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Non-uniform dependence on initial data for the $\mu-b$ equation

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Abstract. This paper is concerned with the non-uniform dependence on initial data for the $\mu-b$ equation on the circle. Using the approximate solution method, we construct two solution sequences to show that the data-to-solution map of the Cauchy problem of the $\mu-b$ equation is not uniformly continuous in $H^s(\mathbb{S})$.

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1. Introduction

The b-equation [4,5,11]

$$u_t - u_{xxt} + u(u_x - u_{xxx}) + bu_x(u - u_{xx}) = 0, \tag{1.1}$$

which in fact comprises a family of equations parameterized by the real value b, models wave phenomena in shallow fluid flow, and has been the subject of considerable interest both to mathematicians and physicists. When $b = 2$ and 3, respectively, (1.1) reduces to the well-known Camassa–Holm (CH) [1] and Degasperis–Procesi (DP) [5] equations, which have interesting connections with the Korteweg–deVries (KdV) equation and have been extensively studied, particularly from the point of view of integrability and geodesic flow on infinite dimensional Lie groups. The Cauchy problem of the b-equation, and in particular its well-posedness and blow-up behavior, has also been well-studied both on the real line and on the circle, see for example [3,6,22,23,25].

In [17], Khesin et al. studied a generalization of the Hunter–Saxton (HS) [12] equation, which is a high-frequency limit of the CH equation and introduced the $\mu-$HS equation. The $\mu-$HS equation, which is now more commonly referred to as the $\mu-$CH equation, describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic field and self-interaction. It was also shown in [17] that the $\mu-$CH equation on the circle is formally integrable, admits a bi-Hamiltonian structure and an infinite hierarchy of conservation laws, and is an Euler equation for geodesic flow on the Lie group of circle diffeomorphisms. Subsequently, in addition to $\mu-$CH, Lenells et al. [20] also introduced the $\mu-$DP as well as $\mu-$Burgers equations, and the $\mu-b$ equation (see also [19]).

In this paper, we consider the Cauchy problem for the (spatially) periodic $\mu-b$ equation:

$$\begin{cases} \mu(u_t) - u_{xxt} - uu_{xxx} + bu_x(\mu(u) - u_{xx}) = 0, & t > 0, \quad x \in \mathbb{S}^1, \\ u(0,x) = u_0(x), & x \in \mathbb{S}^1, \end{cases} \tag{1.2}$$

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where b is a real constant, $u = u(t, x)$ is a time-dependent function on the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \simeq [0, 1)$, and $\mu(u) = \int_{\mathbb{S}^1} u(t, x)\mathrm{d}x$ denotes the mean value of u. We note that under spatial periodicity, $\mu(u_t) = 0 = \mu(u_x)$.

In the case $b = 2$ and 3, respectively, the $\mu-b$ equation reduces to the $\mu-$CH and $\mu-$DP equations. In addition, if $\mu(u) = 0$, they reduce to the HS and $\mu-$Burgers equations, respectively.

The local well-posedness of the $\mu-$CH and $\mu-$DP Cauchy problems have been studied in [17] and [20]. Recently, Fu et al. [7] described precise blow-up scenarios for $\mu-$CH and $\mu-$DP.

Our attention in this paper is on the dependence on initial data. Himonas et al. [8,9] showed that solutions of the CH equation on the real line as well as on the circle do not depend uniformly on the initial data. They made use of the method of approximate solutions [2,10,16,18].

Our work has been inspired by [8,9]. We show non-uniform dependence on initial data of (1.2) by constructing two sequences of solutions in a bounded subset of the Sobolev space $H^s(\mathbb{S}^1)$, whose distance converges to zero at $t = 0$ but is bounded below by a positive constant at any later time. We remark that, unlike the b-equation case, the method of [9] does not apply in a straightforward manner to the $\mu-b$ equation. To be more specific, for the b-equation, one only needs to estimate the H^σ-norm $(1/2 < \sigma < 1)$ of the difference between approximate and actual solutions (as in [13]). For $\mu-b$, however, we need to treat the cases $b = 3$ and $b \neq 3$ separately and estimate the H^σ-norm $(1/2 < \sigma < 1)$ and H^1-norm of the difference, respectively, (see Sect. 3.2 for details). Unfortunately, when $b \neq 3$, we are not yet able to obtain the non-uniform dependence on the initial data for (1.2) with $3/2 < s < 2$ and will study it in a separate paper. We further remark that we have come to be aware that a similar method has been employed to study the non-uniform dependence problem for the b-equation by Yan Li [21].

This paper is organized as follows: In Sect. 2, we recall the well-posedness results of [17] and [20]. We will establish an energy estimate and use it to derive a lower bound for the lifespan of the solution as well as an estimate of the $H^s(\mathbb{S}^1)$ norm of the solution $u(t, x)$ in terms of that of the initial data u_0. In Sect. 3, we consider approximate solutions and estimate their errors in the H^σ-norm $(1/2 < \sigma \leq 1)$ in satisfying the equation (1.2). Further, we estimate the difference between the approximate and actual solutions with the same initial data. The proof of the main result is given in Sect. 4.

