Z. Angew. Math. Phys. 64 (2013), 1327–1336

0044-2275/13/041327-10
published online December 14, 2012
DOI 10.1007/s00033-012-0290-0

Zeitschrift für angewandte Mathematik und Physik ZAMP

On the Stokes problem with data in L^1

Antonio Russo and Alfonsina Tartaglione

Abstract. We consider the steady Stokes equations in bounded and exterior domains Ω of $\mathbb{R}^3$ with boundary data and forces in L^1. We prove existence and uniqueness of a weak solution with gradient in the Iwaniek–Sbordone *grand Lebesgue space* $L^{\frac{3}{2})}$.

Mathematics Subject Classification (2010). Primary 76D05 · 35Q30; Secondary 76D07 · 35J47.

Keywords. Steady-state Stokes equations · Boundary-value problem · Bounded Lipschitz domains.

1. Introduction and statement of the results

Let Ω be a bounded domain of $\mathbb{R}^3$ defined by

$$\Omega = \Omega_0 \setminus \overline{\Omega}', \quad \Omega' = \bigcup_{i=1}^{m} \Omega_i, \tag{1}$$

where Ω_0 and Ω_i are bounded domains of $\mathbb{R}^3$ with connected boundaries such that $\overline{\Omega}_i \subset \Omega_0$ and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset, i \neq j$. Let $\boldsymbol{f}$ be an assigned field on Ω. The classical Stokes problem is to find a solution of the equations[1]

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2}$$

where $\boldsymbol{u} : \Omega \to \mathbb{R}^3, p : \Omega \to \mathbb{R}$ are the (unknown) velocity and pressure fields. It is well-known that if Ω is of class C^1 and $\boldsymbol{f} \in L^t(\Omega)$ $(t > 1)$, then (2) has a unique weak solution $(\boldsymbol{u}, p) \in W^{1,3t/(3-t)}_{\sigma,0}(\Omega) \times L^{3t/(3-t)}(\Omega)$, that is,

$$\int\limits_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi} - \int\limits_\Omega p \operatorname{div} \boldsymbol{\phi} + \int\limits_\Omega \boldsymbol{f} \cdot \boldsymbol{\phi} = 0, \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega),$$

and the following estimate holds [2,7]

$$\|\boldsymbol{u}\|_{W^{1,3t/(3-t)}(\Omega)} + \|p\|_{L^{3t/(3-t)}(\Omega)} \leq c \|\boldsymbol{f}\|_{L^t(\Omega)}. \tag{3}$$

Moreover, if $\boldsymbol{f} \in \mathcal{H}^1(\Omega)$, then one shows that (2) has a unique solution $(\boldsymbol{u}, p) \in [W^{2,1}(\Omega) \cap W^{3/2,2}_{\sigma,0}(\Omega)] \times W^{1,1}(\Omega)$ and

---

[1] Unless otherwise specified we use the notation of [6]; subscript σ in a function space $\mathcal{C}_\sigma(\Omega)$ means that the fields in $\mathcal{C}_\sigma(\Omega)$ are (weakly) divergence free in Ω. $\mathcal{H}^1(\Omega)$ is the space of all functions in $L^1(\Omega)$ whose zero extension to $\mathbb{R}^3$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^3)$. To alleviate notation, we do not distinguish function spaces for scalar and vector (or tensor) valued functions. Thus, for instance, $\boldsymbol{\varphi} \in L^q(\Omega)$ means that every component φ_i of $\boldsymbol{\varphi}$ belongs to $L^q(\Omega)$ and $\|\boldsymbol{\varphi}\|^q_{L^q(\Omega)} = \int_\Omega |\boldsymbol{\varphi}|^q$.

$$\|\boldsymbol{u}\|_{W^{2,1}(\Omega)} + \|p\|_{W^{1,1}(\Omega)} \le c\|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)}. \tag{4}$$

The main purpose of this paper is to prove that in the borderline case $\boldsymbol{f} \in L^1(\Omega)$ and for Lipschtz domains, a solution of (2) exists in a slightly larger space than $W^{1,\frac{3}{2}}_{\sigma,0}(\Omega)$, the so-called *grand Sobolev space* $W^{1,\frac{3}{2})}_{\sigma,0}(\Omega)$, introduced by Iwaniec and Sbordone [10] and defined as the set of all fields $\boldsymbol{u} \in W^{1,1}_{\sigma,0}(\Omega)$ such that[2]

$$\sup_{q\in(1,3/2)} \left\{ \left(\tfrac{3}{2} - q\right) \frac{1}{|\Omega|} \int\limits_{\Omega} |\nabla \boldsymbol{u}|^q \right\}^{\frac{1}{q}} = \|\boldsymbol{u}\|_{W^{1,\frac{3}{2})}(\Omega)} < +\infty. \tag{5}$$

Indeed, we shall prove the following existence and uniqueness theorem.

Theorem 1. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^3$. If $\boldsymbol{f} \in L^1(\Omega)$, then (2) has a unique solution $(\boldsymbol{u}, p) \in W^{1,\frac{3}{2})}_{\sigma,0}(\Omega) \times L^{\frac{3}{2})}(\Omega)$ and*

$$\|\boldsymbol{u}\|_{W^{1,\frac{3}{2})}(\Omega)} + \|p\|_{L^{\frac{3}{2})}(\Omega)} \le c\|\boldsymbol{f}\|_{L^1(\Omega)}. \tag{6}$$

For more regular domains, the above results can be extended to the more general problem

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad &\text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = \gamma \quad &\text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad &\text{on} \quad \partial\Omega, \\ \int\limits_{\Omega} \gamma = \int\limits_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}, & \end{aligned} \tag{7}$$

where $\boldsymbol{n}$ is the unit outward (with respect to Ω) normal to $\partial\Omega$.

