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The multiple nature of concentrated interactions in second-gradient dissipative liquids

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Abstract. It is well known that second-gradient continuum mechanical theories allow for the appearance of concentrated stresses along the edges of piecewise smooth material surfaces, but this is not the sole example of concentrated interaction. Two additional kinds of concentrated interaction are shown to take place in some second-gradient incompressible dissipative fluids: the adherence to one-dimensional immersed bodies and the capability of sustaining concentrated external body forces. These three phenomena turn out to be distinct and independent. This feature is explicitly discussed in two benchmark problems, and the different mathematical origins of each concentrated interaction are explained.

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1. Introduction

A second-gradient theory for viscous fluids has been proposed by Fried and Gurtin in connection with the modeling of small-scale effects [7] and turbulent flows [8]. Such fluids are characterized by an internal power expenditure with dissipative part proportional to both the first and the second gradient of the velocity field. As shown in [11,12], a key feature of that model, related to multiscale interactions, is the possibility of describing the adherence of a three-dimensional fluid body to one-dimensional structures immersed in it. This is an example of concentrated interaction, but it has been pointed out by many authors [3,4,6,9,15,17] that second-gradient theories allow also for the concentration of stresses along the edges of piecewise smooth material surfaces. Nevertheless, Podio-Guidugli and Vianello [16] proved that a constitutive lack of concentrated stresses can appear even in some second-gradient fluid, exemplifying a general feature first noted by dell'Isola and Seppecher [4].

The aim of this paper is to provide evidence of the different nature of the aforementioned non-standard effects, investigating also a third kind of concentrated interaction. In Sect. 2, second-gradient linear isotropic dissipative liquids, which are quasi-Newtonian from a viscometric viewpoint, are presented. The distinction between concentrated stresses and other concentrated interactions, the adherence of a three-dimensional liquid to one-dimensional immersed structures, is emphasized, in Sect. 3, by means of explicit calculations in two benchmark problems. Section 4 aims first to mathematically clarify how the adherence to one-dimensional immersed bodies can be modeled, even in liquids characterized by a constitutive lack of concentrated stresses, and then to introduce a further concentrated interaction, related to the possible concentration of the external forces acting on the liquid. Finally, in Sect. 5, possible applications of theories encompassing concentrated interactions are discussed, and an approach to extending the analysis presented in this note to Nth gradient continua is outlined.

Birkhäuser

2. Second-gradient linear isotropic dissipative liquids

Following [7], I will introduce the model under consideration with an approach based on virtual power. In this approach, the material is characterized by an internal power expenditure, and the action of the environment on it is described by an external power expenditure. The descriptor of the state at any instant t in the time interval $[0,T] \subset \mathbb{R}$ is the Eulerian velocity field $\mathsf{u}(t,x)$. Since the focus of this paper is on liquids, I will assume the *incompressibility condition*, together with the assumption of homogeneity, and I will set the mass density ρ equal to unity, identically, obtaining the first constraint on the velocity:

$$\forall t \in [0,T]: \ \operatorname{div} \mathsf{u} = 0. \tag{1}$$

Since viscous interactions should not occur during a rigid motion, and since I want to model generalized viscous liquids, I require the second-gradient internal power expenditure to vanish on any rigid velocity field. Then, its general form becomes

$$\langle \mathcal{P}^{\text{in}}_{\mathsf{u}}, \mathsf{v} \rangle = \int_\Omega \mathsf{T} \cdot \nabla \mathsf{v} + \int_\Omega \mathbf{G} \cdot \nabla\nabla \mathsf{v}\,, \tag{2}$$

where T is a symmetric tensor field of order 2, representing the classical first-gradient contribution, and $\mathbf{G}$ is a tensor field of order 3, introducing the second-gradient terms. Within this framework, v denotes a *virtual velocity*, that is, any kinematically admissible velocity field.

Linearity and isotropy of the liquid are encoded in the dependence of the tensor fields T and $\mathbf{G}$ on the descriptor u. It is well known that, within incompressible theories,

$$\mathsf{T}_{ij} = \mu(u_{i,j} + u_{j,i}) - p\,\delta_{ij}\,;$$

besides, in [14, Theorem 1.1], it has been shown that

$$\mathbf{G}_{ijk} = \eta_1 u_{i,jk} + \eta_2(u_{j,ki} + u_{k,ij} - u_{i,ss}\delta_{jk}) + \eta_3(u_{j,ss}\delta_{ki} + u_{k,ss}\delta_{ij} - 4u_{i,ss}\delta_{jk}) - p_k\delta_{ij}\,,$$

where $\mu, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$ and δ_{ij} is the usual Kronecker symbol. The fields p and p, scalar- and vector-valued, respectively, enter the definition of the *pressure*, whose role in incompressible theories reduces to that of a Lagrange multiplier of the constraint (1).