In this paper, the symbols $\lesssim$, $\approx$ and $\gtrsim$ are used to denote inequality/equality up to a positive universal constant. For example, $f(x) \lesssim g(x)$ means that $f(x) \leq cg(x)$ for some positive universal constant c. Since all spaces of functions are over $\mathbb{S}^1$, the reference to $\mathbb{S}^1$ will be dropped if no ambiguity arises. $[A, B] = AB - BA$ denotes the commutator of linear operators A and B.

2. Local well-posedness

As $\mu(u_x) = 0$ under spatial periodicity, we can re-write (1.2) as follows:

$$\begin{cases} u_t + uu_x = -\partial_x A^{-1}\left(b\mu(u)u + \dfrac{3-b}{2}u_x^2\right), & t > 0,\ x \in \mathbb{S}^1, \\ u(0, x) = u_0(x), & x \in \mathbb{S}^1, \end{cases} \tag{2.1}$$

where $A = \mu - \partial_x^2$ is an isomorphism between $H^s(\mathbb{S}^1)$ and $H^{s-2}(\mathbb{S}^1)$ with the inverse $v = A^{-1}u$ given by

$$v(x) = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(u) + (x - 1/2)\int_0^1\int_0^y u(s)\mathrm{d}s\mathrm{d}y$$
$$- \int_0^x u(s)\mathrm{d}s\mathrm{d}y + \int_0^1\int_0^y\int_0^s u(r)\mathrm{d}r\mathrm{d}s\mathrm{d}y.$$

Since A^{-1} and ∂_x commute, the following identities hold as follows:

$$A^{-1}\partial_x u(x) = (x-1/2)\int_0^1 u(x)\mathrm{d}x - \int_0^x u(y)\mathrm{d}y + \int_0^1\int_0^x u(y)\mathrm{d}y\mathrm{d}x, \tag{2.2}$$

$$A^{-1}\partial_x^2 u(x) = -u(x) + \int_0^1 u(x)\mathrm{d}x. \tag{2.3}$$

It is easy to show that $\mu(A^{-1}\partial_x u(x)) = 0$.

We recall the following local well-posedness result:

Proposition 2.1. [20, Theorem 5.5] *Let $u_0 \in H^s$, $s > 3/2$. Then, there exist a maximal existence time $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution u to (2.1) such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}).$$

Moreover, the solution depends continuously on the initial data, that is, the map

$$u_0 \mapsto u(\cdot, u_0) : H^s \to C([0,T); H^s) \cap C^1([0,T); H^{s-1})$$

is continuous.

Remark 2.1. *The maximal existence time $T > 0$ in Proposition 2.1 is independent of the Sobolev index $s > 3/2$; this can be proved using the Kato method [14], as in [24].*

Next, we will give an explicit lower bound for the maximal existence time T with respect to the H^s-norm of the initial data. We will also give an estimate of the solution size and show that the H^s-norm of the solution is dominated by that of the initial data.

Lemma 2.1. *Let $s > 3/2$ and let u be the solution of (2.1) with initial data u_0 described in Proposition 2.1. Then, the maximal existence time T satisfies*

$$T \geq T_0 := \frac{1}{2C_s\|u_0\|_{H^s}}, \tag{2.4}$$

where C_s is a constant depending only on s. Also, we have

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad 0 \leq t \leq T_0. \tag{2.5}$$

Proof. First, we note that $\mu(u) = \mu(u_0) = \mu_0$. The proof of the lemma is based on the following differential inequality for the solution u:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \leq C_s\|u(t)\|_{H^s}^3, \quad 0 \leq t < T. \tag{2.6}$$

Integrating this inequality from 0 to t, one obtains

$$\|u(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - C_s\|u_0\|_{H^s}t},$$

which implies that $\|u(t)\|_{H^s} < \infty$ if $C_s\|u_0\|_{H^s}t < 1$. As a result, for $0 \leq t \leq T_0$, we have

$$\|u(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - C_s\|u_0\|_{H^s}T_0} = 2\|u_0\|_{H^s}.$$

It thus remains to prove the inequality (2.6). Note that the product uu_x only has the regularity of H^{s-1} when $u \in H^s$. To deal with this problem, we will consider the following modified equation:

$$(J_\varepsilon u)_t + J_\varepsilon(uu_x) = -\partial_x A^{-1}\left(b\mu_0 J_\varepsilon u + \frac{3-b}{2}J_\varepsilon\left(u_x^2\right)\right), \quad t > 0, \quad x \in \mathbb{S}^1, \tag{2.7}$$

where $\varepsilon \in (0,1]$ and

$$J_\varepsilon f(x) = J_\varepsilon(f)(x) = (j_\varepsilon \times f)(x)$$

is the Friedrichs mollifier with $j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon})$ for a C^∞ function $j(x)$. Let $\Lambda = (1-\partial_x^2)^{1/2}$. Applying the operator Λ^s to (2.7), then multiplying the resulting equation by $\Lambda^s J_\varepsilon u$ and integrating with respect to $x \in \mathbb{S}^1$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|J_\varepsilon u\|_{H^s}^2 = -\left(\Lambda^s J_\varepsilon(uu_x),\, \Lambda^s J_\varepsilon u\right) - \left(\partial_x \Lambda^s A^{-1}\left(b\mu_0 J_\varepsilon u + \frac{3-b}{2} J_\varepsilon(u_x^2)\right),\ \Lambda^s J_\varepsilon u\right), \tag{2.8}$$

where

$$(f,g) := \int fg\mathrm{d}x.$$

We will estimate the right-hand side of (2.8). First, we note that Λ^s and J_ε are commutative and that

$$(J_\varepsilon f, g) = (f,\, J_\varepsilon g), \quad \|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s}\,. \tag{2.9}$$

