(+\infty) + \int_0^{2\pi} p(\tau)\left|\cos\left(\frac{\pi}{2} - \frac{n\tau}{2}\right)\right|\mathrm{d}\tau\right] + o\left(\frac{1}{c}\right). \tag{3.7}$$

If

$$4G(+\infty) - \int_0^{2\pi} p(\tau)\left|\sin\left(\theta + \frac{n\tau}{2}\right)\right|\mathrm{d}\tau > n^2, \quad \theta \in [0, 2\pi],$$

then we get from (3.6) and (3.7) that $\tau_n(r_0, \theta_0) < 2\pi$. Therefore, we have that, for c large enough and $(r_0 \cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n\pi.$$

On the other hand, we know from above estimates that the required time $\tau_{n+1}(r_0, \theta_0)$ for solution $(r(t), \theta(t))$ to complete $n+1$ turns around the origin satisfies $\tau_{n+1}(r_0, \theta_0) = 2\pi + \frac{2\pi}{n} + o(1)$. Consequently, we get that, for c large enough and $(r_0 \cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 > -2(n+1)\pi.$$

According to the Poincaré–Bohl theorem [19], Poincaré map P has at least one fixed point. Consequently, Eq. (1.6′) has at least one 2π-periodic solution.

If

$$4G(+\infty) - \int_0^{2\pi} p(\tau) \left|\sin\left(\theta + \frac{n\tau}{2}\right)\right| \mathrm{d}\tau < n^2, \quad \theta \in [0, 2\pi],$$

then we can prove similarly that Poincaré map P has at least one fixed point. Consequently, Eq. (1.6′) has at least one 2π-periodic solution. □

Proof of Theorem 1.1. Consider the equation equivalent to Eq. (1.6),

$$x'' + \frac{1}{4}n^2 x + \tilde{g}(x) = \tilde{p}(t), \tag{3.8}$$

where $\tilde{g}(x) = g(x+1), \tilde{p}(t) = p(t) - \frac{1}{4}n^2$. Obviously, $\tilde{g}$ satisfies $(h_i')(i = 1, 2, 3)$. Moreover, $\lim_{x\to+\infty} \tilde{g}(x) = g(+\infty)$. From Proposition 3.1, we know that Eq. (3.8) has at least one 2π-periodic solution $x(t)(x(t) > -1)$ provided that the following condition is satisfied,

$$4G(+\infty) - \int_0^{2\pi} \left[p(t) - \frac{1}{4}n^2\right] \left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t \neq n^2, \quad \forall\theta \in \mathbf{R}. \tag{3.9}$$

Since

$$\int_0^{2\pi} \left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t = 4,$$

we know that (3.9) is equivalent to the following inequality,

$$4G(+\infty) - \int_0^{2\pi} p(t) \left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t \neq 0, \quad \forall\theta \in \mathbf{R}. \tag{3.10}$$

Therefore, Eq. (1.6) has at least one positive 2π-periodic solution provided that (3.10) is satisfied. □

4. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following Proposition.

Proposition 4.1. *Assume that conditions $(h_i')(i = 1, 2, 3)$ and $g(x)$ are bounded for $x \in [0, +\infty)$. Then, Eq. (1.6′) has at least one positive 2π-periodic solution provided that either*

$$4\underline{g}(+\infty) - \int_0^{2\pi} p(t) \left|\sin\left(\theta + \frac{nt}{2}\right)\right| dt > n^2, \quad \forall\theta \in \mathbf{R} \tag{4.1}$$

or

$$4\overline{g}(+\infty) - \int_0^{2\pi} p(t)\left|\sin\left(\theta + \frac{nt}{2}\right)\right| dt < n^2, \quad \forall \theta \in \mathbf{R} \tag{4.2}$$

holds.

