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Shallow water equations: viscous solutions and inviscid limit

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Abstract. We establish the inviscid limit of the viscous shallow water equations to the Saint-Venant system. For the viscous equations, the viscosity terms are more degenerate when the shallow water is close to the bottom, in comparison with the classical Navier-Stokes equations for barotropic gases; thus, the analysis in our earlier work for the classical Navier-Stokes equations does not apply directly, which require new estimates to deal with the additional degeneracy. We first introduce a notion of entropy solutions to the viscous shallow water equations and develop an approach to establish the global existence of such solutions and their uniform energy-type estimates with respect to the viscosity coefficient. These uniform estimates yield the existence of measure-valued solutions to the Saint-Venant system generated by the viscous solutions. Based on the uniform energy-type estimates and the features of the Saint-Venant system, we further establish that the entropy dissipation measures of the viscous solutions for weak entropy-entropy flux pairs, generated by compactly supported C^2 test-functions, are confined in a compact set in H^{-1}, which yields that the measure-valued solutions are confined by the Tartar-Murat commutator relation. Then, the reduction theorem established in Chen and Perepelitsa [5] for the measure-valued solutions with unbounded support leads to the convergence of the viscous solutions to a finite-energy entropy solution of the Saint-Venant system with finite-energy initial data, which is relative with respect to the different end-states of the bottom topography of the shallow water at infinity. The analysis also applies to the inviscid limit problem for the Saint-Venant system in the presence of friction.

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1. Introduction

We are concerned with solutions of the viscous shallow water equations:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x + gb_x h = \varepsilon(hu_x)_x, \end{cases} \tag{1.1}$$

with initial data:

$$(h, u)|_{t=0} = (h_0(x), u_0(x)). \tag{1.2}$$

Here, g is the free-fall acceleration and $b(x) \geq 0$ is the function describing the topography of shallow water with possible different end-states:

$$\lim_{x\to\pm\infty} b(x) = b^{\pm},$$

$h(t, x) \geq 0$ is the height of water above the bottom $b(x)$ at the time t and the position x with possible different end-states:

$$\lim_{x\to\pm\infty} h(t, x) = h^{\pm} := \bar{h} - b^{\pm},$$

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$\bar{h} > 0$ is the constant that stands for the height of water at rest, $u(t,x)$ is the velocity of the fluid, and $(h_0(x), u_0(x))$ are the initial values of the height and the velocity.

This system can directly be derived from the balance laws of mass and momentum; it can also be derived from the system of two-dimensional Navier-Stokes equations for incompressible fluids in the shallow water limit and under the assumption that there is no friction at the bottom of the reservoir. See Bouchut [1], Gerbeau and Perthame [14], Mascia [20] and the references cited therein.

Consider the topography function $b(x)$ satisfying

(i) $b \in L^\infty(\mathbb{R})$, $b - b^-\chi_{\{x<0\}} - b^+\chi_{\{x>0\}} \in L^2(\mathbb{R})$, $b_x \in L^1(\mathbb{R}) \cap L^4(\mathbb{R})$, where χ_B is the indication function on the set B, i.e., $\chi_B(x) = 1$ when $x \in B$ and 0 when $x \notin B$;

(ii) $b(x) \geq 0$ and $b^\pm \leq \bar{h}$.

When $\varepsilon = 0$, system (1.1) becomes the Saint-Venant system, i.e., the inviscid shallow water equations:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x + gb_x h = 0, \end{cases} \tag{1.3}$$

or in a short form:

$$U_t + F(U)_x + G(U,x) = 0, \tag{1.4}$$

with $U = (h,m)^\top := (h,hu)^\top, F(U) = (hu, hu^2 + \frac{g}{2}h^2)^\top$, and $G(U,x) = (0, gb_x h)^\top$.

The eigenvalues of system (1.3) are

$$\lambda_j = u + (-1)^j\sqrt{gh}, \qquad j = 1,2. \tag{1.5}$$

From (1.5),

$$\lambda_2 - \lambda_1 = \sqrt{gh} \to 0 \qquad \text{as } h \to 0.$$

Therefore, system (1.3) is strictly hyperbolic when $h > 0$. However, near the vacuum $h = 0$, the two characteristic speeds of (1.3) may coincide and the system be nonstrictly hyperbolic.

A pair of mappings $(\eta, q) : \mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2$ is called an entropy-entropy flux pair (or entropy pair, for short) of system (1.3) if the pair satisfies the 2×2 hyperbolic system:

$$\nabla q(U) = \nabla\eta(U)\nabla F(U). \tag{1.6}$$

Furthermore, $\eta(h,m)$ is called a weak entropy if

$$\eta\Big|_{\substack{h=0 \\ u=m/h \text{ fixed}}} = 0. \tag{1.7}$$

An entropy pair is said convex if the Hessian $\nabla^2\eta(h,m)$ is nonnegative in the region under consideration.

For example, the mechanical energy and the mechanical energy flux

$$\eta^*(h,m) = \frac{1}{2}\frac{m^2}{h} + \frac{1}{2}gh^2, \qquad q^*(h,m) = \frac{1}{2}\frac{m^3}{h^2} + ghm \tag{1.8}$$

form a special entropy pair; $\eta^*(h,m)$ is convex in the region $h \geq 0$.

The relative total mechanical energy over $\mathbb{R}$ for (1.3) with respect to the end-states $(\bar{h} - b^\pm, 0)$ through $(\bar{h} - b(x), 0)$ is

$$E[h,m](t) := \int_{\mathbb{R}} \Big(\eta^*(h,m) - \eta^*(\bar{h} - b(x), 0) - \nabla\eta^*(\bar{h} - b(x), 0)\cdot(h - \bar{h} + b(x), m)\Big)\,\mathrm{d}x \geq 0. \tag{1.9}$$

In the coordinates (h,u), any weak entropy function $\eta(h,hu)$ is governed by the second-order linear wave equation:

$$\begin{cases} \eta_{hh} - \frac{g}{h}\eta_{uu} = 0, \qquad h > 0, \\ \eta|_{h=0} = 0. \end{cases} \tag{1.10}$$

Therefore, any weak entropy pair (η, q) can be represented by

$$\begin{cases} \eta^{\psi}(h, hu) = \int_{\mathbb{R}} \chi(h; s-u)\psi(s)\,\mathrm{d}s, \\ q^{\psi}(h, hu) = \frac{1}{2}\int_{\mathbb{R}} (s+u)\chi(h; s-u)\psi(s)\,\mathrm{d}s \end{cases} \tag{1.11}$$

for any continuous function $\psi(s)$, where the weak entropy kernel $\chi(h, s-u)$ is determined by

$$\begin{cases} \chi_{hh} - \frac{g}{h}\chi_{uu} = 0, \\ \chi(0, u; s) = 0, \quad \chi_h(0, u; s) = \delta_{u=s}, \end{cases} \tag{1.12}$$

with the Dirac mass $\delta_{u=s}$ concentrated at $u = s$.