It holds

Theorem 2. *Let Ω be a bounded domain of $\mathbb{R}^3$ of class C^2. If $\boldsymbol{a} \in L^1(\partial\Omega)$, $\boldsymbol{f} \in L^1(\Omega)$ and $\gamma \in \mathcal{H}^1(\Omega)$, then (7) has a weak solution $(\boldsymbol{u}, p) \in W^{1,\frac{3}{2})}_{\sigma,\mathrm{loc}}(\Omega) \times L^{\frac{3}{2})}_{\mathrm{loc}}(\Omega)$ and*

$$\begin{aligned} \|\boldsymbol{u}\|_{W^{1,\frac{3}{2})}(\Omega')} + \|p\|_{L^{\frac{3}{2})}(\Omega')} + \|\boldsymbol{u}\|_{L^{3)}(\Omega)} \le c\big\{ &\|\boldsymbol{a}\|_{L^1(\partial\Omega)} \\ + &\|\boldsymbol{f}\|_{L^1(\Omega)} + \|\gamma\|_{\mathcal{H}^1(\Omega)} \big\}, \end{aligned} \tag{8}$$

for all $\Omega' \Subset \Omega$, with c depending on Ω and Ω'. Moreover, the solution is unique in the class of all fields $\boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega)$ that satisfy the relation[3]

$$\int\limits_{\Omega} \boldsymbol{u} \cdot \boldsymbol{\phi} = \int\limits_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{T}(\boldsymbol{z}, \vartheta) \cdot \boldsymbol{n} + \int\limits_{\Omega} \gamma\vartheta + \int\limits_{\Omega} \boldsymbol{f} \cdot \boldsymbol{z}, \tag{9}$$

for all $\boldsymbol{\phi} \in C^\infty_0(\Omega)$, where $(\boldsymbol{z}, \vartheta)$ is the solution of

$$\begin{aligned} \Delta \boldsymbol{z} - \nabla \vartheta = \boldsymbol{\phi} \quad &\text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 \quad &\text{in} \quad \Omega, \\ \boldsymbol{z} = \boldsymbol{0} \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{10}$$

and

$$T_{ij}(\boldsymbol{z}, \vartheta) = \partial_j z_i + \partial_i z_j - \vartheta\delta_{ij}$$

is the Cauchy stress tensor.

[2] $W^{1,\frac{3}{2})}_{\sigma,0}(\Omega)$ is a Banach space. For the basic properties of the grand Sobolev spaces we quote [4,9] and [10].

[3] See Remark 3.3.

We shall also consider the problem

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= \gamma \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{a} \quad \text{on} \quad \partial\Omega \\ \boldsymbol{u} &\in L^3(\complement S_{R_0}) \cap L^{3)}_{\text{loc}}(\Omega) \end{aligned} \tag{11}$$

in the exterior domain

$$\Omega = \mathbb{R}^3 \setminus \overline{\Omega'},$$

where Ω' is the domain defined in (1) and $R_0 > \operatorname{diam} \Omega'$, under the assumptions

$$\gamma, \boldsymbol{f} \in \mathcal{H}^1(\Omega), \quad \boldsymbol{a} \in L^1(\partial\Omega). \tag{12}$$

Denote by $\mathfrak{C}$ the linear space of the solutions of the equations

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= \boldsymbol{0} \quad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on} \quad \partial\Omega \\ \boldsymbol{u} &\in D^{1,q}(\Omega), \quad q > 3/2. \end{aligned} \tag{13}$$

It holds

Theorem 3. *Let Ω be an exterior domain of $\mathbb{R}^3$ of class $C^{1,\lambda}$. If $\boldsymbol{a}, \boldsymbol{f}, \gamma$ satisfy (12) and*

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{\psi}' = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}[\boldsymbol{\psi}'] + \int_{\Omega} \gamma P[\boldsymbol{\psi}'], \quad \forall\, \boldsymbol{\psi}' \in \mathfrak{C}, \tag{14}$$

with $\boldsymbol{v}[\boldsymbol{\psi}']$, $P[\boldsymbol{\psi}']$ and $\mathfrak{C}$ defined in section 3, then (11) has a solution $(\boldsymbol{u}, p)$ and

$$\begin{aligned} \|\boldsymbol{u}\|_{L^3(\complement S_{R_0})} + \|\boldsymbol{u}\|_{L^{3)}(\Omega_{R_0})} + \|\nabla \boldsymbol{u}\|_{L^{3/2}(\complement S_{R_0})} + \|p\|_{L^{3/2}(\complement S_{R_0})} \\ \le c\left\{\|\boldsymbol{a}\|_{L^1(\partial\Omega)} + \|\boldsymbol{f}\|_{\mathcal{H}^1(\Omega)} + \|\gamma\|_{\mathcal{H}^1(\Omega)}\right\}. \end{aligned}$$

Moreover, uniqueness holds in the class of all fields $\boldsymbol{u} \in L^3(\complement S_{R_0}) \cap L^1_{\text{loc}}(\Omega)$, that satisfy (9) for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$, with $\boldsymbol{z} \in D^{1,q}(\Omega)(q > 3/2)$ solution of (10). In this function class (14) is also necessary for the existence of a solution of (11).

2. Proof of Theorem 1

We premise the following well-known results.

Lemma 1. [1,13] *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^3$ and let $\boldsymbol{f} = \operatorname{div} \boldsymbol{F}$. There is a positive constant ϵ depending only on Ω such that if $\boldsymbol{F} \in L^q(\Omega)$, with $q \in (-\epsilon + (3+\epsilon)/(2+\epsilon), \epsilon + 3/2)$, then (2) has a unique solution $(\boldsymbol{u}, p) \in W^{1,q}_{\sigma,0}(\Omega) \times L^q(\Omega)$ and*

$$\|\boldsymbol{u}\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \le c\|\boldsymbol{F}\|_{L^q(\Omega)}, \tag{15}$$

with c depending only on Ω and ϵ.

