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Global existence and the low Mach number limit for the compressible magnetohydrodynamic equations in a bounded domain with perfectly conducting boundary

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Abstract. We study the compressible magnetohydrodynamic equations in a bounded smooth domain in $\mathbb{R}^2$ with perfectly conducting boundary, and prove the global existence and uniqueness of smooth solutions around a rest state. Moreover, the low Mach limit of the solutions is verified for all time, provided that the initial data are well prepared.

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1. Introduction

We mainly study the low Mach number limit for an initial boundary value problem of the following two-dimensional resistive magnetohydrodynamic equations of a compressible viscous and conducting fluid:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\epsilon^2}\nabla p(\rho) = \operatorname{div}(2\mu \mathrm{D}(\mathbf{u})) + \lambda \nabla \operatorname{div}\mathbf{u} + (\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{1.2}$$

$$\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\eta \nabla \times \mathbf{H}), \quad \operatorname{div}\mathbf{H} = 0. \tag{1.3}$$

Here, ρ denotes the density of the fluid, $\mathbf{u} = (u_1, u_2)$ the velocity, $\mathbf{H} = (H_1, H_2)$ the magnetic field, $\mathrm{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^t)/2$. The constants μ and λ are the shear and bulk viscosity coefficients of the fluid, respectively, satisfying $\mu > 0$ and $\mu + \lambda > 0$; the constant $\eta > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, ϵ is the Mach number. $p(\rho)$ is the pressure, and in this paper, we consider the case of isentropic flows

$$p(\rho) = a\rho^\gamma, \tag{1.4}$$

where $a > 0$ and $\gamma > 1$ are constants.

Magnetohydrodynamics (MHD) studies the dynamics of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields with a very broad range of applications. In the present paper, we consider the flow in a perfectly conducting container, so that the magnetic field is confined inside and separated from the exterior. The container is assumed to be a bounded and connected domain $\Omega \subset \mathbb{R}^2$ with smooth boundary.

The initial data for the system (1.1)–(1.3) are prescribed as

$$\rho(t = 0) = \rho_0(x), \quad \mathbf{u}(t = 0) = \mathbf{u}_0(x), \quad \mathbf{H}(t = 0) = \mathbf{H}_0(x). \tag{1.5}$$

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We specify the velocity on the boundary with Navier slip boundary conditions, i.e.,

$$\mathbf{u}\cdot\mathbf{n}=0,\quad \tau\cdot \mathrm{T}(\mathbf{u},p)\cdot\mathbf{n}+f\mathbf{u}\cdot\tau=0\quad \text{on }\ \partial\Omega \tag{1.6}$$

with $f(x)\geq 0$, where $\mathrm{T}(\mathbf{u},p)=2\mu\mathrm{D}(\mathbf{u})+(\lambda\mathrm{div}\mathbf{u}-p/\epsilon^2)\mathrm{I}$ is the stress tensor, $\mathbf{n}$ and τ are the normal and tangent vectors on $\partial\Omega$, respectively. The Navier slip boundary conditions describe an interaction between a viscous fluid and a solid wall.

We require that the container be perfectly conducting, that is,

$$\mathbf{n}\times\mathbf{e}=0,\quad \mathbf{n}\cdot\mathbf{H}=0\quad \text{on }\ \partial\Omega, \tag{1.7}$$

where $\mathbf{e}=c^{-1}(\eta\nabla\times\mathbf{H}-\mathbf{u}\times\mathbf{H})$ is the electric field (c is the light speed, see [30]). Furthermore, the conditions (1.7) imply that

$$\mathbf{n}\times(\nabla\times\mathbf{H})=-\frac{1}{\eta}(\mathbf{n}\cdot\mathbf{u})\mathbf{H}\quad \text{on }\ \partial\Omega.$$

Because of (1.6), the boundary conditions (1.7) reduce to

$$\mathbf{n}\times(\nabla\times\mathbf{H})=0,\ \ \mathbf{n}\cdot\mathbf{H}=0\quad \text{on }\ \partial\Omega. \tag{1.8}$$

Notice that in the two-dimensional case, the conditions (1.8) are equivalent to

$$\mathrm{curl}\,\mathbf{H}=0,\ \ \mathbf{n}\cdot\mathbf{H}=0\quad \text{on }\ \partial\Omega, \tag{1.9}$$

where $\mathrm{curl}\,\mathbf{H}=\partial_1 H_2-\partial_2 H_1$.

The MHD equations have recently attracted a lot of attention of applied mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges, see, for example, [4,9,10,14,15,24,27,36] and the references cited therein on the physical background, the well posedness and the vanishing viscosity limit. Recently, the low Mach number limit of local smooth solutions to the full MHD equations with heat conductivity was investigated in [19,20] in the whole space or a torus. The existence of global weak solutions to the MHD equations was established in [16,34], while the low Mach number limit was studied in [17,18]. We remark that the low Mach number limit established in [17–20] for the MHD equations is for the whole space or a torus, consequently, no boundary terms are involved in uniform a priori estimates.

The aim of the present paper is to establish the global well posedness of smooth solutions to the initial boundary value problem (1.1)–(1.6), (1.8) around a rest state, and furthermore, to prove rigorously the corresponding low Mach number limit as $\epsilon\to 0$ of solutions for all time. In [24], Kawashima obtained the global small smooth solution to the three-dimensional Cauchy problem for initial data close to a constant state in $H^3(\mathbb{R}^3)$. In the current paper, we generalize the result of [24] to the initial boundary problem with Navier slip and perfectly conducting boundary conditions, and more important, we establish the uniform estimates of smooth solutions with respect to the Mach number and verify rigorously the incompressible limit for all time when boundary is present.

We also mention that the global smooth small solution to the related compressible isentropic Navier-Stokes (the system (1.1)–(1.3) with $\mathbf{H}\equiv 0$) was obtained, for example, in [32] for the non-slip boundary condition and in [37] for the Navier slip boundary condition, while the existence of global large weak solutions was established in [12,22,23,28] and among others. The corresponding low Mach number limit was investigated extensively in [2,3,6–8,11,13,21,25,26,29,31,33], and in the references cited therein.

In this paper, we shall consider the flow with small density variation, i.e.,

$$\rho=1+\epsilon\sigma.$$

Using the usual vorticity identities together with the constraint $\mathrm{div}\mathbf{H} = 0$, we can write the problem (1.1)–(1.6), (1.8) in the form

$$\partial_t\sigma + \mathrm{div}(\sigma\mathbf{u}) + \frac{1}{\epsilon}\mathrm{div}\mathbf{u} = 0, \tag{1.10}$$

$$\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \frac{1}{\epsilon}p'(1+\epsilon\sigma)\nabla\sigma$$
$$= \mathrm{div}(2\mu\mathrm{D}(\mathbf{u})) + \lambda\nabla\mathrm{div}\mathbf{u} + (\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2, \tag{1.11}$$

$$\partial_t\mathbf{H} + (\mathrm{div}\mathbf{u})\mathbf{H} + (\mathbf{u}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u} = \eta\triangle\mathbf{H}, \quad \mathrm{div}\mathbf{H} = 0, \tag{1.12}$$

$$\sigma(t=0) = \sigma_0(x), \quad \mathbf{u}(t=0) = \mathbf{u}_0(x), \quad \mathbf{H}(t=0) = \mathbf{H}_0(x), \tag{1.13}$$

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \tau\cdot\mathrm{T}(\mathbf{u},p)\cdot\mathbf{n} + f\mathbf{u}\cdot\tau = 0 \quad \text{on } \partial\Omega, \tag{1.14}$$

$$\mathbf{H}\cdot\mathbf{n} = 0, \quad \mathrm{curl}\,\mathbf{H} = 0 \quad \text{on } \partial\Omega. \tag{1.15}$$

Thus, the main results of the present paper read as follows.

Theorem 1.1. *There exists a positive constant α, such that if the initial data $\sigma_0, \mathbf{u}_0, \mathbf{H}_0$ satisfy*

$$\|(\sigma_0, \mathbf{u}_0, \mathbf{H}_0)\|_{\mathbf{H}^2} + \|(\sigma_t, \mathbf{u}_t, \mathbf{H}_t)(0)\|_{H^1} + \|\epsilon(\sigma_{tt}, \mathbf{u}_{tt}, \mathbf{H}_{tt})(0)\|_{L^2} \le \alpha, \tag{1.16}$$

with

$$\mathrm{div}\mathbf{H}_0 = 0, \quad \int_\Omega \sigma_0 \mathrm{d}x = 0 \quad \text{and} \quad 1+\epsilon\sigma_0 \ge m \quad \textit{for some constant } m > 0, \tag{1.17}$$

and the following compatibility conditions

$$\mathbf{u}_0\cdot\mathbf{n} = \mathbf{u}_t(0)\cdot\mathbf{n} = 0, \quad \tau\cdot\mathrm{T}(\mathbf{u}_0,\ p(\rho_0))\cdot\mathbf{n} + f\mathbf{u}_0\cdot\tau = 0 \quad \textit{on } \partial\Omega$$
$$\mathbf{H}_0\cdot\mathbf{n} = \mathbf{H}_t(0)\cdot\mathbf{n} = \mathrm{curl}\,\mathbf{H}_0 = 0 \quad \textit{on } \partial\Omega$$

hold, then for any $\epsilon \in (0, \epsilon_1]$ where $0 < \epsilon_1 < 1$ is some constant, the initial boundary value problem (1.10)–(1.15) admits a unique solution $(\sigma, \mathbf{u}, \mathbf{H})$ in $\Omega\times\bar{\mathbb{R}}^+$, satisfying

$$\sigma \in C(\bar{\mathbb{R}}^+; H^2), \quad (\mathbf{u},\mathbf{H}) \in C(\bar{\mathbb{R}}^+; H^2) \cap L^2(\bar{\mathbb{R}}^+; H^3),$$
$$\sigma_t \in C(\bar{\mathbb{R}}^+; H^1), \quad (\mathbf{u}_t,\mathbf{H}_t) \in C(\bar{\mathbb{R}}^+; H^1) \cap L^2(\bar{\mathbb{R}}^+; H^2),$$
$$\sigma_{tt} \in L^\infty(\bar{\mathbb{R}}^+; L^2), \quad (\mathbf{u}_{tt},\mathbf{H}_{tt}) \in L^\infty(\bar{\mathbb{R}}^+; L^2) \cap L^2(\bar{\mathbb{R}}^+; H^1),$$

where $\bar{\mathbb{R}}^+ = [0, +\infty)$. Furthermore, it holds that:

$$\sup_{0\le s\le t}(\|(\sigma,\mathbf{u},\mathbf{H})(s)\|_{H^2} + \|(\sigma_t,\mathbf{u}_t,\mathbf{H}_t)(s)\|_{H^1})$$
$$+\mathrm{ess}\sup_{0\le s\le t}\|\epsilon(\sigma_{tt},\mathbf{u}_{tt},\mathbf{H}_{tt})(s)\|_{L^2} \le C, \qquad \forall\, t\in\mathbb{R}^+, \tag{1.18}$$

where C is a positive constant independent of ϵ.