This implies that the weak entropy kernel as the unique solution of (1.12) is

$$\chi(h; s-u) = [4gh - (s-u)^2]_+^{1/2}. \tag{1.13}$$

Then, the weak entropy pairs have the form:

$$\begin{aligned} \eta^{\psi}(h, m) = \eta^{\psi}(h, hu) &= \int_{\mathbb{R}} [4gh - (u-s)^2]_+^{1/2}\psi(s)\,\mathrm{d}s \\ &= 4gh\int_{-1}^{1} \psi(u + 2\sqrt{gh}s)[1-s^2]_+^{1/2}\,\mathrm{d}s, \end{aligned} \tag{1.14}$$

$$\begin{aligned} q^{\psi}(h, m) = q^{\psi}(h, hu) &= \frac{1}{2}\int_{\mathbb{R}} (s+u)[4gh - (u-s)^2]_+^{1/2}\psi(s)\,\mathrm{d}s \\ &= 4gh\int_{-1}^{1} (u + \sqrt{gh}s)\psi(u + 2\sqrt{gh}s)[1-s^2]_+^{1/2}\mathrm{d}s. \end{aligned} \tag{1.15}$$

The idea of regarding inviscid fluids as viscous fluids with vanishing physical viscosity dates back the middle of 19th century; see [16,22–24] (also cf. [8]). The first rigorous proof for the inviscid limit of the Navier-Stokes equations to the isentropic Euler equations for polytropic gases for general initial data has been given only until recently in Chen and Perepelitsa [5]. For the Navier-Stokes equations, there exist no natural invariant regions for the equations with the real physical viscosity term so that the uniform sup-norm of solutions with respect to the physical viscosity coefficient may not be directly controllable, and furthermore, convex entropy-entropy flux pairs may not produce signed entropy dissipation measures. For the viscous shallow water equations, the viscosity term is more degenerate when h is close to zero, in comparison with the classical Navier-Stokes equations for barotropic gases; thus, the analysis in [5] does not apply directly in several key steps. These require further new ideas and techniques to deal with the additional degeneracy of the viscosity term for the limiting problem. In particular, the existence of entropy solutions of the viscous shallow water equations (1.1)–(1.2) needs to be established in an appropriate space, and the uniform estimates of the viscous solutions require to be carefully made with the additional degeneracy, on which the inviscid limit is based. See Sects. 2–4.

Now we define a notion of entropy solutions of the viscous shallow water equations (1.1)–(1.2) on which our inviscid limit is also based.

Definition 1.1. *A pair of functions (h, m) is called an entropy solution of (1.1)–(1.2) provided that*

(i) $m \in L^1(\mathbb{R}^2_+)$, $\frac{m^2}{h} \in L^\infty(0, \infty; L^1(\mathbb{R}))$;
(ii) $h \geq 0, h + b - \bar{h} \in L^\infty(0, \infty; L^2(\mathbb{R})), (\sqrt{h})_x \in L^\infty(0, \infty; L^2(\mathbb{R}))$;
(iii) $(\sqrt{h})_t + (\frac{m}{\sqrt{h}})_x \in L^2(\mathbb{R}^2_+)$;

(iv) *for any smooth, compactly supported function* $\psi = \psi(t,x)$ *on* $[0,\infty) \times \mathbb{R}$, *the following identities hold:*

$$\int_0^\infty \int_{-\infty}^\infty \big(h\psi_t + m\psi_x\big)\,\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty h_0(x)\psi(0,x)\,\mathrm{d}x = 0,$$

$$\int_0^\infty \int_{-\infty}^\infty \left(m\psi_t + (\frac{m^2}{h} + \frac{g}{2}h^2 - \varepsilon\sqrt{h}n)\psi_x - gb_x h\psi\right)\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty m_0(x)\psi(0,x)\,\mathrm{d}x = 0,$$

with $n := 2\big((\sqrt{h})_t + (\frac{m}{\sqrt{h}})_x\big)$;

(v) *for any weak entropy pair* (η^ψ, q^ψ) *in* (1.14)–(1.15) *generated by a smooth, compactly supported test-function* $\psi = \psi(\cdot)$, *there exists* $\mu_\varepsilon^\psi \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2_+)$ *such that the following entropy balance equation holds in the distributional sense:*

$$\eta_t^\psi + q_x^\psi + gb_x h\eta_m^\psi - \varepsilon\left(\sqrt{h}\eta_m^\psi n\right)_x + \mu_\varepsilon^\psi = 0. \tag{1.16}$$

In the above conditions, the functions $\sqrt{h}\eta_m^\psi$, $\frac{m}{\sqrt{h}}$ *and* $\frac{m^2}{h}$ *are defined by zero on the set* $\{h = 0\}$, *besides the weak entropy pair* (η^ψ, q^ψ).

A similar weak formulation of equations (1.1) was developed in Mellet and Vasseur [21] even for its multidimensional analog. One of our main concerns in this paper is uniform estimates of global solutions in the sense of Definition 1.1 with respect to the viscosity coefficient ε, which are established in Theorem 1.1 below.

Define

$$B = \|b\|_{L^\infty(\mathbb{R})} + \|b - b^-\chi_{\{x<0\}} - b^+\chi_{\{x>0\}}\|_{L^2(\mathbb{R})} + \|b_x\|_{L^1\cap L^4(\mathbb{R})}, \tag{1.17}$$

and

$$M_0 = \int_{-\infty}^\infty |m_0|\,\mathrm{d}x, \quad E_0 = \int_{-\infty}^\infty \left(\frac{m_0^2}{h_0} + g|h_0 + b - \bar{h}|^2\right)\mathrm{d}x, \quad E_1 = \varepsilon^2\int_{-\infty}^\infty \left|(\sqrt{h_0})_x\right|^2\mathrm{d}x. \tag{1.18}$$

Theorem 1.1 (Main Theorem). *Let* $\varepsilon > 0$. *Let a pair of functions* (h_0, m_0), *with* $h_0 \geq 0$, *be such that*

$$m_0 \in L^1(\mathbb{R}), \qquad \big(\frac{m_0}{\sqrt{h_0}},\ h_0 + b - \bar{h},\ (\sqrt{h_0})_x\big) \in \big(L^2(\mathbb{R})\big)^3.$$

Then, there exists an entropy solution (h,m) *of* (1.1)–(1.2) *on* $\mathbb{R}^2_+$ *with the following properties: For any* $T > 0$, *compact set* $K \subset \mathbb{R}$ *and* $\psi \in C_0^\infty(\mathbb{R}^2_+)$, *there are* $C_i = C_i(T, E_0, E_1, B)$, $i = 1, 2$, $C_3 = C_3(T, K, E_0, E_1, M_0, B)$ *and* $C_4 = C_4(T, E_0, E_1, B, \psi)$ *that are all independent of* $\varepsilon > 0$ *such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{-\infty}^\infty \left(\frac{|m(t,x)|^2}{h(t,x)} + g|h(t,x) + b(x) - \bar{h}|^2\right)\mathrm{d}x \leq E_0, \tag{1.19}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \varepsilon^2\int_{-\infty}^\infty |(\sqrt{h(t,x)})_x|^2\,\mathrm{d}x \leq C_1, \tag{1.20}$$

$$\varepsilon\int_{[0,T]\times\mathbb{R}} (n^2 + |h_x|^2)\,\mathrm{d}x\mathrm{d}t \leq C_2, \tag{1.21}$$

$$\int_0^T\int_K \left(h^3 + \frac{|m|^3}{h^2}\right)\mathrm{d}x\mathrm{d}t \leq C_3, \tag{1.22}$$

$$\varepsilon\int_0^\infty\int_{-\infty}^\infty |\mathrm{d}\mu^\psi| \leq C_4. \tag{1.23}$$

Based on the global existence and uniform estimates established in Theorem 1.1, we study the inviscid limit of the solutions $(h^\varepsilon, m^\varepsilon), \varepsilon > 0$, to a solution of the Cauchy problem of the Saint-Venant system (1.3) with initial data (1.2).

