Z. Angew. Math. Phys. 64 (2013), 1125–1144

0044-2275/13/041125-20
published online October 11, 2012
DOI 10.1007/s00033-012-0269-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global existence and asymptotic behavior of smooth solutions to a bipolar Euler–Poisson equation in a bound domain

Yeping Li

Abstract. In this paper, we present a bipolar hydrodynamic model from semiconductor devices and plasmas, which takes the form of bipolar isentropic Euler–Poisson with electric field and frictional damping added to the momentum equations. We firstly prove the existence of the stationary solutions. Next, we present the global existence and the asymptotic behavior of smooth solutions to the initial boundary value problem for a one-dimensional case in a bounded domain. The result is shown by an elementary energy method. Compared with the corresponding initial data case, we find that the asymptotic state is the stationary solution.

Mathematics Subject Classification (2000). 35B40 · 35M10.

Keywords. Euler–Poisson equation · Bipolar · Energy estimate · Asymptotic behavior · Stationary solution.

1. Introduction

In this paper, we present a bipolar hydrodynamic model from semiconductor devices and plasmas, which takes the form of bipolar isentropic Euler–Poisson with electric field and frictional damping added to the momentum equations. The scaled one-dimensional bipolar Euler–Poisson equation (see [1,2,21,27]) is given by

$$\begin{cases} n_{1t} + (n_1u_1)_x = 0, \\ (n_1u_1)_t + (n_1u_1^2 + P(n_1))_x = n_1\phi_x - n_1u_1, \\ n_{2t} + (n_2u_2)_x = 0, \\ (n_2u_2)_t + (n_2u_2^2 + P(n_2))_x = -n_2\phi_x - n_2u_2, \\ \phi_{xx} = n_1 - n_2 \end{cases} \tag{1.1}$$

for $(x,t) \in \Omega \times \mathbb{R}_+ (\Omega = (0,1))$. Here the unknown variables $n_i, u_i (i = 1,2)$ and ϕ are the charge densities, velocities, pressures and electrostatic potential. The pressures $P(n_i)(i = 1,2)$ are assumed to be functions of the densities given by $P(n_i) = Kn_i^\gamma (K > 0, \gamma \geq 1)$. The case $\gamma = 1$ is important from the physical point of view. The bipolar Euler–Poisson equations (hydrodynamic models) are generally used in the description of charged particle fluids, for example, electrons and holes in semiconductor devices, positively and negatively charged ions in plasma. More details on the semiconductor applications and the applications in plasma physics can be found in [9,20,24], etc.

Recently, many efforts have been made for the one-dimensional bipolar hydrodynamic equations. More precisely, Zhou–Li [28] and Tsuge [26] considered the existence and uniqueness of the stationary solution for the one-dimensional bipolar hydrodynamic model with some proper boundary conditions. Zhu–Hattori [27] proved the stability of steady-state solutions for a recombined one-dimensional bipolar hydrodynamical model with initial data. Natalini [21] and Hsiao–Zhang [8] established the global entropy solutions of the one-dimensional system in the compensated compactness framework on the whole real line and spatial bound domain respectively. Natalini [21] and Hsiao–Zhang [7] studied the relaxation-time limit of the weak solutions for the one-dimensional bipolar hydrodynamic model. Gasser–Marcati [3] also discussed some combined limits of the weak solutions. Gasser–Hsiao–Li [2] investigated the large-time behavior of smooth "small" solutions around diffusion waves for the initial value problem. They observed that the frictional damping is essential to the nonlinear diffusive phenomena of hyperbolic waves. Huang–Li [5] studied the large-time behavior and the quasi-neutral limit of L^∞ solution of the Cauchy problem with large data as well as vacuum. Huang et al. [6] discussed large-time behavior of solution to the bipolar hydrodynamic model for semiconductors with switch-on case. Moreover, we also mention that some authors studied the corresponding multi-dimensional cases, that is, the authors of [1,10–12,17,18], etc. Finally, for the unipolar Euler–Poisson equation, Nishibata and Suzuki [22,23] discuss the stability of the stationary solution for the IBVP for the one-dimensional isentropic and non-isentropic Euler–Poisson equation, respectively. Li [13,14] studied the corresponding multi-dimensional case. As far as we know, no results on the initial boundary value problem of (1.1) can be found. In this paper, we will study the global existence and the asymptotic behavior of smooth solutions to the initial boundary value problem for a one-dimensional bipolar hydrodynamic model (1.1) in a bounded domain. For this, we prescribe the initial and the boundary data as

$$(n_1, u_1, n_2, u_2)(0, x) = (n_{10}, u_{10}, n_{20}, u_{20})(x), \tag{1.2}$$

$$n_1(t, 0) = n_2(t, 0) = n_l > 0, n_1(t, 1) = n_2(t, 1) = n_r > 0, \tag{1.3}$$

$$\phi(t, 0) = 0, \phi(t, 1) = \phi_r > 0, \tag{1.4}$$

satisfying the following compatibility conditions:

$$\begin{aligned} n_1(0, 0) = n_2(0, 0) = n_l, n_1(0, 1) = n_2(0, 1) = n_r, \\ (n_1u_1)_x(0, 0) = (n_2u_2)_x(0, 0) = 0 = (n_1u_1)_x(0, 1) = (n_2u_2)_x(0, 1). \end{aligned} \tag{1.5}$$

For the sake of simplicity, we only consider the subsonic solutions as in [15,22,23]. Thus, we assume that this initial boundary value problem is considered in the region where the subsonic condition and positivity of the density hold.

$$\inf_{x\in\Omega}(P'(n_i) - u_i^2) > 0, \ \inf_{x\in\Omega} n_i > 0, \quad i = 1, 2, \tag{1.6}$$

further, we need to suppose that the initial data satisfy these condition

$$\inf(P'(n_{i0}) - u_{i0}^2) > 0, \ \inf n_{i0} > 0, \quad i = 1, 2. \tag{1.7}$$

We will establish the solution in the neighborhood of the initial data (1.7) as the conditions (1.6) hold. For convenience, introducing the current densities $j_i = n_iu_i (i = 1, 2)$, then the initial boundary value problem for $(n_1, j_1, n_2, j_2, \phi)$ can be written as

$$\begin{cases} n_{1t} + j_{1x} = 0, \\ j_{1t} + (P'(n_1) - \frac{j_1^2}{n_1^2})n_{1x} + \frac{2j_1}{n_1}j_{1x} = n_1\phi_x - j_1, \\ n_{2t} + j_{2x} = 0, \\ j_{2t} + (P'(n_2) - \frac{j_2^2}{n_2^2})n_{2x} + \frac{2j_2}{n_2}j_{2x} = -n_2\phi_x - j_2, \\ \phi_{xx} = n_1 - n_2, \end{cases} \tag{1.8}$$

with the initial data $(n_{10}, j_{10}, n_{20}, j_{20})(x) = (n_{10}, n_{10}u_{10}, n_{20}, n_{20}u_{20})(x)$, which is derived from (1.2), and (1.3)–(1.4). Apparently, (1.1) is equivalent to (1.8), provided that the density $n_i (i = 1, 2)$ are positive. Moreover, integrating $(1.8)_5$ and using the boundary condition (1.4), we obtain an explicit formula of the electrostatic potential:

$$\begin{aligned} \phi(t, x) &= \Phi(n_1, n_2)(t, x), \\ &:= \int_0^x \int_0^y (n_1 - n_2)(t, z)\mathrm{d}z\mathrm{d}y + \left(\phi_r - \int_0^1 \int_0^y (n_1 - n_2)(t, z)\mathrm{d}x\mathrm{d}y\right) x. \end{aligned} \tag{1.9}$$

Compared with the initial value problem in [2], here we believe that the asymptotic state is the stationary solutions. That is, when $n_{1t} = j_{1t} = n_{2t} = j_{2t} = 0$, we have the corresponding stationary state problem:

$$\begin{cases} \tilde{j}_{1x} = 0, \\ \frac{\partial F}{\partial n_1}(\tilde{n}_1, \tilde{j}_1)\tilde{n}_{1x} = \tilde{\phi}_x - \frac{\tilde{j}_1}{n_1}, \\ \tilde{j}_{2x} = 0, \\ \frac{\partial F}{\partial n_2}(\tilde{n}_2, \tilde{j}_2)\tilde{n}_{2x} = -\tilde{\phi}_x - \frac{\tilde{j}_2}{n_2}, \\ \tilde{\phi}_{xx} = \tilde{n}_1 - \tilde{n}_2, \end{cases} \tag{1.10}$$

with the boundary data

$$\tilde{n}_1(t,0) = \tilde{n}_2(t,0) = n_l > 0, \tilde{n}_1(t,1) = \tilde{n}_2(t,1) = n_r > 0, \tag{1.11}$$

$$\tilde{\phi}(t,0) = 0, \tilde{\phi}(t,1) = \phi_r > 0. \tag{1.12}$$

Here $F(\tilde{n}_i, \tilde{j}_i) = \frac{\tilde{j}_i^2}{2\tilde{n}_i^2} + h(\tilde{n}_i)$ $(i = 1, 2)$ and $h(\xi) = \int_1^\xi \frac{P'(n)}{n}\mathrm{d}n$. Further, differentiating $(1.10)_2$ and $(1.10)_4$ in x yield that

$$\left(\frac{\partial F}{\partial n_1}(\tilde{n}_1, \tilde{j}_1)\tilde{n}_{1x}\right)_x - \frac{\tilde{j}_1}{\tilde{n}_1^2}\tilde{n}_{1x} - \tilde{n}_1 + \tilde{n}_2 = 0, \tag{1.13}$$

$$\left(\frac{\partial F}{\partial n_2}(\tilde{n}_2, \tilde{j}_2)\tilde{n}_{2x}\right)_x - \frac{\tilde{j}_2}{\tilde{n}_2^2}\tilde{n}_{2x} + \tilde{n}_1 - \tilde{n}_2 = 0. \tag{1.14}$$

From $(1.10)_2$ and $(1.10)_4$, we have the following current–voltage relationships:

$$\phi_r = F(n_r, \tilde{j}_1) - F(n_l, \tilde{j}_1) + \tilde{j}_1 \int_0^1 \frac{1}{\tilde{n}_1}\mathrm{d}x, \tag{1.15}$$

$$-\phi_r = F(n_r, \tilde{j}_2) - F(n_l, \tilde{j}_2) + \tilde{j}_2 \int_0^1 \frac{1}{\tilde{n}_2}\mathrm{d}x, \tag{1.16}$$

which mean

$$F(n_r, \tilde{j}_1) - F(n_l, \tilde{j}_1) + \tilde{j}_1 \int_0^1 \frac{1}{\tilde{n}_1}\mathrm{d}x + F(n_r, \tilde{j}_2) - F(n_l, \tilde{j}_2) + \tilde{j}_2 \int_0^1 \frac{1}{\tilde{n}_2}\mathrm{d}x = 0.$$

Moreover, owing to Eq. $(1.10)_5$, $\widetilde{\phi}$ is given by the formula

$$\tilde{\phi}(x) = \int_0^x \int_0^y (\tilde{n}_1 - \tilde{n}_2)(z)\mathrm{d}z\mathrm{d}y + \left(\phi_r - \int_0^1 \int_0^y (\tilde{n}_1 - \tilde{n}_2)(z)\mathrm{d}z\mathrm{d}y\right)x, \tag{1.17}$$

which corresponds to (1.9) for the non-stationary problem.