Lemma 2. [4] *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^3$. For all $\boldsymbol{f} \in L^1(\Omega)$, there is $\boldsymbol{F} \in L^{\frac{3}{2})}(\Omega)$ such that $\operatorname{div} \boldsymbol{F} = \boldsymbol{f}$ and for all $q \in [1, 3/2)$*

$$[3(1-q) + q] \int_{\Omega} |\boldsymbol{F}|^q \le c|\Omega|^{(3(1-q)+q)/3} \|\boldsymbol{f}\|^q_{L^1(\Omega)}, \tag{16}$$

where c is an absolute positive constant.

Proof. We recall the proof in [4], since we shall need it in the sequel. A solution of $\operatorname{div}\boldsymbol{F}=\boldsymbol{f}$ is given by the gradient of the Newtonian potential

$$\boldsymbol{F}(x)=\frac{1}{4\pi}\int\limits_{\Omega}\frac{(x-y)\otimes\boldsymbol{f}(y)}{|x-y|^3}\mathrm{d}v_y.$$

Hence, by the Minkowski inequality,

$$\begin{aligned}\|\boldsymbol{F}\|_{L^q(\Omega)}&\le\frac{1}{4\pi}\int\limits_{\Omega}\left\|\frac{1}{|x-y|}\right\|_{L^q(\Omega)}|\boldsymbol{f}(y)|\mathrm{d}y\\&\le\frac{1}{4\pi}\sup_{y\in\Omega}\left\|\frac{1}{|x-y|}\right\|_{L^q(\Omega)}\|\boldsymbol{f}\|_{L^1(\Omega)}.\end{aligned}\tag{17}$$

Hence, the desired result follows, taking into account that [4]

$$\left\|\frac{1}{|x-y|}\right\|_{L^q(\Omega)}\le\frac{(4\pi)^{2/3}|\Omega|^{(3(1-q)+q)/(3q)}}{(3(1-q)+q)^{1/q}}.$$

□

Proof of Theorem 1. By Lemma 2, there is a sequence $\boldsymbol{F}_k$ such that $\boldsymbol{f}_k=\operatorname{div}\boldsymbol{F}_k$, $\boldsymbol{f}_k\to\boldsymbol{f}$ strongly in $L^1(\Omega)$ and

$$\left(\tfrac{3}{2}-q\right)\int\limits_{\Omega}|\boldsymbol{F}_k|^q\le c\|\boldsymbol{f}_k\|^q_{L^1(\Omega)},\tag{18}$$

for q in a small left neighborhood of $3/2$. The field $\boldsymbol{u}_k$ satisfies the relation

$$\int\limits_{\Omega}\nabla\boldsymbol{u}_k\cdot\nabla\boldsymbol{\varphi}=\int\limits_{\Omega}\boldsymbol{f}_k\cdot\boldsymbol{\varphi},\quad\forall\boldsymbol{\varphi}\in C^\infty_{\sigma,0}(\Omega).\tag{19}$$

To $\boldsymbol{u}_k$, we can associate a pressure field p_k which satisfies the estimate

$$\|p_k\|_{L^q(\Omega)}\le c\|\nabla\boldsymbol{u}_k\|_{L^q(\Omega)}.\tag{20}$$

By Lemma 1, the sequence $\boldsymbol{u}_k$ of the solutions to (2) with data $\boldsymbol{f}_k$ satisfies

$$\|\boldsymbol{u}_k-\boldsymbol{u}_h\|_{W^{1,q}(\Omega)}\le c\|\boldsymbol{F}_k-\boldsymbol{F}_h\|_{L^q(\Omega)}.\tag{21}$$

Putting together (18), (21), we have

$$\left(\tfrac{3}{2}-q\right)\|\boldsymbol{u}_k-\boldsymbol{u}_h\|_{W^{1,q}(\Omega)}\le c\|\boldsymbol{f}_k-\boldsymbol{f}_h\|_{L^1(\Omega)}.\tag{22}$$

Therefore, $\boldsymbol{u}_k$ is a Cauchy sequence in $W^{1,q}(\Omega)$ for $q<3/2$ so that it converges to a field $\boldsymbol{u}\in W^{1,q}(\Omega)$. Letting $k\to+\infty$ and taking into account (19), (22), we see that $\boldsymbol{u}$ is the solution of (2) and

$$\left(\tfrac{3}{2}-q\right)\|\boldsymbol{u}_k-\boldsymbol{u}\|_{W^{1,q}(\Omega)}\le c\|\boldsymbol{f}_k-\boldsymbol{f}\|_{L^1(\Omega)}.\tag{23}$$

Hence, it follows that $\boldsymbol{u}\in W^{1,\frac{3}{2})}_{\sigma,0}(\Omega)$. Moreover, from (20), it follows that $p\in L^{\frac{3}{2})}(\Omega)$ and (6) holds. To prove uniqueness, we have to show that (2) with $\boldsymbol{f}=\boldsymbol{0}$ has only the trivial solution. To this end, denote by $(\boldsymbol{u},p)\in W^{1,\frac{3}{2})}_{\sigma,0}(\Omega)\times L^{\frac{3}{2})}(\Omega)$ a solution of (2). By virtue of Lemma 1, (2) with $\boldsymbol{f}\in C^\infty_0(\Omega)$ has a solution $(\boldsymbol{v},Q)\in W^{1,q}_{\sigma,0}(\Omega)\times L^q(\Omega)$ for some $q>3$. Thus, an integration by parts yields

$$\int\limits_{\Omega}\boldsymbol{u}\cdot\boldsymbol{f}=0,\quad\forall\,\boldsymbol{f}\in C^\infty_0(\Omega).$$

Hence, the desired result follows.