Theorem 1.2. *Let the assumptions in Theorem 1.1 be satisfied, and $(\mathbf{u}, \mathbf{H})$ be the global solution established in Theorem 1.1. Assume the initial data $(\mathbf{u}_0, \mathbf{H}_0) \to (\mathbf{v}_0, \mathbf{B}_0)$ as $\epsilon \to 0$ in H^s for any $0 \le s < 2$. Then as $\epsilon \to 0, (\mathbf{u}, \mathbf{H}) \to (\mathbf{v}, \mathbf{B})$ in $C(\bar{\mathbb{R}}^+_{\mathrm{loc}}; H^s)$ for any $0 \le s < 2$. And there exists a function $P(x, t)$, such that $(\mathbf{v}, \mathbf{B}, P)$ is the unique smooth solution of the following initial boundary value problem for the incompressible magnetohydrodynamic equations:*

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P = \mu\triangle\mathbf{v} + (\nabla\times\mathbf{B})\times\mathbf{B}, \quad \mathrm{div}\,\mathbf{v} = 0,$$
$$\mathbf{B}_t + \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v} = \eta\triangle\mathbf{B}, \quad \mathrm{div}\mathbf{B} = 0,$$

with initial and boundary conditions

$$\begin{aligned}&\mathbf{v}(x,0)=\mathbf{v}_0(x), \quad \mathbf{B}(x,0)=\mathbf{B}_0(x), \quad x\in\Omega,\\&\mathbf{v}\cdot\mathbf{n}=0, \quad \tau\cdot D(\mathbf{v})\cdot\mathbf{n}+f\mathbf{v}\cdot\tau=0 \quad \textit{on } \partial\Omega,\\&\mathbf{B}\cdot\mathbf{n}=\operatorname{curl}\mathbf{B}=0 \quad \textit{on } \partial\Omega.\end{aligned}$$

In the next section, we shall prove Theorems 1.1 and 1.2. Roughly speaking, Theorems 1.1 and 1.2 are proved based on the uniform estimates of solutions in Sobolev norms which do not depend on time t and the Mach number ϵ. As aforementioned, compared with the Cauchy or spatially periodic problem, due to the presence of boundary here, some difficulties involved with controlling the boundary terms, in particular for the low Mach number limit, arise. To circumvent such difficulties, we exploit the fact that the vorticity of the velocity and magnetic fields is a scalar in the two-dimensional case to estimate carefully the vorticity and the divergence of the velocity as well as the vorticity of the magnetic field. More precisely, to control the vorticity of the velocity and magnetic field, we exploit the fact that both the Navier slip and the perfectly conducting boundary conditions make it possible to give a representation of the vorticity on the boundary in the two-dimensional case (cf. (2.5) and (1.9)), while the estimate of the divergence of the velocity is obtained by using the equality $\Delta\mathbf{u}=\nabla\operatorname{div}\mathbf{u}-\overrightarrow{\operatorname{curl}}\operatorname{curl}\mathbf{u}$ with $\overrightarrow{\operatorname{curl}}=(\partial_2,-\partial_1)^t$ and $\operatorname{curl}\mathbf{u}=\partial_1u_2-\partial_2u_1$ that helps us separate the divergence and vorticity. On the other hand, the coupling of the hydrodynamic motion and the magnetic fields makes the derivation of the uniform a priori estimates more complicated and delicate in comparison with the hydrodynamic case. Therefore, we have to deal with the estimates involving these coupling terms carefully, especially on higher-order spatial derivatives.

We remark that the effect of a boundary layer on the propagation of the acoustic waves will not appear in the convergence of solutions under the Navier slip boundary condition (1.14) and perfectly conducting boundary condition (1.15) in the present paper (see Chapter 7 in [11]). This is different from the case of Dirichlet boundary condition. As shown in [8], the acoustic waves are asymptotically damped due to the formation of a thin boundary layer for a viscous flow in a bounded domain with Dirichlet boundary condition. The effect of boundary layers due to the Dirichlet boundary condition on the velocity is our future study.

Before ending this section, we give the notations used throughout this paper. We use the letter C (or C_δ) to denote various positive constants independent of ϵ (or to emphasize the dependence on δ). For simplicity, we denote by H^m and $\|\cdot\|_{H^m}$ the standard Sobolev space $H^m(\Omega)$ and its norm, by L^p and $\|\cdot\|_{L^p}$ the Lebesgue space $L^p(\Omega)$ and its norm.

2. Proof of Theorems 1.1 and 1.2

To prove Theorem 1.1, we first establish the local existence for the problem (1.10)–(1.15) with an arbitrary but fixed ϵ. Assume that the assumptions in Theorem 1.1 are satisfied. Then, one can show by the Galerkin method (see [37]) that there exists a $T^\star>0$, such that for $T\le T^\star$ the problem (1.10)–(1.15) admits a solution satisfying

$$\begin{aligned}\sigma&\in C([0,T],H^2), \quad (\mathbf{u},\mathbf{H})\in C([0,T],H^2)\cap L^2(0,T;H^3),\\\sigma_t&\in C([0,T],H^1), \quad (\mathbf{u}_t,\mathbf{H}_t)\in C([0,T],H^1)\cap L^2(0,T;H^2),\\\sigma_{tt}&\in L^\infty(0,T;L^2), \quad (\mathbf{u}_{tt},\mathbf{H}_{tt})\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1).\end{aligned}$$

The boundary conditions (1.14) and (1.15) are "complementing" boundary conditions in the sense of Agmon et al. [1]. This fact can be verified as in [1]. Therefore, the regularity theory can be used in the proof. We omit the details of the proof of the local existence here.

To extend the local solution globally in time, we shall establish a differential inequality which provides us the uniform estimates of solutions for both time and the Mach number. Suppose that $(\sigma,\mathbf{u},\mathbf{H})$ is the local solution to the initial boundary value problem (1.10)–(1.15) in $\Omega\times(0,T)$, for $0<T<\infty$. Moreover, we assume that $1/c\le\rho=1+\epsilon\sigma\le c$ for some constant $c>1$.

2.1. L^2 Estimate

First, we obtain from the continuity equation (1.10) and the boundary condition $\mathbf{u}\cdot\mathbf{n}=0$ that

$$\int_\Omega \sigma\,\mathrm{d}x=\int_\Omega \sigma_0\,\mathrm{d}x=0.$$

Lemma 2.1. *For the solution to (1.10)–(1.15), we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}(\|\sqrt{p'(1)}\sigma\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{H}\|_{L^2}^2)+\gamma_0(\|\mathbf{u}\|_{H^1}^2+\|\mathbf{H}\|_{H^1}^2)\\&\quad\le C\|\mathbf{u}\|_{H^1}(\|\sigma\|_{H^1}^2+\|\mathbf{H}\|_{H^1}^2),\end{aligned}\tag{2.1}$$

where γ_0 and C are positive constants independent of ϵ.

Proof. Throughout this section, we denote the inner product in $L^2(\Omega)$ by

$$\langle f,g\rangle:=\int_\Omega fg\mathrm{d}x.$$

By taking $\langle(1.10),p'(1)\sigma\rangle$, we see that

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\sqrt{p'(1)}\sigma\|_{L^2}^2-\frac{p'(1)}{\epsilon}\int_\Omega \mathbf{u}\cdot\nabla\sigma\,\mathrm{d}x=-p'(1)\int_\Omega\sigma\mathrm{div}(\sigma\mathbf{u})\,\mathrm{d}x\le C\|\mathbf{u}\|_{H^1}\|\sigma\|_{H^1}^2.$$

Using (1.14) and Korn's inequality, one gets

$$\begin{aligned}&-\int_\Omega(\mathrm{div}(2\mu\mathrm{D}(\mathbf{u}))+\lambda\nabla\mathrm{div}\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\\&\quad=\int_\Omega(2\mu|\mathrm{D}(\mathbf{u}))|^2+\lambda(\mathrm{div}\mathbf{u})^2)\,\mathrm{d}x+\int_{\partial\Omega}f(\mathbf{u}\cdot\tau)^2\mathrm{d}S\ge\iota_0\|\mathbf{u}\|_{H^1}^2\end{aligned}$$

for some constant $\iota_0>0$. Thus, we take $\langle(1.11),\mathbf{u}\rangle$ to derive that