Before stating the main results, we introduce some notations. For a nonnegative integer $l \geq 0, H^l$ denotes the usual Sobolev space in the L^2 sense, equipped with the norm $||\cdot||_l$, in particular, $||\cdot||_0 = ||\cdot||$. $C^k([0,T]; H^l(\Omega))$ denotes the space of the k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\Omega)$. For a nonnegative integer $k \geq 0, \mathcal{B}^k(\bar{\Omega})$ denotes the space of the functions whose derivatives up to k-th order are continuous and bounded over $\bar{\Omega}$, equipped with the norm

$$|f|_k := \sum_{l=0}^k \sup_{x\in\bar{\Omega}} |\partial_x^l f(x)|.$$

Moreover, we also introduce the function space:

$$\mathfrak{X}_i^k([0,T]) = \cap_{l=0}^{i} C^l([0,T], H^{k+i-l}(\Omega)), \quad \text{for } i,k = 0,1,2,$$
$$\mathfrak{X}_i([0,T]) := \mathfrak{X}_i^0([0,T]) \quad \text{for } i = 0,1,2.$$

Throughout the rest of this paper, C always denotes generic positive constant.

The unique existence of the stationary solution $(\tilde{n}_1, \tilde{u}_1, \tilde{n}_2, \tilde{u}_2, \tilde{\phi})$ is stated in the next Lemma.

Lemma 1.1. *Let the boundary data satisfy condition (1.3) and (1.4). For an arbitrary n_l, there exists a positive constant δ_0 such that if $|n_r - n_l| + \phi_r \le \delta_0$, then the stationary problem (1.10), (1.11) and (1.12) has a unique solution $(\tilde{n}_1, \tilde{u}_1, \tilde{n}_2, \tilde{u}_2, \tilde{\phi})$ satisfying the conditions (1.6).*

Next, the global existence and asymptotic behavior of smooth solution for the initial boundary value problem (1.1)–(1.4) is summarized in the following Theorem.

Theorem 1.2. *Let $(\tilde{n}_1, \tilde{u}_1, \tilde{n}_2, \tilde{u}_2, \tilde{\phi})$ be the stationary solution of (1.10), (1.11) and (1.12). Suppose that the initial data $(n_{10}, u_{10}, n_{20}, u_{20})(x) \in H^2(\Omega)$ and the boundary data n_l, n_r and ϕ_r satisfy (1.3), (1.4), (1.5) and (1.7). Then there exists a positive constant ϵ, such that if $|n_r - n_l| + \phi_r + \|(n_{10} - \tilde{n}_1, u_{10} - \tilde{u}_1, n_{20} - \tilde{n}_2, u_{10} - \tilde{u}_2)\|_2 \le \epsilon$, the initial boundary value problem (1.1)–(1.4) has a unique solution $(n_1, u_1, n_2, u_2, \phi)(t,x) \in \mathfrak{X}_2([0,\infty))^4 \times \mathfrak{X}_2^2([0,\infty))$. Moreover the solution $(n_1, u_1, n_2, u_2, \phi)(t,x)$ satisfies the decay estimate*

$$\begin{aligned} &\|(n_1 - \tilde{n}_1, u_1 - \tilde{u}_1, n_2 - \tilde{n}_2, u_2 - \tilde{u}_2)(\cdot,t)\|_2 + \|(\phi - \tilde{\phi})(\cdot,t)\|_4 \\ &\quad \le C\|(n_{10} - \tilde{n}_1, u_{10} - \tilde{u}_1, n_{20} - \tilde{n}_2, u_{20} - \tilde{u}_2)\|_2 e^{-\alpha t}, \end{aligned} \tag{1.18}$$

where C and α are positive constants independent of a time variable t.

Remark 1.3. *Here we obtain the similar results for the initial boundary value problem of the one-dimensional bipolar Euler–Poisson equation without the doping profile. This assumption is to overcome the interaction of the two particles. As to more general case, it is left for us in the future.*

The idea of the proof is outlined as follows. First, we show the unique existence of the stationary solution by Schauder fixed-point principle. In this procedure, the key point is the bound of the stationary densities. We cannot use the maximal principle here, which is different from the unipolar case in [16,22,23]. Next, based on the local existence and the a priori estimates, the continuum arguments can be applied to showing global existence and asymptotic behavior of smooth solution for the nonlinear problem. The a priori estimate can be derived by the elaborate energy methods. That is, we first find that the spatial derivatives of the perturbed variables can be controlled by the temporal derivatives of the perturbed variables with the help of the special structure of the perturbed equation. Next, we show the estimates of the temporal derivatives of the perturbed variables. However, due to the interaction of the two particles, we cannot directly derive the estimate of the densities by the electric field (ϕ_x) as in [16,22,23]. Thus, we need to make some elaborated treatments of the perturbed density. See Lemmas 4.3, 4.4 and 4.5.

The remaining part of the present paper is organized as follows. In Sect. 2, we begin detailed discussion with the proof of the existence and the uniqueness of the stationary solution. We state the local existence and reformulation of the original problem in Sect. 3. Section 4 is the core, in which we present the global existence and the asymptotic behavior of the smooth solution.

2. The stationary solution

This section is devoted to the discussion on the unique existence and the properties of the stationary solution of the problem (1.10)–(1.12). That is, we mainly present the following results.

Lemma 2.1. *Under the assumptions of Lemma 1.1, the stationary problem (1.10), (1.11) and (1.12) has a unique solution* $(\tilde{n}_1, \tilde{j}_1, \tilde{n}_2, \tilde{j}_2, \tilde{\phi}) \in \mathcal{B}^2(\bar{\Omega})$ *satisfying the condition (1.6) and*

$$N_1 \leq \tilde{n}_1(x), \tilde{n}_2(x) \leq N_2, \quad x \in [0,1], \tag{2.1}$$

$$|\tilde{\phi}| \leq C, |\tilde{\phi}_x,\ \tilde{\phi}_{xx}| \leq C\delta_0 \tag{2.2}$$

$$|\tilde{j}_i| \leq C\delta_0, \quad j = 1, 2, \tag{2.3}$$

$$|\tilde{n}_{1x}|, |\tilde{n}_{2x}|, |\tilde{n}_{1xx}|, |\tilde{n}_{2xx}| \leq C\delta_0, \tag{2.4}$$

$$|\tilde{u}_{1x}|, |\tilde{u}_{2x}|, |\tilde{u}_{1xx}|, |\tilde{u}_{2xx}| \leq C\delta_0, \tag{2.5}$$

where $N_1 = \min(n_l, n_r),\ N_2 = \max(n_l, n_r)$.

Proof. First, we give a closed convex set $W := \{f \in H^1(\Omega) : N_1 \leq f \leq N_2\}$. Taking $(m_1, m_2) \in W$, and solving the current–voltage relationship (1.15) and (1.16) with (m_1, m_2), we have

$$\bar{J}_{m_1} = 2C_b^1 \left(\int_0^1 m_1^{-1} \mathrm{d}x \pm \sqrt{\left(\int_0^1 m_1^{-1} \mathrm{d}x \right)^2 + 2C_b^1 (n_r^{-2} - n_l^{-2})} \right)^{-1},$$

$$\bar{J}_{m_2} = 2C_b^2 \left(\int_0^1 m_2^{-1} \mathrm{d}x \pm \sqrt{\left(\int_0^1 m_2^{-1} \mathrm{d}x \right)^2 + 2C_b^2 (n_r^{-2} - n_l^{-2})} \right)^{-1},$$

where

$$C_b^1 = \phi_r - (h(n_r) - h(n_l)),\ C_b^2 = -\phi_r - (h(n_r) - h(n_l)).$$

□

Then, choosing a proper positive number δ_1 such that when $|n_r - n_l| + \phi_r \leq \delta_1$, we can define

$$J_{m_1} = 2C_b^1 \left(\int_0^1 m_1^{-1} \mathrm{d}x + \sqrt{\left(\int_0^1 m_1^{-1} \mathrm{d}x \right)^2 + 2C_b^1 (n_r^{-2} - n_l^{-2})} \right)^{-1}, \tag{2.6}$$

$$J_{m_2} = 2C_b^2 \left(\int_0^1 m_2^{-1} \mathrm{d}x + \sqrt{\left(\int_0^1 m_2^{-1} \mathrm{d}x \right)^2 + 2C_b^2 (n_r^{-2} - n_l^{-2})} \right)^{-1}. \tag{2.7}$$

The details can be found in [22,26]. Further, we define the mapping $S : (m_1, m_2) \to (M_1, M_2)$ over $W := \{f \in H^1(\Omega) : N_1 \leq f \leq N_2\}$ by solving the linear problem

$$\begin{cases} \left(\frac{\partial F}{\partial m_1}(m_1, J_{m_1}) M_{1x} \right)_x - \frac{J_{m_1}}{m_1^2} M_{1x} - M_1 + M_2 = 0, \\ \left(\frac{\partial F}{\partial m_2}(m_2, J_{m_2}) M_{2x} \right)_x - \frac{J_{m_2}}{m_2^2} M_{2x} + M_1 - M_2 = 0, \\ M_1(0) = n_l = M_2(0), M_1(1) = n_r = M_2(1), \end{cases} \tag{2.8}$$

Apparently, we can choose δ_2 such that for $|n_r - n_l| + \phi_r \leq \delta_2$, the pairs $(m_i, J_{m_i})(i = 1, 2)$ satisfy the subsonic condition (1.6), which implies the Eqs. $(2.8)_1$ and $(2.8)_2$ are elliptic. Thus, Lax-Milgram's theorem guarantees the existence of a unique $H^1(0,1) \times H^1(\Omega)$-solution (M_1, M_2). Next, it is easy to see that S is precompact in $C(\bar{\Omega})$ from Sobolev's imbedding theorem. Moreover, using the standard arguments, we know that S is continuous. In order to apply the Schauder fixed-point theorem [4], it remains to prove that $N_1 \leq M_1, M_2 \leq N_2$.