Remark 2.1. Taking into account the results of [3], we have that if Ω is Lipschitz and

$$\boldsymbol{a} \in L^2(\partial\Omega), \quad \int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{f} \in L^1(\Omega),$$

then the equations

$$\begin{aligned} \Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad &\text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad &\text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{24}$$

have a weak solution $(\boldsymbol{u}, p) \in W^{1,\frac{3}{2})}_{\sigma,\mathrm{loc}}(\Omega) \times L^{\frac{3}{2})}_{\mathrm{loc}}(\Omega)$ and

$$\|\boldsymbol{u}\|_{L^{3)}(\Omega)} \le c\left\{\|\boldsymbol{a}\|_{L^2(\partial\Omega)} + \|\boldsymbol{f}\|_{L^1(\Omega)}\right\}. \tag{25}$$

If Ω is of class C^1, then we can take $\boldsymbol{a} \in L^q(\partial\Omega)$ [2], $q > 1$, and it holds

$$\|\boldsymbol{u}\|_{L^{3)}(\Omega)} \le c\left\{\|\boldsymbol{a}\|_{L^q(\partial\Omega)} + \|\boldsymbol{f}\|_{L^1(\Omega)}\right\}. \tag{26}$$

□

3. Proof of Theorem 2

The equations

$$\begin{aligned} \Delta\boldsymbol{u} - \nabla p = \boldsymbol{0} \\ \operatorname{div}\boldsymbol{u} = 0 \end{aligned} \tag{27}$$

admit the fundamental solution $(\boldsymbol{U}(x-y), \boldsymbol{q}(x-y))$, with

$$\begin{aligned} \boldsymbol{U}(\boldsymbol{t}) &= -\frac{1}{8\pi|\boldsymbol{t}|}\left\{\boldsymbol{1} + \frac{\boldsymbol{t}\otimes\boldsymbol{t}}{|\boldsymbol{t}|^2}\right\}, \\ \boldsymbol{q}(\boldsymbol{t}) &= -\frac{\boldsymbol{t}}{4\pi|\boldsymbol{t}|^3}. \end{aligned}$$

The simple and double Stokes layer potential with densities $\boldsymbol{\psi}$ and $\boldsymbol{\varphi} \in L^q(\partial\Omega)$ are the pairs defined, respectively, by [11]

$$\begin{aligned} \boldsymbol{v}[\boldsymbol{\psi}](x) &= \int_{\partial\Omega} \boldsymbol{U}(x-\zeta)\cdot\boldsymbol{\psi}(\zeta)\mathrm{d}\sigma_\zeta, \\ P[\boldsymbol{\psi}](x) &= \int_{\partial\Omega} \boldsymbol{q}(x-\zeta)\cdot\boldsymbol{\psi}(\zeta)\mathrm{d}\sigma_\zeta, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{w}[\boldsymbol{\varphi}](x) &= \int_{\partial\Omega} \boldsymbol{T}'(\boldsymbol{U},\boldsymbol{q})(x-\zeta)\cdot(\boldsymbol{\varphi}\otimes\boldsymbol{n})(\zeta)\mathrm{d}\sigma_\zeta, \\ \varpi[\boldsymbol{\varphi}](x) &= -2\operatorname{div}\int_{\partial\Omega} [\boldsymbol{q}(x-\zeta)\cdot\boldsymbol{\varphi}(\zeta)]\boldsymbol{n}(\zeta)\mathrm{d}\sigma_\zeta, \end{aligned}$$

where $T'_{ij}(\boldsymbol{U},\boldsymbol{q})(x-\zeta) = [(\partial_{\zeta_i}U_{jk} + \partial_{\zeta_k}U_{kj}) + q_k](x-\zeta)$. They are analytical solutions of (27) in $\mathbb{R}^3 \setminus \partial\Omega$ and, if Ω is of class $C^{1,\lambda}$ for some $\lambda \in (0,1)$, then the limits [11,12]

$$\begin{aligned} &\lim_{\alpha\to 0^+} \boldsymbol{w}[\boldsymbol{\varphi}](x-\alpha\boldsymbol{n}(\xi)) = \mathcal{W}[\boldsymbol{\varphi}](\xi) = (\tfrac{1}{2}\mathcal{I} + \mathcal{K})[\boldsymbol{\varphi}](\xi) \in L^1(\partial\Omega), \\ &\lim_{\alpha\to 0^+} \{\boldsymbol{T}(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])\cdot\boldsymbol{n}\}(x-\alpha\boldsymbol{n}(\xi)) = -(\tfrac{1}{2}\mathcal{I} + \mathcal{K}')[\boldsymbol{\varphi}](\xi) \in L^\infty(\partial\Omega) \end{aligned}$$

exist for almost all $\xi \in \partial\Omega$, where

$$\mathcal{K}' : L^\infty(\partial\Omega) \to C^{0,\mu}(\partial\Omega)$$

($\mu < \lambda$) is the adjoint map of $\mathcal{K}$ [11,12]. Hence, it follows that $\mathcal{K} : L^1(\partial\Omega) \to L^1(\partial\Omega)$ is completely continuous so that the operator $\mathcal{W} : L^1(\partial\Omega) \to L^1(\partial\Omega)$ is Fredholm with index zero and Kern $(\frac{1}{2}\mathcal{I}+\mathcal{K}') = \mathrm{sp}\{\boldsymbol{n}\} \otimes \mathfrak{F}$, where $\mathfrak{F}$ is the $n(n+1)m/2$ dimensional space

$$\mathfrak{F} = \{\boldsymbol{\psi} : \ \boldsymbol{v}[\boldsymbol{\psi}]_{|\Omega_i} = \text{rigid motions}, \boldsymbol{v}[\boldsymbol{\psi}]_{|\complement\Omega_0} = \mathbf{0}, P[\boldsymbol{\psi}]_{|\Omega_i \cup \complement\Omega_0} = 0\} \subset C^{1,\lambda}(\partial\Omega),$$

$i = 1 \ldots, m$ [17,15,16,18]. If Ω is of class C^1 by virtue of the results of [2], we have that $\mathcal{K}$ is compact from $L^q(\partial\Omega)$ into itself and from $W^{1,q}(\partial\Omega)$ into itself for all $q \in (1, +\infty)$.