To estimate the first integral in the right-hand side of (2.8), we write it as follows:

$$\begin{aligned}\big(\Lambda^s J_\varepsilon(uu_x),\, \Lambda^s J_\varepsilon u\big) &= \big(J_\varepsilon \Lambda^s(uu_x),\, \Lambda^s J_\varepsilon u\big) = \big(\Lambda^s(uu_x),\ J_\varepsilon \Lambda^s J_\varepsilon u\big)\\ &= ([\Lambda^s, u]u_x,\, J_\varepsilon \Lambda^s J_\varepsilon u) + (u\Lambda^s u_x,\, J_\varepsilon \Lambda^s J_\varepsilon u).\end{aligned}$$

Using [15, Lemma X1] and (2.9), we obtain

$$\|([\Lambda^s, u]u_x,\, J_\varepsilon \Lambda^s J_\varepsilon u)\| \le \|[\Lambda^s, u]u_x\|_2\, \|J_\varepsilon \Lambda^s J_\varepsilon u\|_2 \le C_s\, \|u_x\|_\infty\, \|u\|_{H^s}^2\,, \tag{2.10}$$

where we have used the fact that $\|u\|_{H^s} = \|\Lambda^s u\|_2$. Noting that, by [8, Lemma 2],

$$\|[J_\varepsilon, u]f_x\|_2 \le C\, \|u_x\|_\infty\, \|f\|_2\,,$$

and integrating by parts, we obtain

$$\begin{aligned}|(u\Lambda^s u_x,\, J_\varepsilon \Lambda^s J_\varepsilon u)| &= |(J_\varepsilon u \Lambda^s u_x,\, \Lambda^s J_\varepsilon u)|\\ &= |([J_\varepsilon, u]\, \partial_x \Lambda^s u,\, \Lambda^s J_\varepsilon u) + (uJ_\varepsilon \partial_x \Lambda^s u,\, \Lambda^s J_\varepsilon u)|\\ &\le \|[J_\varepsilon, u]\partial_x \Lambda^s u\|_2 \|u\|_{H^s} + \frac{1}{2}\big|(u_x \Lambda^s J_\varepsilon u,\, \Lambda^s J_\varepsilon u)\big|\\ &\le C_s \|u_x\|_\infty \|u\|_{H^s}^2.\end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$$\big|\big(\Lambda^s J_\varepsilon(uu_x),\, \Lambda^s J_\varepsilon u\big)\big| \le C_s \|u_x\|_\infty \|u\|_{H^s}^2. \tag{2.12}$$

For the second integral in the right-hand side of (2.8), we note

$$\big|\big(\partial_x \Lambda^s A^{-1}\left(b\mu_0 J_\varepsilon u\right),\ \Lambda^s J_\varepsilon u\big)\big| \le C|\mu_0| \cdot \|A^{-1}\partial_x J_\varepsilon u\|_{H^s} \|u\|_{H^s}, \tag{2.13}$$

$$\left|\left(\partial_x \Lambda^s A^{-1}\left(\frac{3-b}{2} J_\varepsilon(u_x^2)\right),\ \Lambda^s J_\varepsilon u\right)\right| \le C\|A^{-1}\partial_x J_\varepsilon(u_x^2)\|_{H^s}\|u\|_{H^s}. \tag{2.14}$$

Since $\mu_0 = \mu(u)$, we have

$$|\mu_0| = \left|\int_{\mathbb{S}^1} u(x,t)\mathrm{d}x\right| \le \|u\|_2 \le \|u\|_{H^s}. \tag{2.15}$$

By (2.2) and (2.3), and using the Sobolev embedding theorem, we obtain

$$\begin{aligned}\|A^{-1}\partial_x J_\varepsilon u\|_{H^s} \le \|A^{-1}\partial_x u\|_{H^s} &\le \|A^{-1}\partial_x u\|_2 + \|A^{-1}\partial_x^2 u\|_{H^{s-1}}\\ &\le 3\|u\|_2 + \left\|u - \int u(x)\mathrm{d}x\right\|_{H^{s-1}}\\ &\le 3\|u\|_2 + 2\|u\|_{H^{s-1}}\\ &\le 5\|u\|_{H^{s-1}}, \qquad (2.16)\\ \|A^{-1}\partial_x J_\varepsilon (u_x^2)\|_{H^s} \le \|A^{-1}\partial_x u_x^2\|_{H^s} &\le \|A^{-1}\partial_x u_x^2\|_2 + \|A^{-1}\partial_x^2 u_x^2\|_{H^{s-1}}\\ &\le 3\|u_x\|_2^2 + \left\|u_x^2 - \int u_x^2(x)\mathrm{d}x\right\|_{H^{s-1}}\\ &\le 3\|u\|_2^2 + 2\|u_x^2\|_{H^{s-1}}\\ &\le 5\|u_x\|_\infty \|u\|_{H^s}. \qquad (2.17)\end{aligned}$$