Indeed, using $(M_1-N_2)^+=\max(M_1-N_2,0)$ and $(M_2-N_2)^+=\max(M_2-N_2,0)$ as two test functions in the weak formulations of $(2.8)_1$ and $(2.8)_2$. Integration by parts leads to

$$\int_0^1\left(-\frac{J_{m_1}^2}{m_1^3}+\frac{1}{m_1}P'(m_1)\right)|(M_1-N_2)_x^+|^2\mathrm{d}x+\int_0^1\frac{J_{m_1}}{m_1^2}(M_1-N_2)_x^+(M_1-N_2)\mathrm{d}x$$
$$+\int_0^1(M_1-M_2)(n_1-N_2)^+\mathrm{d}x=0, \tag{2.9}$$

and

$$\int_0^1\left(-\frac{J_{m_2}^2}{m_2^3}+\frac{1}{m_2}P'(m_2)\right)|(M_2-N_2)_x^+|^2\mathrm{d}x+\int_0^1\frac{J_{m_2}}{m_2^2}(M_2-N_2)_x^+(M_2-N_2)\mathrm{d}x$$
$$-\int_0^1(M_1-M_2)(M_2-N_2)^+\mathrm{d}x=0. \tag{2.10}$$

By means of the Cauchy–Schwarz and Poincaré inequalities, we have

$$\begin{aligned}\int_0^1\frac{J_{m_1}}{m_1^2}(M_1-N_2)_x^+(M_1-N_1)\mathrm{d}x&\le C|J_{m_1}|\left(\int_0^1|(M_1-N_2)_x^+|^2\mathrm{d}x\right)^{\frac12}\left(\int_0^1|(M_1-N_2)^+|^2\mathrm{d}x\right)^{\frac12}\\&\le C|J_{m_1}|\int|(M_1-N_2)_x^+|^2\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\int_0^1\frac{J_{m_2}}{m_2}(M_2-N_2)_x^+(M_2-N_2)\mathrm{d}x&\le C|J_{m_2}|\left(\int_0^1|(M_2-N_2)_x^+|^2\mathrm{d}x\right)^{\frac12}\left(\int_0^1|(M_2-N_2)^+|^2\mathrm{d}x\right)^{\frac12}\\&\le C|J_{m_2}|\int_0^1|(M_2-N_2)_x^+|^2\mathrm{d}x.\end{aligned}$$

Thanks to the definition of N_2, we have

$$\begin{aligned}&\int_0^1(M_1-M_2)(M_1-N_2)^+\mathrm{d}x-\int_0^1(M_1-M_2)(M_2-N_2)^+\mathrm{d}x\\&=\int_0^1(M_1-N_2+N_2-M_2)(M_1-N_2)^+\mathrm{d}x\\&\quad-\int_0^1(M_1-N_2+N_2-M_2)(M_2-N_2)^+\mathrm{d}x\ge0.\end{aligned}$$

Further, putting the above relations into (2.9) and (2.10), there is a positive constant δ_3 such that for $|n_r-n_l|+\phi_r\le\delta_3$,

$$\int_0^1|(M_1-N_2)_x^+|^2\mathrm{d}x+\int_0^1|(M_2-N_2)_x^+|^2\mathrm{d}x\le0.$$

Hence, we can obtain

$$M_1, M_2 \le N_2.$$

In complete similar way, there is a positive constant δ_3 such that for $|n_r - n_l| + |\phi_r| \le \delta_3$, such that we can prove

$$M_1, M_2 \ge N_1.$$

By Schauder's fixed-point theorem, we have a fixed-point $(\tilde{n}_1, \tilde{n}_2) = S(\tilde{n}_1, \tilde{n}_2) \in H^1(\Omega) \times H^1(\Omega)$ satisfying $N_1 \le \tilde{n}_1(x), \tilde{n}_2(x) \le N_2$. Apparently, the function $(\tilde{n}_1, \tilde{n}_2)$ is a solution to the system (1.13) and (1.14) with the boundary data (1.11). We construct the solution to (1.10), (1.11) and (1.12) from $\tilde{n}_1$ and $\tilde{n}_2$ as follows. Define two constants $\tilde{j}_i = J_{\tilde{n}_i} (i = 1, 2)$ by (2.6) and (2.7) and define a function $\tilde{\phi}$ by the formula (1.17). Finally, it is a straightforward computation to confirm that $(\tilde{n}_1, \tilde{j}_1, \tilde{n}_2, \tilde{j}_2, \tilde{\phi})$ is a desired solution to the stationary problem (1.10), (1.11) and (1.12). Choosing proper small δ_4 for $|n_r - n_l| + |\phi_r| \le \delta_4$, we can show the uniqueness of the stationary solution. Since the procedure is similar as those in [28], we can omit the details here. Thus, choosing $\delta_0 =: \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, the proof of unique existence of (1.10) is completed.

Next, we discuss some properties of $(\tilde{n}_1, \tilde{j}_1, \tilde{n}_2, \tilde{j}_2, \tilde{\phi})$. First, note that $\tilde{\phi}$ is given by the formula (1.17) or equivalently

$$\tilde{\phi}(x) = \int_x^1 \int_y^1 (\tilde{n}_1 - \tilde{n}_2)(z) \mathrm{d}z \mathrm{d}y + \phi_r x - \left(\int_0^1 \int_y^1 (\tilde{n}_1 - \tilde{n}_2)(z) \mathrm{d}z \mathrm{d}y \right) (1 - x), \tag{2.11}$$

By estimating the formula (1.17) for $x \in [0, \frac{1}{2}]$ and the formula (2.11) for $x \in [\frac{1}{2}, 1]$, we obtain the first estimate in (2.2), due to (2.1). Next, due to (2.1) and the subsonic condition (1.6), we have

$$|\tilde{j}_i| \le C\delta_0, \quad i = 1, 2, \tag{2.12}$$

with the aid of (1.15) and (1.16). Finally, we take $n_b(x) = n_r + (n_l - n_r)(1 - x)$ in $x \in [0, 1]$. We observe that $(\tilde{n}_1 - n_b)(x)$ and $(\tilde{n}_2 - n_b)(x)$ vanishes at $x = 0$ and $x = 1$, furthermore, multiply (1.13) and (1.14) by $\tilde{n}_1 - n_b$ and $\tilde{n}_2 - n_{2b}$ respectively, to have

$$\begin{aligned} &\int_0^1 \left(-\frac{\tilde{j}_1^2}{\tilde{n}_1^3} + \frac{1}{\tilde{n}_1} P'(\tilde{n}_1) \right) (\tilde{n}_1 - n_b)_x^2 \mathrm{d}x + \int_0^1 \frac{\tilde{j}_1}{\tilde{n}_1} (\tilde{n}_1 - n_b)_x \mathrm{d}x \\ &\quad + \int_0^1 (\tilde{n}_1 - \tilde{n}_2)(\tilde{n}_1 - n_b) \mathrm{d}x = (n_l - n_r) \int_0^1 \left(-\frac{\tilde{j}_1^2}{\tilde{n}_1^3} + \frac{1}{\tilde{n}_1} P'(\tilde{n}_1) \right) (\tilde{n}_1 - n_b)_x \mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 \left(-\frac{\tilde{j}_2^2}{\tilde{n}_2^3} + \frac{1}{\tilde{n}_2} P'(\tilde{n}_2) \right) (\tilde{n}_2 - n_b)_x^2 \mathrm{d}x + \int_0^1 \frac{\tilde{j}_2}{\tilde{n}_2} (\tilde{n}_2 - n_b)_x \mathrm{d}x \\ &\quad - \int_0^1 (\tilde{n}_1 - \tilde{n}_2)(\tilde{n}_2 - n_b) \mathrm{d}x = (n_l - n_r) \int_0^1 \left(-\frac{\tilde{j}_2^2}{\tilde{n}_2^3} + \frac{1}{\tilde{n}_2} P'(\tilde{n}_2) \right) (\tilde{n}_2 - n_b)_x \mathrm{d}x. \end{aligned}$$

By means of the Cauchy–Schwarz and Poincaré inequalities, and noting that

$$\int_0^1 (\tilde{n}_1 - \tilde{n}_2)(\tilde{n}_1 - n_b) \mathrm{d}x - \int_0^1 (\tilde{n}_1 - \tilde{n}_2)(\tilde{n}_2 - n_b) \mathrm{d}x \ge 0,$$

we can derive

$$\|\tilde{n}_{1x}\| + \|\tilde{n}_{2x}\| \le C(|\tilde{j}_1| + |\tilde{j}_2| + |n_l - n_r|) \le C\delta_0. \tag{2.13}$$

Similarly, we can obtain

$$\|\tilde{n}_{1xx}\|_1 + \|\tilde{n}_{2xx}\|_1 \le C\delta_0,$$

which together with (2.13) implies (2.4). From (2.4) and (2.12), we immediately have (2.5). Moreover, using $(1.10)_2$ or $(1.10)_4$, and (2.4), we can show the second inequality in (2.2).