Lemma 3. *Let Ω be a bounded domain of $\mathbb{R}^3$ of class $C^{1,\lambda}$, for some $\lambda > 0$. If $\boldsymbol{a} \in L^1(\partial\Omega)$ satisfies*

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} = 0, \tag{28}$$

then the equations

$$\begin{aligned} \Delta \boldsymbol{v} - \nabla p &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{v} &= \boldsymbol{a} \quad \text{on} \quad \partial\Omega \end{aligned} \tag{29}$$

have a solution expressed by

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{v}[\boldsymbol{\psi}], \\ p &= \varpi[\boldsymbol{\psi}] + P[\boldsymbol{\psi}], \end{aligned} \tag{30}$$

for some $\boldsymbol{\varphi} \in L^1(\partial\Omega)$ and $\boldsymbol{\psi} \in \mathfrak{F}$. $\boldsymbol{u}, p$ are analytical in Ω and $\boldsymbol{u}$ takes the value $\boldsymbol{a}$ pointwise almost everywhere, that is,

$$\lim_{\alpha \to 0^+} \boldsymbol{u}(x - \alpha \boldsymbol{n}(\xi)) = \boldsymbol{a}(\xi) \tag{31}$$

for almost all $\xi \in \partial\Omega$, and

$$\|\boldsymbol{u}\|_{L^{3)}(\Omega)} \le c\|\boldsymbol{a}\|_{L^1(\partial\Omega)}. \tag{32}$$

If Ω is of class C^1 and $\boldsymbol{a} \in L^q(\partial\Omega)$, for some $q > 1$, then the above result hold with $\boldsymbol{\varphi} \in L^q(\partial\Omega)$ and

$$\|\boldsymbol{u}\|_{L^3(\Omega)} \le c\|\boldsymbol{a}\|_{L^q(\partial\Omega)}. \tag{33}$$

Proof. By a simple application of Fredholm's alternative, we see that the functional equation

$$\mathcal{W}[\boldsymbol{\varphi}] = \boldsymbol{a} - \boldsymbol{v}[\boldsymbol{\psi}]_{|\partial\Omega'} \tag{34}$$

has a solution $\boldsymbol{\psi} \in L^1(\partial\Omega)$ for some $\boldsymbol{\psi} \in \mathfrak{F}$. Recall that $\boldsymbol{v}[\boldsymbol{\psi}] \in C^{1,\lambda}(\overline{\Omega})$ [12]. By Gagliardo's trace theorem [5], there is a field $\boldsymbol{\omega} \in W^{1,1}(\Omega)$ such that

$$\boldsymbol{u}(x) = \int_\Omega \boldsymbol{T}(\boldsymbol{w}[\boldsymbol{\varphi}], \varpi[\boldsymbol{\varphi}])(x - y) \cdot \nabla \boldsymbol{\omega}(y) \mathrm{d}v_y.$$

Hence,

$$|\boldsymbol{u}(x)| \le c \int_\Omega \frac{|\nabla \boldsymbol{\omega}(y)|}{|x - y|^2} \mathrm{d}v_y.$$

Therefore, (32) follows by repeating the argument used in the proof of Lemma 2. The last part of the lemma is a consequence of the regularity properties of the layer potentials. □

Remark 3.2. The operator $\mathcal{K}$ maps subspaces of $L^1(\partial\Omega)$ in more regular spaces with natural estimates (see [11, Ch. 3] and [12, Sections 14, 15]). For instance, $\mathcal{K}[L^q(\partial\Omega)] = C^{0,\mu}(\partial\Omega)$ $(q > 2/\lambda)$ for all $\mu < \lambda - 2/q$. Hence, it follows that, if $\boldsymbol{a} \in C^{0,\mu}(\partial\Omega)$, $\mu \in [0, \lambda]$, then

$$\|\boldsymbol{u}\|_{C^{0,\mu}(\overline{\Omega})} \le c\|\boldsymbol{a}\|_{C^{0,\mu}(\partial\Omega)}. \tag{35}$$

Moreover,

(i) if $\boldsymbol{a} \in C^{1,\mu}(\partial\Omega)$, $\mu > 1 - \lambda$, then

$$\|\boldsymbol{u}\|_{C^{1,\mu}(\overline{\Omega})} + \|p\|_{C^{0,\mu}(\overline{\Omega})} \le c\|\boldsymbol{a}\|_{C^{1,\mu}(\partial\Omega)};$$

(ii) if $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $q \in (1, +\infty)$, then

$$\|\boldsymbol{u}\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \le c\|\boldsymbol{a}\|_{W^{1-1/q,q}(\partial\Omega)}.$$

Property (ii) also holds for domains of class C^1 and for $q \in ((3+\epsilon)/(2+\epsilon), 3+\epsilon)$ in Lipschitz domains, where ϵ is a positive number depending only on Ω [1]. Other classical regularity results as well as natural estimates can be find in [13].