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega\mathbf{u}\cdot\nabla\sigma\mathrm{d}x+\gamma_0\|\mathbf{u}\|_{H^1}^2\\&\le\int_\Omega\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\cdot\mathbf{u}\,\mathrm{d}x+\int_\Omega\left(\mathbf{H}\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2\right)\mathbf{u}\,\mathrm{d}x\\&\le C(\|\mathbf{u}\|_{H^1}\|\sigma\|_{H^1}^2+\|\mathbf{u}\|_{H^1}\|\mathbf{H}\|_{H^1}^2)\qquad\text{for some constant }\gamma_0>0.\end{aligned}$$

To deal with the magnetic equation, we denote $\overrightarrow{\mathrm{curl}}=(\partial_2,-\partial_1)^t$. Then, the Eq. (1.12) can be written as

$$\partial_t\mathbf{H}+(\mathrm{div}\mathbf{u})\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}=-\eta\,\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\mathbf{H}.\tag{2.2}$$

Taking $\langle(2.2),\mathbf{H}\rangle$ and using (1.15), we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\|\mathbf{H}\|_{L^2}^2+\eta\|\mathrm{curl}\mathbf{H}\|_{L^2}^2&=\int_\Omega[(\mathrm{div}\mathbf{u})\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}]\mathbf{H}\,\mathrm{d}x\\&\le C\|u\|_{H^1}\|\mathbf{H}\|_{H^1}^2.\end{aligned}$$

Putting the above estimates together and keeping in mind that

$$\|\mathbf{F}\|_{H^1} \le C\|\nabla \mathbf{F}\|_{L^2} \le C(\|\mathrm{div}\mathbf{F}\|_{L^2} + \|\mathrm{curl}\mathbf{F}\|_{L^2}), \tag{2.3}$$

for any vector $\mathbf{F} \in H^1(\Omega)$ with $\mathbf{F}\cdot\mathbf{n} = 0$, we obtain the estimate (2.1). □

2.2. Estimates of first-order derivatives

Now, we derive the estimates of the first-order temporal and spatial derivatives of $(\sigma, \mathbf{u}, \mathbf{H})$, based on their L^2 estimates.

Lemma 2.2. *For the solution to* (1.10)–(1.15)*, we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\left[2\mu\|D(\mathbf{u})\|_{L^2}^2 + \lambda\|\mathrm{div}\mathbf{u}\|_{L^2}^2 + \eta\|\mathrm{curl}\,\mathbf{H}\|_{L^2}^2\right] \\
&\quad + \frac{d}{\mathrm{d}t}\int_\Omega \rho\mathbf{u}_t\mathbf{u}\,\mathrm{d}x + \|\sqrt{p'(1)}\sigma_t\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2 \\
&\le C(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}_t\|_{H^1}\|\mathbf{u}\|_{H^1}^2 + \|\sigma_t\|_{H^1}\|\sigma\|_{H^1}\|\mathbf{u}\|_{H^1} + \|\mathbf{H}_t\|_{H^1}\|\mathbf{H}\|_{H^1}\|\mathbf{u}\|_{H^1}).
\end{aligned}$$

Proof. First, differentiating (1.11) with respect to t and multiplying the resulting equation by $\mathbf{u}$ in L^2, integrating by parts and using the boundary conditions (1.14), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\left[(2\mu\|D(\mathbf{u})\|_{L^2}^2 + \lambda\|\mathrm{div}\mathbf{u}\|_{L^2}^2) + \int_{\partial\Omega} f(\mathbf{u}\cdot\tau)^2\,\mathrm{d}S\right] \\
&\quad + \frac{d}{\mathrm{d}t}\int_\Omega \rho\mathbf{u}_t\mathbf{u}\,\mathrm{d}x + \frac{p'(1)}{\epsilon}\int_\Omega \nabla\sigma_t\cdot\mathbf{u}\,\mathrm{d}x \\
&= \int_\Omega \left[\frac{p'(1) - p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\right]_t \cdot\mathbf{u}\,\mathrm{d}x + \int_\Omega (\rho\mathbf{u}_t^2 - \rho(\mathbf{u}_t\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_t))\cdot\mathbf{u}\,\mathrm{d}x \\
&\quad + \int_\Omega (\mathbf{H}\cdot\nabla\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2)_t\cdot\mathbf{u}\,\mathrm{d}x \\
&\le C(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}\|\mathbf{u}\|_{H^1}^2 + \|\sigma_t\|_{H^1}\|\sigma\|_{H^1}\|\mathbf{u}\|_{H^1} + \|\mathbf{H}_t\|_{H^1}\|\mathbf{H}\|_{H^1}\|\mathbf{u}\|_{H^1}).
\end{aligned}$$

We apply $\langle(1.10), p'(1)\sigma_t\rangle$ and $\langle(2.2), \mathbf{H}_t\rangle$ to infer that

$$\|\sqrt{p'(1)\sigma_t}\|_{L^2}^2 + \frac{p'(1)}{\epsilon}\int_\Omega \sigma_t\mathrm{div}\mathbf{u}\,\mathrm{d}x \le C\|\sigma_t\|_{H^1}\|\sigma\|_{H^1}\|\mathbf{u}\|_{H^1}$$

and

$$\begin{aligned}
\frac{\eta}{2}\frac{d}{\mathrm{d}t}\|\mathrm{curl}\mathbf{H}\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2 &= \int_\Omega (\mathbf{H}\mathrm{div}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{u})\mathbf{H}_t\mathrm{d}x \\
&\le C\|\mathbf{u}\|_{H^1}\|\mathbf{H}\|_{H^1}\|\mathbf{H}_t\|_{H^1},
\end{aligned}$$

respectively. Summing up the above estimates and using the boundary conditions (1.14) again, we obtain the lemma. □

Lemma 2.3. *For the solution to* (1.10)–(1.15)*, we have*

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\|\nabla\sigma\|_{L^2}^2 + \gamma_1\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 \le \delta\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2 \\
&\quad + C_\delta(\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^4 + \|\mathbf{H}\|_{H^2}^4), \quad 0 < \delta < 1,
\end{aligned}$$

where γ_1 is a positive constants independent of ϵ.

Proof. We take $\langle \nabla(1.10), \nabla\sigma\rangle$ to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla\sigma\|_{L^2}^2 + \frac{1}{\epsilon}\int\limits_{\Omega} \nabla \mathrm{div}\mathbf{u}\cdot\nabla\sigma \mathrm{d}x \\ = -\int\limits_{\Omega}((\mathbf{u}\cdot\nabla)\nabla\sigma + \nabla\mathbf{u}\nabla\sigma + \nabla\sigma\mathrm{div}\mathbf{u} + \sigma\nabla\mathrm{div}\mathbf{u})\nabla\sigma\mathrm{d}x \\ \le C(\|\mathbf{u}\|_{H^1}\|\sigma\|_{H^2}^2 + \|\nabla\mathrm{div}\mathbf{u}\|_{L^2}\|\sigma\|_{H^2}^2).\end{aligned}$$

To bound $\|\nabla\mathrm{div}\mathbf{u}\|_{L^2(0,t;L^2)}$, we rewrite the Eq. (1.11) as

$$\begin{aligned}\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \frac{1}{\epsilon}P'(1+\epsilon\sigma)\nabla\sigma \\ = (2\mu+\lambda)\nabla\mathrm{div}\mathbf{u} - \mu\overrightarrow{\mathrm{curl}}\mathrm{curl}\mathbf{u} + (\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2.\end{aligned} \tag{2.4}$$

Now, we apply $\langle(2.4), p'(1)^{-1}\nabla\mathrm{div}\mathbf{u}\rangle$ to derive that

$$\begin{aligned}(2\mu+\lambda)\|\sqrt{p'(1)^{-1}}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 - \frac{1}{\epsilon}\int\limits_{\Omega}\nabla\mathrm{div}\mathbf{u}\cdot\nabla\sigma\mathrm{d}x \\ = \frac{1}{p'(1)}\int\limits_{\Omega}\rho[\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - (\mathbf{H}\cdot\nabla\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2)]\cdot\nabla\mathrm{div}\mathbf{u}\,\mathrm{d}x \\ + \frac{1}{p'(1)}\int\limits_{\Omega}\frac{p'(1+\epsilon\sigma)-p'(1)}{\epsilon}\nabla\sigma\cdot\nabla\mathrm{div}\mathbf{u}\,\mathrm{d}x + p'(1)^{-1}\mu\int\limits_{\Omega}\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u}\nabla\mathrm{div}\mathbf{u}\,\mathrm{d}x.\end{aligned}$$

Differentiating $(1.14)_1$ with respect to the length parameter (see [5]), we obtain

$$\mathrm{curl}\,\mathbf{u} = \left(2\chi - \frac{f}{\mu}\right)\mathbf{u}\cdot\tau \quad \text{on } \partial\Omega, \tag{2.5}$$

where χ is the curvature of $\partial\Omega$. In view of (2.5) and the trace theorem, we find that

$$\begin{aligned}\int\limits_{\Omega}\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u}\nabla\mathrm{div}\mathbf{u}\,\mathrm{d}x &= \int\limits_{\partial\Omega}\mathrm{curl}\,\mathbf{u}\nabla\mathrm{div}\mathbf{u}\cdot\tau\,\mathrm{d}S \\ &= \int\limits_{\partial\Omega}\left(2\chi - \frac{f}{\mu}\right)(\mathbf{u}\cdot\tau)(\nabla\mathrm{div}\mathbf{u}\cdot\tau)\,dS \\ &\le \delta\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2 + C_\delta\|\mathbf{u}\|_{H^1}^2\end{aligned}$$

for some small number $\delta > 0$. Therefore,

$$\begin{aligned}(2\mu+\lambda)\|\sqrt{p'(1)^{-1}}\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2 - \frac{1}{\epsilon}\int\limits_{\Omega}\nabla\mathrm{div}\mathbf{u}\cdot\nabla\sigma\mathrm{d}x \\ \le \delta\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2 + C_\delta\|\mathbf{u}\|_{H^1}^2 + C\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}(\|\mathbf{u}_t\|_{L^2} + \|\mathbf{u}\|_{H^2}\|\mathbf{u}\|_{H^1} \\ + \|\sigma\|_{H^2}^2 + \|\mathbf{H}\|_{H^2}\|\mathbf{H}\|_{H^1}).\end{aligned} \tag{2.6}$$

Putting the above estimates together and using Young's inequality, we obtain the lemma. □

Lemma 2.4. *For the solution to* (1.10)–(1.15), *we have*

$$\begin{aligned}&\frac{d}{\mathrm{d}t}(\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{H}_t\|_{L^2}^2)+\gamma_2(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^2)\\&\le C_\delta(\|\sigma_t\|_{H^1}^2(\|\mathbf{u}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^4+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{H^1}^2)\\&\quad+C_\delta(\|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^4)+C\epsilon^2\|\sigma_t\|_{L^2}^2+\delta\|\mathbf{u}\|_{H^1}^2,\end{aligned}\tag{2.7}$$

where $0<\delta<1$, *and* γ_2 *is a positive constant independent of* ϵ.