Using (2.12)–(2.17) in (2.8), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|J_\varepsilon u\|_{H^s}^2 \le C_s\big(\|u\|_{H^s} + \|u_x\|_\infty\big)\|u\|_{H^s}^2.$$

Letting $\varepsilon \to 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 \le C_s\big(\|u\|_{H^s} + \|u_x\|_\infty\big)\|u\|_{H^s}^2,$$

or

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \le C_s\big(\|u\|_{H^s} + \|u\|_{C^1}\big)\|u(t)\|_{H^s}^2. \qquad (2.18)$$

Since $s > 3/2$, by Sobolev's inequality,

$$\|u(t)\|_{C^1} \le C_s \|u(t)\|_{H^s},$$

and the desired inequality (2.6) follows. □

3. Approximate solutions

In this section, following [8,9], we consider a two-parameter family of approximate solutions. We will first estimate the error of the approximate solutions in satisfying (2.1) in the H^σ-norm $(1/2 < \sigma \le 1)$, and then, estimate the difference between the approximate solution and actual solution having the same initial data.

We consider approximate solutions of the form

$$u^{\omega,\lambda} = \omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x - \omega t),$$

where $s \in \mathbb{R}$, ω is in a bounded set in $\mathbb{R}$ and $\lambda = 2\pi z$ with $z \in \mathbb{Z}_+$. We note that $u^{\omega,\lambda} \in H^s$.

3.1. Error in satisfying (2.1)

Now, we estimate the H^σ-norm of the error of $u^{\omega,\lambda}$ is satisfying (2.1), which is given by

$$\begin{aligned}E &= \left[u_t^{\omega,\lambda} + u^{\omega,\lambda}u_x^{\omega,\lambda}\right] + \partial_x A^{-1}\left(b\mu\left(u^{\omega,\lambda}\right)u^{\omega,\lambda} + \frac{3-b}{2}\left(u_x^{\omega,\lambda}\right)^2\right)\\ &=: E_1 + E_2 + E_3.\end{aligned}$$

A direct calculation shows that

$$E_1 := u_t^{\omega,\lambda} + u^{\omega,\lambda} u_x^{\omega,\lambda} = -\frac{1}{2}\lambda^{-2s+1}\sin 2(\lambda x - \omega t).$$

Using (2.2), we have

$$\begin{aligned}
E_2 &:= b\mu(u^{\omega,\lambda})\partial_x A^{-1} u^{\omega,\lambda} \\
&= b\mu(u^{\omega,\lambda}) \left[\left(x - \frac{1}{2}\right) \int_0^1 u^{\omega,\lambda}(t,x)\mathrm{d}x - \int_0^x u^{\omega,\lambda}(t,y)\mathrm{d}y + \int_0^1 \int_0^x u^{\omega,\lambda}(t,y)\mathrm{d}y\mathrm{d}x \right] \\
&= b\lambda^{-s-1}\left[\omega\lambda^{-1} + \lambda^{-s-1}(\sin(\lambda - \omega t) + \sin(\omega t))\right] \\
&\quad \times \left[\left(x - \frac{1}{2}\right)\left[\sin(\lambda - \omega t) + \sin(\omega t)\right] - \sin(\lambda x - \omega t) + \frac{\cos(\omega t) - \cos(\lambda - \omega t)}{\lambda} \right]; \\
E_3 &:= \frac{3-b}{2}\partial_x A^{-1}(u_x^{\omega,\lambda})^2 \\
&= \frac{3-b}{2} \left[\left(x - \frac{1}{2}\right) \int_0^1 (u_x^{\omega,\lambda}(t,x))^2\mathrm{d}x - \int_0^x (u_x^{\omega,\lambda}(t,y))^2\mathrm{d}y + \int_0^1 \int_0^x (u_x^{\omega,\lambda}(t,y))^2\mathrm{d}y\mathrm{d}x \right] \\
&= \frac{b-3}{4}\lambda^{-2s+1} \left[\left(x - \frac{1}{2}\right)\left[\sin(2\lambda - 2\omega t) + \sin(2\omega t)\right] - \sin(2\lambda x - 2\omega t) \right. \\
&\quad \left. + \frac{\cos(2\omega t) - \cos(2\lambda - 2\omega t)}{2\lambda} \right].
\end{aligned}$$

Lemma 3.1. *Let $s > 3/2$ and $\sigma \in (1/2, 1]$. When ω is in a bounded set in $\mathbb{R}$ and $\lambda \gg 1$, we have*

$$\|E\|_{H^\sigma} \lesssim \lambda^{-r_s} \tag{3.1}$$

for $0 < t < T$, where $r_s = 2s - \sigma - 1 > 0$.