Proof. We put ourselves into the same situation as in the proof of Proposition 4.1. Under the present conditions, we can prove that, for $c \to +\infty$,

$$\tau_n(r_0, \theta_0) \leq 2\pi - \frac{4}{n^2 c}\left[-n^2 + 4\underline{g}(+\infty) - \int_0^{2\pi} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau\right] + o\left(\frac{1}{c}\right), (\cos\theta_0 \geq 0),$$

$$\tau_n(r_0, \theta_0) \leq 2\pi - \frac{4}{n^2 c}\left[-n^2 + 4\underline{g}(+\infty) - \int_0^{2\pi} p(\tau)\left|\cos\left(\frac{\pi}{2} - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau\right] + o\left(\frac{1}{c}\right), (\cos\theta_0 \leq 0),$$

and

$$\tau_n(r_0, \theta_0) \geq 2\pi - \frac{4}{n^2 c}\left[-n^2 + 4\overline{g}(+\infty) - \int_0^{2\pi} p(\tau)\left|\cos\left(\theta_0 - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau\right] + o\left(\frac{1}{c}\right), (\cos\theta_0 \geq 0),$$

$$\tau_n(r_0, \theta_0) \geq 2\pi - \frac{4}{n^2 c}\left[-n^2 + 4\overline{g}(+\infty) - \int_0^{2\pi} p(\tau)\left|\cos\left(\frac{\pi}{2} - \frac{n\tau}{2}\right)\right| \mathrm{d}\tau\right] + o\left(\frac{1}{c}\right), (\cos\theta_0 \leq 0).$$

If (4.1) holds, then we have that, for $c > 0$ large enough, $\tau_n(r_0, \theta_0) < 2\pi$. If (4.2) holds, then we have that, for $c > 0$ large enough, $\tau_n(r_0, \theta_0) > 2\pi$. Using the same method as proving Proposition 3.1, we can prove that Poincaré map P has at least one fixed point. Consequently, Eq. (1.6′) has at least one 2π-periodic solution. □

Proof of Theorem 1.2. We also consider Eq. (3.8). According to Proposition 4.1, Eq. (3.8) has at least one 2π-periodic solution provided that either

$$4\underline{g}(+\infty) - \int_0^{2\pi} \left[p(t) - \frac{1}{4}n^2\right]\left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t > n^2, \quad \forall \theta \in \mathbf{R}$$

or

$$4\overline{g}(+\infty) - \int_0^{2\pi} \left[p(t) - \frac{1}{4}n^2\right]\left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t < n^2, \quad \forall \theta \in \mathbf{R}$$

holds. Since

$$\int_0^{2\pi} \left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t = 4,$$

we know that Eq. (3.8) has at least one 2π-periodic solution provided that either

$$4\underline{g}(+\infty) - \int_0^{2\pi} p(t)\left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t > 0, \quad \forall \theta \in \mathbf{R} \tag{4.3}$$

or

$$4\overline{g}(+\infty) - \int_0^{2\pi} p(t)\left|\sin\left(\theta + \frac{nt}{2}\right)\right| \mathrm{d}t < 0, \quad \forall \theta \in \mathbf{R} \tag{4.4}$$

holds. Consequently, Eq. (1.6) has at least one 2π-periodic solution provided that either (4.3) or (4.4) holds. □

5. Appendix

In this section, we shall give the proofs of (3.2), (3.4) and (3.5). Let us put ourselves into the same situation considered as in the proof of Proposition 3.1.

Lemma 5.1. *For $c \to +\infty$, the following estimates*

$$\int_{-\frac{\pi}{2}}^{\theta_0} g(r\cos\theta)\cos\theta\, d\theta = (1+\sin\theta_0)G(+\infty) + o(1),$$

and for $i = 1, 2, \dots, n$,

$$\int_{-2(i-1)\pi-\frac{\pi}{2}}^{-2(i-1)\pi+\frac{\pi}{2}} g(r\cos\theta)\cos\theta\, d\theta = 2G(+\infty) + o(1),$$

and

$$\int_{-2n\pi+\theta_0}^{-2n\pi+\frac{\pi}{2}} g(r\cos\theta)\cos\theta\, d\theta = (1-\sin\theta_0)G(+\infty) + o(1),$$

hold uniformly with respect to (r_0, θ_0) satisfying $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $(r_0\cos\theta_0, \frac{n}{2}r_0\sin\theta_0) \in \Gamma_c$, where $r = r(t(\theta)) = r(t(\theta), r_0, \theta_0)$.