Proof. Taking $\langle\partial_t(1.10),p'(1)\sigma_t\rangle$, we get

$$\begin{aligned}&\frac12\frac{d}{\mathrm{d}t}\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega\sigma_t\mathrm{div}\mathbf{u}_t\mathrm{d}x\\&=-p'(1)\int_\Omega(\mathbf{u}\cdot\nabla\sigma_t+\mathbf{u}_t\cdot\nabla\sigma+\sigma_t\mathrm{div}\mathbf{u}+\sigma\mathrm{div}\mathbf{u}_t)\sigma_t\mathrm{d}x\\&\le\delta(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2)+C_\delta(\|\sigma\|_{H^1}^4+\|\sigma_t\|_{H^1}^4),\end{aligned}$$

while taking $\langle(1.11)_t,\mathbf{u}_t\rangle$ and using the boundary conditions (1.14), we find that

$$\begin{aligned}&\frac12\frac{d}{\mathrm{d}t}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+2\mu\|D(\mathbf{u}_t)\|_{L^2}^2+\lambda\|\mathrm{div}\mathbf{u}_t\|_{L^2}^2\\&\quad+\int_{\partial\Omega}f(\mathbf{u}_t\cdot\tau)^2\,dS+\frac{p'(1)}{\epsilon}\int_\Omega\nabla\sigma_t\cdot\mathbf{u}_t\mathrm{d}x\\&=\int_\Omega\left[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\right]_t\cdot\mathbf{u}_t-\int_\Omega[\rho_t\mathbf{u}_t+\epsilon\sigma_t\mathbf{u}\cdot\nabla\mathbf{u}+\rho(\mathbf{u}\cdot\nabla\mathbf{u})_t\\&\quad+\frac12\nabla(|\mathbf{H}|^2)_t-H_t\cdot\nabla H-H\cdot\nabla H_t]\cdot\mathbf{u}_t\mathrm{d}x\\&\le\delta\|\mathbf{u}_t\|_{H^1}^2+C_\delta\left(\|\sigma_t\|_{H^1}^2(\|(\mathbf{u}_t,\nabla\sigma)\|_{L^2}^2+\|\mathbf{u}\|_{H^1}^4)+\|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^1}^2\right.\\&\quad\left.+\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^1}^2\right)+C\epsilon^2\|\sigma_t\|_{L^2}^2.\end{aligned}$$

If we differentiate (2.2) with respect to t, we obtain

$$\begin{aligned}&\mathbf{H}_{tt}+\mathrm{div}\mathbf{u}_t\mathbf{H}+\mathrm{div}\mathbf{u}\mathbf{H}_t+\mathbf{u}_t\cdot\nabla\mathbf{H}+\mathbf{u}\cdot\nabla\mathbf{H}_t-\mathbf{H}_t\cdot\nabla\mathbf{u}-\mathbf{H}\cdot\nabla\mathbf{u}_t\\&\quad=-\eta\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{H}_t.\end{aligned}\tag{2.8}$$

And from taking $\langle(2.8),\mathbf{H}_t\rangle$ and using the boundary condition (1.15), we get

$$\begin{aligned}&\frac12\frac{d}{\mathrm{d}t}\|\mathbf{H}_t\|_{L^2}^2+\eta\|\mathrm{curl}\mathbf{H}_t\|_{L^2}^2\\&=-\int_\Omega(\mathrm{div}\mathbf{u}_t\mathbf{H}+\mathrm{div}\mathbf{u}\mathbf{H}_t+\mathbf{u}_t\cdot\nabla\mathbf{H}+\mathbf{u}\cdot\nabla\mathbf{H}_t-\mathbf{H}_t\cdot\nabla\mathbf{u}-\mathbf{H}\cdot\nabla\mathbf{u}_t)\cdot\mathbf{H}_t\mathrm{d}x\\&\le\delta(\|\mathbf{u}_t\|_{H^1}^2+\|\nabla\mathbf{u}\|_{L^2}^2)+C_\delta(\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^4).\end{aligned}$$

Hence, by choosing δ appropriately small, we obtain the estimate (2.7). □

Next, we estimate the vorticity of the velocity and magnetic fields, which are denoted by ω and φ, respectively. By virtue of (1.10) and (1.11), it is easy to see that ω and φ satisfy the following systems

$$\rho\omega_t+\rho\mathbf{u}\cdot\nabla\omega-\mu\triangle\omega=g,\tag{2.9}$$

$$\omega=\left(2\chi-\frac{f}{\mu}\right)\mathbf{u}\cdot\tau\quad\text{on }\partial\Omega,\tag{2.10}$$

and

$$\partial_t\varphi - \eta\triangle\varphi = \mathrm{curl}(\mathrm{div}\mathbf{u}\mathbf{H} + \mathbf{u}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{u}), \tag{2.11}$$

$$\varphi = 0 \quad \text{on } \partial\Omega, \tag{2.12}$$

where

$$g = -\rho\omega\mathrm{div}\mathbf{u} - \frac{\epsilon}{\rho}\nabla\sigma\times(\mu\triangle\mathbf{u} + (\mu+\lambda)\nabla\mathrm{div}\mathbf{u}) - \frac{\epsilon}{\rho}\nabla\sigma\times(\mathbf{H}\cdot\nabla\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2) + \mathrm{curl}(\mathbf{H}\cdot\nabla\mathbf{H}).$$

Then, we have

Lemma 2.5.

$$\begin{aligned}&\frac{d}{\mathrm{d}t}\|(\sqrt{\rho}\omega,\varphi)\|_{L^2}^2 + \mu\|\nabla\omega\|_{L^2}^2 + \eta\|\nabla\varphi\|_{L^2}^2\\ &\quad\le \delta\|\omega\|_{H^2}^2 + C_\delta\|\mathbf{u}\|_{H^1}^2 + C_\delta\|\mathbf{u}\|_{H^2}^2(\|\mathbf{u}\|_{H^1}^2 + \epsilon^2\|\sigma\|_{H^2}^2)\\ &\quad + C(\|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\epsilon\sigma\|_{H^2}^2 + \|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2), \qquad 0<\delta<1.\end{aligned} \tag{2.13}$$

Proof. Multiplying (2.9) by ω, and using the boundary condition (2.10), we infer that

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\sqrt{\rho}\omega\|_{L^2}^2 + \mu\|\nabla\omega\|_{L^2}^2 = \int_\Omega g\omega\,\mathrm{d}x + \int_{\partial\Omega}\omega\nabla\omega\cdot\mathbf{n}\,dS. \tag{2.14}$$

It is easy to verify that

$$\int_\Omega g\omega\,\mathrm{d}x \le \delta\|\nabla\omega\|_{L^2}^2 + C_\delta(\|\mathbf{u}\|_{H^2}^2(\|\mathbf{u}\|_{H^1}^2 + \epsilon^2\|\sigma\|_{H^2}^2) + \|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\epsilon\sigma\|_{H^2}^2 + \|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2).$$

Employing (2.10) and the trace theorem, we arrive at

$$\int_{\partial\Omega}\omega\nabla\omega\cdot\mathbf{n}\,dS = \int_{\partial\Omega}\left[\left(2\chi - \frac{f}{\mu}\right)\mathbf{u}\cdot\tau\right]\nabla\omega\cdot\mathbf{n}\,dS \le \delta\|\omega\|_{H^2}^2 + C_\delta\|\mathbf{u}\|_{H^1}^2.$$

Inserting the above two inequalities into (2.14), we conclude that

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\|\sqrt{\rho}\omega\|_{L^2}^2 + \mu\|\nabla\omega\|_{L^2}^2 &\le \delta\|\omega\|_{H^2}^2 + C_\delta\|\mathbf{u}\|_{H^1}^2 + C_\delta(\|\mathbf{u}\|_{H^2}^2(\|\mathbf{u}\|_{H^1}^2\\ &\quad + \epsilon^2\|\sigma\|_{H^2}^2) + \|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\epsilon\sigma\|_{H^2}^2 + \|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2).\end{aligned} \tag{2.15}$$

Taking $\langle(2.11),\varphi\rangle$, one finds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\varphi\|_{L^2}^2 + \eta\|\nabla\varphi\|_{L^2}^2\\ &\quad\le C\int_\Omega(\nabla\mathrm{div}\mathbf{u}\mathbf{H} + \mathrm{div}\mathbf{u}\varphi + \mathbf{u}\cdot\nabla\varphi + \nabla(\mathbf{u}\cdot\nabla)\varphi + \mathbf{H}\cdot\nabla\omega + \nabla(\mathbf{H}\cdot\nabla)\mathbf{u})\varphi\,\mathrm{d}x\\ &\quad\le \delta(\|\nabla\omega\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2) + C_\delta\|\mathbf{u}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2.\end{aligned} \tag{2.16}$$