3. Local-in-time existence and reformulation of the original problem

In this section, we firstly present the unique existence of the solution locally in time to the initial boundary problem (1.8), (1.2)–(1.4), then reformulate the original problem (1.1)–(1.4). At first, applying Theorem$-A1$ in [25], we can obtain the local existence of the linearized equation:

$$\begin{cases} \partial_t \hat{n}_1 + \partial_x \hat{j}_1 = 0, \\ \hat{j}_{1t} + (P'(n_1) - \frac{j_1^2}{n_1^2})\hat{n}_{1x} + 2\frac{j_1}{n_1}\hat{j}_{1x} = n_1\phi_x - j_1, \\ \partial_t \hat{n}_2 + \partial_x \hat{j}_2 = 0, \\ \hat{j}_{2t} + (P'(n_2) - \frac{j_2^2}{n_2^2})\hat{n}_{2x} + 2\frac{j_2}{n_2}\hat{j}_{2x} = -n_2\phi_x - j_2, \\ (\hat{n}_1, \hat{j}_1, \hat{n}_2, \hat{j}_2)(x,0) = (n_{10}, j_{10}, n_{20}, j_{20})(x), \\ \hat{n}_1(0,t) = n_l = \hat{n}_2(0,t), \hat{n}_1(1,t) = n_r = \hat{n}_1(1,t). \end{cases} \tag{3.1}$$

where $\phi = \Phi(n_1, n_2)$ by (1.9). That is,

Lemma 3.1. *Suppose that the initial data $(n_{10}, j_{10}, n_{20}, j_{20})(x) \in H^2(\Omega)$ and the boundary data n_l and n_l satisfy (1.7) and (1.3)–(1.4). In addition, assume the compatible conditions (1.5) holds. Then there exist positive constants T, m, k and M satisfying the following property: if $(n_1, j_1, n_2, j_2) \in \mathfrak{X}_2([0,T])$, and*

$$\begin{aligned} &(n_1, j_1, n_2, j_2)(x,0) = (n_{10}, j_{10}, n_{20}, j_{20})(x), \\ &n_i(x,t) \ge m, \ P(n_i) - \frac{j_i^2}{n_i^2} \ge k \quad \text{for } i = 1,2, \ (t,x) \in [0,T] \times \Omega, \\ &\|(n_1, j_1, n_2, j_2)(t)\|_2 + \|(n_{1t}, j_{1t}, n_{2t}, j_{2t})(t)\|_1 + \|(n_{1tt}, j_{1tt}, n_{2tt}, j_{2tt})(t)\| \le M \quad \text{for } t \in [0,T], \end{aligned}$$

then the problem (3.1) admits a unique solution $(\hat{n}_1, \hat{j}_1, \hat{n}_2, \hat{j}_2)(x,t)$ in the same set $\mathfrak{X}_2([0,T])$ satisfying

$$\begin{aligned} &\hat{n}_i(x,t) \ge m, \ P(\hat{n}_i) - \frac{\hat{j}_i^2}{\hat{n}_i^2} \ge k \quad \text{for } i = 1,2, \ (t,x) \in [0,T] \times \Omega, \\ &\|(\hat{n}_1, \hat{j}_1, \hat{n}_2, \hat{j}_2)(t)\|_2 + \|(\hat{n}_{1t}, \hat{j}_{1t}, \hat{n}_{2t}, \hat{j}_{2t})(t)\|_1 + \|(\hat{n}_{1tt}, \hat{j}_{1tt}, \hat{n}_{2tt}, \hat{j}_{2tt})(t)\| \le M \quad \text{for } t \in [0,T]. \end{aligned}$$

Since the procedure is similar as those in [22,23], we omit the details here.

Applying Lemma 3.1, and using the standard iterative arguments and energy estimates, we can prove the following result without the details (we can refer to [22,23]).

Lemma 3.2. *Suppose that the initial data $(n_{10}, j_{10}, n_{20}, j_{20}) \in H^2(\Omega)$ and the boundary data n_l, n_r and ϕ_r satisfy (1.7), (1.3), (1.4) and (1.5). Then there exists a constant $T_1 > 0$ such that the initial boundary value problem (1.1)–(1.4) has a unique solution $(n_1, j_1, n_2, j_2, \phi) \in \mathfrak{X}_2([0,T]_1)^4 \times \mathfrak{X}_2^2([0,T]_1)$ satisfying the condition (1.6) for $(t,x) \in [0,T_1] \times \Omega$.*

Next, in order to prove the global existence and asymptotic behavior of the smooth solutions in Theorem 1.2, we regard the solution $(n_1, u_1, n_2, u_2, \phi)(x,t)$ as a perturbation from the stationary solution

$(\tilde{n}_1, \tilde{u}_1, \tilde{n}_2, \tilde{u}_2, \tilde{\phi})$. That is, we need to reformulate the original problem in terms of the perturbed variables in the following. For this, multiplying $(1.8)_2$ and $(1.8)_4$ by $\frac{1}{n_1}$ and $\frac{1}{n_2}$, respectively, and using the Eqs. $(1.8)_1$ and $(1.8)_3$, we have

$$u_{1t} + u_1 u_{1x} + h(n_1)_x = \phi_x - u_1, \;\; u_{2t} + u_2 u_{2x} + h(n_1)_x = -\phi_x - u_2. \tag{3.2}$$

Similarly, we have from $(1.10)_{2,4}$ that

$$\tilde{u}_1 \tilde{u}_{1x} + h(\tilde{n}_1)_x = \tilde{\phi}_x - \tilde{u}_1, \;\; \tilde{u}_2 \tilde{u}_{2x} + h(\tilde{n}_1)_x = -\tilde{\phi}_x - \tilde{u}_2. \tag{3.3}$$

Then, we introduce new unknown functions as

$$\varphi_1 = n_1 - \tilde{n}_1, \varphi_2 = n_2 - \tilde{n}_2, \psi_1 = u_1 - \tilde{u}_1, \psi_2 = u_2 - \tilde{u}_2, w = \phi - \tilde{\phi}.$$

Further, from $(1.10)_{1,3,5}$ and (3.2)–(3.3), we have

$$\begin{cases} \varphi_{1t} + [(\tilde{n}_1 + \varphi_1)(\tilde{u}_1 + \psi_1) - \tilde{n}_1 \tilde{u}_1]_x = 0, \\ \psi_{1t} + \frac{1}{2}[(\tilde{u}_1 + \psi_1)^2 - \tilde{u}_1^2]_x + [h(\tilde{n}_1 + \varphi_1) - h(\tilde{n}_1)]_x - w_x + \psi_1 = 0, \\ \varphi_{2t} + [(\tilde{n}_2 + \varphi_2)(\tilde{u}_2 + \psi_2) - \tilde{n}_2 \tilde{u}_2]_x = 0, \\ \psi_{2t} + \frac{1}{2}[(\tilde{u}_2 + \psi_2)^2 - \tilde{u}_2^2]_x + [h(\tilde{n}_2 + \varphi_2) - h(\tilde{n}_2)]_x + w_x + \psi_2 = 0, \\ w_{xx} = \varphi_1 - \varphi_2. \end{cases} \tag{3.4}$$

The corresponding initial and the boundary condition are derived from (1.2), (1.3) and (1.4) as

$$\varphi_i(x, 0) = \varphi_{i0}(x) = n_{i0}(x) - \tilde{n}_i(x), \psi_i(x, 0) = \psi_{i0}(x) = u_{i0}(x) - \tilde{u}_i(x), \quad i = 1, 2, \tag{3.5}$$

$$\varphi_i(t, 0) = \varphi_i(t, 1) = 0, \quad i = 1, 2, \;\; w(t, 0) = w(t, 1) = 0. \tag{3.6}$$

From Lemma 3.2, we have

Corollary 3.3. *Suppose that the initial data* $(\varphi_{10}, \psi_{10}, \varphi_{20}, \psi_{20})(x) \in H^2(\Omega)$ *and* $(\tilde{n}_i + \varphi_{i0}, \tilde{u}_i + \psi_{i0})(i = 1, 2)$ *satisfy* (1.7). *Then there exists a constant* $T_2 > 0$ *such that the initial boundary value problem* (3.4)–(3.6) *has a unique solution* $(\varphi_1, \psi_1, \varphi_2, \psi_2, w)(x, t) \in \mathfrak{X}([0, T_2])^4 \times \mathfrak{X}_2^2([0, T_2])$ *with the property that* $(\tilde{n}_i + \varphi_i, \tilde{u}_i + \psi_i)$ *satisfying* (1.6).

4. Global existence and asymptotic behavior

In this section, we mainly present the global existence and asymptotic behavior of smooth solutions. Owing to Corollary 3.3, it suffices to derive an a priori estimate in order to show the existence and asymptotic behavior of the global-in-time solution. For convenience, we introduce

$$E(t) = \sum_{l=0}^{2} (\|\partial_t^l \varphi_1(t)\|_{2-l}^2 + \|\partial_t^l \psi_1(t)\|_{2-l}^2 + \|\partial_t^l \varphi_2(t)\|_{2-l}^2 + \|\partial_t^l \psi_2(t)\|_{2-l}^2 + \|\partial_t^l w(t)\|_{4-l}^2),$$

$$E_1(t) = \sum_{l=0}^{2} (\|\partial_t^l \varphi_1(t)\|^2 + \|\partial_t^l \psi_1(t)\|^2 + \|\partial_t^l \varphi_2(t)\|^2 + \|\partial_t^l \psi_2(t)\|^2 + \|\partial_t^l w_x(t)\|^2).$$

The main aim of this part is to show the following Theorem.

Theorem 4.1. *There exists* $\varepsilon > 0$ *such that if* $E(0) + \delta \leq \varepsilon$, *then there is a unique smooth solution* $(\varphi_1, \psi_1, \varphi_2, \psi_2, w)(t, x) \in \mathfrak{X}_2([0, \infty))^4 \times \mathfrak{X}_2^2([0, \infty))$ *to* (3.4) *and* (3.5)–(3.6), *and there are positive numbers* C *and* β, *which are independent of* t, *such that it holds that*

$$E(t) \leq C(E(0) + \delta) e^{-\beta t}, \tag{4.1}$$

where $\delta := |n_r - n_l| + \phi_r$.

From Theorem 4.1, we immediately have Theorem 1.2.

The proof of Theorem 4.1 is based on several steps of careful energy estimates which are stated as a sequence of Lemmas. Firstly, from $(3.4)_5$ and the boundary data $w(t,0) = w(t,1) = 0$, we can derive the estimate of w as follows.

Lemma 4.2. *Assume that the assumptions in Theorem 4.1 hold. Then the following estimates hold:*

$$\|w(t)\|_4^2 \le C(\|\varphi_1\|_2^2 + \|\varphi_2\|_2^2), \tag{4.2}$$

$$\|\partial_t w(t)\|_3^2 \le C(\|\partial_t \varphi_1\|_1^2 + \|\partial_t^l \varphi_2\|_1^2), \tag{4.3}$$

$$\|\partial_t^2 w(t)\|_2^2 \le C(\|\partial_t^2 \varphi_1\|^2 + \|\partial_t^2 \varphi_2\|^2), \tag{4.4}$$

where C is a positive constant independent of t.

The details of the proofs can be found in [19,22,23] and can be omitted here.

The next lemma plays an important role in the proof of Theorem 4.1.