Proof of Theorem 2. For $\gamma \in \mathcal{H}^1(\Omega)$, the field

$$\mathcal{V}[\gamma](x) = \frac{1}{4\pi}\int\limits_{\Omega} \frac{(x-y)\gamma(y)}{|x-y|^3}\,\mathrm{d}v_y$$

belongs to $W^{1,1}(\Omega)$ so that its trace satisfies

$$\|\mathrm{tr}\,\mathcal{V}[\gamma]_{\partial\Omega}\|_{L^1(\partial\Omega)} \le c\|\mathcal{V}[\gamma]\|_{W^{1,1}(\Omega)} \le c\|\gamma\|_{\mathcal{H}^1(\Omega)}. \tag{36}$$

By Lemma 3, the equations

$$\begin{aligned} \Delta\boldsymbol{v} - \nabla Q &= \boldsymbol{0} && \text{in} \quad \Omega,\\ \operatorname{div}\boldsymbol{v} &= 0 && \text{in} \quad \Omega,\\ \boldsymbol{v} &= \boldsymbol{a} - \mathrm{tr}\,\mathcal{V}[\gamma]_{\partial\Omega} && \text{on} \quad \partial\Omega,\\ \int\limits_{\partial\Omega}(\boldsymbol{a} - \mathrm{tr}\,\mathcal{V}[\gamma]_{\partial\Omega})\cdot\boldsymbol{n} &= 0 \end{aligned} \tag{37}$$

have a solution $(\boldsymbol{v}, Q)$, and denoting by $(\boldsymbol{v}_f, p_f)$, the solution of (2) given by Theorem 1, it is obvious that

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{v} + \boldsymbol{v}_f + \mathcal{V}[\gamma],\\ p &= Q + p_f + \gamma \end{aligned}$$

satisfies (7) and (8).

If $(\boldsymbol{u}_k, p_k)$ is the solution of (7) for a regular $\boldsymbol{a}_k$, an integration by parts yields

$$\int\limits_{\Omega}\boldsymbol{u}_k\cdot\boldsymbol{\phi} = \int\limits_{\partial\Omega}\boldsymbol{a}_k\cdot\boldsymbol{T}(\boldsymbol{z},\vartheta)\cdot\boldsymbol{n} + \int\limits_{\Omega}\gamma\vartheta + \int\limits_{\Omega}\boldsymbol{f}\cdot\boldsymbol{z} \tag{38}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$, with $(\boldsymbol{z}, \varpi)$ solution of (10). Let $\boldsymbol{a}_k \to \boldsymbol{a}$ strongly in $L^1(\partial\Omega)$. Since

$$\|\boldsymbol{u} - \boldsymbol{u}_k\|_{L^q(\Omega)} \le c\|\boldsymbol{a} - \boldsymbol{a}_k\|_{L^1(\partial\Omega)}$$

for some $q > 1$, we can let $k \to +\infty$ in (38) to see that $\boldsymbol{u}$ satisfies (9). Calling $(\boldsymbol{u}, p)$, a very weak solution of (2) (in the sense of J. Nečas [14]), we have that (7) has a unique very weak solution. □

The following problem

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= \gamma \quad \text{in} \quad \Omega,\\ \boldsymbol{v} &= \boldsymbol{a} \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{39}$$

is of some interest in the theory of the Navier–Stokes equations (see Ch. III of [6]). As an immediate consequence of Theorem 2, we have

Corollary 1. *Let Ω be a bounded domain of $\mathbb{R}^3$ of class C^2. If $\gamma \in \mathcal{H}^1(\Omega)$ and $\boldsymbol{a} \in L^1(\partial\Omega)$, then (39) has a weak solution $\boldsymbol{v} \in W^{1,3)}_{\mathrm{loc}}(\Omega)$ and*

$$\|\boldsymbol{v}\|_{L^{3)}(\Omega)} \le c\left\{\|\boldsymbol{a}\|_{L^1(\partial\Omega)} + \|\gamma\|_{\mathcal{H}^1(\Omega)}\right\}. \tag{40}$$

Remark 3.3. Note that a very weak solution of (7) (in the sense of J. Nečas [14]) can also be defined as a field $\boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega)$ which satisfies[4]

$$\int\limits_\Omega \boldsymbol{u}\cdot\boldsymbol{\phi} = \int\limits_{\partial\Omega} \boldsymbol{a}\cdot(\partial_n \boldsymbol{z} - \vartheta\boldsymbol{n}) + \int\limits_\Omega \vartheta\gamma + \int\limits_\Omega \boldsymbol{f}\cdot\boldsymbol{z} \tag{41}$$

for all $\boldsymbol{\phi} \in W^{1,\infty}_0(\Omega)$. Now, choosing first $\boldsymbol{\phi} = \Delta\boldsymbol{\eta}$, with $\boldsymbol{\eta} \in C^2_{\sigma,0}(\Omega) = \{\boldsymbol{\eta} \in C^2_\sigma(\overline{\Omega}) : \boldsymbol{\eta}_{|\partial\Omega} = \boldsymbol{0}\}$, then $\boldsymbol{\phi} = \nabla\omega$ with $\omega \in C^1(\overline{\Omega})$, we see that $\boldsymbol{u}$ also satisfies the relations

$$\begin{aligned}
&\int\limits_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\eta} = \int\limits_{\partial\Omega} \boldsymbol{a}\cdot\partial_n\boldsymbol{\eta} + \int\limits_\Omega \boldsymbol{f}\cdot\boldsymbol{\eta}, \quad \forall\,\boldsymbol{\eta} \in C^1_{\sigma,0}(\Omega)\cap C^2(\overline{\Omega}),\\
&\int\limits_\Omega \boldsymbol{u}\cdot\nabla\omega = \int\limits_{\partial\Omega} \omega\boldsymbol{a}\cdot\boldsymbol{n} - \int\limits_\Omega \gamma\omega, \quad \forall\,\omega \in C^1(\overline{\Omega}),
\end{aligned} \tag{42}$$

that represent a more popular definition of a very weak solution to (7) [8]. Note that, in particular, (42) yields

$$\begin{aligned}
&\int\limits_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\eta} = \int\limits_\Omega \boldsymbol{f}\cdot\boldsymbol{\eta}, \quad \forall\,\boldsymbol{\eta} \in C^\infty_{\sigma,0}(\Omega),\\
&\int\limits_\Omega \boldsymbol{u}\cdot\nabla\omega = -\int\limits_\Omega \gamma\omega, \quad \forall\,\omega \in C^\infty_0(\Omega),
\end{aligned} \tag{43}$$

$i.e$, $\boldsymbol{u}$ satisfies (7) in the sense of the distributions.