Proof. We only deal with the case $i = 1$. The other cases can be treated similarly. Let us write

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(r\cos\theta)\cos\theta \mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{0} g(r\cos\theta)\cos\theta \mathrm{d}\theta + \int_{0}^{\frac{\pi}{2}} g(r\cos\theta)\cos\theta \mathrm{d}\theta. \tag{5.1}$$

Since the limits

$$\lim_{c\to+\infty} \frac{1}{c} r(t(\theta))\cos\theta = \cos\theta, \qquad \lim_{c\to+\infty} \frac{1}{c}\left[r(t(\theta))\cos\theta\right]' = -\sin\theta$$

hold uniformly with respect to (r_0, θ_0) satisfying $(r_0\cos\theta_0, \frac{n}{2}r_0\sin\theta_0) \in \Gamma_c$ and $\theta \in [-\frac{\pi}{2}, 0]$, there exists a unique mapping $\sigma : [-\frac{\pi}{2}, 0] \to \mathrm{R}$ (which depends on c, r_0, θ_0, θ) such that

$$r(t(\theta))\cos\theta = c\cos\sigma(\theta), \quad \theta \in \left[-\frac{\pi}{2}, 0\right].$$

Moreover, we have that, for $c \to +\infty$,

$$\sigma(\theta) \to \theta, \quad \sigma'(\theta) \to 1$$

hold uniformly for (r_0, θ_0) satisfying $(r_0 \cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$ and $\theta \in [-\frac{\pi}{2}, 0]$. Then, we get

$$\int_{-\frac{\pi}{2}}^{0} g(r\cos\theta)\cos\theta \mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{0} g(c\cos\sigma(\theta))\cos\theta \mathrm{d}\theta.$$

Taking $\tau = \sigma(\theta)$, we derive that, for $c \to +\infty$, the estimate

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{0} g(c\cos\sigma(\theta))\cos\theta \mathrm{d}\theta &= \int_{\sigma(-\frac{\pi}{2})}^{\sigma(0)} \frac{g(c\cos\tau)\cos[\sigma^{-1}(\tau)]}{\sigma'(\sigma^{-1}(\tau))}\mathrm{d}\tau \\ &= \int_{-\frac{\pi}{2}}^{0} g(c\cos\tau)\cos\tau \mathrm{d}\tau + o(1)\end{aligned}$$

holds uniformly for (r_0, θ_0) satisfying $(r_0 \cos\theta_0, \frac{n}{2} r_0 \sin\theta_0) \in \Gamma_c$. In what follows, we shall prove

$$\lim_{c\to+\infty} \int_{-\frac{\pi}{2}}^{0} g(c\cos\tau)\cos\tau \mathrm{d}\tau = G(+\infty).$$

Let us take a sufficiently small constant $\varepsilon > 0$. Then, we have that, for $c \to +\infty$,

$$\begin{aligned}\int_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} g(c\cos\tau)\cos\tau \mathrm{d}\tau &= -\int_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} \frac{\cos\tau}{c\sin\tau}\mathrm{d}G(c\cos\tau) \\ &= -\frac{\cos^2\tau}{\sin\tau}\frac{G(c\cos\tau)}{c\cos\tau}\Big|_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} - \int_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} \frac{G(c\cos\tau)}{c\cos\tau}\frac{\cos\tau}{\sin^2\tau}\mathrm{d}\tau \\ &= -G(+\infty)\left[\frac{\cos^2\tau}{\sin\tau}\Big|_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} + \int_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} \frac{\cos\tau}{\sin^2\tau}\mathrm{d}\tau\right] + o(1) \\ &= -G(+\infty)\int_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} \frac{\cos\tau}{\sin\tau} d\cos\tau + o(1) \\ &= G(+\infty)\int_{-\frac{\pi}{2}+\varepsilon}^{-\varepsilon} \cos\tau \mathrm{d}\tau + o(1) \\ &= G(+\infty)(\cos\varepsilon - \sin\varepsilon) + o(1).\end{aligned} \tag{5.2}$$

Since g is bounded, there exists a constant $\varrho > 0$ such that, for any $c > 0$,

$$\left|\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\varepsilon} g(c\cos\tau)\cos\tau \mathrm{d}\tau\right| \le \varrho\varepsilon, \quad \left|\int_{-\varepsilon}^{0} g(c\cos\tau)\cos\tau \mathrm{d}\tau\right| \le \varrho\varepsilon. \tag{5.3}$$

From (5.2) and (5.3), we can derive

$$\lim_{c\to+\infty}\int_{-\frac{\pi}{2}}^{0} g(c\cos\tau)\cos\tau\mathrm{d}\tau = G(+\infty).$$