Defining the *symmetric part* of a tensor X of order m as

$$\operatorname{Sym} \mathsf{X} := \frac{1}{m!} \sum_\sigma \mathsf{X}_{\sigma(i_1 \dots i_m)}\,,$$

where σ runs over the group of permutations of m elements, denoting by Δ the Laplace operator, and setting $\mathsf{I} = (\delta_{ij})$, the previous relations can be written in intrinsic notation as

$$\mathsf{T} = 2\mu \operatorname{Sym} \nabla \mathsf{u} - p\,\mathsf{I}\,, \tag{3}$$

$$\mathbf{G} = (\eta_1 - \eta_2)\nabla\nabla\mathsf{u} + 3\eta_2 \operatorname{Sym} \nabla\nabla\mathsf{u} - (\eta_2 + 5\eta_3)\Delta\mathsf{u} \otimes \mathsf{I} + 3\eta_3 \operatorname{Sym}(\Delta\mathsf{u} \otimes \mathsf{I}) - \mathsf{I} \otimes \mathsf{p}. \tag{4}$$

Following these definitions, since also the virtual velocities must obey the constraint (1), the internal power expenditure for a second-gradient linear isotropic dissipative liquid can be expressed as

$$\begin{aligned} \langle \mathcal{P}^{\text{in}}_{\mathsf{u}}, \mathsf{v} \rangle = 2\mu \int_\Omega \operatorname{Sym} \nabla\mathsf{u} \cdot \nabla\mathsf{v} + (\eta_1 - \eta_2) \int_\Omega \nabla\nabla\mathsf{u} \cdot \nabla\nabla\mathsf{v} \\ + 3\eta_2 \int_\Omega \operatorname{Sym} \nabla\nabla\mathsf{u} \cdot \nabla\nabla\mathsf{v} - (\eta_2 + 4\eta_3) \int_\Omega \Delta\mathsf{u} \cdot \Delta\mathsf{v}. \end{aligned} \tag{5}$$

On the basis of thermodynamical considerations, the instantaneous dissipation must be non-negative for any flow; it has been proved in [12] that this constraint is satisfied if and only if

$$\mu \geq 0\,, \quad \eta_1 + 2\eta_2 \geq 0\,, \quad \eta_1 - \eta_2 \geq 0\,, \quad \eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} \geq 0. \tag{6}$$

Notice that a thermodynamically consistent choice with $\eta_1 = 0$ requires both $\eta_2 = 0$ and $\eta_3 \leq 0$.

By the definition of a second-gradient power expenditure, it is clear that $\mathbf{G}$ carries dimensions of mass per unit time squared. Moreover, the constitutive prescriptions (3) and (4) imply that the ratios $\sqrt{\eta_i/\mu}$ $(i = 1, 2, 3)$ have dimensions of length. Hence, many characteristic length scales could be defined combining the four material parameters; one of those combinations, which will be used later, leads to

$$L := \sqrt{\frac{\eta_1 - \eta_2 - 4\eta_3}{\mu}}\,, \tag{7}$$

where the constraints (6) imply $L \geq 0$.

It is important to decompose the stresses $\mathsf{T}, \mathbf{G}$ into *active contributions* $\mathsf{T}_{\rm a}, \mathbf{G}_{\rm a}$ and *pressure contributions* $\mathsf{T}_{\rm p}, \mathbf{G}_{\rm p}$. Such a decomposition of T is obvious, while a comparison between expressions (4) and (5) helps in understanding that pressure contributions contain terms that do not expend power on divergence-free virtual velocities; this also clarifies the meaning of the active part. Summarizing, we have

$$\begin{aligned}
\mathsf{T}_{\rm a} &= 2\mu\, \mathrm{Sym}\, \nabla \mathsf{u}\,, \quad \mathsf{T}_{\rm p} = -p\,\mathsf{I}\,,\\
\mathbf{G}_{\rm a} &= (\eta_1 - \eta_2)\nabla\nabla \mathsf{u} + 3\eta_2\ \mathrm{Sym}\,\nabla\nabla \mathsf{u} - (\eta_2 + 4\eta_3)\Delta \mathsf{u} \otimes \mathsf{I}\,,\\
\mathbf{G}_{\rm p} &= 3\eta_3\ \mathrm{Sym}(\Delta \mathsf{u} \otimes \mathsf{I}) - \eta_3 \Delta \mathsf{u} \otimes \mathsf{I} - \mathsf{I} \otimes \mathsf{p}\,,
\end{aligned}$$

with

$$\int_\Omega \mathsf{T}_{\rm p} \cdot \nabla \mathsf{v} = 0 = \int_\Omega \mathbf{G}_{\rm p} \cdot \nabla\nabla \mathsf{v}$$

for any virtual velocity v.