4. Proof of Theorem 3

We can repeat the classical argument of the potential theory we outlined in Sect. 3 to see that the problem

$$\begin{aligned}
\Delta\boldsymbol{u} - \nabla p &= \boldsymbol{0} \quad \text{in} \quad \Omega,\\
\operatorname{div}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega,\\
\boldsymbol{u} &= \boldsymbol{a} \quad \text{on} \quad \partial\Omega
\end{aligned} \tag{44}$$

has a solution expressed by (30) for some $\boldsymbol{\varphi} \in L^1(\partial\Omega)$ and $\boldsymbol{\psi} \in \mathfrak{G}$, with

$$\mathfrak{G} = \{\boldsymbol{\psi} : \ \boldsymbol{v}[\boldsymbol{\psi}]_{|\Omega_i} = \text{rigid motions}, P[\boldsymbol{\psi}]_{|\Omega_i} = 0, i = 1\ldots, m\} \subset C^{1,\lambda}(\partial\Omega).$$

Let $\mathfrak{C}$ be the linear subspace of all $\boldsymbol{\psi} \in \mathfrak{G}$ such that $\boldsymbol{v}[\boldsymbol{\psi}]_{|\Omega'} = \text{constant}$[5]. A well-known argument (see, *e.g*, [17, 15, 16, 18]) assures that $\dim\mathfrak{C} = 3$ and if $\{\boldsymbol{\psi}_i\}$ is a basis of $\mathfrak{C}$, then $\int_{\partial\Omega}\boldsymbol{\psi}_i$ is a basis of $\mathbb{R}^3$. Therefore, there is $\bar{\boldsymbol{\psi}} \in \mathfrak{C}$ such that

$$\int\limits_{\partial\Omega} (\boldsymbol{\psi} + \bar{\boldsymbol{\psi}}) = \boldsymbol{0}$$

[4] Note that this corresponds to a different decomposition of the Stokes operator.
[5] Clearly, the pairs $(\boldsymbol{v}[\boldsymbol{\psi}], P[\boldsymbol{\psi}])$ are the solutions of (13).

and, putting

$$\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi} + \bar{\boldsymbol{\psi}}, \quad \boldsymbol{\kappa} = -\boldsymbol{v}[\bar{\boldsymbol{\psi}}]_{|\partial\Omega},$$

the pair

$$\begin{aligned} \boldsymbol{u}' &= \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{v}[\tilde{\boldsymbol{\psi}}] + \boldsymbol{\kappa}, \\ p' &= \varpi[\boldsymbol{\psi}] + P[\tilde{\boldsymbol{\psi}}] \end{aligned} \tag{45}$$

is an analytical solution of (44) such that $\boldsymbol{u}' - \boldsymbol{\kappa} \in L^3$ in a neighborhood of infinity. A simple integration yields

$$\int_{\partial\Omega} (\boldsymbol{a} - \boldsymbol{\kappa}) \cdot \boldsymbol{\psi}' = 0, \quad \forall\, \boldsymbol{\psi}' \in \mathfrak{C}.$$

Hence, it follows that if

$$\int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{\psi}' = 0, \quad \forall\, \boldsymbol{\psi}' \in \mathfrak{C}, \tag{46}$$

then $\boldsymbol{u}$ satisfies $(11)_4$. Completing the standard procedure of "adding" to $(\boldsymbol{u}', p')$ suitable volume potentials, we see that (11) has a solution expressed by

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{v}[\boldsymbol{\psi}] + \mathcal{E}[\boldsymbol{f}] + \mathcal{V}[\gamma], \\ p &= \varpi[\boldsymbol{\psi}] + P[\boldsymbol{\psi}] + \mathcal{Q}[\boldsymbol{f}] + \gamma \end{aligned} \tag{47}$$

for some $\boldsymbol{\varphi} \in L^1(\partial\Omega)$ and $\boldsymbol{\psi} \in C^{1,\lambda}(\partial\Omega)$ such that $\int_{\partial\Omega} \boldsymbol{\psi} = \mathbf{0}$, where

$$\begin{aligned} \mathcal{E}[\boldsymbol{f}] &= \int_{\Omega} \boldsymbol{U}(x-y) \cdot \boldsymbol{f}(y) \mathrm{d}v_y, \\ \mathcal{Q}[\boldsymbol{f}] &= \int_{\Omega} \boldsymbol{q}(x-y) \cdot \boldsymbol{f}(y) \mathrm{d}v_y. \end{aligned} \tag{48}$$

Let $\boldsymbol{a}_k$ be a sequence of regular fields on $\partial\Omega$ which converges strongly to $\boldsymbol{a}$ in $L^1(\partial\Omega)$ and let $(\boldsymbol{u}_k, p_k)$ be the solution of (11) with data $(\boldsymbol{a}_k, \boldsymbol{f}, \gamma)$. Let g be a regular function in $\mathbb{R}^3$, vanishing outside S_{2R}, equal to 1 in S_R and such that $|\nabla g| \le cR^{-1}$. By an integration by parts, we have

$$\begin{aligned} \int_{\Omega} g\boldsymbol{u}_k \cdot \boldsymbol{\phi} &= \int_{\partial\Omega} \boldsymbol{a}_k \cdot \boldsymbol{T}(\boldsymbol{z}, \vartheta) \cdot \boldsymbol{n} + \int_{\Omega} g\vartheta\gamma + \int_{\Omega} g\boldsymbol{f} \cdot \boldsymbol{z} \\ &\quad - \int_{\Omega} [\boldsymbol{u}_k \cdot \boldsymbol{T}(\boldsymbol{z}, \vartheta) - \boldsymbol{z} \cdot \boldsymbol{T}(\boldsymbol{u}_k, p_k)] \cdot \nabla g. \end{aligned} \tag{49}$$