Definition 1.2. *Let (h_0, m_0) be given initial data with relative finite energy with respect to the end-states $(h^{\pm}, 0) := (\bar{h} - b^{\pm}, 0)$ at infinity, i.e.,*

$$E[h_0, m_0] \leq E_0 < \infty.$$

A pair of measurable functions $(h, m) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is called a finite-energy entropy solution of the Cauchy problem (1.3) with Cauchy data (1.2) provided that

(i) *The relative total energy with respect to the end-states $(h^{\pm}, 0)$ is uniformly bounded in time: there exists a bounded function $C_T(E_0)$ such that, for a.e. $t > 0$,*

$$E[h, m](t) \leq C_T(E_0) \qquad \text{for a.e. } t \in [0, T];$$

(ii) *The entropy inequality:*

$$\eta^{\psi}(h, m)_t + q^{\psi}(h, m)_x + g\eta^{\psi}_m(h, m)b_x h \leq 0 \tag{1.24}$$

is satisfied in the sense of distributions for any test-function $\psi(s) \in \{\pm 1, \pm s, s^2\}$ in $[0, T] \times \mathbb{R}$;

(iii) *The initial data functions (h_0, m_0) are attained in the sense of distributions.*

Then, as a corollary of Theorem 1.1 (Main Theorem) and the compactness framework established in Chen and Perepelitsa [5], we conclude

Theorem 1.2. *Let $(h^{\varepsilon}_0, m^{\varepsilon}_0)$ be a sequence of initial data functions for problem (1.2)–(1.3) which satisfy the assumptions of Theorem 1.1 with the constants M_0, E_0 and E_1, independent of ε. Moreover, assume that $(h^{\varepsilon}_0, m^{\varepsilon}_0) \to (h_0, m_0)$ a.e. $x \in \mathbb{R}$. Then, for the global solutions $(h^{\varepsilon}, m^{\varepsilon})$ established in Theorem 1.1, when $\varepsilon \to 0$, there exists a subsequence of $(h^{\varepsilon}, m^{\varepsilon})$ that converges almost everywhere to a relative finite-energy entropy solution (h, m) to the Cauchy problem (1.3) with Cauchy data (1.2) in the sense of Definition 1.2. Moreover, there exists a bounded Radon measure $\mu(t, x; s)$ on $\mathbb{R}^2_+ \times \mathbb{R}$ such that*

$$\mu(U \times \mathbb{R}) \geq 0$$

for any open set $U \subset \mathbb{R}^2_+$, and the corresponding entropy kernel $\chi(h, s - u)$ defined by (1.13) satisfies

$$\partial_t \chi(h, s - u) + \frac{1}{2}\partial_x\big((s + u)\chi(h, s - u)\big) - gb_x \partial_s \chi(h, s - u) = \partial^2_s \mu \tag{1.25}$$

in the sense of distributions on $\mathbb{R}^2_+ \times \mathbb{R}$.

When the Saint-Venant system describes the motion of shallow water in the presence of friction, an additional term $r(h, u)hu$, called the friction term, is present:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x + gb_x h + r(h, u)hu = 0. \end{cases} \tag{1.26}$$

The interaction between the dynamics of the shallow water and the geometry of the physical domain may generate the formation of remarkable structures. In Sect. 6, for the case that $r(h, u)$ is positive constant, we show how our analysis can also be applied to the inviscid limit of the viscous solutions to the viscous shallow water equations with the friction term to the corresponding Saint-Venant system (1.26).

We remark that, through straightforward arguments, the analysis also applies to a range $\alpha > 0$ and $\gamma > 1$ for more general singular viscosity term: $\varepsilon(h^{\alpha}u_x)_x$ replacing $\varepsilon(hu_x)_x$ and more general pressure laws $p(h) = \kappa h^{\gamma}$ replacing $h^2/2$.

The further organization of the paper is as follows. In Sect. 2, we introduce and construct a nondegenerate viscosity approximation of global solutions to the viscous shallow water equations and establish the basic uniform estimates. Further essential uniform apriori estimates are made for the approximate solutions in Sect. 3. In Sect. 4, we establish Theorem 1.1 for the existence and uniform estimates of the viscous solutions with respect to the viscosity coefficient ε. Finally, in Sect. 5, we complete the proof of Theorem 1.2 by combining the uniform estimates in Theorem 1.1 with the reduction theorem of Young measures with unbounded support established in Chen and Perepelitsa [5].

Finally, we further remark that the reduction theorem of Young measures with bounded support for the isentropic Euler equations was established by DiPerna [13], Chen [3,4], Ding et al. [9–12], Lions et al. [19] and Lions et al. [18] for the γ-law gases, and Chen and LeFloch [6,7] for general pressure-law gases. Also see LeFloch and Westdickenberg [17] for the reduction theorem for the case $1 < \gamma \leq 5/3$ and Chen and Perepelitsa [5] for the general case $\gamma > 1$ with a simpler proof for the Young measures with unbounded support.

2. Nondegenerate viscosity approximation

To establish Theorem 1.1, we first introduce and analyze an approximate system of equations for which the viscosity coefficient is bounded away from the vacuum. Let $\delta > 0$. Consider

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x + gb_x h = \varepsilon((h+\delta)u_x)_x. \end{cases} \tag{2.1}$$

We first follow Hoff [15] to show that (2.1) has a unique, global, smooth solution with $h(t,x) > 0$ provided that the initial data functions (h_0, u_0) are smooth, $h_0 > 0$, and $b^{\pm} < \bar{h}$, as stated below. Meanwhile, we carefully make uniform estimates of the solutions with respect to both parameters $\delta > 0$ and $\varepsilon > 0$, which are essential to establish the inviscid limit for Theorem 1.2.

Theorem 2.1. *Let (h_0, u_0) be such that*

$$h_0 + b - \bar{h} \in L^2(\mathbb{R}), \quad \operatorname*{ess\,inf}_{x\in\mathbb{R}} h_0(x) > 0, \quad \sqrt{h_0}(\sigma_0)_x \in L^2(\mathbb{R}), \quad u_0 \in H^1(\mathbb{R}),$$

where $\sigma_0 = \varepsilon\frac{h_0 \ln h_0 - \delta}{h_0}$. Then there exists a unique strong solution (h, u) of (2.1) on $[0,\infty)\times\mathbb{R}$ with the initial data (h_0, u_0) and such that, for any $T > 0$,

$$\begin{aligned} &h, u \in C([0,T]; L^2_{\text{loc}}(\mathbb{R})), \quad h, h^{-1} \in L^\infty([0,T]\times\mathbb{R}), \\ &h_x \in L^\infty(0,T; L^2(\mathbb{R})), \quad h_t \in L^2(0,T; L^2(\mathbb{R})), \\ &u \in L^\infty(0,T; H^1(\mathbb{R})), \quad u_t,\, u_{xx} \in L^2(0,T; L^2(\mathbb{R})). \end{aligned}$$

Proof. We first assume that (h, u) is a smooth solution of (2.1) on $Q_T := [0,T)\times\mathbb{R}$, with sufficiently rapid decay as $|x| \to \infty$ and with $h(t,x) \geq \underline{h} > 0$ for all $(t,x) \in Q$. In the estimate below, we also assume that $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$ for some fixed ε_0 and δ_0.

1. For a smooth solution (h, u) of (2.1), multiplying the second equation in (2.1) by $2u$, using the first equation in (2.1), the fact that

$$\begin{aligned} \big(g(h^2)_x + 2ghb_x\big)u &= 2g(h+b)_x hu = 2g(h+b-\bar{h})_x hu \\ &= 2g\big((h+b-\bar{h})hu\big)_x + 2g(h+b-\bar{h})h_t \\ &= 2g\big((h+b-\bar{h})hu\big)_x + g\big((h+b-\bar{h})^2\big)_t \end{aligned} \tag{2.2}$$

and integrating the result over $(0,t)\times\mathbb{R}$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big(h(t,x)|u(t,x)|^2 + |h(t,x)+b(x)-\bar{h}|^2\Big)\,\mathrm{d}x + \varepsilon\int_0^t\int_{-\infty}^{\infty} 2(h+\delta)|u_x|^2\,\mathrm{d}x\mathrm{d}t \\ &\quad = \int_{-\infty}^{\infty}\Big(h_0(x)|u_0(x)|^2 + |h_0(x)+b(x)-\bar{h}|^2\Big)\,\mathrm{d}x \leq E_0. \end{aligned} \tag{2.3}$$

2. The next estimate was first observed by Bresch and Desjardins [2]. However, our main concern here is whether the estimate is uniformly bounded, independent of the viscosity coefficient ε. That is,

$$\int_{-\infty}^{\infty} h(t,x)|\sigma_x(t,x)|^2\,\mathrm{d}x + \varepsilon g\int_0^t\int_{-\infty}^{\infty}\frac{h+\delta}{h}|h_x|^2\,\mathrm{d}x\mathrm{d}t \leq C_0, \qquad t\in[0,T], \tag{2.4}$$

where $\sigma = \varepsilon \frac{h \ln h - \delta}{h}$, and $C_0 = C_0(T, E_0, E_1, B)$ is independent of both $\delta > 0$ and $\varepsilon > 0$.