Notice that the material coefficient η_3 enters both the active and the pressure contributions $\mathbf{G}_{\rm a}$ and $\mathbf{G}_{\rm p}$ to the second-gradient stress $\mathbf{G}$. This is a striking difference from the first-gradient case, where active and pressure contributions are completely independent. The term multiplied by η_3 in $\mathbf{G}_{\rm p}$ originates from the projection of the third-order tensor $\Delta \mathsf{u} \otimes \mathsf{I}$ onto the space of completely symmetric tensors. Hence, the form of $\mathbf{G}_{\rm p}$ suggests that, within incompressible theories, there is no point in taking the symmetric part of $\Delta \mathsf{u} \otimes \mathsf{I}$. Nonetheless, the dual role played by η_3 is somewhat puzzling and deserves further analysis.

Since the assumed form of the internal power of a continuous body limits the range of internal interactions that body can experience, I assume that the external power expenditure is also a second-gradient power. This ensures that the external world can act only in ways consistent with the interactions sustainable by a second-gradient liquid. Applying D'Alembert's principle, it is customary to include within the external power expenditure the inertial term

$$-\int_\Omega \rho\, \dot{\mathsf{u}} \cdot \mathsf{v} := -\int_\Omega \rho \left(\frac{\partial \mathsf{u}}{\partial t} + (\mathsf{u} \cdot \nabla)\mathsf{u} \right) \cdot \mathsf{v}\,,$$

and thereby make it possible to formulate evolution equations as a consequence of the principle of virtual power.

3. Some explicit calculations

The aim of this section is to analyze the effects of the higher-order material parameters η_1, η_2, and η_3 in some examples, with the objective of providing some insight into their relationship with concentrated interactions. It will become clear that η_1 is strictly related to the presence of concentrated stresses, while η_3 is only responsible for the adherence to one-dimensional immersed structures. The role of η_2 is less obvious: it must vanish if η_1 vanishes, but it can be otherwise set equal to η_1 to make $\operatorname{Sym}\nabla\nabla\mathsf{u}$ and $\Delta\mathsf{u}$ the sole relevant terms affecting second-gradient dissipation, as can be inferred by the form of (5).

Concentrated stresses appear in second-gradient theories on considering an equivalent representation of the internal power (2), expended on a piecewise smooth domain Ω, in terms of interaction fields. Indeed, as shown in [7,16], there exist vector fields $\hat{\mathsf{b}}, \hat{\mathsf{t}}, \hat{\mathsf{m}}$, and $\hat{\mathsf{k}}$ such that

$$\langle \mathcal{P}^{\rm in}_{\mathsf{u}}, \mathsf{v}\rangle = \int_\Omega \hat{\mathsf{b}}\cdot\mathsf{v} + \int_{\mathcal{S}} \hat{\mathsf{t}}\cdot\mathsf{v} + \int_{\mathcal{S}} \hat{\mathsf{m}}\cdot\frac{\partial\mathsf{v}}{\partial n} + \int_{\mathcal{E}} \hat{\mathsf{k}}\cdot\mathsf{v}\,,$$

where $\mathcal{S}$ and $\mathcal{E}$ are, respectively, the regular and the singular part of $\partial\Omega$, n is the unit outer normal to $\mathcal{S}$, and $\partial/\partial n$ denotes the normal derivative on $\mathcal{S}$. Moreover, the following relations hold:

$$\begin{aligned}
\hat{\mathsf{b}} &= -\operatorname{div}\mathsf{T} + \operatorname{div}\operatorname{div}\mathbf{G}\,,\\
\hat{\mathsf{t}} &= [\mathsf{T} - \operatorname{div}\mathbf{G}]\mathsf{n} + \operatorname{div}_{\mathcal{S}}[\mathbf{G}\mathsf{n}]\,,\\
\hat{\mathsf{m}} &= [\mathbf{G}\mathsf{n}]\mathsf{n}\,,\\
\hat{\mathsf{k}} &= [\mathbf{G}\mathsf{n}_a]\mathsf{e}_a + [\mathbf{G}\mathsf{n}_b]\mathsf{e}_b\,,
\end{aligned}$$

where $\operatorname{div}_{\mathcal{S}}$ denotes the projection onto $\mathcal{S}$ of the divergence operator, n_a and n_b are the limits of n coming from the two sides of an edge in $\mathcal{E}$, and e_a and e_b are unit vector fields orthogonal to $\mathcal{E}$ and to n_a and n_b, respectively, and pointing outward the a and b faces, respectively.