Adding (2.15) to (2.16) and choosing δ appropriately small, we obtain (2.13). □

Now, defining two functions

$$\Psi_1(t) := \int_\Omega\rho\mathbf{u}\mathbf{u}_t\,\mathrm{d}x + \|(\nabla\sigma,\sigma_t,\mathbf{u}_t,\mathbf{H}_t)\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\mathrm{curl}\,\mathbf{H}\|_{L^2}^2,$$

$$\Phi_1(t) := \|\sigma_t\|_{L^2}^2 + \|(\nabla\mathrm{curl}\,\mathbf{u},\nabla\mathrm{div}\mathbf{u},\nabla\mathrm{curl}\,\mathbf{H})\|_{L^2}^2 + \|(\mathbf{u}_t,\mathbf{H}_t)\|_{H^1}^2,$$

we conclude from Lemmas 2.2–2.5 that for small ϵ, there are a positive constant M_1 and a sufficiently small constant δ, such that

$$\frac{d}{dt}\Psi_1(t)+\Phi_1(t)\le M_1(\|\sigma_t\|_{H^1}^2(\|(\sigma,\mathbf{u})\|_{H^2}^2+\|\mathbf{u}\|_{H^2}^4+\|\sigma_t\|_{H^1}^2+\|\mathbf{u}_t\|_{L^2}^2)+\|\sigma\|_{H^2}^4 +\|\mathbf{u}\|_{H^2}^2(\|\mathbf{u}_t\|_{H^1}^2+\|(\sigma,\mathbf{u})\|_{H^2}^2)+\|\mathbf{u}_t\|_{L^2}^2)+M_1(\|\mathbf{H}_t\|_{H^1}^2(\|\mathbf{H}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^2) +\|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2(1+\|\epsilon\sigma\|_{H^2}^2))+\delta(\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2+\|\omega\|_{H^2}^2). \tag{2.17}$$

2.3. Boundedness of second-order derivatives

First, we show the following lemma.

Lemma 2.6. *For the solution to (1.10)–(1.15), we have*

$$(2\mu+\lambda)\frac{d}{dt}\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2-2\frac{d}{dt}\int_\Omega\rho\mathbf{u}_t\cdot\nabla\mathrm{div}\mathbf{u}dx+\|\sqrt{p'(1)}\nabla\sigma_t\|_{L^2}^2 \le\delta(\|\nabla\mathrm{div}\mathbf{u}_t\|_{L^2}^2+\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2)+C_\delta(\|(\mathbf{u}_t,\nabla\mathrm{div}\mathbf{u})\|_{L^2}^2+\|\sigma_t\|_{H^1}^2\|\sigma\|_{H^2}^2 +\|\mathbf{u}\|_{H^2}^2(\epsilon^2\|\sigma_t\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|\mathbf{u}_t\|_{H^1}^2+\|\sigma\|_{H^2}^2)+\|\mathbf{H}\|_{H^2}^2\|\mathbf{H}_t\|_{H^1}^2), \tag{2.18}$$

where $\delta>0$ is a small constant.

Proof. Differentiating (2.4) with respect to t, multiplying then by $\nabla\mathrm{div}\mathbf{u}$ in L^2, we arrive at

$$\frac{2\mu+\lambda}{2}\frac{d}{dt}\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2-\frac{d}{dt}\int_\Omega\rho\mathbf{u}_t\cdot\nabla\mathrm{div}\mathbf{u}dx-\frac{p'(1)}{\epsilon}\int_\Omega\nabla\sigma_t\cdot\nabla\mathrm{div}\mathbf{u}dx$$
$$=\int_\Omega\left[\left(\frac{p'(1+\epsilon\sigma)-p'(1)}{\epsilon}\nabla\sigma\right)_t+\epsilon\sigma_t\mathbf{u}\cdot\nabla\mathbf{u}+\rho(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t)\right.$$
$$\left.+\left(\frac{1}{2}(\nabla|\mathbf{H}|^2)_t-\mathbf{H}_t\cdot\nabla\mathbf{H}-\mathbf{H}\cdot\nabla\mathbf{H}_t\right)\right]\cdot\nabla\mathrm{div}\mathbf{u}dx-\int_\Omega\rho\mathbf{u}_t\cdot\nabla\mathrm{div}\mathbf{u}_tdx$$
$$+\mu\int_\Omega\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u}_t\nabla\mathrm{div}\mathbf{u}\,dx$$
$$\le\delta(\|(\nabla\mathrm{div}\mathbf{u}_t,\nabla^2\mathrm{div}\mathbf{u},\nabla\sigma_t)\|_{L^2}^2)+C_\delta(\|\mathbf{u}_t\|_{\mathbf{H}^1}^2+\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\|\sigma_t\|_{\mathbf{H}^1}^2\|\sigma\|_{H^2}^2 +\|\mathbf{u}\|_{H^2}^2(\epsilon^2\|\sigma_t\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|\mathbf{u}_t\|_{H^1}^2)+\|\mathbf{H}\|_{H^2}^2\|\mathbf{H}_t\|_{H^1}^2),$$

where we have used the estimate

$$\int_\Omega\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u}_t\nabla\mathrm{div}\mathbf{u}\,dx=\int_{\partial\Omega}\left(2\chi-\frac{f}{\mu}\right)(\mathbf{u}_t\cdot\tau)(\mathbf{n}\times\nabla\mathrm{div}\mathbf{u})\,dS \le\delta\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2+C_\delta\|\mathbf{u}_t\|_{H^1}^2.$$

Similarly, we take $\langle\nabla(1.10),p'(1)\nabla\sigma_t\rangle$ to infer that

$$\|\sqrt{p'(1)}\nabla\sigma_t\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega\nabla\sigma_t\cdot\nabla\mathrm{div}\mathbf{u}dx$$
$$=-p'(1)\int_\Omega((\mathbf{u}\cdot\nabla)\nabla\sigma+\nabla\mathbf{u}\nabla\sigma+\nabla\sigma\mathrm{div}\mathbf{u}+\sigma\nabla\mathrm{div}\mathbf{u})\cdot\nabla\sigma_tdx$$
$$\le\delta\|\nabla\sigma_t\|_{L^2}^2+C_\delta\|\mathbf{u}\|_{H^2}^2\|\sigma\|_{H^2}^2.$$

Putting the above inequalities together and choosing δ small, we get the estimate (2.18). □

Lemma 2.7. *We have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla^2\sigma\|_{L^2}^2+(2\mu+\lambda)\|\sqrt{p'(\rho)^{-1}}\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2\\&\quad\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\nabla^2\mathrm{curl}\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{H^1}^2)\\&\qquad+C_\delta(\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^2}^4+\|\sigma\|_{H^2}^2(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4)+\|\mathbf{H}\|_{H^2}^4),\quad 0<\delta<1.\end{aligned}\tag{2.19}$$

Proof. First, we differentiate (1.10) twice with respect to x to have

$$\nabla^2\sigma_t+\mathbf{u}\cdot\nabla(\nabla^2\sigma)+2\nabla\mathbf{u}\cdot\nabla(\nabla\sigma)+\nabla^2\mathbf{u}\cdot\nabla\sigma+\nabla^2(\sigma\mathrm{div}\mathbf{u})+\frac{1}{\epsilon}\nabla^2\mathrm{div}\mathbf{u}=0.\tag{2.20}$$

If one takes $\langle(2.20),\nabla^2\sigma\rangle$, one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla^2\sigma\|_{L^2}^2+\frac{1}{\epsilon}\int\limits_\Omega\nabla^2\mathrm{div}\mathbf{u}\nabla^2\sigma\mathrm{d}x\\&\quad=-\int\limits_\Omega[(\mathbf{u}\cdot\nabla(\nabla^2\sigma)+2\nabla\mathbf{u}\cdot\nabla(\nabla\sigma)+\nabla^2\mathbf{u}\cdot\nabla\sigma+\nabla^2(\sigma\mathrm{div}\mathbf{u}))]\nabla^2\sigma\mathrm{d}x\\&\quad\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\sigma\|_{H^2}^4.\end{aligned}$$

Then, we differentiate (2.4) with respect to x to get

$$\begin{aligned}&(2\mu+\lambda)\nabla^2\mathrm{div}\mathbf{u}-\mu\nabla\overrightarrow{\mathrm{curl}}\mathrm{curl}\mathbf{u}-\frac{1}{\epsilon}p'(1)\nabla^2\sigma\\&=\nabla\left[\frac{p'(1+\epsilon\sigma)-p'(1)}{\epsilon}\nabla\sigma\right]+\rho(\nabla\mathbf{u}_t+\nabla\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla^2\mathbf{u})+\epsilon\nabla\sigma(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})\\&\quad+\nabla\mathbf{H}\cdot\nabla\mathbf{H}+\mathbf{H}\cdot\nabla(\nabla\mathbf{H})-\frac{1}{2}\nabla^2|\mathbf{H}|^2,\end{aligned}$$

which, by multiplying $p'(1)^{-1}\nabla^2\mathrm{div}\mathbf{u}$ in L^2, gives

$$\begin{aligned}&(2\mu+\lambda)\|\sqrt{p'(1)^{-1}}\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2-\frac{1}{\epsilon}\int\limits_\Omega\nabla^2\mathrm{div}\mathbf{u}\nabla^2\sigma\mathrm{d}x\\&\quad\le\delta\|\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2+C_\delta\left[\|\nabla^2\mathrm{curl}\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4)\right.\\&\qquad\left.+C_\delta(\|\sigma\|_{H^2}^4+\|\sigma\|_{H^2}^2(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4)+\|\mathbf{H}\|_{H^2}^4\right],\end{aligned}$$

where we have used the fact that $\|\nabla\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{u}\|_{L^2}\le\|\nabla^2\mathrm{curl}\mathbf{u}\|_{L^2}$.