Lemma 4.3. *Let $(\varphi_1, \psi_1, \varphi_2, \psi_2, w)$ be the solution of (3.4)–(3.6). If $E(t)^{\frac{1}{2}} + \delta$ is small enough, then there exists a constant $C_1 > 0$ such that*

$$E(t) \le C_1 E_1(t). \tag{4.5}$$

Proof. From the velocity equations $(3.4)_2$ and $(3.4)_4$, we have

$$h'(\tilde{n}_1 + \theta_1\varphi_1)\varphi_{1x} - w_x = -\psi_{1t} - \frac{1}{2}((\tilde{u}_1 + \psi_1)^2 - \tilde{u}_1^2)_x - \psi_1 - (h'(\tilde{n}_1 + \theta_1\varphi_1))_x\varphi_1, \tag{4.6}$$

and

$$h'(\tilde{n}_2 + \bar{\theta}_1\varphi_2)\varphi_{2x} + w_x = -\psi_{2t} - \frac{1}{2}((\tilde{u}_2 + \psi_2)^2 - \tilde{u}_2^2)_x - \psi_2 - (h'(\tilde{n}_2 + \bar{\theta}_1\varphi_2))_x\varphi_2, \tag{4.7}$$

where $0 < \theta_1, \bar{\theta}_1 < 1$. Taking the inner products of (4.6) by φ_{1x} and (4.7) by φ_{2x}, respectively, then summing their resultant equations, and noting

$$-\int_0^1 (w_x\varphi_{1x} - w_x\varphi_{2x})\mathrm{d}x = \int_0^1 w_{xx}(\varphi_1 - \varphi_2)\mathrm{d}x = \int_0^1 (\varphi_1 - \varphi_2)^2\mathrm{d}x,$$

we have

$$\begin{aligned}
&\int_0^1 (h'(\tilde{n}_1 + \theta_1\varphi_1)\varphi_{1x}^2 + h'(\tilde{n}_2 + \bar{\theta}_1\varphi_2)\varphi_{2x}^2)\mathrm{d}x + \int_0^1 (\varphi_1 - \varphi_2)^2\mathrm{d}x \\
&\quad \le C(\|\psi_1\|^2 + \|\psi_{1t}\|^2 + \|\psi_2\|^2 + \|\psi_{2t}\|^2) + C(\|\tilde{n}_{1x}\|_{L^\infty}\|\varphi_{1x}\|\|\varphi_1\| + \|\varphi_1\|_{L^\infty}\|\varphi_{1x}\|^2) \\
&\qquad + C(\|\tilde{n}_{2x}\|_{L^\infty}\|\varphi_{2x}\|\|\varphi_2\| + \|\varphi_2\|_{L^\infty}\|\varphi_{2x}\|^2) + C(\|\tilde{u}_1\|_{L^\infty}\|\varphi_{1x}\|\|\psi_{1x}\| + \|\tilde{u}_{1x}\|_{L^\infty}\|\varphi_{1x}\|\|\psi_1\| \\
&\qquad + \|\psi_1\|_{L^\infty}\|\varphi_{1x}\|\|\psi_{1x}\|) + C(\|\tilde{u}_2\|_{L^\infty}\|\varphi_{2x}\|\|\psi_{2x}\| + \|\tilde{u}_{1x}\|_{L^\infty}\|\varphi_{2x}\|\|\psi_2\| + \|\psi_2\|_{L^\infty}\|\varphi_{2x}\|\|\psi_{2x}\|) \\
&\quad \le C(\|\psi_1\|^2 + \|\psi_{1t}\|^2 + \|\psi_2\|^2 + \|\psi_{1t}\|^2) + C(E(t)^{\frac{1}{2}} + \delta)E(t),
\end{aligned}$$

with the help of (2.4) and (2.5).

Moreover, the continuity equations $(3.4)_2$ and $(3.4)_4$ imply

$$\psi_{ix} = -\frac{1}{\varphi_i + \tilde{n}_i}[\varphi_{it} + (\varphi_i + \tilde{n}_i)_x\psi_i + (\tilde{u}_i\varphi_i)_x], \quad i = 1, 2. \tag{4.8}$$

Therefore, from (2.4) and (2.5), we obtain

$$\|\psi_{1x}\|^2 + \|\psi_{2x}\|^2 \le C(\|\varphi_{1t}\|^2 + \|\varphi_{2t}\|^2) + C(E(t)^{\frac{1}{2}} + \delta)E(t).$$

Next, we take time derivatives of (4.6), (4.7) and (4.8). It is clear that every time derivative and spatial derivative of $\varphi_{1x}, \varphi_{2x}, \psi_{1x}$ and ψ_{2x} is again bounded by $E_1(t)$. By noting Lemma 4.2, we finally deduce (4.5). This completes the proof of Lemma 4.3.

Lemma 4.3 reduce the estimates of $E(t)$ to those of $E_1(t)$, then our next goal is dealt with the estimates of $E_1(t)$. First, from the expansion of the conservation of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{1}{2}n_1u_1^2+P(n_1)n_1^2+\frac{1}{2}n_2u_2^2+P(n_2)n_2^2+\frac{1}{2}\phi_x^2\right)\mathrm{d}x+\int_0^1(n_1u_1^2+n_2u_2^2)\mathrm{d}x\equiv 0,$$

around the steady state $(\tilde{n}_1,\tilde{u}_1,\tilde{n}_2,\tilde{u}_2,\tilde{\phi})(x,t)$, we can show the following basic estimates. The procedures are standard and the details can be omitted here.

Lemma 4.4. *Let $(\varphi_1,\psi_1,\varphi_2,\psi_2,w)$ be the solution of (3.4)–(3.6), then there is a constant $C>0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)+2(\|\psi_1\|^2+\|\psi_2\|^2)\le C(E(t)^{\frac{1}{2}}+\delta)E(t), \tag{4.9}$$

where

$$\begin{aligned}E_2(t)&=h'(\tilde{n}_1+\theta_1\varphi_1)\|\varphi_1\|^2+(\tilde{n}_1+\varphi_1)\|\psi_1\|^2+h'(\tilde{n}_2+\bar{\theta}_1\varphi_2)\|\varphi_2\|^2\\&\quad+(\varphi_2+\tilde{n}_2)\|\psi_2\|^2+\|w_x(t)\|^2,\ 0<\theta_1,\bar{\theta}_1<1.\end{aligned}$$

In the following, we derive the higher-order estimates. It is necessary to justify these computations by the discussion using the mollifier with respect to time variable t since the regularity of the solution $(\varphi_1,\psi_1,\varphi_2,\psi_2)(x,t)$ constructed in Corollary 3.3 is not enough. However, we omit this discussion as it is a well-known argument. Differentiating $(3.4)_2$ and $(3.4)_4$ with respect to t, we have the following equation for $l=1,2$

$$\partial_t^l\psi_{1t}+(\tilde{u}_1+\psi_1)\partial_t^l\psi_{1x}+(h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1)_x-\partial_t^lw_x+\partial_t^l\psi_1=F_1^l, \tag{4.10}$$
$$\partial_t^l\psi_{2t}+(\tilde{u}_2+\psi_2)\partial_t^l\psi_{2x}+(h'(\tilde{n}_2+\varphi_2)\partial_t^l\varphi_2)_x+\partial_t^lw_x+\partial_t^l\psi_2=F_2^l, \tag{4.11}$$

where for $i=1,2$,

$$F_i^1=-(\tilde{u}_i+\psi_i)_x\psi_{it},\ F_i^2=-(\tilde{u}_i+\psi_i)_x\psi_{it}-2\psi_{it}\psi_{ixt}-(h''(\tilde{n}_i+\varphi_i)(\varphi_{it})^2)_x.$$

The absolute values of $F_i^1(i=1,2)$ and $F_i^2(i=1,2)$ are estimated as

$$|F_i^1|\le C(E(t)^{\frac{1}{2}}+\delta)|\psi_{it}|, \tag{4.12}$$
$$|F_i^2|\le C(E(t)^{\frac{1}{2}}+\delta)E(t)^{\frac{1}{2}}(|\psi_{it}|+|\psi_{itx}|+|\varphi_{itx}|), \tag{4.13}$$

where C is a positive constant independent of t. In deriving (4.13), we have also used the estimates (2.4), (2.5) and the following inequality

$$|\varphi_{it}(t)|_{L^\infty}+|\psi_{it}(t)|_{L^\infty}\le CE(t)^{\frac{1}{2}}, \tag{4.14}$$

where C is a positive constant independent of t. In fact, we see that $(\varphi_1,\psi_1,\varphi_2,\psi_2)\in\mathfrak{X}_2([0,t])$ satisfies (4.14) by applying the Sobolev inequality on the Eqs. $(3.4)_{1,2,3,4}$ with using the estimate (2.4) and (2.5). Next, differentiating $(3.4)_1$ and $(3.4)_3$ with respect to t, we have for $l=0,1,2$,

$$((\tilde{n}_1+\varphi_1)\partial_t^l\psi_1)_x=-\partial_t^l\varphi_{1t}-(\tilde{u}_1+\psi_1)\partial_t^l\varphi_{1x}+G_1^l, \tag{4.15}$$
$$((\tilde{n}_2+\varphi_2)\partial_t^l\psi_2)_x=-\partial_t^l\varphi_{2t}-(\tilde{u}_2+\psi_2)\partial_t^l\varphi_{2x}+G_2^l, \tag{4.16}$$

where for $i=1,2$,

$$G_i^0=-\tilde{u}_{ix}\varphi_i+\psi_i\varphi_{ix},G_i^1=-(\tilde{u}_i+\psi_i)_x\varphi_{it},\ G_i^2=-(\tilde{u}_i+\psi_i)_x\varphi_{itt}-2(\varphi_{it}\psi_{it})_x.$$

The estimate (2.4), (2.5) and (4.14) give that

$$|G_i^0| \le C(E(t)^{\frac{1}{2}} + \delta)(|\psi_i| + |\varphi_i|), \tag{4.17}$$

$$|G_i^1| \le C(E(t)^{\frac{1}{2}} + \delta)|\varphi_{it}|, \tag{4.18}$$

$$|G_i^2| \le C(E(t)^{\frac{1}{2}} + \delta)(|\varphi_{itt}| + |\psi_{itx}| + |\varphi_{itx}|), \tag{4.19}$$

where C is a positive constant independent of t.