The previous representation was exploited by Fried and Gurtin [7] to arrive at various boundary conditions. Using their terminology, I will assume a no-slip condition with *weak adherence* (*i.e.* $\hat{\mathsf{m}} = 0$) to solid walls. I will consider steady flows and neglect body forces, in which case the differential problem given by the balance of internal and external power expenditures is

$$\begin{aligned}
&2\mu\int_\Omega \operatorname{Sym}\nabla\mathsf{u}\cdot\nabla\mathsf{v} + (\eta_1-\eta_2)\int_\Omega \nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v}\\
&\quad+3\eta_2\int_\Omega \operatorname{Sym}\nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v} - (\eta_2+4\eta_3)\int_\Omega \Delta\mathsf{u}\cdot\Delta\mathsf{v} = -\int_\Omega \rho\,(\mathsf{u}\cdot\nabla)\mathsf{u}\cdot\mathsf{v}\,,
\end{aligned}\tag{8}$$

for any virtual velocity v. If the velocity field u is sufficiently regular, the local form of Eq. (8) reads

$$\rho\,(\mathsf{u}\cdot\nabla)\mathsf{u} - \mu\Delta\mathsf{u} + (\eta_1-\eta_2-4\eta_3)\Delta\Delta\mathsf{u} = 0.\tag{9}$$

3.1. Dragged flow in a cylinder

Consider now a liquid in a concentric annular pipe with radii $R_1 < R_2$; the flow is driven by moving the inner cylinder at constant speed U along the axial direction e_z. We look for cylindrically symmetric stationary solutions, *i.e.* $\mathsf{u} = u(r)\mathsf{e}_z$, where r is the cylindrical radius. In the case of Newtonian liquids with viscosity μ and the usual adherence condition on the walls, one easily finds the following profile for the axial component of the velocity as a function of r:

$$U\frac{\log R_2 - \log r}{\log R_2 - \log R_1}\,.$$

It is clear that this solution has no continuous extension to the case $R_1 = 0$.

However, with a second-order linear isotropic viscous liquid, this problem leads to the equation

$$\Delta u - L^2\Delta\Delta u = 0.\tag{10}$$

We have now the family of solutions

$$u(r) = \alpha_1 + \alpha_2\,\mathrm{I}_0\left(\frac{r}{L}\right) + \alpha_3\log\left(\frac{r}{L}\right) + \alpha_4\,\mathrm{K}_0\left(\frac{r}{L}\right)\,,\tag{11}$$

where $\alpha_i, i = 1, \ldots, 4$, are constants (depending on R_1 and R_2) fixed by the boundary conditions, I_s and $\mathrm{K}_s, s \in \mathbb{N}$, are Bessel functions of imaginary argument (see [13, Sect. 5.7]), and the parameter L is defined as in (7). If we now set $R_1 = 0$, the solution remains bounded provided that $\alpha_3 = \alpha_4$, since

$$c_0 := \lim_{r\to 0}\left(\log\left(\frac{r}{L}\right) + \mathrm{K}_0\left(\frac{r}{L}\right)\right) < +\infty\,;$$

besides, one can still meet the prescribed boundary conditions by a suitable choice of the constants.

Actually, true boundary conditions can now be imposed only at $r = R_2$ that is on the outer boundary of the annular pipe, while at $r = 0$, I will impose the value of the velocity field to be equal to U—that is, to the speed at which the one-dimensional degenerate cylinder moves in the e_z direction. Hence, letting n denotes the outer unit normal to the boundary, the three conditions are:

$$u(0) = U\,, \tag{12}$$

$$u(R_2) = 0\,, \tag{13}$$

$$(\mathbf{G}\mathsf{n})\mathsf{n}|_{r=R_2} = \left[\eta_1\frac{\partial^2 u}{\partial r^2} - (\eta_2 + 4\eta_3)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_{r=R_2} = 0. \tag{14}$$