Putting the above two inequalities together and choosing δ suitably small, we get the estimate (2.19). □

Now, we differentiate (2.4) with respect to t and divide the resulting equations by ρ to arrive at

$$\begin{aligned}&\mathbf{u}_{tt}-\rho^{-1}((2\mu+\lambda)\nabla\mathrm{div}\mathbf{u}_t-\mu\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\mathbf{u}_t)+\frac{1}{\epsilon}\rho^{-1}p'(1)\nabla\sigma_t\\&=\rho^{-1}\left[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\right]_t-\epsilon\rho^{-1}\sigma_t(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})-\mathbf{u}_t\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{u}_t\\&\quad+\rho^{-1}\left((\mathbf{H}_t\cdot\nabla)\mathbf{H}+\mathbf{H}\cdot\nabla\mathbf{H}_t-\frac{1}{2}\nabla(|\mathbf{H}|^2)_t\right).\end{aligned}\tag{2.21}$$

Moreover, we take $\partial_t\nabla(1.10)$ to get

$$\begin{aligned}&(\nabla\sigma_t)_t+\mathbf{u}\cdot\nabla^2\sigma_t+\nabla\mathbf{u}\nabla\sigma_t+\nabla^2\sigma\mathbf{u}_t+\nabla\mathbf{u}_t\nabla\sigma+\nabla\sigma\mathrm{div}\mathbf{u}_t\\&\quad+\sigma\nabla\mathrm{div}\mathbf{u}_t+\nabla\sigma_t\mathrm{div}\mathbf{u}+\sigma_t\nabla\mathrm{div}\mathbf{u}+\frac{1}{\epsilon}\nabla\mathrm{div}\mathbf{u}_t=0.\end{aligned}\tag{2.22}$$

It is to see that $\mathbf{u}_{tt}\cdot\mathrm{n}|_{\partial\Omega}=0$, thus it holds that

$$\int_\Omega \mathbf{u}_{tt}\nabla\mathrm{div}\mathbf{u}_t\,\mathrm{d}x=-\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega |\mathrm{div}\mathbf{u}_t|^2\,\mathrm{d}x.$$

Finally, we take $\langle(2.21),\nabla\mathrm{div}\mathbf{u}_t\rangle+\langle(2.22),\rho^{-1}p'(1)\nabla\sigma_t\rangle$ to obtain the following lemma.

Lemma 2.8. *For the solution to* (1.10)–(1.15)*, we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}(\|\mathrm{div}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho^{-1}}\nabla\sigma_t\|_{L^2}^2)+(2\mu+\lambda)\|\sqrt{\rho^{-1}}\nabla\mathrm{div}\mathbf{u}_t\|_{L^2}^2\\&\quad\le\delta(\|\nabla\mathrm{div}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^3}^2)+C_\delta\|\nabla\mathrm{curl}\,\mathbf{u}_t\|_{L^2}^2+C_\delta\left(\|\sigma\|_{H^2}^2\|\mathrm{curl}\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2\right.\\&\qquad\left.+\|\sigma_t\|_{H^1}^2(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{H^1}^2)+\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2\right),\quad 0<\delta<1.\end{aligned}$$

Next, we estimate the derivatives of curl**u** and curl**H**. To this end, we apply ∂_t to (2.9) to see that

$$\rho\omega_{tt}+\rho\mathbf{u}\cdot\nabla\omega_t-\mu\triangle\omega_t=h,\tag{2.23}$$

where

$$h:=-\epsilon\sigma_t(\omega_t+\mathbf{u}\cdot\nabla\omega)-\rho\mathbf{u}_t\nabla\omega+g_t.$$

Furthermore, the boundary condition for (2.23) reads as

$$\omega_t=\left(2\chi-\frac{f}{\mu}\right)\mathbf{u}_t\cdot\tau\quad\text{on }\ \partial\Omega.\tag{2.24}$$

Therefore, by virtue of (2.24) and the trace theorem,

$$\begin{aligned}\int_\Omega\Delta\omega_t\omega_t\,\mathrm{d}x&=-\int_\Omega|\nabla\omega_t|^2\,\mathrm{d}x+\int_{\partial\Omega}\omega_t\nabla\omega_t\cdot\mathbf{n}\,\mathrm{d}S\\&\le-\int_\Omega|\nabla\omega_t|^2\,\mathrm{d}x+C\|\mathbf{u}_t\|_{H^2}^2.\end{aligned}$$

Multiplying (2.23) by ω_t in L^2, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\sqrt{\rho}\omega_t\|_{L^2}^2+\mu\|\nabla\omega_t\|_{L^2}^2\\&\quad\le C\|\mathbf{u}_t\|_{H^2}^2+C_\delta(\epsilon^2\|\sigma_t\|_{H^1}^2(\|\omega_t\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^2\|\nabla\omega\|_{L^2}^2)+\|\mathbf{u}_t\|_{H^1}^2\|\nabla\omega\|_{L^2}^2\\&\qquad+\epsilon^2\|\sigma_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{H^1}^4+\|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\epsilon^4\|\sigma_t\|_{H^1}^2\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^2}^2\\&\qquad+\epsilon^2(\|\sigma_t\|_{H^1}^2\|\nabla^2\mathbf{u}\|_{H^1}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^2}^2))+\delta\|\nabla\omega_t\|_{L^2}^2\\&\qquad+\int_\Omega\left(-\frac{\epsilon}{\rho}\nabla\sigma\times(\mathbf{H}\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2)+\mathrm{curl}(\mathbf{H}\cdot\nabla\mathbf{H})\right)_t\nabla\omega_t\mathrm{d}x,\end{aligned}\tag{2.25}$$

where the last term on the right-hand side can be bounded as follows:

$$\begin{aligned}&\int\limits_{\Omega}\left(-\frac{\epsilon}{\rho}\nabla\sigma\times(\mathbf{H}\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2)+\mathrm{curl}(\mathbf{H}\cdot\nabla\mathbf{H})\right)_t\nabla\omega_t\mathrm{d}x\\&=\int\limits_{\Omega}\left\{\frac{\epsilon^2\sigma_t}{\rho^2}(\nabla\sigma\times(\mathbf{H}\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2))-\frac{\epsilon}{\rho}(\nabla\sigma\times(H\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2))_t\right.\\&\quad\left.+\mathrm{curl}(\mathbf{H}\cdot\nabla\mathbf{H})_t\right\}\nabla\omega_t\mathrm{d}x\\&\le C_\delta(\epsilon^4\|\sigma_t\|_{H^1}^2\|\sigma\|_{H^2}^2\|\mathbf{H}\|_{H^2}^4+\epsilon^2(\|\sigma_t\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^3}^2\\&\quad+\|\sigma\|_{H^2}^2\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}_t\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\sigma\|_{H^2}^2)\\&\quad+\|\mathbf{H}_t\|_{H^1}^2\|\varphi\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^2\|\varphi_t\|_{H^1}^2+\|\mathbf{H}_t\|_{H^2}^2\|\mathbf{H}\|_{H^2}^2)+\delta\|\nabla\omega_t\|_{L^2}^2.\end{aligned}$$

Substituting the above inequality into (2.25) and taking δ small enough, we obtain the following estimate.

Lemma 2.9.

$$\begin{aligned}&\frac{d}{\mathrm{d}t}\|\sqrt{\rho}\omega_t\|_{L^2}^2+\frac{\mu}{2}\|\nabla\omega_t\|_{L^2}^2\\&\le C\|\mathbf{u}_t\|_{H^2}^2+C\left(\|\sigma_t\|_{H^1}^2(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4+\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^2}^2)\right.\\&\quad\left.+\|\mathbf{u}_t\|_{H^2}^2(\|\mathbf{u}\|_{H^2}^2+\|\sigma\|_{H^2}^2)\right)+C\left(\epsilon^4\|\sigma_t\|_{H^1}^2\|\sigma\|_{H^2}^2\|\mathbf{H}\|_{H^2}^4\right.\\&\quad+\epsilon^2(\|\sigma_t\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2\|\mathbf{H}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}_t\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\sigma\|_{H^2}^2)\\&\quad\left.+\|\mathbf{H}_t\|_{H^1}^2\|\varphi\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^2\|\varphi_t\|_{H^1}^2+\|\mathbf{H}_t\|_{H^2}^2\|\mathbf{H}\|_{H^2}^2\right).\end{aligned}$$

By applying ∂_t to (2.11), we get that

$$\partial_t\varphi_t-\nu\triangle\varphi_t=\mathrm{curl}(\mathrm{div}\mathbf{u}\mathbf{H}+\mathbf{u}\cdot\nabla\mathbf{H}-\mathbf{H}\cdot\nabla\mathbf{u})_t \tag{2.26}$$

with $\varphi_t=0$ on $\partial\Omega$.

If we multiply (2.26) by φ_t in L^2 and integrate by parts, we get the following result.