Proof. First, we recall from [9, Lemma 1] that

$$\|u^{\omega,\lambda}\|_{H^\sigma} = \|\omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x - \omega t)\|_{H^\sigma} \lesssim \lambda^{-1} + \lambda^{-s+\sigma}.$$

The following was also obtained in [9]:

$$\|E_1(t)\|_{H^\sigma} = \left\| \frac{1}{2}\lambda^{-2s+1}\sin 2(\lambda x - \omega t) \right\|_{H^\sigma} \lesssim \lambda^{-2s+1+\sigma}.$$

Finally, direct calculations show that

$$\begin{aligned}
\|E_2(t)\|_{H^\sigma} &= \lambda^{-s-1}|b|\big|\omega\lambda^{-1} + \lambda^{-s-1}(\sin(\lambda-\omega t)+\sin(\omega t))\big| \\
&\quad \times \left\| \left(x-\frac{1}{2}\right)\big[\sin(\lambda-\omega t)+\sin(\omega t)\big] - \sin(\lambda x-\omega t) + \frac{\cos(\omega t)-\cos(\lambda-\omega t)}{\lambda} \right\|_{H^\sigma} \\
&\lesssim \lambda^{-s-2}\left(\left\| x-\frac{1}{2}\right\|_{H^\sigma} + \|\sin(\lambda x-\omega t)\|_{H^\sigma} + \lambda^{-1}\right) \\
&\lesssim \lambda^{-s-2+\sigma}, \\
\|E_3(t)\|_{H^\sigma} &= \left|\frac{b-3}{4}\right| \lambda^{-2s+1}\Big\| (x-1/2)\big[\sin(2\lambda-2\omega t)+\sin(2\omega t)\big] - \sin(2\lambda x-2\omega t) \\
&\quad + \frac{\cos(2\omega t)-\cos(2\lambda-2\omega t)}{2\lambda}\Big\|_{H^\sigma} \\
&\lesssim \lambda^{-2s+1}\left(\|x-1/2\|_{H^\sigma} + \|\sin(2\lambda x-2\omega t)\|_{H^\sigma} + \lambda^{-1}\right) \\
&\lesssim \lambda^{-2s+1+\sigma}.
\end{aligned}$$

□

3.2. Difference between approximate and actual solutions

We now estimate the difference between the approximate and actual solutions.

Let $u_{\omega,\lambda}(t,x)$ be the solution to (2.1) with initial data $u^{\omega,\lambda}(0,x)$, that is, $u_{\omega,\lambda}(t,x)$ satisfies

$$\begin{cases} \partial_t u_{\omega,\lambda} + u_{\omega,\lambda}\partial_x u_{\omega,\lambda} + \partial_x A^{-1}\left(b\mu(u_{\omega,\lambda})u_{\omega,\lambda} + \dfrac{3-b}{2}(\partial_x u_{\omega,\lambda})^2\right) = 0, \quad t>0, \quad x\in\mathbb{S}^1, \\ u_{\omega,\lambda}(0,x) = u^{\omega,\lambda}(0,x) = \omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x), \quad x\in\mathbb{S}^1. \end{cases} \tag{3.2}$$

Then, by Proposition 2.1, Lemma 2.1 and [9, Lemma 1], $u_{\omega,\lambda}\in C([0,T];H^s)$ is the unique solution of (3.2) with

$$T \gtrsim \frac{1}{\|u_{\omega,\lambda}(0,x)\|_{H^s}} = \frac{1}{\|u_0^{\omega,\lambda}\|_{H^s}} \gtrsim \frac{1}{\lambda^{-1+1}} \gtrsim 1, \quad \lambda\gg 1.$$

To estimate the difference between the approximate and actual solutions, we let

$$v = u^{\omega,\lambda} - u_{\omega,\lambda}.$$

Then, for $t>0$ and $x\in\mathbb{S}^1$, v satisfies

$$\begin{cases} v_t = E - \dfrac{1}{2}\partial_x\big[(u^{\omega,\lambda}+u_{\omega,\lambda})v\big] - \partial_x A^{-1}\left[b\mu(u_{\omega,\lambda})v + \dfrac{3-b}{2}\partial_x(u^{\omega,\lambda}+u_{\omega,\lambda})\partial_x v\right], \\ v(0,x)=0. \end{cases} \tag{3.3}$$

We treat the cases $b=3$ and $b\neq 3$ separately.

Lemma 3.2. *Let $b=3$. Let $s>3/2$, $\sigma\in(1/2,s-1]$, ω be in a bounded set in $\mathbb{R}$, and $\lambda\gg 1$. Then*

$$\|v(t)\|_{H^\sigma} = \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^\sigma} \lesssim \lambda^{-r_s} \tag{3.4}$$

for $0\le t<T$ where $r_s = 2s-1-\sigma>0$.

Proof. Applying to both sides of (3.3) with $b = 3$ the operator Λ^σ, multiplying the resulting equation by $\Lambda^\sigma v$ and integrating, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^\sigma}^2 = (\Lambda^\sigma E,\, \Lambda^\sigma v) - \frac{1}{2}\left(\Lambda^\sigma\partial_x\left[(u^{\omega,\lambda} + u_{\omega,\lambda})v\right],\ \Lambda^\sigma v\right) + 3\mu(u_{\omega,\lambda})\left(\Lambda^\sigma\partial_x A^{-1}v,\, \Lambda^\sigma v\right). \tag{3.5}$$

By the Cauchy–Schwarz inequality, we have

$$\left|\left(\Lambda^\sigma E,\, \Lambda^\sigma v\right)\right| \le \|\Lambda^\sigma E\|_2\|\Lambda^\sigma v\|_2 \le \|E\|_{H^\sigma}\|v\|_{H^\sigma}. \tag{3.6}$$