Thus, it follows that α_1, α_2, and α_3 obey

$$\begin{aligned}
\alpha_1 &= U - \alpha_2 - \alpha_3 c_0\,,\\
\alpha_2 &= \frac{1}{1-\mathrm{I}_0\left(\frac{R_2}{L}\right)}\left[U + \alpha_3\left(\log\left(\frac{R_2}{L}\right) + \mathrm{K}_0\left(\frac{R_2}{L}\right) - c_0\right)\right],\\
\alpha_3 &= \frac{1}{B}\cdot\frac{U}{1-\mathrm{I}_0\left(\frac{R_2}{L}\right)}\left[\frac{\eta_1-\eta_2-4\eta_3}{2L^2}\left(\mathrm{I}_0\left(\frac{R_2}{L}\right) + \mathrm{I}_2\left(\frac{R_2}{L}\right)\right) - \frac{\eta_2+4\eta_3}{LR_2}\mathrm{I}_1\left(\frac{R_2}{L}\right)\right],
\end{aligned}$$

with

$$\begin{aligned}
B = {} & \frac{\eta_1-\eta_2-4\eta_3}{2L^2}\left[\frac{\log\left(\frac{R_2}{L}\right) + \mathrm{K}_0\left(\frac{R_2}{L}\right) - c_0}{1-\mathrm{I}_0\left(\frac{R_2}{L}\right)}\left(\mathrm{I}_0\left(\frac{R_2}{L}\right) + \mathrm{I}_2\left(\frac{R_2}{L}\right)\right)\right]\\
&+\frac{\eta_1-\eta_2-4\eta_3}{2L^2}\left[\mathrm{K}_0\left(\frac{R_2}{L}\right) + \mathrm{K}_2\left(\frac{R_2}{L}\right) - \frac{L^2}{R_2^2}\right]\\
&-\frac{\eta_2+4\eta_3}{LR_2}\left[\frac{\log\left(\frac{R_2}{L}\right) + \mathrm{K}_0\left(\frac{R_2}{L}\right) - c_0}{1-\mathrm{I}_0\left(\frac{R_2}{L}\right)}\mathrm{I}_1\left(\frac{R_2}{L}\right) + \frac{L}{R_2} - \mathrm{K}_1\left(\frac{R_2}{L}\right)\right].
\end{aligned}$$

Using the foregoing expressions for α_1, α_2, and α_3 in (11) determines the axial velocity in terms of the geometric and material parameters.

The existence and uniqueness results proved in [11] require $\eta_1 > 0$. However, the example treated in this section admits a solution also when $\eta_1 = \eta_2 = 0$ and $\eta_3 < 0$, in which case the liquid is still a non-simple one and no thermodynamic requirement is violated. In fact, a general result of existence and uniqueness of solution with the latter choice for the material parameters can be proved, as outlined in Sect. 4.1.

So far, there is no clear distinction between the effects of the presence of the three parameters. Let us now calculate the concentrated force on the edge of a cylindrical wedge, corresponding to the point E on the section depicted in Fig. 1. Notice that there

$$\hat{\mathsf{k}} = \mathbf{G}(\mathsf{e}_x\otimes\mathsf{e}_y + \mathsf{e}_y\otimes\mathsf{e}_x) = 2\mathbf{G}(\mathsf{e}_x\otimes\mathsf{e}_y)\,;$$

indeed,

$$\mathbf{G}_{112} = \mathbf{G}_{212} = 0\,,\quad \text{and}\quad \mathbf{G}_{312} = \eta_1\frac{\partial^2 u}{\partial x\partial y}.$$

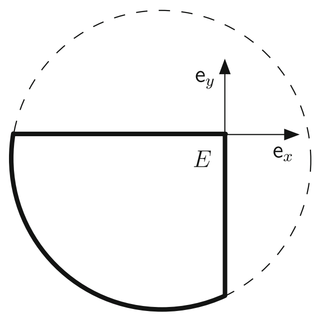

FIG. 1. Section of the cylindrical wedge

Remembering that $r = \sqrt{x^2 + y^2}$, we have

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{xy}{r^3}\left[\frac{\alpha_2}{L}\,\mathrm{I}_1\left(\frac{r}{L}\right) + \frac{\alpha_3}{r} - \frac{\alpha_3}{L}\,\mathrm{K}_1\left(\frac{r}{L}\right)\right] + \frac{xy}{r^2}\left[\frac{\alpha_2}{2L^2}\left(\mathrm{I}_0\left(\frac{r}{L}\right) + \mathrm{I}_2\left(\frac{r}{L}\right)\right) - \frac{\alpha_3}{r^2} + \frac{\alpha_3}{2L^2}\left(\mathrm{K}_0\left(\frac{r}{L}\right) + \mathrm{K}_2\left(\frac{r}{L}\right)\right)\right]; \tag{15}$$

u being a radial function, the expression in (15) vanishes whenever $x = 0$ or $y = 0$ but is generically non-zero when calculated at a point $E = (x_0, y_0)$ with both $x_0 \neq 0$ and $y_0 \neq 0$ and thus gives rise to a concentrated stress along the corresponding edge.