Lemma 2.10. *We have*

$$\begin{aligned}&\frac{d}{\mathrm{d}t}\|\varphi_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla\varphi_t\|_{L^2}^2\\&\le\delta(\|\varphi_t\|_{H^1}^2+\|\omega_t\|_{H^1}^2)+C_\delta\left(\|\mathbf{u}_t\|_{H^2}^2(\|\mathbf{H}\|_{H^1}^2+\|\varphi\|_{L^2}^2+\|\varphi\|_{H^1}^2)\right.\\&\quad+\|\mathbf{u}\|_{H^2}^2(\|\mathbf{H}_t\|_{H^1}^2+\|\varphi_t\|_{L^2}^2+\|\varphi_t\|_{H^1}^2)+\|\mathbf{H}_t\|_{H^1}^2(\|\omega\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^2)\\&\quad\left.+\|\mathbf{H}\|_{H^1}^2(\|\varphi_t\|_{H^1}^2+\|\mathbf{u}_t\|_{H^2}^2)\right),\quad 0<\delta<1.\end{aligned}$$

On the other hand, we take $\langle(2.9),\omega_t-\eta_1\triangle\omega\rangle$ (in which η_1 is a positive constant to be chosen later) to get that

$$\begin{aligned}&\frac{(\mu+\eta_1)}{2}\frac{d}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+\|\sqrt{\rho}\omega_t\|_{L^2}^2+\mu\eta_1\|\triangle\omega\|_{L^2}^2\\&\le\delta(\|\sqrt{\rho}\omega_t\|_{L^2}^2+\|\omega\|_{H^2}^2)+C_\delta(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}\|_{H^2}^2\|\nabla\omega\|_{L^2}^2)\\&\quad+\eta_1^2\|\rho\|_{L^\infty}^2\|\triangle\omega\|_{L^2}^2+\eta_1^2\|\triangle\omega\|_{L^2}^2+C(\|\mathbf{u}\|_{H^2}^2\|\nabla\omega\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^4+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2)\\&\quad+C(\|\sigma\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4),\end{aligned}$$

where we have used the following estimate on the magnetic field $\mathbf{H}$

$$\begin{aligned}&\int_\Omega \left(-\frac{\epsilon}{\rho}\nabla\sigma\times(\mathbf{H}\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla\mathbf{H}^2)+\mathrm{curl}(\mathbf{H}\cdot\nabla\mathbf{H})\right)(w_t-\eta_1\triangle)w\mathrm{d}x\\&\quad\le\delta\|\sqrt{\rho}w_t\|_{L^2}^2+C_\delta(\|\sigma\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4)\\&\qquad+\eta_1^2\|\triangle w\|_{L^2}^2+\frac{1}{4}(\|\sigma\|_{H^2}^2\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4),\end{aligned}$$

and the boundary term is bounded as follows:

$$\begin{aligned}-\int_\Omega\Delta\omega\omega_t\ \mathrm{d}x&=\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2-\int_{\partial\Omega}\left(2\chi-\frac{f}{\mu}\right)\mathbf{u}_t\cdot\tau(\nabla\omega\cdot\mathbf{n})\,ds\\&\le\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+\delta\|\omega\|_{H^2}^2+C_\delta\|\mathbf{u}_t\|_{H^1}^2.\end{aligned}$$

Thus, we choose $\eta_1=10^{-1}\mu\le\mu(2(1+\|\rho\|_{L^\infty}))^{-1}$ and δ suitably small to conclude that

Lemma 2.11. *For the solution to (1.10)–(1.15), we have*

$$\begin{aligned}&\mu\frac{d}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+\|\sqrt{\rho}\omega_t\|_{L^2}^2+\frac{\mu^2}{20}\|\triangle\omega\|_{L^2}^2\le\delta\|\omega\|_{H^2}^2+C\left(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^4\right.\\&\qquad\left.+\|\sigma\|_{H^2}^2(\|\mathbf{u}\|_{H^3}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2)+\|\mathbf{H}\|_{H^2}^4\right),\quad 0<\delta<1.\end{aligned}$$

Now, if we take $\langle(2.11),\varphi_t-\eta_2\triangle\varphi\rangle$, with η_2 being a positive constant to be determined later, we deduce that

$$\begin{aligned}&\frac{\eta+\eta_2}{2}\frac{d}{\mathrm{d}t}\|\nabla\varphi\|_{L^2}^2+\|\varphi_t\|_{L^2}^2+\eta\eta_2\|\triangle\varphi\|_{L^2}^2\\&\quad=\int_\Omega\mathrm{curl}(\mathrm{div}\mathbf{u}\mathbf{H}+\mathbf{u}\cdot\nabla\mathbf{H}-\mathbf{H}\cdot\nabla\mathbf{u})(\varphi_t-\eta\triangle\varphi)\mathrm{d}x\\&\quad\le C\int_\Omega\{\nabla\mathrm{div}\mathbf{u}H+\mathrm{div}\mathbf{u}\varphi+\mathbf{u}\cdot\nabla\varphi+\nabla(\mathbf{u}\cdot\nabla)\varphi\\&\qquad+\mathbf{H}\cdot\nabla\omega+\nabla(\mathbf{H}\cdot\nabla)\mathbf{u}\}(\varphi_t-\eta_2\triangle\varphi)\mathrm{d}x\\&\quad\le\delta(\|\varphi_t\|_{L^2}^2+\|\varphi\|_{H^2}^2+\|\omega\|_{H^2}^2)+C_\delta(\|\mathbf{u}\|_{H^2}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{u}\|_{H^2}^2\|\varphi\|_{H^1}^2\\&\qquad+\|\mathbf{u}\|_{H^2}^2\|\varphi_t\|_{H^1}^2+\|\mathbf{H}\|_{H^2}^2\|\varphi_t\|_{H^1}^2)+\eta_2^2\|\triangle\varphi\|_{L^2}^2\\&\qquad+C(\|\mathbf{u}\|_{H^2}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^2\|\nabla\omega\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^2\|\varphi\|_{H^2}^2),\end{aligned}$$

which, recalling $\|\varphi\|_{H^2}\le C\|\Delta\varphi\|_{L^2}$ and choosing $\eta_2=\eta/10$, yields that

Lemma 2.12.

$$\begin{aligned}&\eta\frac{d}{\mathrm{d}t}\|\nabla\varphi\|_{L^2}^2+\|\varphi_t\|_{L^2}^2+\frac{\eta^2}{20}\|\triangle\varphi\|_{L^2}^2\\&\quad\le\delta\|\omega\|_{H^2}^2+C_\delta(\|\mathbf{u}\|_{H^2}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{u}\|_{H^2}^2\|\varphi\|_{H^1}^2+\|\mathbf{u}\|_{H^2}^2\|\varphi_t\|_{H^1}^2+\|H\|_{H^2}^2\|\varphi_t\|_{H^1}^2)\\&\qquad+C_\delta(\|\mathbf{u}\|_{H^2}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^2\|\nabla w\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^2\|\varphi\|_{H^2}^2),\quad 0<\delta<1.\end{aligned}$$

We apply the elliptic regularity theory to the Eq. (2.9) to obtain that

$$\|\omega\|_{H^2}^2\le C(\|g\|_{L^2}^2+\|\rho\|_{H^2}^2(\|\omega_t\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^2\|\nabla\omega\|_{L^2}^2)+\|\mathbf{u}\cdot\tau\|_{H^{3/2}(\partial\Omega)}^2).\tag{2.27}$$

We proceed to derive more estimates. For simplicity, we define

$$\Psi = \sum_{i=0}^{2} \Psi_i(t) + \|\epsilon(\sigma_{tt}, \sqrt{\rho}\mathbf{u}_{tt}, \mathbf{H}_{tt})\|_{L^2}^2,$$
$$\Phi = \sum_{i=0}^{2} \Phi_i(t) + \|\epsilon\sigma_{tt}\|_{L^2}^2 + \|\epsilon(\mathbf{u}_{tt}, \mathbf{H}_{tt})\|_{H^1}^2 + \|\sigma\|_{H^2}^2, \tag{2.28}$$

where

$$\Psi_0(t) = \|(\sigma, \mathbf{u}, \mathbf{H})\|_{L^2}^2, \qquad \Phi_0(t) = \|(\mathbf{u}, \mathbf{H})\|_{H^1}^2,$$

$\Psi_1(t)$ and $\Phi_1(t)$ are defined by (2.2), and

$$\begin{aligned}
\Psi_2(t) =& \|\nabla \mathrm{div}\mathbf{u}\|_{L^2}^2 - 2\int_\Omega \rho\mathbf{u}_t \cdot \nabla \mathrm{div}\mathbf{u}\mathrm{d}x + \|\nabla^2\sigma\|_{L^2}^2 + \|\mathrm{div}\mathbf{u}_t\|_{L^2}^2 \\
&+ \|\nabla\sigma_t\|_{L^2}^2 + \|\nabla \mathrm{curl}\mathbf{u}\|_{L^2}^2 + \|\mathrm{curl}\mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathrm{curl}\mathbf{H}\|_{L^2}^2 + \|\mathrm{curl}\mathbf{H}_t\|_{L^2}^2, \\
\Phi_2(t) =& (\|\nabla^2 \mathrm{div}\mathbf{u}\|_{L^2}^2 + \|\nabla \mathrm{div}\mathbf{u}_t\|_{L^2}^2) + \|\nabla\sigma_t\|_{L^2}^2 + \|\nabla \mathrm{curl}\mathbf{u}_t\|_{L^2}^2 \\
&+ \|\nabla^2 \mathrm{curl}\mathbf{u}\|_{L^2}^2 + \|\mathrm{curl}\mathbf{H}_t\|_{H^1}^2 + \|\nabla^2 \mathrm{curl}\mathbf{H}\|_{L^2}^2.
\end{aligned}$$

The next lemma gives a bound on $\|\epsilon\sigma_{tt}\|$ and $\|\epsilon(\mathbf{u}_{tt}, \mathbf{H}_{tt})\|$.