Indeed, from the first equation in (2.1), we obtain

$$\sigma_t + u\sigma_x = -\varepsilon \frac{h+\delta}{h} u_x,$$

and

$$\sigma_{xt} + u\sigma_{xx} = -\varepsilon \frac{h+\delta}{h^2} h_x u_x - \varepsilon \left(\frac{h+\delta}{h} u_x\right)_x. \tag{2.5}$$

Multiplying (2.5) by h yields

$$(h\sigma_x)_t + (hu\sigma_x)_x = -\varepsilon\big((h+\delta)u_x\big)_x.$$

Adding this to the momentum equation, we have

$$\big(h(u+\sigma_x)\big)_t + \big(hu(u+\sigma_x)\big)_x + gh(h+b)_x = 0,$$

and then

$$\big(h(u+\sigma_x)^2\big)_t + \big(hu(u+\sigma_x)^2\big)_x + 2gh(h+b)_x\sigma_x + 2gh(h+b)_x u = 0.$$

We integrate this over $(0,t)\times\mathbb{R}$ and use (2.2) to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(h|u+\sigma_x|^2 + g|h+b-\bar{h}|^2\right) \mathrm{d}x + 2g\varepsilon \int_0^t \int_{-\infty}^{\infty} \frac{h+\delta}{h} |h_x|^2 \,\mathrm{d}x\mathrm{d}t + 2g \int_0^t \int_{-\infty}^{\infty} b_x h \sigma_x \,\mathrm{d}x\mathrm{d}t \\
&\quad = \int_{-\infty}^{\infty} \left(h_0|u_0+\sigma_{0,x}|^2 + |h_0+b-\bar{h}|^2\right) \mathrm{d}x.
\end{aligned} \tag{2.6}$$

Moreover, the last term on the left of (2.6) can be estimated as

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} b_x h\sigma_x \,dxdt\right| \\
&\quad \le \int_0^t \int_{-\infty}^{\infty} \left(h|\sigma_x|^2 + (h+b-\bar{h})|b_x|^2 + |b-\bar{h}||b_x|^2\right) dxdt \\
&\quad \le \int_0^t \int_{-\infty}^{\infty} \left(h|\sigma_x|^2 + |h+b-\bar{h}|^2\right) dxdt + \|b_x\|_{L^4(\mathbb{R})}^4 + \|b-\bar{h}\|_{L^\infty(\mathbb{R})}\|b_x\|_{L^2(\mathbb{R})}^2.
\end{aligned} \tag{2.7}$$

Then, (2.4) follows from both (2.3) and the assumptions on the initial data and $b(x)$.

3. There exists $C = C(T, E_0, E_1, B, \varepsilon, \delta)$, such that

$$\sup_{[0,T]\times\mathbb{R}} \big(h(t,x) + h^{-1}(t,x)\big) \le C.$$

We first show the uniform lower bound for h. For any $x, y \in \mathbb{R}$,

$$\left|\frac{1}{\sqrt{h(t,x)}} - \frac{1}{\sqrt{h(t,y)}}\right| \le \int_x^y \left|\left(\frac{1}{\sqrt{h(t,x)}}\right)_x\right| \mathrm{d}x.$$

Then, using the energy estimate (2.4), there exists $C = C(T, E_0, E_1, B)$ such that

$$\left|\frac{1}{\sqrt{h(t,x)}} - \frac{1}{\sqrt{h(t,y)}}\right| \le C|x-y|^{\frac{1}{2}}. \tag{2.8}$$

From the properties of $b(x)$, it follows that there exists $\beta_0 > 0$ such that $\bar{h} - b(x) \ge \beta_0$ for all large $|x|$. Define $\mathcal{B}$ as an open interval such that

$$\mathcal{B}^c \subset \{x \,:\, \bar{h} - b(x) \ge \beta_0 > 0\}.$$

Let

$$\mathcal{A}(t) = \left\{ x \ : \ h(t,x) < \frac{\beta_0}{2} \right\}.$$

Then, from the first energy estimate, it follows that

$$|\mathcal{A}(t) \cap \mathcal{B}^c| \le C_1$$

for some $C_1 = C_1(T, E_0, E_1, B)$. Then, for any $x \in \mathcal{A}(t)$, the interval $(x - 2(C_1 + |B|), x + 2(C_1 + |B|))$ contains a point $x_0(t)$ such that $h(t, x_0) > \frac{\beta_0}{2}$. From (2.8) with $y = x_0$, we obtain

$$\inf_{x \in \mathcal{A}(t)} h(t,x) \ge C > 0$$

for some $C = C(T, E_0, E_1)$. On the other hand, $h(t,x) > \frac{\beta_0}{2}$ on $\mathcal{A}(t)^c$. This yields a uniform bound for h in (t,x). The uniform upper bound is obtained in a similar fashion.

4. Higher-order derivative estimates: Let, in addition to the conditions on the data used above, $u_{0,x} \in L^2(\mathbb{R})$. Then, there exists $C = C(T, E_0, E_1, B, \varepsilon, \delta)$ such that

$$\sup_{t \in [0,T]} \int_{-\infty}^{\infty} |u_x(t,x)|^2 \, \mathrm{d}x + \int_0^T \int_{-\infty}^{\infty} |u_{xx}|^2 \, \mathrm{d}x\mathrm{d}t \le C \left(1 + \int_{-\infty}^{\infty} |u_{0,x}|^2 \, \mathrm{d}x \right). \tag{2.9}$$

We will repeatedly use the inequality

$$\|u\|_{L^\infty(\mathbb{R})} \le \sqrt{2} \|u\|^{1/2}_{L^2(\mathbb{R})} \|u_x\|^{1/2}_{L^2(\mathbb{R})}.$$

Differentiating the momentum equation with x yields

$$h(u_{xt} + u u_{xx} + u_x u_x) + h_x(u_t + u u_x) + (h^2)_{xx} + (g b_x h)_x - \varepsilon \big((h+\delta) u_x \big)_{xx} = 0.$$

Multiply this equation by u_x, integrate over $\mathbb{R}$ and notice that

$$\int_{-\infty}^{\infty} \big((h+\delta) u_x \big)_{xx} u_x \, \mathrm{d}x = - \int_{-\infty}^{\infty} \big(h_x u_x u_{xx} + (h+\delta) |u_{xx}|^2 \big) \, \mathrm{d}x.$$

Then, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{\mathrm{d}t} \int_{-\infty}^{\infty} h |u_x|^2 \, \mathrm{d}x + \varepsilon \int_{-\infty}^{\infty} (h+\delta) |u_{xx}|^2 \, \mathrm{d}x \\ &\quad = \int_{-\infty}^{\infty} \varepsilon h_x u_x u_{xx} \, \mathrm{d}x - \int_{-\infty}^{\infty} h |u_x|^3 \, \mathrm{d}x - \int_{-\infty}^{\infty} h_x (u_t + u u_x) u_x \, \mathrm{d}x \\ &\qquad - \int_{-\infty}^{\infty} (h^2)_x u_{xx} \, \mathrm{d}x - \int_{-\infty}^{\infty} (g b_x h)_x u_x \, \mathrm{d}x \\ &\quad = I_1 + \cdots + I_5. \end{aligned} \tag{2.10}$$