Moreover, it is clear that the parameters η_2 and η_3, entering (15) via the constants α_i, cannot determine the presence of the concentrated force $\hat{\mathsf{k}}$, since no concentrated force can appear if $\eta_1 = 0$. This result is in agreement with observations of Podio-Guidugli and Vianello [16, Sect. 4]. Indeed, the second-order stress would take the form

$$\mathsf{G} = -5\eta_3 \Delta \mathsf{u} \otimes \mathsf{I} + 3\eta_3\ \mathrm{Sym}(\Delta \mathsf{u} \otimes \mathsf{I}) + \mathsf{I} \otimes \mathsf{p}\,,$$

whose active part, given the incompressibility constraint, is $\mathsf{G}_{\mathrm{a}} = -4\eta_3 \Delta \mathsf{u} \otimes \mathsf{I}$, which cannot develop concentrated stresses, since $\mathsf{I} \cdot (\mathsf{e}_a \otimes \mathsf{n}_a) = \mathsf{e}_a \cdot \mathsf{n}_a = 0$ for any surface label a, e_a, and n_a being a pair of orthogonal vectors.

The results of the present section show that with particular constitutive choices, it is possible to define a liquid which is capable of adhering to one-dimensional objects without developing concentrated stresses. This is the case when $\eta_1 = \eta_2 = 0$ and $\eta_3 < 0$. Such a result is a bit surprising. But we have to properly understand the meaning of concentrated stresses; what we find for $\eta_1 = \eta_2 = 0$ and $\eta_3 < 0$ is that the adherence to a one-dimensional structure can be represented, on any material surface containing it, by diffuse surface interactions. On the other hand, for $\eta_1 > 0$, the representation of internal stresses can have a concentrated part on the singular edges of the surface of some subbody, even if there are no manifestly concentrated interactions on any part of the boundary or the interior of the body.

3.2. Pressure-driven flow

Consider now a pipe with square section in the (x, y)-plane, a velocity field $\mathsf{u} = w(x, y)\mathsf{e}_z$, and a constant and uniform pressure gradient $C\mathsf{e}_z$, which drives the flow. To emphasize the appearance of concentrated stresses, I set $\eta_2 = \eta_3 = 0$. The nonlinear term in (9) vanishes again, thanks to the chosen geometry, and the differential equation for the steady flow of a second-order liquid with viscosity μ becomes

$$\mu \Delta w - \eta_1 \Delta\Delta w = C\,,$$

with $w = 0$ and $(\nabla\nabla w)\mathsf{nn} = 0$ on the boundary of the pipe. The previous equation can be written as

$$-\Delta(w - L^2\Delta w) = -\frac{C}{\mu}\,,$$

and since $w - L^2\Delta w = 0$ on the boundary, it is permissible to set $u = w - L^2\Delta w$, obtaining

$$\begin{cases} -\Delta u = -C/\mu =: \widetilde{C} \\ u = 0 \quad \text{on the boundary.} \end{cases} \tag{16}$$

Take now the section of the pipe to be $[0, \pi] \times [0, \pi]$, so that we can expand on the basis given by the eigenfunctions of the Laplace operator on that square:

$$X_{h,k} = \frac{4}{\pi^2} \sin hx \sin ky,$$

with eigenvalues $\lambda_{h,k} = h^2 + k^2$, with $h, k \in \mathbb{N}$. It follows that

$$\langle \widetilde{C}, X_{h,k} \rangle = \begin{cases} f_{h,k} := \frac{8\widetilde{C}}{\pi^2} hk & \text{for } h, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the solution of (16) is

$$\tilde{u} = \sum_{h,k \text{ odd}} \frac{f_{h,k}}{\lambda_{h,k}} X_{h,k}. \tag{17}$$

It remains to solve

$$-L^2\Delta w + w = \tilde{u}.$$

The operator $(-L^2\Delta + 1)$ has again $X_{h,k}$ as eigenfunctions, with eigenvalues $L^2\lambda_{h,k} + 1$; thus,

$$w(x, y) = \sum_{h,k \text{ odd}} \frac{f_{h,k}}{\lambda_{h,k}(L^2\lambda_{h,k} + 1)} X_{h,k} = -\frac{32C}{\mu\pi^4} \sum_{h,k \text{ odd}} \frac{hk}{(h^2 + k^2)(L^2h^2 + L^2k^2 + 1)} \sin hx \sin ky. \tag{18}$$

Now, the concentrated traction along the edges of the pipe can be evaluated. Since

$$\hat{\mathsf{k}} = 2\mathbf{G}(\mathsf{e}_x \otimes \mathsf{e}_y)\,, \quad \mathbf{G}_{112} = \mathbf{G}_{212} = 0\,, \quad \text{and} \quad \mathbf{G}_{312} = \eta_1 \frac{\partial^2 w}{\partial x \partial y}\,,$$

the concentrated stress density at the origin is given by

$$\hat{\mathsf{k}}(0, 0) = 2\eta_1 \frac{\partial^2 w}{\partial x \partial y}(0, 0)\mathsf{e}_z = -2\eta_1 \frac{32C}{\mu\pi^4} \sum_{h,k \text{ odd}} \frac{h^2k^2}{(h^2 + k^2)(L^2h^2 + L^2k^2 + 1)} \mathsf{e}_z \neq \mathsf{0}.$$

Thanks to the symmetry of the problem, identical results hold on the remaining edges.