Lemma 4.5. *Let $(\varphi_1, \psi_1, \varphi_2, \psi_2, w)$ be the solution of (3.4)–(3.6), then the following estimate holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_3(t) + C_2\sum_{l=1}^{2}\|(\partial_t^l\varphi_1, \partial_t^l\psi_{1t}, \partial_t^l\varphi_2, \partial_t^l\psi_{2t}, \partial_t^l w_x)(t)\|^2 \le C(E(t)^{\frac{1}{2}} + \delta)E(t), \tag{4.20}$$

where C_2 and C are two positive constants independent of t, and

$$E_3(t) = \sum_{l=1}^{2}\Big((\tilde{n}_1 + \varphi_1)(\partial_t^l\psi_1)^2 + h'(\tilde{n}_1 + \varphi_1)(\partial_t^l\varphi_1)^2 + (\tilde{n}_2 + \varphi_2)(\partial_t^l\psi_2)^2 + h'(\tilde{n}_2 + \varphi_2)(\partial_t^l\varphi_2)^2$$
$$+(\partial_t^l w_x)^2 + (\tilde{n}_1 + \varphi_1)\partial_t^l\psi_1\partial_t^{l-1}\psi_1 - (\tilde{u}_1 + \psi_1)\partial_t^{l-1}\psi_1\partial_t^l\varphi_1 + \frac{1}{2}(\tilde{n}_1 + \varphi_1)(\partial_t^{l-1}\psi_1)^2$$
$$+(\tilde{n}_2 + \varphi_2)\partial_t^l\psi_2\partial_t^{l-1}\psi_2 - (\tilde{u}_2 + \psi_2)\partial_t^{l-1}\psi_2\partial_t^l\varphi_2 + \frac{1}{2}(\tilde{n}_2 + \varphi_2)(\partial_t^{l-1}\psi_2)^2\Big).$$

Proof. Multiplying (4.10) by $(\tilde{n}_1 + \varphi_1)\partial_t^{l-1}\psi_1$ for $l = 1, 2$ and integrate the resultant equality over Ω to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1[(\tilde{n}_1 + \varphi_1)\partial_t^l\psi_1\partial_t^{l-1}\psi_1 - (\tilde{u}_1 + \psi_1)\partial_t^{l-1}\psi_1\partial_t^l\varphi_1 + \frac{1}{2}(\tilde{n}_1 + \varphi_1)(\partial_t^{l-1}\psi_1)^2]\mathrm{d}x$$
$$+\int_0^1 h'(\tilde{n}_1 + \varphi_1)(\partial_t^l\varphi_1)^2\mathrm{d}x - \int_0^1 \partial_t^l w_x(\tilde{n}_1 + \varphi_1)\partial_t^{l-1}\psi_1\mathrm{d}x$$
$$= \int_0^1(\tilde{n}_1 + \varphi_1)(\partial_t^l\psi_1)^2\mathrm{d}x + H_1^l, \tag{4.21}$$

where

$$H_1^l(t) = \int_0^1(\tilde{n}_1 + \varphi_1)_t\partial_t^{l-1}\psi_1\partial_t^l\psi_1\mathrm{d}x - \int_0^1[(\tilde{u}_1 + \psi_1)\partial_t^{l-1}\psi_1]_t\partial_t^l\varphi_1\mathrm{d}x$$
$$-\int_0^1((\tilde{u}_1 + \psi_1)\partial_t^{l-1}\psi_1)_x\partial_t^l\varphi_1\mathrm{d}x + \int_0^1(\tilde{u}_1 + \psi_1)(\tilde{n}_1 + \varphi_1)_x\partial_t^{l-1}\psi_1\partial_t^l\psi_1\mathrm{d}x$$
$$-\int_0^1(\tilde{u}_1 + \psi_1)\partial_t^{l-1}\psi_1 G_1^l\mathrm{d}x - \int_0^1 h'(\tilde{n}_1 + \varphi_1)(\tilde{u}_1 + \psi_1)\partial_t^l\varphi_1\partial_t^{l-1}\varphi_{1x}\mathrm{d}x$$
$$+\int_0^1\frac{1}{2}\varphi_{1t}(\partial_t^{l-1}\psi_1)^2\mathrm{d}x + \int_0^1 h'(\tilde{n}_1 + \varphi_1)G_1^{l-1}\partial_t^l\varphi_1\mathrm{d}x + \int_0^1(\tilde{n}_1 + \varphi_1)F_1^l\partial_t^{l-1}\psi_1\mathrm{d}x.$$

In this procedure, we have used

$$
\begin{aligned}
&\int_0^1 [\partial_t^l\psi_{1t} + (\tilde{u}_1+\psi_1)\partial_t^l\psi_{1x} + \partial_t^l\psi_1](\tilde{n}_1+\varphi_1)\partial_t^{l-1}\psi_1 \mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tilde{n}_1+\varphi_1)\partial_t^l\psi_1\partial_t^{l-1}\psi_1\mathrm{d}x - \int_0^1 [(\tilde{n}_1+\varphi_1)\partial_t^{l-1}\psi_1]_t\partial_t^l\psi_1\mathrm{d}x \\
&\qquad - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\tilde{u}_1+\psi_1)\partial_t^{l-1}\psi_1\partial_t^l\varphi_1\mathrm{d}x + \int_0^1 ((\tilde{u}_1+\psi_1)\partial_t^{l-1}\psi_1)_t\partial_t^l\varphi_1\mathrm{d}x \\
&\qquad + \int_0^1 [(\tilde{u}_1+\psi_1)\partial_t^{l-1}\psi_1]_x\partial_t^l\varphi_1\mathrm{d}x + \int_0^1 (\tilde{u}_1+\psi_1)\partial_t^{l-1}\psi_1 G_1^l\mathrm{d}x \\
&\qquad - \int_0^1 (\tilde{n}_1+\varphi_1)_x(\tilde{u}_1+\psi_1)\partial_t^{l-1}\psi_1\partial_t^l\psi_1\mathrm{d}x \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}(\tilde{n}_1+\varphi_1)(\partial_t^{l-1}\psi_1)^2\mathrm{d}x - \int_0^1 \frac{1}{2}\varphi_{1t}(\partial_t^{l-1}\psi_1)^2\mathrm{d}x,
\end{aligned}
$$

with the chain rule and (4.15), and

$$
\begin{aligned}
\int_0^1 (h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1)_x(\tilde{n}_1+\varphi_1)\partial_t^{l-1}\psi_1\mathrm{d}x &= -\int_0^1 ((\tilde{n}_1+\varphi_1)\partial_t^{l-1}\psi_1)_x h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1\mathrm{d}x \\
&= \int_0^1 [h'(\tilde{n}_1+\varphi_1)(\partial_t^l\varphi_1)^2 + h'(\tilde{n}_1+\varphi_1)(\tilde{u}_1+\psi_1) \\
&\quad \times \partial_t^l\varphi_1\partial_t^{l-1}\varphi_{1x} - h'(\tilde{n}_1+\varphi_1)G_1^{l-1}\partial_t^l\varphi_1]\mathrm{d}x,
\end{aligned}
$$

with the help of integration by parts and (4.15).

Similarly, treating (4.11), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [(\tilde{n}_2+\varphi_2)\partial_t^l\psi_2\partial_t^{l-1}\psi_2 - (\tilde{u}_2+\psi_2)\partial_t^{l-1}\psi_2\partial_t^l\varphi_2 + \frac{1}{2}(\tilde{n}_2+\varphi_2)(\partial_t^{l-1}\psi_2)^2]\mathrm{d}x \\
&\quad + \int_0^1 h'(\tilde{n}_2+\varphi_2)(\partial_t^l\varphi_2)^2\mathrm{d}x + \int_0^1 \partial_t^l w_x(\tilde{n}_2+\varphi_2)\partial_t^{l-1}\psi_2\mathrm{d}x \\
&= \int_0^1 (\tilde{n}_2+\varphi_2)(\partial_t^l\psi_2)^2\mathrm{d}x + H_2^l,
\end{aligned}
\tag{4.22}
$$

with

$$H_2^l(t) = \int_0^1 (\tilde{n}_2+\varphi_2)_t \partial_t^{l-1}\psi_2 \partial_t^l \psi_2 \mathrm{d}x - \int_0^1 [(\tilde{u}_2+\psi_2)\partial_t^{l-1}\psi_2]_t \partial_t^l \varphi_2 \mathrm{d}x$$
$$- \int_0^1 ((\tilde{u}_2+\psi_2)\partial_t^{l-1}\psi_2)_x \partial_t^l \varphi_2 \mathrm{d}x + \int_0^1 (\tilde{u}_2+\psi_2)(\tilde{n}_2+\varphi_2)_x \partial_t^{l-1}\psi_2 \partial_t^l \psi_2 \mathrm{d}x$$
$$- \int_0^1 (\tilde{u}_2+\psi_2)\partial_t^{l-1}\psi_2 G_2^l \mathrm{d}x - \int_0^1 h'(\tilde{n}_2+\varphi_2)(\tilde{u}_2+\psi_2)\partial_t^l \varphi_2 \partial_t^{l-1}\varphi_{2x} \mathrm{d}x$$
$$+ \int_0^1 \frac{1}{2}\varphi_{2t}(\partial_t^{l-1}\psi_2)^2 \mathrm{d}x + \int_0^1 h'(\tilde{n}_2+\varphi_2) G_2^{l-1} \partial_t^l \varphi_2 \mathrm{d}x + \int_0^1 (\tilde{n}_2+\varphi_2) F_2^l \partial_t^{l-1}\psi_2 \mathrm{d}x.$$

From Cauchy–Schwartz inequality, (4.12)–(4.14), and (4.17)–(4.19), we have

$$|H_1^l(t)| \le C(E(t)^{\frac{1}{2}} + \delta)E(t), \tag{4.23}$$

and

$$|H_2^l(t)| \le C(E(t)^{\frac{1}{2}} + \delta)E(t). \tag{4.24}$$

Moreover, from (4.15) and (4.16), we have

$$- \int_0^1 \partial_t^l w_x (\tilde{n}_1+\varphi_1)\partial_t^{l-1}\psi_1 \mathrm{d}x + \int_0^1 \partial_t^l w_x (\tilde{n}_2+\varphi_2)\partial_t^{l-1}\psi_2 \mathrm{d}x$$
$$= - \int_0^1 \partial_t^l w (\partial_t^{l-1}\varphi_{1t} + (\tilde{u}_1+\psi_1)\partial_t^{l-1}\varphi_{1x} + G_1^l - \partial_t^{l-1}\varphi_{2t} - (\tilde{u}_2+\psi_2)\partial_t^{l-1}\varphi_{2x} - G_2^l)\mathrm{d}x$$
$$\ge \int_0^1 (\partial_t^l w_x)^2 \mathrm{d}x - C(E(t)^{\frac{1}{2}} + \delta)E(t).$$

Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [(\tilde{n}_1+\varphi_1)\partial_t^l\psi_1\partial_t^{l-1}\psi_1 - (\tilde{u}_1+\psi_1)\partial_t^{l-1}\psi_1\partial_t^l\varphi_1 + \frac{1}{2}(\tilde{n}_1+\varphi_1)(\partial_t^{l-1}\psi_1)^2]\mathrm{d}x$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [(\tilde{n}_2+\varphi_2)\partial_t^l\psi_2\partial_t^{l-1}\psi_2 - (\tilde{u}_2+\psi_2)\partial_t^{l-1}\psi_2\partial_t^l\varphi_2 + \frac{1}{2}(\tilde{n}_2+\varphi_2)(\partial_t^{l-1}\psi_2)^2]\mathrm{d}x$$
$$+ \int_0^1 [h'(\tilde{n}_1+\varphi_1)(\partial_t^l\varphi_1)^2 + h'(\tilde{n}_2+\varphi_2)(\partial_t^l\varphi_2)^2 + (\partial_t^l w_x)^2]\mathrm{d}x$$
$$- \int_0^1 [(\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2 + (\tilde{n}_2+\varphi_2)(\partial_t^l\psi_2)^2]\mathrm{d}x \le C(E(t)^{\frac{1}{2}} + \delta)E(t). \tag{4.25}$$

Next, multiplying (4.10) by $(\tilde{n}_1+\varphi_1)\partial_t^l\psi_1$, for $l = 1, 2$ and integrate the resultant equality over Ω to obtain that

$$\int_0^1 (\tilde{n}_1+\varphi_1)\partial_t^l\psi_1\partial_t^l\psi_{1t}\mathrm{d}x + \int_0^1 (\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)\partial_t^l\psi_1\partial_t^l\psi_{1x}\mathrm{d}x + \int_0^1 (\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2\mathrm{d}x$$
$$+\int_0^1 (\tilde{n}_1+\varphi_1)(h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1)_x\partial_t^l\psi_1\mathrm{d}x - \int_0^1 (\tilde{n}_1+\varphi_1)\partial_t^l w_x\partial_t^l\psi_1\mathrm{d}x$$
$$= \int_0^1 (\tilde{n}_1+\varphi_1)F_1^l\partial_t^l\psi_1\mathrm{d}x. \tag{4.26}$$

In the following, we treat the terms in (4.26) one by one. Firstly, the chain rule leads to

$$\int_0^1 (\tilde{n}_1+\varphi_1)\partial_t^l\psi_1\partial_t^l\psi_{1t}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}(\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2\mathrm{d}x - \int_0^1 \frac{1}{2}\varphi_{1t}(\partial_t^l\psi_1)^2\mathrm{d}x,$$

and integration by parts yields

$$\int_0^1 (\tilde{n}_1+\varphi_1)(h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1)_x\partial_t^l\psi_1\mathrm{d}x = -\int_0^1 [(\tilde{n}_1+\varphi_1)\partial_t^l\psi_1]_x h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1\mathrm{d}x$$
$$= \int_0^1 [h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1\partial_t^l\varphi_{1t} + h'(\tilde{n}_1+\varphi_1)(\tilde{u}_1+\psi_1)\partial_t^l\varphi_{1x}\partial_t^l\varphi_1 - h'(\tilde{n}_1+\varphi_1)G_1^l\partial_t^l\varphi_1]\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}h'(\tilde{n}_1+\varphi_1)(\partial_t^l\varphi_1)^2\mathrm{d}x - \int_0^1 \frac{1}{2}h''(\tilde{n}_1+\varphi_1)\varphi_{1t}(\partial_t^l\varphi_1)^2\mathrm{d}x$$
$$-\frac{1}{2}\int_0^1 (h'(\tilde{n}_1+\varphi_1)(\tilde{u}_1+\psi_1))_x(\partial_t^l\varphi_1)^2\mathrm{d}x + \int_0^1 h'(\tilde{n}_1+\varphi_1)G_1^l\partial_t^l\varphi_1\mathrm{d}x,$$

with the help of (4.15). Moreover, using (4.15) and chain rule, we have

$$\int_0^1 (\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)\partial_t^l\psi_1\partial_t^l\psi_{1x}\mathrm{d}x$$
$$= -\int_0^1 (\tilde{u}_1+\psi_1)\partial_t^l\psi_1(\partial_t^l\varphi_{1t} + (\tilde{u}_1+\psi_1)\partial_t^l\varphi_{1x} + (\tilde{n}_1+\varphi_1)_x\partial_t^l\psi_1 - G_1^l)\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 -(\tilde{u}_1+\psi_1)\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x + \int_0^1 ((\tilde{u}_1+\psi_1)\partial_t^l\psi_1)_t\partial_t^l\varphi_1\mathrm{d}x$$
$$+\int_0^1 ((\tilde{u}_1+\psi_1)^2\partial_t^l\psi_1)_x\partial_t^l\varphi_1 - (\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)_x(\partial_t^l\psi_1)^2 + (\tilde{u}_1+\psi_1)G_1^l\partial_t^l\psi_1\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 -(\tilde{u}_1+\psi_1)\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x + \int_0^1 [((\tilde{u}_1+\psi_1)^2)_x + \psi_{1t}]\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x$$

$$
\begin{aligned}
&-\int_0^1[(\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)_x(\partial_t^l\psi_1)^2-(\tilde{u}_1+\psi_1)G_1^l\partial_t^l\psi_1]\mathrm{d}x\\
&+\int_0^1[\partial_t^l\psi_{1t}+(\tilde{u}_1+\psi_1)\partial_t^l\psi_{1x}](\tilde{u}_1+\psi_1)\partial_t^l\varphi_1\mathrm{d}x\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1-(\tilde{u}_1+\psi_1)\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x+\int_0^1[((\tilde{u}_1+\psi_1)^2)_x+\psi_{1t}]\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x\\
&-\int_0^1[(\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)_x(\partial_t^l\psi_1)^2-(\tilde{u}_1+\psi_1)G_1^l\partial_t^l\psi_1]\mathrm{d}x\\
&+\int_0^1(\tilde{u}_1+\psi_1)[(-h'(\tilde{n}_1+\varphi_1)\partial_t^l\varphi_1)_x+\partial_t^lw_x-\partial_t^i\psi_1+F_1^l]\partial_t^l\varphi_1\mathrm{d}x\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1-(\tilde{u}_1+\psi_1)\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x+\int_0^1[((\tilde{u}_1+\psi_1)^2)_x+\psi_{1t}]\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x\\
&-\int_0^1[(\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)_x(\partial_t^l\psi_1)^2-(\tilde{u}_1+\psi_1)G_1^l\partial_t^l\psi_1]\mathrm{d}x\\
&+\int_0^1\{\frac{1}{2}[h'(\tilde{n}_1+\varphi_1)(\tilde{u}_1+\psi_1)]_x(\partial_t^l\varphi_1)^2-h''(\tilde{n}_1+\varphi_1)(\tilde{n}_1+\varphi_1)_x(\tilde{u}_1+\psi_1)(\partial_t^l\varphi_1)^2\}\mathrm{d}x\\
&+\int_0^1(\tilde{u}_1+\psi_1)(\partial_t^lw_x-\partial_t^l\psi_1+F_1^l)\partial_t^l\varphi_1\mathrm{d}x.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1[\frac{1}{2}(\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2+\frac{1}{2}h'(\tilde{n}_1+\varphi_1)(\partial_t^l\varphi_1)^2-(\tilde{u}_1+\psi_1)\partial_t^l\varphi_1\partial_t^l\psi_1]\mathrm{d}x\\
&\quad+\int_0^1(\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2\mathrm{d}x-\int_0^1(\tilde{n}_1+\varphi_1)\partial_t^lw_x\partial_t^l\psi_1\mathrm{d}x=J_1^l,
\end{aligned}
\tag{4.27}
$$

where

$$
\begin{aligned}
J_1^l=&\int_0^1\frac{1}{2}\varphi_{1t}(\partial_t^l\psi_1)^2\mathrm{d}x-\int_0^1[((\tilde{u}_1+\psi_1)^2)_x+\psi_{1t}]\partial_t^l\psi_1\partial_t^l\varphi_1\mathrm{d}x\\
&+\int_0^1[(\tilde{u}_1+\psi_1)(\tilde{n}_1+\varphi_1)_x(\partial_t^l\psi_1)^2-(\tilde{u}_1+\psi_1)G_1^l\partial_t^l\psi_1]\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^1\{\frac{1}{2}[h'(\tilde{n}_1+\varphi_1)(\tilde{u}_1+\psi_1)]_x(\partial_t^l\varphi_1)^2-h''(\tilde{n}_1+\varphi_1)(\tilde{n}_1+\varphi_1)_x(\partial_t^l\varphi_1)^2\}\mathrm{d}x\\
&-\int_0^1(\tilde{u}_1+\psi_1)(\partial_t^l w_x-\partial_t^l\psi_1+F_1^l)\partial_t^l\varphi_1\mathrm{d}x+\frac{1}{2}\int_0^1 h''(\tilde{n}_1+\varphi_1)\varphi_{1t}(\partial_t^l\varphi_1)^2\mathrm{d}x\\
&+\int_0^1\{\frac{1}{2}[h'(\tilde{n}_1+\varphi_1)(\tilde{u}_1+\psi_1)]_x(\partial_t^l\varphi_1)^2+h'(\tilde{n}_1+\varphi_1)G_1^l\partial_t^l\varphi_1\}\mathrm{d}x\\
&+\int_0^1(\tilde{n}_1+\varphi_1)F_1^l\partial_t^l\psi_1\mathrm{d}x.
\end{aligned}
$$

In the complete similar way, from (4.11), we can obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1[\frac{1}{2}(\tilde{n}_2+\varphi_2)(\partial_t^l\psi_2)^2+\frac{1}{2}h'(\tilde{n}_2+\varphi_2)(\partial_t^l\varphi_2)^2-(\tilde{u}_2+\psi_2)\partial_t^l\varphi_2\partial_t^l\psi_2]\mathrm{d}x\\
&\quad+\int_0^1(\tilde{n}_2+\varphi_2)(\partial_t^l\psi_2)^2\mathrm{d}x+\int_0^1(\tilde{n}_2+\varphi_2)\partial_t^l w_x\partial_t^l\psi_2\mathrm{d}x=J_2^l, \qquad (4.28)
\end{aligned}
$$

where

$$
\begin{aligned}
J_2^l=&\int_0^1\frac{1}{2}\varphi_{2t}(\partial_t^l\psi_2)^2\mathrm{d}x-\int_0^1[((\tilde{u}_2+\psi_2)^2)_x+\psi_{2t}]\partial_t^l\psi_2\partial_t^l\varphi_2\mathrm{d}x\\
&+\int_0^1[(\tilde{u}_2+\psi_2)(\tilde{n}_2+\varphi_2)_x(\partial_t^l\psi_2)^2-(\tilde{u}_2+\psi_1)G_2^l\partial_t^l\psi_2]\mathrm{d}x\\
&-\int_0^1\{\frac{1}{2}[h'(\tilde{n}_2+\varphi_2)(\tilde{u}_2+\psi_2)]_x(\partial_t^l\varphi_2)^2-h''(\tilde{n}_2+\varphi_2)(\tilde{n}_2+\varphi_2)_x(\partial_t^l\varphi_2)^2\}\mathrm{d}x\\
&-\int_0^1(\tilde{u}_2+\psi_2)(-\partial_t^l w_x-\partial_t^l\psi_2+F_2^l)\partial_t^l\varphi_2\mathrm{d}x+\frac{1}{2}\int_0^1 h''(\tilde{n}_2+\varphi_2)\varphi_{2t}(\partial_t^l\varphi_2)^2\mathrm{d}x\\
&+\int_0^1\{\frac{1}{2}[h'(\tilde{n}_2+\varphi_2)(\tilde{u}_2+\psi_2)]_x(\partial_t^l\varphi_2)^2+h'(\tilde{n}_2+\varphi_2)G_2^l\partial_t^l\varphi_2\}\mathrm{d}x\\
&+\int_0^1(\tilde{n}_2+\varphi_2)F_2^l\partial_t^l\psi_1\mathrm{d}x
\end{aligned}
$$