Let $\kappa > 0$ be a number to be chosen later. We have

$$
\begin{aligned}
|I_1| &\le \kappa \int_{-\infty}^{\infty} |u_{xx}|^2 \,\mathrm{d}x + C_\kappa \|u_x\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} |h_x|^2 \,\mathrm{d}x \\
&\le \kappa \int_{-\infty}^{\infty} |u_{xx}|^2 \,\mathrm{d}x + C_\kappa \left(\int_{-\infty}^{\infty} |u_{xx}|^2 \,\mathrm{d}x \int_{-\infty}^{\infty} |u_x|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} |h_x|^2 \,\mathrm{d}x \\
&\le \kappa \int_{-\infty}^{\infty} |u_{xx}|^2 \,\mathrm{d}x + C_\kappa(T_1, E_0, E_1) \int_{-\infty}^{\infty} |u_x|^2 \,\mathrm{d}x; \\
|I_2| &\le C \|u_x\|^{\frac{5}{2}}_{L^2(\mathbb{R})} \|u_{xx}\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \le \kappa \|u_{xx}\|^2_{L^2(\mathbb{R})} + C_\kappa \|u_x\|^{\frac{10}{3}}_{L^2(\mathbb{R})} \\
&= \kappa \|u_{xx}\|^2_{L^2(\mathbb{R})} + C_\kappa \|u_x\|^2_{L^2(\mathbb{R})} \|u_x\|^{\frac{4}{3}}_{L^2(\mathbb{R})}; \\
I_3 &= \int_{-\infty}^{\infty} h_x u_x \Big(\frac{\varepsilon}{h} ((h+\delta) u_x)_x - 2h_x - b_x \Big) \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} h_x u_x \Big(\varepsilon (\ln h)_x u_x + \varepsilon \frac{h+\delta}{h} u_{xx} - 2h_x - g b_x \Big) \,\mathrm{d}x \\
&= J_1 + \cdots + J_4,
\end{aligned}
$$

where

$$
\begin{aligned}
|J_1| &\le C \int_{-\infty}^{\infty} \frac{|h_x|^2 |u_x|^2}{h} \,\mathrm{d}x \\
&\le C \|u_x\|^2_{L^\infty(\mathbb{R})} \|h_x\|^2_{L^2(\mathbb{R})} \\
&\le C \|u_x\|_{L^2(\mathbb{R})} \|u_{xx}\|_{L^2(\mathbb{R})} \\
&\le \kappa \|u_{xx}\|^2_{L^2(\mathbb{R})} + C_\kappa \|u_x\|^2_{L^2(\mathbb{R})}, \\
|J_2| &\le C \|u_x\|_{L^\infty(\mathbb{R})} \|h_x\|_{L^2(\mathbb{R})} \|u_{xx}\|_{L^2(\mathbb{R})} \\
&\le C \|u_x\|_{L^\infty(\mathbb{R})} \|u_{xx}\|_{L^2(\mathbb{R})} \\
&\le \kappa \|u_{xx}\|^2_{L^2(\mathbb{R})} + C_\kappa \|u_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}, \\
|J_3| &\le C \int_{-\infty}^{\infty} |h_x|^2 |u_x| \,\mathrm{d}x \\
&\le C \|u_x\|_{L^\infty(\mathbb{R})} \|h_x\|^2_{L^2(\mathbb{R})} \\
&\le C \|u_x\|_{L^\infty(\mathbb{R})} \\
&\le \kappa \|u_{xx}\|_{L^2(\mathbb{R})} + C_\kappa \|u_x\|_{L^2(\mathbb{R})},
\end{aligned}
$$

and similarly,

$$|J_4| \le \kappa \|u_{xx}\|_{L^2(\mathbb{R})} + C_\kappa \|b_x\|_{L^2(\mathbb{R})} \|u_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}.$$

Thus,

$$|I_3| \le \kappa \|u_{xx}\|^2_{L^2(\mathbb{R})} + C_\kappa \Big(\|u_x\|^2_{L^2(\mathbb{R})} + \|u_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})} \Big).$$

Similarly, we have

$$|I_4| + |I_5| \le \kappa \|u_{xx}\|^2_{L^2(\mathbb{R})} + C_\kappa.$$

Now we combine all the above estimates in (2.10). Since $h(t,x)$ has a lower bound, we can choose γ sufficiently small so that (2.10) implies

$$\int_{-\infty}^{\infty} |u_x(t,x)|^2 \,\mathrm{d}x + \int_0^t \int_{-\infty}^{\infty} |u_{xx}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t$$
$$\leq C\|u_{0,x}\|^2_{L^2(\mathbb{R})} + C\int_0^t \left(1 + \|u_x\|^2_{L^2(\mathbb{R})}\|u_x\|^{\frac{4}{3}}_{L^2(\mathbb{R})} + \|u_x\|^2_{L^2(\mathbb{R})} + \|u_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}\right) \mathrm{d}t. \tag{2.11}$$

From the first energy estimates, we know that

$$\int_0^T \|u_x\|^2_{L^2(\mathbb{R})} \,\mathrm{d}t \leq C(E_0).$$

Thus, by Gronwall's inequality and from (2.11), we obtain the expected estimate.

Then, the global existence of strong solutions with the uniform apriori estimates is established by extending the local solutions that can be obtained directly via the fixed point argument. Moreover, the solution in this class of functions is unique. □

3. Further apriori estimates

In this section, we derive further uniform estimates of the global strong solution (h, u) of (2.1), which are needed to pass to the limit $\delta \to 0$ to establish Theorem 1.1, especially to analyze the inviscid limit as $\varepsilon \to 0$ later on.

A direct corollary of the energy estimate (2.4) is the following uniform L^∞ bound for h, independent of $\delta \in (0, \delta_0]$:

$$\|h\|_{L^\infty(0,T;L^\infty(K))} \leq C(\varepsilon, K, E_0, E_1, B) \tag{3.1}$$

for every compact $K \subset \mathbb{R}$.

Estimates (2.3) and (2.4) allow us to obtain the uniform higher integrability of h and u.

Higher Integrability I: Given a compact set $K \subset \mathbb{R}$ and $T > 0$, there exists $C_1 = C_1(K, T, E_0, B)$, independent of ε, such that

$$\int_0^T \int_K h^3 \,\mathrm{d}x\mathrm{d}t \leq C_1. \tag{3.2}$$

The proof of this estimate is identical to the proof of Lemma 3.3 in [5].

Higher Integrability II: For K as above, there exists $C_2 = C_2(K, T, E_0, E_1, B)$, independent of ε, such that

$$\int_0^T \int_K h|u|^3 \,\mathrm{d}x\mathrm{d}t \leq C_2. \tag{3.3}$$

The estimate is analogous to the estimate in Lemma 3.4 in [5], but requires few technical modifications that we explain now.

Let $(\eta^\sharp, q^\sharp)$ be the weak entropy pair, generated by the function $\psi_\sharp(w) = \frac{1}{2}w|w|$, through the formulas:

$$\eta^\sharp = 4gh \int_{-1}^{1} \psi_\sharp(u + 2\sqrt{gh}s)[1 - s^2]_+^{\frac{1}{2}} \,\mathrm{d}s,$$
$$q^\sharp = 4gh \int_{-1}^{1} \left(u + \sqrt{gh}\right) \psi_\sharp(u + 2\sqrt{gh}s)[1 - s^2]_+^{\frac{1}{2}} \,\mathrm{d}s.$$

They satisfy the following estimates (see [5,18]):

$$|\eta^\sharp(h, m)| \leq C\big(h|u|^2 + h^2\big), \qquad q^\sharp(h, m) \geq C^{-1}\big(h|u|^3 + h^{3/2}\big), \tag{3.4}$$

$$|\eta^\sharp_m(h,m)| \le C(|u| + \sqrt{h}), \qquad |\eta^\sharp_{mm}(h,m)| \le Ch^{-1}, \tag{3.5}$$

and, regarding $\eta^\sharp_m$ in the coordinates (h,u),

$$|\eta^\sharp_{mu}(h,hu)| \le C, \qquad |\eta^\sharp_{mh}(h,hu)| \le \frac{C}{\sqrt{h}} \tag{3.6}$$

for all $h \ge 0$ and $u \in \mathbb{R}$.