4. Concentrated interactions without concentrated stress

In the previous section, a clear distinction between concentrated stresses and concentrated adherence interactions has been exemplified. The aim of what follows is to present the crucial mathematical facts that make it possible to model the adherence to one-dimensional immersed bodies, regardless of whether concentrated stresses are allowed, and to introduce a third kind of concentrated interaction, related to the possible concentration of the external forces acting on the liquid.

4.1. The sole adherence

Existence and uniqueness results for the solution of the differential problem associated with a second-gradient dissipative liquid that adheres to one-dimensional immersed structures have been proved in [11, 12]. An essential hypothesis for those proofs is

$$\mu > 0\,, \quad \eta_1 + 2\eta_2 > 0\,, \quad \eta_1 - \eta_2 > 0\,, \quad \eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} > 0\,, \tag{19}$$

which implies, in particular, $\eta_1 > 0$. Such an assumption is related to the coercivity of the bilinear form, defined by the internal power expenditure (5), on a suitable subspace of the Sobolev space $H^2(\Omega;\mathbb{R}^3)$; this is a basic ingredient needed to establish the existence of solutions. Moreover, the embedding of $H^2(\Omega;\mathbb{R}^3)$ into a space of Hölder continuous functions is needed to obtain both uniqueness of the solution and adherence to one-dimensional rigid bodies (for the theory of Sobolev spaces see [1]).

Actually, conditions (19) are stronger than the thermodynamical constraints (6), and as anticipated in Sect. 3.1, it should be possible to prove existence and uniqueness of flows for a second-gradient liquid interacting with a line, even when $\eta_1 = \eta_2 = 0$ and $\eta_3 < 0$, which is a choice consistent with (6). Indeed, such a result can be established with minor modifications of the proofs contained in [11,12], as discussed below.

In this case, the internal power expenditure becomes

$$A(\mathsf{u},\mathsf{v}) := \langle \mathcal{P}^{\rm in}_{\mathsf{u}}, \mathsf{v}\rangle = 2\mu \int\limits_{\Omega} \operatorname{Sym}\nabla\mathsf{u}\cdot\nabla\mathsf{v} - 4\eta_3 \int\limits_{\Omega} \Delta\mathsf{u}\cdot\Delta\mathsf{v}\,, \tag{20}$$

which defines the bilinear form A. Assuming $\mathsf{u},\mathsf{v} \in X$, where X is a suitable Banach space, the bilinear form A is coercive if there exists $\nu > 0$ such that $\|\mathsf{u}\|_X^2 \le \nu A(\mathsf{u},\mathsf{u})$ for every $\mathsf{u} \in X$.

Now, if the boundary of the domain Ω is regular enough, A is coercive on a subspace of $H^2(\Omega;\mathbb{R}^3)$. But if $\partial\Omega$ is only piecewise smooth (in which case the presence of concentrated stresses could be detected!), A cannot be coercive on $H^2(\Omega;\mathbb{R}^3)$, whereas the form defined by (5), with $\eta_1 - \eta_2 > 0$, is coercive. Nevertheless, thanks to the fact that A is coercive on $\mathcal{D}$, existence of solutions for the steady flow problem can be proved in the space

$$\mathcal{D} := \left\{\mathsf{u} \in H^1_0(\Omega;\mathbb{R}^3) : \operatorname{div}\mathsf{u} = 0 \text{ and } \Delta\mathsf{u} \in L^2(\Omega;\mathbb{R}^3)\right\}. \tag{21}$$

Moreover, standard regularity theorems for elliptic second-order partial differential equations guarantee that functions belonging to $\mathcal{D}$ are essentially bounded and Hölder continuous [10, Theorems 8.16 and 8.22]. Essential boundedness provides the estimates needed to establish the uniqueness of any solution, while the continuity of the solutions makes it possible to assign the value of the velocity field u on a one-dimensional set contained in the interior of Ω, imposing adherence. It thus transpires that a second-gradient liquid can undergo a concentrated adherence interaction in generic situations, even when the choice of the constitutive parameters rules out the development of concentrated stresses.