Similarly to [9], for $3/2 < \rho \le s$ and $\sigma + 1 \le \rho$, one can show that

$$\begin{aligned}\left|\left(\Lambda^\sigma\partial_x[(u^{\omega,\lambda} + u_{\omega,\lambda})v],\ \Lambda^\sigma v\right)\right| &\lesssim \left(\|\partial_x u^{\omega,\lambda}\|_\infty + \|\partial_x u_{\omega,\lambda}\|_\infty\right)\|v\|_{H^\sigma}^2 + \left(\|u^{\omega,\lambda}\|_{H^\rho} + \|u_{\omega,\lambda}\|_{H^\rho}\right)\|v\|_{H^\sigma}^2 \\ &\lesssim \left(\|u^{\omega,\lambda}\|_{H^\rho} + \|u_{\omega,\lambda}\|_{H^\rho}\right)\|v\|_{H^\sigma}^2.\end{aligned} \tag{3.7}$$

By using (2.2) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\mu(u_{\omega,\lambda})\left(\Lambda^\sigma\partial_x A^{-1}v,\, \Lambda^\sigma v\right)x\right| &= \left|\mu(u_{\omega,\lambda}(0))\left(\Lambda^\sigma\partial_x A^{-1}v,\, \Lambda^\sigma v\right)x\right| \\ &\le |\omega\lambda^{-1} + \lambda^{-s-1}\sin\lambda|\cdot\|\Lambda^\sigma\partial_x A^{-1}v\|_2\|\Lambda^\sigma v\|_2 \\ &\lesssim \lambda^{-1}\|v\|_{H^\sigma}\left\|\left(x - \frac{1}{2}\right)\int_0^1 v(x)\mathrm{d}x - \int_0^x v(y)\mathrm{d}y + \int_0^1\int_0^x v(y)\mathrm{d}y\mathrm{d}x\right\|_{H^\sigma} \\ &\lesssim \lambda^{-1}\|v\|_{H^\sigma}\left(\left\|x - \frac{1}{2}\right\|_{H^\sigma}\int_0^1 |v(x)|\mathrm{d}x + \|v\|_{H^{\sigma-1}} + \int_0^1\int_0^x |v(y)|\mathrm{d}y\mathrm{d}x\right) \\ &\lesssim \lambda^{-1}\|v\|_{H^\sigma}^2.\end{aligned} \tag{3.8}$$

Substituting (3.6)–(3.8) into (3.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^\sigma}^2 \lesssim \left(\lambda^{-1} + \|u^{\omega,\lambda}\|_{H^\rho} + \|u_{\omega,\lambda}\|_{H^\rho}\right)\|v\|_{H^\sigma}^2 + \|E\|_{H^\sigma}\|v\|_{H^\sigma}.$$

Noting that $\|u^{\omega,\lambda}(t)\|_{H^\rho} \lesssim \lambda^{-1} + \lambda^{\rho-s}$ for $3/2 < \rho < s$, and, by Lemma 2.1, that

$$\|u_{\omega,\lambda}(t)\|_{H^\rho} \lesssim \|u_{\omega,\lambda}(0)\|_{H^\rho} = \|u^{\omega,\lambda}(0)\|_{H^\rho} \lesssim \lambda^{-1} + \lambda^{\rho-s},$$

it follows from Lemma 3.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^\sigma}^2 \lesssim \left(\lambda^{-1} + \lambda^{\rho-s}\right)\|v\|_{H^\sigma}^2 + \lambda^{-r_s}\|v\|_{H^\sigma},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^\sigma} \lesssim \|v\|_{H^\sigma} + \lambda^{-r_s}. \tag{3.9}$$

As $v(0) = 0$, solving the differential inequality (3.9), we obtain (3.4). □

Now, we consider the case $b \ne 3$. In this case, the Eq. (3.5) will have an extra term:

$$I := -\frac{3-b}{2}\int \partial_x\Lambda^\sigma A^{-1}\left[\partial_x\left(u^{\omega,\lambda} + u_{\omega,\lambda}\right)\partial_x v\right]\cdot\Lambda^\sigma v\mathrm{d}x. \tag{3.10}$$

It is easy to see that

$$|I| \lesssim \left\|\partial_x A^{-1}\left[\partial_x(u^{\omega,\lambda} + u_{\omega,\lambda})\partial_x v\right]\right\|_{H^\sigma}\|v(t)\|_{H^\sigma}. \tag{3.11}$$

By (2.2), we know that

$$\|\partial_x A^{-1}w\|_{H^\sigma} = \left\| \left(x-\frac{1}{2}\right)\int_0^1 w(x)\mathrm{d}x - \int_0^x w(y)\mathrm{d}y + \int_0^1\int_0^x w(y)\mathrm{d}y\mathrm{d}x \right\|_{H^\sigma}$$

$$\lesssim \left(\left\| x-\frac{1}{2}\right\|_{H^\sigma} \int_0^1 |w(x)|\mathrm{d}x + \|w\|_{H^{\sigma-1}} + \int_0^1\int_0^x |w(y)|\mathrm{d}y\mathrm{d}x \right), \tag{3.12}$$

where $w = \partial_x(u^{\omega,\lambda} + u_{\omega,\lambda})\partial_x v$. When $3/2 < s < 2$, we are unable to show that $\int_0^1 |w(x)|\mathrm{d}x$ is bounded. So, we shall limit the scope of s in the following result:

Lemma 3.3. *Let $b \neq 3$. Let $s \geq 2$, ω be in a bounded set in $\mathbb{R}$, and $\lambda \gg 1$. Then*

$$\|v(t)\|_{H^1} = \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^1} \lesssim \lambda^{-\theta_s} \tag{3.13}$$

for $0 \leq t < T$ where $\theta_s = 2(s-1) > 0$.