By the properties of the function g, Hölder inequality, the summability properties of $(\boldsymbol{u}_k, p_k)$ and the behavior at infinity of $(\boldsymbol{z}, \vartheta)$

$$\begin{aligned} \left| \int_{\Omega} \boldsymbol{u}_k \cdot \boldsymbol{T}(\boldsymbol{z}, \vartheta) \cdot \nabla g \right| &\le \|\boldsymbol{u}_k\|_{L^3(S_{2R}\setminus S_R)} \big\{ \|\nabla \boldsymbol{z}\|_{L^{3/2}(S_{2R}\setminus S_R)} + \|\vartheta\|_{L^{3/2}(S_{2R}\setminus S_R)} \big\}, \\ &\le c\|\boldsymbol{u}_k\|_{L^3(S_{2R}\setminus S_R)} \left\{ \int_R^{2R} \frac{\mathrm{d}r}{r} \right\}^{2/3} \le c\|\boldsymbol{u}_k\|_{L^3(S_{2R},\setminus S_R)} \\ \left| \int_{\Omega} \boldsymbol{z} \cdot \boldsymbol{T}(\boldsymbol{u}_k, p_k) \cdot \nabla g \right| &\le c \big\{ \|\nabla \boldsymbol{u}_k\|_{L^{3/2}(S_{2R}\setminus S_R)} + \|p_k\|_{L^{3/2}(S_{2R}\setminus S_R)} \big\}. \end{aligned}$$

Therefore, letting $R \to +\infty$ in (49) yields

$$\int_{\Omega} \boldsymbol{u}_k \cdot \boldsymbol{\phi} = \int_{\partial\Omega} \boldsymbol{a}_k \cdot \boldsymbol{T}(\boldsymbol{z}, \vartheta) \cdot \boldsymbol{n} + \int_{\Omega} \vartheta\gamma + \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{z}.$$

Hence, (9) follows by letting $k \to +\infty$.

To prove the last part of the theorem, it is sufficient to choose in (9) every pair $(\boldsymbol{v}[\boldsymbol{\psi}'], P[\boldsymbol{\psi}'])$, with $\boldsymbol{\psi}' \in \mathfrak{C}$.

References

1. Brown, R.M., Shen, Z.: Estimates for the Stokes operator in Lipschitz domains. Indiana Univ. Math. J. **44**, 1183–1206 (1995)
2. Fabes, E.B., Jodeit, M. Jr.., Rivière, N.M.: Potential techniques for boundary value problems on C^1 domains. Acta Math. **141**, 165–186 (1978)
3. Fabes, E.B., Kenig, C.E., Verchota, G.C.: Boundary value problems for the Stokes system on Lipschitz domains. Duke Math. J. **57**, 769–793 (1988)
4. Fiorenza, A., Sbordone, C.: Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1. Studia Math. **127**, 223–231 (1998)
5. Gagliardo, E.: Caratterizzazione delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabli. Rend. Sem. Mat. Padova **27**, 284–305 (1957)
6. Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady State Problems, 2nd edn, Springer Monographs in Mathematics, Springer, Berlin (2011)
7. Galdi, G.P., Simader, C.G., Sohr, H.: On the Stokes problem in Lipschitz domains. Ann. Mat. Pura Appl. **167**, 147–163 (1994)
8. Galdi, G.P., Simader, C.G., Sohr, H.: A class of solutions to stationary Stokes and Navier–Stokes equations with boundary data in $W^{-1/q,q}$. Math. Ann. **331**, 41–74 (2005)
9. Greco, L., Iwaniec, T., Sbordone, C.: Inverting the p-harmonic operator. Manuscripta Math. **92**, 249–258 (1997)
10. Iwaniec, T., Sbordone, C.: On the integrability of the Jacobian under minimal hypotheses. Arch. Ration. Mech. Anal. **119**, 129–143 (1992)
11. Ladyzhenskaia, O.A.: The Mathematical Theory of Viscous Incompressible Fluid. Gordon and Breach, London (1969)
12. Miranda, C.: Partial Differential Equations of Elliptic Type. Springer, Berlin (1970)
13. Mitrea, M., Taylor, M.: Navier–Stokes equations on Lipschitz domains in Riemannian manifolds. Math. Ann. **321**, 955–987 (2001)
14. Nečas, J.: Les méthodes directes en théorie des équations élliptiques. Masson-Paris and Academie-Prague (1967)
15. Russo, R.: On the existence of solutions to the stationary Navier–Stokes equations. Ricerche Mat. **52**, 285–348 (2003)
16. Russo, R., Tartaglione, A.: On the Robin problem for Stokes and Navier–Stokes systems. Math. Models Methods Appl. Sci. **19**, 701–716 (2006)
17. Russo, A., Tartaglione, A.: On the Oseen and Navier-Stokes systems with a slip boundary condition. Appl. Math. Lett. **22**, 674–678 (2009)
18. Tartaglione, A.: On the Stokes problem with slip boundary conditions. CAIM **1**, 186–205 (2010)

Antonio Russo
Largo Aldifreda 9
81100 Caserta
Italy

Alfonsina Tartaglione
Dipartimento di Matematica e Fisica
Seconda Università degli Studi di Napoli
Via Vivaldi, 43
81100 Caserta, Italy
e-mail: alfonsina.tartaglione@unina2.it

(Received: July 10, 2012)