4.2. Concentrated body force

By the definition (2) of the internal power expenditure, it is clear that $\mathcal{P}^{\rm in}_{\mathsf{u}}$ is a linear continuous form on the space of virtual velocities and can balance the linear continuous form representing the external power $\mathcal{P}^{\rm ex}_{\mathsf{u}}$. Considering steady flows in a domain Ω with piecewise smooth boundary, it is permissible to assume that virtual velocities belong to the space $\mathcal{D}$ defined by (21). Since $\mathcal{D}$ is a Hilbert space, the Riesz representation theorem ensures that there exist vector fields $\mathsf{b},\mathsf{t},\mathsf{m}$, and k such that

$$\langle \mathcal{P}^{\rm ex}_{\mathsf{u}}, \mathsf{v}\rangle = \int\limits_{\Omega} \mathsf{b}\cdot\mathsf{v} + \int\limits_{\mathcal{S}} \mathsf{t}\cdot\mathsf{v} + \int\limits_{\mathcal{S}} \mathsf{m}\cdot\frac{\partial\mathsf{v}}{\partial n} + \int\limits_{\mathcal{E}} \mathsf{k}\cdot\mathsf{v}.$$

The interaction fields t, m, and k belong to suitable trace spaces, and b belongs to $\mathcal{D}'$, the topological dual of $\mathcal{D}$. It is implicit in the proofs of the existence result contained in [12, Theorem 3] that the steady flow problem admits a solution for any body force field $\mathsf{b} \in \mathcal{D}'$.

Since functions in $\mathcal{D}$ are Hölder continuous, $\mathcal{D}'$ contains also distributions with support concentrated in points or along lines laying in the interior of Ω. This implies that internal stresses can balance concentrated external forces and thereby represents a third kind of concentrated interaction.

5. Discussion

Within the present paper, three different kinds of concentrated interaction have been shown to be possible in linearly viscous second-gradient liquids: concentrated stresses along the edges of piecewise smooth bounding surfaces, adhesion to one-dimensional immersed bodies, and concentrated external body forces. Those interactions, absent in simple (*i.e.,* first-gradient) liquids, should be regarded as available tools when modeling non-standard interactions, and they can be particularly useful whenever the coupling between physical phenomena occurring at very different length scales is considered.

A relevant example of such a context arises in the modeling of *nanofluids* (see [2] for a comprehensive introduction), that is, suspensions where the typical size of the solid particles is of the order of ten nanometers, six or more orders of magnitude smaller than the typical length scale of the relevant flows. Such composite fluids display unexpectedly enhanced heat and mass transfer properties, relative to those of the base fluid, and the understanding of their behavior is still an open problem. In this case, a theory that makes it possible to model the dispersed phase as lower dimensional, providing effective interactions with the fluid phase, could serve as a good reduced model to investigate nanofluids.

The coupling between a three-dimensional environment and structures that can be effectively considered as lower dimensional is relevant also while modeling biological systems, such as blood vessels within soft tissues or alveoli in the lungs. It is therefore important to know what kind of interactions can be included in the mathematical model. In particular, concentrated body forces are necessary to encompass any non-trivial mechanical behavior of the lower dimensional structures. Indeed, in many situations, such structures actively react to various stimuli, giving rise to concentrated body forces.

In a recent paper, dell'Isola et al. [5] clarify the structure of concentrated stresses in Nth gradient continua, giving also their representation in terms of the shape of Cauchy cuts. However, it could be interesting to perform an analysis of further concentrated interactions allowed in Nth order materials, akin to that presented above for second-gradient liquids. Such a project could take advantage of a fact, which has been exploited also in this note: the structure of the internal power expenditure and the availability of each kind of concentrated interaction is strictly related to the (generalized) kinematics of the system, as explained below.

By *generalized kinematics*, I mean the functional space $\mathcal{U}$ to which virtual velocities belong. If powers are considered to be linear continuous forms on $\mathcal{U}$, it is clear that their integral representation depends on the regularity of functions in $\mathcal{U}$. As the simplest example, an Nth gradient power is associated with a linear continuous form on the Sobolev space $H^N(\Omega; \mathbb{R}^3)$, a space where Nth order weak derivatives are defined. But when it comes to interactions, finer distinctions are called for. Indeed, many functional spaces that are 'intermediate' between $H^{N-1}(\Omega; \mathbb{R}^3)$ and $H^N(\Omega; \mathbb{R}^3)$ can be defined. Abstract results, such as the Riesz representation theorem, suggest a suitable integral form of the internal power, and the increased regularity of the virtual velocities allows for new interactions.

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