Similar as (4.23) and (4.24), we have

$$
|J_i^l(t)|\le C(E(t)^{\frac{1}{2}}+\delta)E(t),\ i=1,2.
$$

Moreover, note that

$$\begin{aligned}
&-\int_0^1 \partial_t^l w_x(\tilde{n}_1+\varphi_1)\partial_t^l\psi_1 \mathrm{d}x + \int_0^1 \partial_t^l w_x(\tilde{n}_2+\varphi_2)\partial_t^l\psi_1 \mathrm{d}x \\
&\quad = \int_0^1 ((\tilde{n}_1+\varphi_1)\partial_t^l\psi_1)_x \partial_t^l w \mathrm{d}x - \int_0^1 ((\tilde{n}_1+\varphi_1)\partial_t^l\psi_1)_x \partial_t^l w \mathrm{d}x \\
&\quad = -\int_0^1 \partial_t^l w(\partial_t^l\varphi_{1t} + (\tilde{u}_1+\psi_1)\partial_t^l\varphi_{1x} - G_1^l - \partial_t^l\varphi_{2t} - (\tilde{u}_2+\psi_2)\partial_t^l\varphi_{2x} + G_2^l)\mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}(\partial_t^l w_x)^2 \mathrm{d}x + \int_0^1 [(\tilde{u}_1+\psi_1)\partial_t^l w]_x \partial_t^l\varphi_1 \mathrm{d}x \\
&\qquad - \int_0^1 [(\tilde{u}_2+\psi_2)\partial_t^l w]_x \partial_t^l\varphi_2 \mathrm{d}x + \int_0^1 (G_1^l - G_2^l)\partial_t^l w \mathrm{d}x.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 [(\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2 + h'(\tilde{n}_1+\varphi_1)(\partial_t^l\varphi_1)^2 + (\tilde{n}_2+\varphi_2)(\partial_t^l\psi_2)^2 \\
&\qquad + h'(\tilde{n}_2+\varphi_2)(\partial_t^l\varphi_2)^2 + (\partial_t^l w_x)^2]\mathrm{d}x + \int_0^1 [(\tilde{n}_1+\varphi_1)(\partial_t^l\psi_1)^2 + (\tilde{n}_2+\varphi_2)(\partial_t^l\psi_2)^2]\mathrm{d}x \\
&\leq C(E(t)^{\frac{1}{2}} + \delta)E(t).
\end{aligned} \tag{4.29}$$

Multiplying (4.29) by 2, adding the resulting inequality to (4.25), we arrive at the desired estimate (4.20). This completes the proof of Lemma 4.5

Proof of Theorem 4.1. From (4.9) and (4.20), there exists a positive constant C_3 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_2(t)+E_3(t)) + C_3E_1(t) \leq C(E(t)^{\frac{1}{2}} + \delta)E(t). \tag{4.30}$$

From the definitions of $E_1(t), E_2(t), E_3(t)$ and Lemma 4.2, we can easily see that $E_1(t)$ and $E_2(t)+E_3(t)$ is equivalent. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_1(t)) + C_4E_1(t) \leq C(E(t)^{\frac{1}{2}} + \delta)E(t). \tag{4.31}$$

On the other hand, using Lemma 4.3, it is easy to see that

$$E(t) \leq C_1E_1(t). \tag{4.32}$$

Thus, for $E(t)^{\frac{1}{2}} + \delta$ sufficiently small, (4.31) and (4.32) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + \frac{C_4}{2}E_1(t) \leq 0. \tag{4.33}$$

which yields the exponential decaying of $E_1(t)$. Finally, the exponential decay of $E(t)$ follows from (4.32) and Lemma 4.3. This completes the proof of Theorem 4.1.

Acknowledgments

The authors would like to express sincere thanks to the referee for the suggestive comments on this paper. The research is partially supported by the National Science Foundation of China (Grant No. 11171223) and the Innovation Program of Shanghai Municipal Education Commission (Grant No. 13ZZ109).

References

1. Ali, G., Jüngel, A.: Global smooth solutions to the multi-dimensional hydrodynamic model for two-carrier plasma. J. Differ. Equ. **190**, 663–685 (2003)
2. Gasser, I., Hsiao, L., Li, H.-L.: Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors. J. Differ. Equ. **192**, 326–359 (2003)
3. Gasser, I., Marcati, P.: The combined relaxation and vanishing Debye length limit in the hydrodynamic model for semiconductors. Math. Methods Appl. Sci. **24**, 81–92 (2001)
4. Gilbarg, D., Trüdinger, N.S.: Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer, Berlin (1983)
5. Huang, F.-M., Li, Y.-P.: Large time behavior and quasineutral limit of solutions to a bipolar hydrodynamic model with large data and vacuum. Dis. Cont. Dyn. Sys. A **24**, 455–470 (2009)
6. Huang, F., Mei, M., Wang, Y.: Large time behavior of solution to n-dimensional bipolar hydrodynamic model for semiconductors. SIAM J. Math. Anal. **43**, 1595–1630 (2011)
7. Hsiao, L., Zhang, K.-J.: The relaxation of the hydrodynamic model for semiconductors to the drift-diffusion equations. J. Differ. Equ. **165**, 315–354 (2000)
8. Hsiao, L., Zhang, K.-J.: The global weak solution and relaxation limits of the initial boundary value problem to the bipolar hydrodynamic model for semiconductors. Math. Models Methods Appl. Sci. **10**, 1333–1361 (2000)
9. Jüngel, A.: Quasi-hydrodynamic Semiconductor Equations, Progress in Nonlinear Differential Equations and their Applications. Birkhäuser, Basel (2001)
10. Ju, Q.-C.: Global smooth solutions to the multidimensional hydrodynamic model for plasmas with insulating boundary conditions. J. Math. Anal. Appl. **336**, 888–904 (2007)
11. Lattanzio, C.: On the 3-D bipolar isentropic Euler–Poisson model for semiconductors and the drift-diffusion limit. Math. Models Methods Appl. Sci. **10**, 351–360 (2000)
12. Li, Y.-P.: Diffusion relaxation limit of a bipolar isentropic hydrodynamic model for semiconductors. J. Math. Anal. Appl. **336**, 1341–1356 (2007)
13. Li, Y.-P.: The Cauchy–Neumann problem for a multidimensioal nonisentropic hydrodynamic semiconductor model. Nonlinearity **18**, 959–980 (2005)
14. Li, Y.-P.: The Cauchy–Neumann problem for a multidimensioal isentropic hydrodynamic semiconductor model. Math. Methods Appl. Sci. **28**, 1955–1975 (2005)
15. Luo, T., Natalini, R., Xin, Z.: Large time behavior of the solutions to a hydrodynamic model for semiconductors. SIAM J. Appl. Math. **59**, 810–830 (1998)
16. Li, H.-L., Markowich, P.A., Mei, M.: Asymptotic behavior of solutions of the hydrodynmaic model of semiconductors. Proc. R. Soc. Edinb. Sect. A **132**, 773–796 (2002)
17. Li, Y.-P., Yang, X.-F.: Global existence and asymptotic behavior of the solutions to the three dimensional bipolar Euler–Poisson systems. J. Differ. Equ. **252**, 768–791 (2012)
18. Li, Y.-P., Zhang, T.: Relaxation-time limit of the multidimensional bipolar hydrodynamic model in Besov space. J. Differ. Equ. **251**, 3143–3162 (2011)
19. Mao, J.-F., Li, Y.-P.: Global existence and asymptotic behavior of smooth solutions to the IBVP for the 3D bipolar Euler–Poisson system. Acta Math. Sci., Series B (2012, to appear)
20. Markowich, P.A., Ringhofev, C.A., Schmeiser, C.: Semiconductor equations. Springer, Wien (1990)
21. Natalini, R.: The bipolar hydrodynamic model for semiconductors and the drift-diffusion equation. J. Math. Anal. Appl. **198**, 262–281 (1996)
22. Nishibata, S., Suzuki, M.: Asymptotic stability of a stationary solution to a hydrodynamic model of semiconductors. Osaka J. Math. **44**, 639–665 (2007)
23. Nishibata, S., Suzuki, M.: Asymptotic stability of a stationary solution to a thermal hydrodynamic model for semiconductors. Arch. Ration. Mech. Anal. **244**, 836–874 (2008)
24. Sitnko, A., Malnev, V.: Plasma Physics Theory. Chapman & Hall, London (1995)
25. Schochet, S.: The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit. Commun. Math. Phys. **104**, 49–75 (1986)

26. Tsuge, N.: Existence and uniqueness of stationary solutions to a one-dimensional bipolar hydrodynamic models of semiconductors. Nonlinear Anal. **TMA73**, 779–787 (2010)
27. Zhu, C., Hattori, H.: Stability of steady state solutions for an isentropic hydrodynamic model of semiconductors of two species. J. Differ. Equ. **166**, 1–32 (2000)
28. Zhou, F., Li, Y.-P.: Existence and some limits of stationary solutions to a one-dimensional bipolar Euler–Poisson system. J. Math. Anal. Appl. **351**, 480–490 (2009)

Yeping Li
Department of Mathematics
Shanghai Normal University
Shanghai 200234
People's Republic China
e-mail: ypleemei@yahoo.com.cn

(Received: March 21, 2012; revised: September 17, 2012)