We multiply the first equation in (1.3) by $\eta^\sharp_h$, the second by $\eta^\sharp_m$, and add them and integrate over $(0,t) \times (-\infty, x)$ to obtain

$$\begin{aligned}\int_0^t q^\sharp \,\mathrm{d}t = tq^{\sharp,-} &+ \int_{-\infty}^x \big(\eta^\sharp(h_0,m_0) - \eta^\sharp(h,m)\big)\,\mathrm{d}y \\ &+ \varepsilon \int_0^t \eta^\sharp_m (h+\delta)u_x \,\mathrm{d}t + \varepsilon \int_0^t \int_{-\infty}^x \Big(\eta^\sharp_{mu}(h+\delta)|u_x|^2 + \eta^\sharp_{mh}(h+\delta)h_x\Big)\,\mathrm{d}y\mathrm{d}t \\ &- g\int_0^t \int_{-\infty}^x h\eta^\sharp_m b_x \,\mathrm{d}y\mathrm{d}t \\ &= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned} \tag{3.7}$$

where $q^{\sharp,-} = q^\sharp(h^-, 0)$ is the left-end-state of $q^\sharp$.

Consider the term I_2. First, as argued in [5], there exists a constant $\alpha > 0$ such that

$$\eta^\sharp = \alpha\sqrt{h}m + r_2(h,m), \qquad |r_2| \le chu^2.$$

We write

$$\begin{aligned}\alpha\sqrt{h}m &= \alpha(\sqrt{h} - \sqrt{h^-})hu + \alpha\sqrt{h^-}hu \\ &\le \alpha h|u|^2 + \alpha h\big(\sqrt{h} - \sqrt{h^-}\big) + \alpha\sqrt{h^-}hu.\end{aligned} \tag{3.8}$$

Without loss of generality, we assume $h^- > 0$ (the case $h^- = 0$ can be treated similarly). Then

$$h(\sqrt{h} - \sqrt{h^-}) \le C\,(h-h^-)^2 \le C(h+b-\bar{h})^2 + (b-b^-)^2$$

since $\bar{h} = h^- + b^-$. It follows that

$$|I_2| \le C\int_{-\infty}^x \big(hu^2 + (h+b-\bar{h})^2 + (b-b^-)^2\big)\,\mathrm{d}y + \alpha\sqrt{h^-}\left|\int_{-\infty}^x hu\,\mathrm{d}y\right|.$$

Thus, for a compact set K,

$$\int_K |I_2|\,\mathrm{d}x \le C(K, E_0, B) + \alpha\sqrt{h^-}\int_K \left|\int_{-\infty}^x hu\,\mathrm{d}y\right|\,\mathrm{d}y.$$

Assume that $M_0 = \int h_0|u_0|\,\mathrm{d}x < \infty$. Then, we can obtain as in [5]:

$$\int_K \left|\int_{-\infty}^x hu\,\mathrm{d}y\right|\,\mathrm{d}y \le C(T, E_0, B, M_0),$$

and consequently,

$$\int_K |I_2|\,\mathrm{d}x \le C(T, K, E_0, B, M_0). \tag{3.9}$$

From the pointwise estimate (3.6) on $(\eta^\sharp_{mh}, \eta^\sharp_{mu})$ and using (2.3) and (2.4), we have

$$\left|\varepsilon\int_0^t\int_{-\infty}^x \check{\eta}_{mu}(h+\delta)|u_x|^2\,\mathrm{d}y\mathrm{d}t\right| \le C(E_0), \tag{3.10}$$

$$\left|\varepsilon\int_0^t\int_{-\infty}^x \check{\eta}_{mh}(h+\delta)h_xu_x\,\mathrm{d}y\mathrm{d}t\right| \le C(T, E_0, E_1, B). \tag{3.11}$$

It follows that

$$\int_K |I_4|\,\mathrm{d}x \le C(T,K,E_0,E_1,B).$$

Furthermore, we have

$$\int_K |I_3|\,\mathrm{d}x \le C(T,K,E_0,E_1,B)$$

by the same arguments as in the proof of Lemma 3.4 in [5].

Finally,

$$\begin{aligned}|I_5| &= \left|\int_0^t\int_{-\infty}^x g b_x h\eta_m^\sharp\,\mathrm{d}y\mathrm{d}t\right|\\ &\le C\int_0^t\int_{-\infty}^x g|b_x|h(u+\sqrt{h})\,\mathrm{d}y\mathrm{d}t\\ &\le \int_0^t\int_{-\infty}^x \big(h|u|^2+|h+b-\bar h|^2\big)\,\mathrm{d}y\mathrm{d}t + C(B).\end{aligned}\tag{3.12}$$

Integrating (3.7) over K and using (3.4) and the estimates on $I_j, j=1,\dots,5$, we conclude the proof of (3.3).

4. Proof of Theorem 1.1: existence and estimates of the viscous solutions

In this section, we establish Theorem 1.1, that is, the existence of global solutions to the Cauchy problem (1.1)–(1.2) in the sense of Definition 1.1 and the necessary uniform estimates for the inviscid limit to the Saint-Venant system as $\varepsilon\to 0$.

Let $b(x)$ be as described in the Introduction. Define $b^\delta(x)=\min\{\bar h-\delta,\, b(x)\}$. Notice that b^δ measured in the norms in (1.17) does not increase B. Let (h_0,m_0) be as in Theorem 1.1, and let (h_0^δ,u_0^δ), with $\operatorname{ess\,inf}_{\mathbb{R}} h_0^\delta>0$ and $u_0^\delta\in H^1(\mathbb{R})$, be such that $(h_0^\delta,h_0^\delta u_0^\delta)\to(h_0,m_0)$ a.e. $x\in\mathbb{R}$ with the norms

$$\int_{\mathbb{R}}\big(h_0^\delta|u_0^\delta|^2+g|h_0^\delta+b^\delta-\bar h|^2\big)\,\mathrm{d}x,\quad \int_{\mathbb{R}}|(\sqrt{h_0^\delta})_x|^2\,\mathrm{d}x$$

bounded independently of $\delta\in(0,\delta_0]$. Let (h^δ,u^δ) be the global strong solution of (2.1), with $b^\delta(x)$ replacing $b(x)$ and with initial data (h_0^δ,u_0^δ), constructed in Theorem 2.1.

We will make use of the following equation that holds for a smooth solution (h^δ,u^δ) :

$$(\sqrt{h^\delta})_t+(\sqrt{h^\delta}u^\delta)_x=\frac{1}{2}\sqrt{h^\delta}u_x^\delta.\tag{4.1}$$

In particular, we will use this equation to define the limit of the left-hand side as a suitable distribution obtained in the limit as $\delta\to 0$.

Using estimates (2.3), (2.4), (3.1), and Eq. (4.1), we conclude

(i) $\sqrt{h^\delta}$ uniformly bounded in $L^\infty(0,T;L^\infty_{\mathrm{loc}}(\mathbb{R}))$;
(ii) $(\sqrt{h^\delta})_x$ uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}))$;
(iii) $(\sqrt{h^\delta})_t$ uniformly bounded in $L^2(0,T;H^{-1}_{\mathrm{loc}}(\mathbb{R}))$ by using (4.1).