Proof. In the proof of Lemma 3.2, if we take $\sigma = 1$, the Eq. (3.5) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^1} = (\Lambda E,\, \Lambda v) - \frac{1}{2}\left(\Lambda\partial_x\left[\left(u^{\omega,\lambda} + u_{\omega,\lambda}\right)v\right],\ \Lambda v\right)$$

$$+ 3\mu(u_{\omega,\lambda})\left(\Lambda\partial_x A^{-1}v,\, \Lambda v\right) + I, \tag{3.14}$$

where I is given by (3.10) with $\sigma = 1$. We will show that the estimates (3.6)–(3.8) with $\sigma = 1$ remain true. In (3.7), we take $\rho = 2$.

Now, we estimate $\int_0^1 |w(x)|\mathrm{d}x$ and $\|w\|_2$ with $w = \partial_x(u^{\omega,\lambda} + u_{\omega,\lambda})\partial_x v$. A direct calculation yields

$$\int_0^1 |w(x)|\mathrm{d}x = \int_0^1 \left|\partial_x\left(u^{\omega,\lambda} + u_{\omega,\lambda}\right)\partial_x v\right| \mathrm{d}x$$

$$\leq \|v\|_{H^1}\|u^{\omega,\lambda} + u_{\omega,\lambda}\|_{H^1}, \tag{3.15}$$

$$\|w\|_2 = \left(\int \left|\partial_x(u^{\omega,\lambda} + u_{\omega,\lambda})\right|^2 |\partial_x v|^2\, \mathrm{d}x\right)^{1/2}$$

$$\leq \left(\|\partial_x u^{\omega,\lambda}\|_\infty + \|\partial_x u_{\omega,\lambda}\|_\infty\right)\|v\|_{H^1}$$

$$\lesssim \left(\|u^{\omega,\lambda}\|_{H^2} + \|u_{\omega,\lambda}\|_{H^2}\right)\|v\|_{H^1}. \tag{3.16}$$

Substituting (3.12), (3.15) and (3.16) into (3.11), we have

$$|I| \lesssim \|v\|^2_{H^1}\left(\|u^{\omega,\lambda}\|_{H^1} + \|u_{\omega,\lambda}\|_{H^1} + \|u^{\omega,\lambda}\|_{H^2} + \|u_{\omega,\lambda}\|_{H^2}\right)$$

$$\lesssim \|v\|^2_{H^1}\left(\|u^{\omega,\lambda}\|_{H^2} + \|u_{\omega,\lambda}\|_{H^2}\right). \tag{3.17}$$

Applying (3.6)–(3.8) with $\sigma = 1$ and $\rho = 2$, and (3.17), it follows from (3.14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^1} \lesssim \left(\lambda^{-1} + \|u^{\omega,\lambda}\|_{H^2} + \|u_{\omega,\lambda}\|_{H^2}\right)\|v\|^2_{H^1} + \|E\|_{H^1}\|v\|_{H^1}. \tag{3.18}$$

Similarly to the proof of Lemma 3.2,

$$\|u^{\omega,\lambda}(t)\|_{H^2} \lesssim \lambda^{-1} + \lambda^{2-s}, \qquad \|u_{\omega,\lambda}(t)\|_{H^2} \lesssim \lambda^{-1} + \lambda^{2-s}. \tag{3.19}$$

Noting that $s \geq 2$ and applying Lemma 3.1 with $\sigma = 1$, it follows from (3.18) and (3.19) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^1} \lesssim \left(1 + \lambda^{-1}\right)\|v\|^2_{H^1} + \lambda^{-2(s-1)}\|v\|_{H^1}.$$

The rest of the proof is the same as that of Lemma 3.2. □

4. Non-uniform dependence

In this section, we prove our main result, namely,

Theorem 4.1. *Let*

$$\begin{cases} s > 3/2 & \text{if } \ b = 3, \\ s \geq 2 & \text{if } \ b \neq 3. \end{cases}$$

Then, the data-to-solution map $u(0) \mapsto u(t)$ for (2.1) is not uniformly continuous from any bounded subset of H^s into $C([-T,T];H^s)$. More precisely, there exist two sequences of solutions $u_\lambda(t)$ and $\tilde{u}_\lambda(t)$ to (2.1) in $C([-T,T];H^s)$ such that

$$\|u_\lambda(t)\|_{H^s} + \|\tilde{u}_\lambda(t)\|_{H^s} \lesssim 1, \quad \lim_{\lambda\to\infty} \|u_\lambda(0) - \tilde{u}_\lambda(0)\|_{H^s} = 0, \tag{4.1}$$

and

$$\liminf_{\lambda\to\infty} \|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} \gtrsim |\sin t|, \quad |t| < T \leq 1. \tag{4.2}$$

Proof. Let $u_\lambda(t) = u_{1,\lambda}(t,x)$ and $\tilde{u}_\lambda(t) = u_{-1,\lambda}(t,x)$, where $u_{1,\lambda}(t,x)$ and $u_{-1,\lambda}(t,x)$ are the unique solution to the Cauchy problem (3.2) with initial data $u^{1,\lambda}(0,x)$ and $u^{-1,\lambda}(0,x)$, respectively.