Therefore, we have that, for $c \to +\infty$, the estimate

$$\int_{-\frac{\pi}{2}}^{0} g(r\cos\theta)\cos\theta\mathrm{d}\theta = G(+\infty) + o(1) \tag{5.4}$$

holds uniformly with respect to (r_0, θ_0) satisfying $(r_0\cos\theta_0, \frac{n}{2}r_0\sin\theta_0) \in \Gamma_c$. Similarly, we obtain

$$\int_{0}^{\frac{\pi}{2}} g(r\cos\theta)\cos\theta\mathrm{d}\theta = G(+\infty) + o(1). \tag{5.5}$$

Consequently, we know from (5.1), (5.4), and (5.5) that, for $c \to +\infty$, the estimate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(r\cos\theta)\cos\theta\mathrm{d}\theta = 2G(+\infty) + o(1)$$

holds uniformly with respect to (r_0, θ_0) satisfying $(r_0\cos\theta_0, \frac{n}{2}r_0\sin\theta_0) \in \Gamma_c$. The proof is complete. □

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References

1. Capietto, A., Dambrosio, W., Liu, B.: On the boundedness of solutions to a nonlinear singular oscillator. Z. Angew. Math. Phys. **60**, 1007–1034 (2009)
2. Bonheure, D., Fabry, C., Smets, D.: Periodic solutions of forced isochronous oscillator at resonance. Discrete Contin. Dyn. Syst. **8**, 907–930 (2002)
3. Del Pino, M., Manásevich, R.: Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity. J. Differ. Equ. **103**, 260–277 (1993)
4. Del Pino, M., Manásevich, R., Montero, A.: T-periodic solutions for some second order differential equations with singularities. Proc. R. Soc. Edinb. A **120**, 231–243 (1992)
5. Ding, T.: Applications of Qualitative Methods of Ordinary Differential Equations. Higher Education Press, Beijing (2004)
6. Boscaggin, A., Fonda, A., Garrione, M.: A multiplicity result for periodic solutions of second order differential equations with a singularity. Nonlinear Anal. **75**, 4457–4470 (2012)
7. Fonda, A., Manásevich, R., Zanolin, F.: Subharmonic solutions for some second order differential equations with singularities. SIAM J. Math. Anal. **24**, 1294–1311 (1993)
8. Habets, P., Sanchez, L.: Periodic solutions of some Liénard equations with singularities. Proc. Am. Math. Soc. **109**, 1035–1044 (1990)
9. Hakl, R., Torres, P., Zamora, M.: Periodic solutions of singular second order differential equations: upper and lower functions. Nonlinear Anal. **74**, 7078–7093 (2011)
10. Jiang, D., Chu, J., Zhang, M.: Multiplicity of positive periodic solutions to superlinear repulsive singular equations. J. Differ. Equ. **211**, 282–302 (2005)
11. Li, X., Zhang, Z.: Periodic solutions for second order differential equations with a singular nonlinearity. Nonlinear Anal. **69**, 3866–3876 (2008)

12. Rachunková, I., Tvrdý, M., Vrkoč, I.: Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems. J. Differ. Equ. **176**, 445–469 (2001)
13. Torres, P.J.: Existence of one-signed periodic solutions of some second order differential equations via a Krasnoselskii fixed point theorem. J. Differ. Equ. **190**, 643–662 (2003)
14. Wang, Z.: Periodic solutions of the second order differential equations with singularities. Nonlinear Anal. **58**, 319–331 (2004)
15. Zhang, M.: Periodic solutions of Liénard equations with singular forces of repulsive type. J. Math. Anal. Appl. **203**, 254–269 (1996)
16. Lazer, A.C., Solimini, S.: On periodic solutions of nonlinear differential equations with singularities. Proc. Am. Math. Soc. **88**, 109–114 (1987)
17. Wang, Z., Ma, T.: Existence and multiplicity of periodic solutions of semilinear resonant Duffing equations with singularities. Nonlinearity **25**, 279–307 (2012)
18. Lazer, A.C., Leach, D.E.: Bounded perturbations of forced harmonic oscillators at resonance. Ann. Math. Pura. Appl. **83**, 49–68 (1969)
19. Krasnosel'skii, M., Zabreiko, P.: Geometrical Methods of Nonlinear Analysis. Springer, Berlin (1984)

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