Also,

$$m_x=2(\sqrt{h})_x\sqrt{h}u+\sqrt{h}\sqrt{h}u_x,$$

and, using estimates (2.3), (2.4), (3.1) and (3.3),

$$(\sqrt{h^\delta})_x\sqrt{h^\delta}u^\delta \qquad \text{is uniformly bounded in } L^3(0,T;L^{\frac{6}{5}}_{\mathrm{loc}}(\mathbb{R})),$$

and

$$\sqrt{h^\delta}\sqrt{h^\delta}u_x^\delta \qquad \text{is uniformly bounded in } L^2(0,T;L^2_{\text{loc}}(\mathbb{R})).$$

On the other hand,

$$(m^\delta)_t = -\big(h^\delta(u^\delta)^2 + \frac{g}{2}(h^\delta)^2\big)_x - gb_x h^\delta + \varepsilon\big((h^\delta+\delta)u_x^\delta\big)_x,$$

and is uniformly bounded in $L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}_{\text{loc}}(\mathbb{R}))$.

We now summarize the estimates on m^δ:

(i) m^δ is uniformly bounded in $L^3(0,T;L^3_{\text{loc}}(\mathbb{R}))$;
(ii) m_x^δ is uniformly bounded in $L^2(0,T;L^{\frac{6}{5}}(\mathbb{R}))$;
(iii) m_t^δ is uniformly bounded in $L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}_{\text{loc}}(\mathbb{R}))$.

Moreover,

$$\sqrt{h^\delta}u_x^\delta \qquad \text{is uniformly bounded in } L^2(\mathbb{R}^2_+).$$

We conclude (by Aubin's lemma) that

$$\begin{aligned} h^\delta &\quad \text{is compact in } L^p_{\text{loc}}(\mathbb{R}^2_+),\ p\in[1,\infty),\\ m^\delta &\quad \text{is compact in } L^3_{\text{loc}}(\mathbb{R}^2_+),\\ \sqrt{h^\delta}u_x^\delta &\ \text{is compact in } L^2_{weak}(\mathbb{R}^2_+). \end{aligned}$$

Then there exists $(h,m)\in L^\infty_{\text{loc}}(\mathbb{R}^2_+)\times L^3_{\text{loc}}(\mathbb{R}^2_+)$ such that, on a suitable subsequence,

$$(h^\delta,m^\delta)\to(h,m) \qquad \text{in } L^p(\mathbb{R}^2_+)\times L^3_{\text{loc}}(\mathbb{R}^2_+),\ p\in[1,\infty),\ \text{and } a.e.\ (t,x)\in\mathbb{R}^2_+,$$

and

$$\sqrt{h^\delta}u_x^\delta \qquad \text{converges weakly to } n \text{ in } L^2(\mathbb{R}^2_+).$$

To pass to the limit in Eqs. (1.1) and (4.1), we have to deal with the functions that involve the negative powers of h, which are not defined on the vacuum set. We will use repeatedly the following simple lemma.

Lemma 4.1. *Let (h^δ,m^δ), with $h^\delta>0$, converge to (h,m) a.e. $(t,x)\in\mathbb{R}^2_+$. Let $\phi(h,m)$ be a measurable function defined on $\{(h,m)\in\mathbb{R}^2_+,\ h>0\}$. Suppose that, for some $p_0>1$ and any compact $K\subset\mathbb{R}^2_+$,*

$$\begin{aligned} &\|\phi(h^\delta,m^\delta)\|_{L^{p_0}(K)}\le C(K),\\ &\|\phi(h^\delta,m^\delta)\chi_{\{h=0\}}\|_{L^{p_0}(K)}\to 0 \qquad \textit{with } \delta\to 0. \end{aligned}$$

Then

$$\phi(h^\delta,m^\delta)\to\phi(h,m) \qquad \textit{in } L^p(K),\ \forall p\in[1,p_0),\ \textit{as } \delta\to 0,$$

where $\phi(h,m)$ is defined by zero on $\{h=0\}$.

We apply this lemma for $\phi=\frac{m^2}{h}$ and $\phi=\frac{m}{\sqrt{h}}$. Estimates (3.1) and (3.3) imply

$$\begin{aligned} \frac{m^\delta}{\sqrt{h^\delta}} &\quad \text{is uniformly bounded in } L^3(0,T;L^3_{\text{loc}}(\mathbb{R})),\\ \frac{(m^\delta)^2}{h^\delta} &\quad \text{is uniformly bounded in } L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}_{\text{loc}}(\mathbb{R})). \end{aligned}$$

Lemma 4.1 yields

$$\begin{aligned} \frac{m^\delta}{\sqrt{h^\delta}}\to\frac{m}{\sqrt{h}} &\qquad \text{in } L^p(0,T;L^p_{\text{loc}}(\mathbb{R})),\ p\in[1,3),\\ \frac{(m^\delta)^2}{h^\delta}\to\frac{m^2}{h} &\qquad \text{in } L^p(0,T;L^p_{\text{loc}}(\mathbb{R})),\ p\in[1,\frac{3}{2}). \end{aligned}$$

By taking the limit in (2.1) and (4.1), we recover (1.1). To obtain (1.16), we find from (1.3) that

$$\eta_t^{\psi,\delta} + q_x^{\psi,\delta} + gb_x h^\delta \eta_m^{\psi,\delta} - \varepsilon\big((h^\delta+\delta)\eta_m^{\psi,\delta} u_x^\delta\big)_x + \varepsilon(h^\delta+\delta)u_x^\delta(\eta_m^{\psi,\delta})_x = 0. \tag{4.2}$$

For any compactly supported C^2-function ψ, it holds (see [5]):

$$|\eta^\psi(h,m)| + |q^\psi(h,m)| \le C_\psi h,$$

and

$$|\eta_m^\psi(h,m)| \le C_\psi$$

uniformly in (h,m).

Using Lemma 4.1 and the uniform apriori estimates, it is straightforward to deduce that

$$(\eta^{\psi,\delta}, q^{\psi,\delta}) \to (\eta^\psi, q^\psi) \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2_+),\ p>1,$$
$$\sqrt{h^\delta+\delta}\,\eta_m^{\psi,\delta} \to \sqrt{h}\,\eta^\psi \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2_+),\ p>1.$$

It follows then that

$$(h^\delta+\delta)\eta_m^{\psi,\delta}u_x^\delta = \sqrt{h^\delta+\delta}\,\eta_m^{\psi,\delta}\sqrt{h^\delta+\delta}\,u_x^\delta \to \sqrt{h}\,\eta_m^\psi n \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2_+),$$

and we can pass to the limit in the terms of Eq. (4.2), except the last one. However, the last term equals

$$(h^\delta+\delta)u_x^\delta(\eta_m^{\psi,\delta})_x = (h^\delta+\delta)\eta_{mu}^{\psi,\delta}|u_x^\delta|^2 + (h^\delta+\delta)\eta_{mh}^{\psi,\delta}h_x^\delta u_x^\delta.$$

From the uniform estimates,

$$|\eta_{mu}^\psi| \le C_\psi, \qquad |\eta_{mh}^\psi| \le C_\psi h^{-\frac{1}{2}},$$

we obtain

$$|(h^\delta+\delta)\eta_{mu}^{\psi,\delta}|u_x^\delta|^2| \le C_\psi(h^\delta+\delta)|u_x^\delta|^2,$$
$$|(h^\delta+\delta)\eta_{mh}^{\psi,\delta}h_x^\delta u_x^\delta| \le C_\psi(h^\delta+\delta)|u_x^\delta|^2 + C_\psi\frac{h^\delta+\delta}{h^\delta}|h_x|^2,$$

both of which are uniformly bounded in $L^1(0,T;L^1(\mathbb{R}))$, by the energy estimates. This means that $(h^\delta+\delta)u_x^\delta(\eta_m^{\psi,\delta})_x$ converges to some $\mu^\psi \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^2_+)$ and

$$\varepsilon\|\mu^\psi\|_{\mathcal{M}_{\mathrm{loc}}(K)} \le C(K)$$

for every compact $K \subset \mathbb{R}^2_+$.