By Proposition 2.1, these solutions lie in $C([0,T];H^s)$. By Lemma 2.1, we know that the existence time T can be chosen to be independent of λ provided $\lambda \gg 1$. Moreover, for $0 \leq t \leq T_0$,

$$\|u_{\pm 1,\lambda}(t)\|_{H^s} \leq 2\|u_{\pm 1,\lambda}(0)\|_{H^s} = 2\|u^{\pm 1,\lambda}(0)\|_{H^s}.$$

If λ is large enough and $k > s-1$, then, by [9, Lemma 1],

$$\|u^{\pm 1,\lambda}(t)\|_{H^k} \lesssim \lambda^{-1} + \lambda^{k-s} \lesssim \lambda^{k-s}.$$

It follows that

$$\|u_{\pm 1,\lambda}(t)\|_{H^k} \lesssim \lambda^{k-s}, \quad \lambda \gg 1,$$

and

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^k} \lesssim \lambda^{k-s}, \quad \lambda \gg 1. \tag{4.3}$$

We first consider the case $b = 3$. Lemma 3.2 implies that

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^\sigma} \lesssim \lambda^{-r_s}, \quad \lambda \gg 1. \tag{4.4}$$

Now, applying the interpolation inequality

$$\|\varphi\|_{H^s} \leq \|\varphi\|_{H^{s_1}}^{(s_2-s)/(s_2-s_1)} \|\varphi\|_{H^{s_2}}^{(s-s_1)/(s_2-s_1)}$$

with $s_1 = \sigma$ and $s_2 = 2s - \sigma = k$ and using (4.3) and (4.4), we get

$$\begin{aligned} \|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^s} &\leq \left(\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^\sigma} \|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^{2s-\sigma}}\right)^{1/2} \\ &\lesssim \left(\lambda^{-r_s}\lambda^{s-\sigma}\right)^{1/2} \lesssim \lambda^{-(s-1)/2}, \quad \lambda \gg 1. \end{aligned}$$

Hence,

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^s} \lesssim \lambda^{-\varepsilon_s}, \quad \lambda \gg 1, \tag{4.5}$$

where $\varepsilon_s = (s-1)/2$.

We can now prove (4.1) and (4.2). It follows from the definition of $u^{\omega,\lambda}$ that

$$\|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s} = \|u^{1,\lambda}(0) - u^{-1,\lambda}(0)\|_{H^s} = 2\lambda^{-1} \to 0 \quad \text{as } \lambda \to \infty,$$

which implies that (4.1) holds. Obviously,

$$\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \geq \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} - \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s} - \|u^{-1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s}.$$

It follows from (4.5) that

$$\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \geq \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} - c\lambda^{-\varepsilon_s}, \quad \lambda \gg 1,$$

which implies that

$$\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \geq \liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s}. \tag{4.6}$$

Using the identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

we obtain

$$u^{1,\lambda}(t) - u^{-1,\lambda}(t) = 2\lambda^{-1} + 2\lambda^{-s}\sin(\lambda x)\sin t.$$

Thus,

$$\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} \geq 2\lambda^{-s}\|\sin(\lambda x)\|_{H^s}|\sin t| - 2\lambda^{-1}\|1\|_{H^s}.$$

Letting $\lambda \to \infty$ in the above inequality, we have

$$\liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} \gtrsim |\sin t|. \tag{4.7}$$

Summing (4.6) and (4.7) yields (4.2).

Next, we consider the case $b \neq 3$. Lemma 3.3 implies that

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^1} \lesssim \lambda^{-\theta_s}, \quad \lambda \gg 1. \tag{4.8}$$

Applying the interpolation inequality

$$\|\varphi\|_{H^s} \leq \|\varphi\|_{H^{s_1}}^{(s_2-s)/(s_2-s_1)}\|\varphi\|_{H^{s_2}}^{(s-s_1)/(s_2-s_1)}$$

with $s_1 = 1$ and $s_2 = [s] + 2 = k$ and using (4.3) and (4.8), we get

$$\begin{aligned}\left\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\right\|_{H^s} &\leq \left\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\right\|_{H^1}^{(k-s)/(k-1)}\left\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\right\|_{H^k}^{(s-1)/(k-1)} \\ &\lesssim \lambda^{-\theta_s(k-s)/(k-1)}\lambda^{(k-s)(s-1)/(k-1)} \\ &\lesssim \lambda^{-(\theta_s - s+1)(k-s)/(k-1)}, \quad \lambda \gg 1.\end{aligned}$$

Hence,

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^s} \lesssim \lambda^{-\epsilon_s}, \quad \lambda \gg 1,$$

where $\epsilon_s = (s-1)(k-s)/(k-1)$. The rest of the proof is similar to the case $b = 3$. This completes the proof of the theorem. □

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