Lemma 2.13. *For the solution to* (1.10)–(1.15), *we have*

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\|\epsilon(\sqrt{p'(1)}\sigma_{tt}, \sqrt{\rho}\mathbf{u}_{tt}, \mathbf{H}_{tt})\|_{L^2}^2 + \gamma_3\|\epsilon(\mathbf{u}_{tt}, \mathbf{H}_{tt})\|_{H^1}^2 \\
&\quad \le C\Phi(t)(\Psi(t) + \Psi^2(t)) + \epsilon^2\|\sigma_t\|_{H^1}^2 + \delta(\|\mathbf{u}\|_{H^3}^2 + \|\mathbf{u}_t\|_{H^2}^2), \qquad 0 < \delta < 1,
\end{aligned} \tag{2.29}$$

where γ_3 is a positive constant independent of ϵ.

Proof. We take $\langle \partial_{tt}(1.10), \epsilon^2 p'(1)\sigma_{tt}\rangle + \langle \partial_{tt}(2.4), \epsilon^2\mathbf{u}_{tt}\rangle$ to get that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}(\|\epsilon\sqrt{p'(1)}\sigma_{tt}\|_{L^2}^2 + \|\epsilon\sqrt{\rho}\mathbf{u}_{tt}\|_{L^2}^2) + (2\mu+\lambda)\|\epsilon \mathrm{div}\mathbf{u}_{tt}\|_{L^2}^2 + \mu\|\epsilon \mathrm{curl}\mathbf{u}_{tt}\|_{L^2}^2 \\
&\quad \le \delta(\|\mathrm{div}\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^2}^2 + \|\epsilon\mathbf{u}_{tt}\|_{H^1}^2) + C_\delta(\|\epsilon\sigma_{tt}\|_{L^2}^2 + \|\sigma_t\|_{H^1}^2 + \|\sigma\|_{H^2}^2)\|\epsilon\sigma_{tt}\|_{L^2}^2 \\
&\qquad + \|\epsilon\mathbf{u}_{tt}\|_{L^2}(\|\sigma_t\|_{H^1}\|\nabla\sigma_t\|_{L^2} + \epsilon\|\sigma_t\|_{H^1}^2\|\nabla\sigma\|_{H^1}^2 + \|\epsilon\sigma_{tt}\|_{L^2}(\|\nabla\sigma\|_{H^1} + \|\mathbf{u}_t\|_{H^1}) \\
&\qquad + \|\sigma_t\|_{H^1}\|\epsilon\mathbf{u}_{tt}\|_{L^2} + \|\nabla\mathbf{u}\|_{H^1}(\|\epsilon\mathbf{u}_{tt}\|_{L^2} + \|\epsilon\sigma_{tt}\|_{L^2}\|\mathbf{u}\|_{H^2} + \epsilon\|\sigma_t\|_{H^1}\|\mathbf{u}_t\|_{H^1}) \\
&\qquad + \|\nabla\mathbf{u}_t\|_{L^2}(\epsilon\|\sigma_t\|_{H^1}\|\mathbf{u}\|_{H^2} + \|\mathbf{u}_t\|_{H^1})) + \langle[(\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2]_{tt}, \epsilon^2\mathbf{u}_{tt}\rangle \\
&\quad \le \delta(\|\mathrm{div}\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^2}^2 + \|\epsilon\mathbf{u}_{tt}\|_{H^1}^2) + C\Phi(t)(\Psi(t) + \Psi^2(t)),
\end{aligned} \tag{2.30}$$

where we have used the following estimate

$$\begin{aligned}
&\langle[(\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla\mathbf{H}^2]_{tt}, \epsilon^2\mathbf{u}_{tt}\rangle \\
&\quad = \int_\Omega (\mathbf{H}_{tt}\cdot\nabla\mathbf{H} + 2\mathbf{H}_t\cdot\nabla\mathbf{H}_t + \mathbf{H}\cdot\nabla\mathbf{H}_{tt} - \nabla\mathbf{H}_{tt}\cdot\mathbf{H} - 2\nabla\mathbf{H}_t\cdot\mathbf{H}_t \\
&\qquad - \nabla\mathbf{H}\cdot\mathbf{H}_{tt})\epsilon^2\mathbf{u}_{tt}\mathrm{d}x \\
&\quad \le \delta\|\epsilon\mathbf{u}_{tt}\|_{L^2}^2 + C_\delta(\|\nabla\mathbf{H}\|_{H^1}^2\|\epsilon\mathbf{H}_{tt}\|_{L^2}^2 + \|\mathbf{H}_t\|_{H^1}^4 + \|\mathbf{H}\|_{H^2}^2\|\epsilon\mathbf{H}_{tt}\|_{H^1}^2).
\end{aligned}$$

And again we take $\langle \partial_{tt}(2.2), \epsilon^2 \mathbf{H}_{tt} \rangle$ and use the boundary condition $\mathbf{H}_{tt}|_{\partial\Omega} = 0$ to deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\epsilon\mathbf{H}_{tt}\|_{L^2}^2 + \eta\|\epsilon\nabla\mathbf{H}_{tt}\|_{L^2}^2 \\
&\quad = -\int_\Omega (\mathrm{div}\mathbf{u}_t \cdot \mathbf{H} + \mathrm{div}\mathbf{u}\mathbf{H}_t + \mathbf{u}_t \cdot \nabla\mathbf{H} + \mathbf{u}\cdot\nabla\mathbf{H}_t - \mathbf{H}_t\cdot\nabla\mathbf{u} - \mathbf{H}\cdot\nabla\mathbf{u}_t)_t \cdot \epsilon^2\mathbf{H}_{tt}\mathrm{d}x \\
&\quad = -\int_\Omega (\mathrm{div}\mathbf{u}_{tt}\cdot\mathbf{H} + 2\mathrm{div}\mathbf{u}_t\cdot\mathbf{H}_t + \mathrm{div}\mathbf{u}\cdot\mathbf{H}_{tt} + \mathbf{u}_{tt}\cdot\nabla\mathbf{H} + 2\mathbf{u}_t\cdot\nabla\mathbf{H}_t + \mathbf{u}\cdot\nabla\mathbf{H}_{tt} \\
&\qquad -\mathbf{H}_{tt}\cdot\nabla\mathbf{u} - 2\mathbf{H}_t\cdot\nabla\mathbf{u}_t - \mathbf{H}\cdot\nabla\mathbf{u}_{tt})\cdot\epsilon^2\mathbf{H}_{tt}\mathrm{d}x \\
&\quad \le \delta\|\epsilon\mathbf{H}_{tt}\|_{H^1}^2 + C\Phi(t)(\Psi(t)+\Psi^2(t)). \qquad (2.31)
\end{aligned}
$$

Thus, taking δ suitably small in (2.30) and (2.31), we obtain the estimate (2.29). □

In order to close the established estimates, we have to control $\|\epsilon\sigma_{tt}\|_{L^2}$ and $\|\sigma\|_{H^2}$. To this end, we differentiate (1.10) with respect to t to see that

$$
\|\epsilon\sigma_{tt}\|_{L^2}^2 \le C\epsilon^2\Phi(t)\Psi(t) + \|\mathrm{div}\mathbf{u}_t\|_{L^2}^2. \qquad (2.32)
$$

On the other hand, from the equations (1.11), we have

$$
\|\sigma\|_{H^2}^2 \le C\epsilon^2(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2) + C\epsilon^2\Phi(t)(\Psi(t)+\Psi^2(t)). \qquad (2.33)
$$

In addition, we use the following fact to control the terms $\|\mathbf{u}_t\|_{H^2}$ and $\|\mathbf{u}\|_{H^3}$,

$$
\|\mathbf{w}\|_{W^{s,p}} \le C(\|\mathrm{div}\mathbf{w}\|_{W^{s-1,p}} + \|\mathrm{curl}\,\mathbf{w}\|_{W^{s-1,p}} + \|\mathbf{w}\cdot\mathbf{n}\|_{W^{s-1/p,p}(\partial\Omega)} + \|\mathbf{w}\|_{W^{s-1,p}}) \qquad (2.34)
$$

for all $\mathbf{w} \in W^{s,p}(\Omega)$ and $1 < p < +\infty$.

Combining Lemma 2.1, Lemmas 2.6–2.13 with the estimates (2.17) and (2.32)–(2.34) and choosing ϵ and δ small enough, we finally conclude that

$$
\frac{d}{\mathrm{d}t}\Psi(t) + \Phi(t) \le c_0\Phi(t)\left(\Psi(t)+\Psi^2(t)\right), \qquad (2.35)
$$

where $c_0 \ge 1$ is a constant independent of ϵ.

Now, employing (2.35), and following the analysis in [35], we obtain the following uniform estimate.

Lemma 2.14. *Suppose $\Psi(0) \le \beta/(2c_0)$ for some $\beta \in (0, 1/2]$ where c_0 is the same as in (2.17). Then there is an $\epsilon_1 > 0$, such that for any $\epsilon \in (0, \epsilon_1]$, we have $c^{-1} \le 1 + \epsilon\sigma \le c$ for some $c > 1$, and $\Psi(t) \le \beta/(2c_0)$ for all $t \in [0, T]$.*

2.4. Proof of Theorems 1.1 and 1.2

Now, recalling the definition (2.28) of $(\Psi(t), \Phi(t))$, we can use the uniform a priori estimate established in Lemma 2.14 to continue the local solution $(\sigma, \mathbf{u}, \mathbf{H})$ globally in time by applying the standard extension techniques (see, for example, [37]), and obtain therefore a global solution. Furthermore, we can employ the uniform estimate given in Lemma 2.14 and Arzelà-Ascoli's theorem to easily show the strong convergence of $(\sigma, \mathbf{u}, \mathbf{H})$ to the solution of the corresponding incompressible magnetohydrodynamic equations as $\epsilon \to 0$. This completes the proof of Theorems 1.1 and 1.2.

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