Finally, we notice that the estimates (2.3), (2.4), (3.2) and (3.3) hold in the limit $\delta \to 0$ for the limiting solution (h,m) as well. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2: H^{-1}-compactness and inviscid limit

For the sequence of solutions of (1.1) constructed in Theorem 1.1, we can establish the compactness of entropy dissipation measures. The following proposition is the straightforward generalization of Proposition 4.1 in [5].

Proposition 5.1. *Let $\psi:\mathbb{R}\to\mathbb{R}$ be any compactly supported C^2-function. Let (η^ψ, q^ψ) be a weak entropy pair generated by ψ. Then, for the solutions $(h^\varepsilon, m^\varepsilon)$ with $m^\varepsilon = h^\varepsilon u^\varepsilon$ of equations (1.1), the entropy dissipation measures*

$$\eta^\psi(h^\varepsilon,m^\varepsilon)_t + q^\psi(h^\varepsilon,m^\varepsilon)_x \quad \text{are confined in a compact subset of } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+). \tag{5.1}$$

The proof basically repeats the arguments from [5]. We conclude that the sequence of solutions $(h^\varepsilon, m^\varepsilon)$ of (1.1)–(1.2), with the initial data $(h_0^\varepsilon, m_0^\varepsilon)$, satisfying the uniform bounds in Theorem 1.1 converges (on a subsequence) almost everywhere to a finite-energy entropy solution (h,m) to problem (1.3) with initial data (1.2). This completes the proof of Theorem 1.2.

6. Shallow water equations in the presence of friction

When the friction is taken into consideration, an additional term, called the friction term, is present in the viscous shallow water equations:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x + gb_x h + r(h,u)hu = \varepsilon(hu_x)_x. \end{cases} \tag{6.1}$$

The interaction between the dynamics of the fluid and the geometry of the physical domain may generate the formation of remarkable structures; see [20,25] and the references cited therein.

Now we consider the case $r(h,u) = \alpha > 0$ and show how our analysis can also be applied to the inviscid limit of the viscous solutions of the viscous shallow water equations with the friction term to the corresponding Saint-Venant system.

Theorem 6.1. *Let $(h_0^\varepsilon, m_0^\varepsilon)$ be a sequence of initial data functions for problem (1.2) and (6.1) which satisfy the assumptions of Theorem 1.1 with the constants M_0, E_0 and E_1, independent of ε. Moreover, assume that $(h_0^\varepsilon, m_0^\varepsilon) \to (h_0, m_0)$ a.e. $x \in \mathbb{R}$. Then there exists an entropy solution (h,m) of problem (1.2) and (6.1) on $\mathbb{R}_+^2$ for $r(h,u) = \alpha > 0$ with the following properties: For any $T > 0$, compact set $K \subset \mathbb{R}$ and $\psi \in C_0^\infty(\mathbb{R}_+^2)$, there are $C_i = C_i(T, E_0, E_1, B)$, $i = 1, 2$, $C_3 = C_3(T, K, E_0, E_1, M_0, B)$ and $C_4 = C_4(T, E_0, E_1, B, \psi)$ that are all independent of $\varepsilon > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{-\infty}^{\infty} \left(\frac{|m(t,x)|^2}{h(t,x)} + g|h(t,x) + b(x) - \bar{h}|^2 \right) \mathrm{d}x \le E_0,$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \varepsilon^2 \int_{-\infty}^{\infty} \left| \left(\sqrt{h(t,x)} \right)_x \right|^2 \mathrm{d}x \le C_1,$$

$$\varepsilon \int_{[0,T]\times\mathbb{R}} (n^2 + |h_x|^2)\, \mathrm{d}x\mathrm{d}t \le C_2,$$

$$\int_0^T \int_K \left(h^3 + \frac{|m|^3}{h^2} \right) \mathrm{d}x\mathrm{d}t \le C_3,$$

$$\varepsilon \int_0^\infty \int_{-\infty}^\infty |\mathrm{d}\mu^\psi| \le C_4.$$

Furthermore, when $\varepsilon \to 0$, there exists a subsequence of $(h^\varepsilon, m^\varepsilon)$ that converges almost everywhere to a relative finite-energy entropy solution (h,m) to the Cauchy problem (1.26) with Cauchy data (1.2) in the sense of Definition 1.2 in which the entropy inequality (1.24) is replaced by

$$\eta^\psi(h,m)_t + q^\psi(h,m)_x + \eta_m^\psi(h,m)\big(gb_x h + \alpha hu\big) \le 0$$

in the sense of distributions for any test-function $\psi(s) \in \{\pm 1, \pm s, s^2\}$ in $\mathbb{R}_+^2$. Moreover, there exists a bounded Radon measure $\mu(t,x;s)$ on $\mathbb{R}_+^2 \times \mathbb{R}$ such that

$$\mu(U \times \mathbb{R}) \ge 0$$

for any open set $U \subset \mathbb{R}_+^2$, and the corresponding entropy kernel $\chi(h, s-u)$ defined by (1.13) satisfies

$$\partial_t \chi(h, s-u) + \frac{1}{2}\partial_x\big((s+u)\chi(h,s-u)\big) - (gb_x + \alpha u)\partial_s \chi(h, s-u) = \partial_s^2 \mu \tag{6.2}$$

in the sense of distributions on $\mathbb{R}_+^2 \times \mathbb{R}$.

As before, we first conclude the following approximate system for $\delta > 0$:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x + gb_x h + \alpha hu = \varepsilon\big((h+\delta)u_x\big)_x. \end{cases} \tag{6.3}$$

First, the energy estimate (2.3) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\Big(h(t,x)u(t,x)^2+g(h(t,x)+b(x)-\bar{h})^2\Big)\,\mathrm{d}x\\ +\varepsilon\int_0^t\int_{-\infty}^{\infty}2(h+\delta)|u_x|^2\,\mathrm{d}x\mathrm{d}t+\alpha\int_0^t\int_{-\infty}^{\infty}h|u|^2\,\mathrm{d}x\mathrm{d}t\\ =\int_{-\infty}^{\infty}\Big(h_0(x)|u_0(x)|^2+g\big(h_0(x)+b(x)-\bar{h}\big)^2\Big)\,\mathrm{d}x\le E_0.\end{aligned} \tag{6.4}$$

It follows by multiplying the second equation in (6.1) by $2u$, using the first equation of (6.1) and the fact (2.2).

Then, under the assumptions on the initial data and $b(x)$, we still have

$$\int_{-\infty}^{\infty}h(t,x)|\sigma_x(t,x)|^2\,\mathrm{d}x+\varepsilon g\int_0^t\int_{-\infty}^{\infty}\frac{h+\delta}{h}|h_x|^2\,\mathrm{d}x\mathrm{d}t\le C_0,\qquad t\in[0,T], \tag{6.5}$$

where $\sigma=\varepsilon\frac{h\ln h-\delta}{h}$.

This can be proved as follows: Indeed, from the first equation of (6.1), we obtain

$$\sigma_t+u\sigma_x=-\varepsilon\frac{h+\delta}{h}u_x,$$

and then

$$\sigma_{xt}+u\sigma_{xx}=-\varepsilon\frac{h+\delta}{h^2}h_xu_x-\varepsilon\left(\frac{h+\delta}{h}u_x\right)_x.$$

Multiplying it by h:

$$(h\sigma_x)_t+(hu\sigma_x)_x=-\varepsilon((h+\delta)u_x)_x.$$

and adding this to the second equation of (6.1), we have

$$(h(u+\sigma_x))_t+(hu(u+\sigma_x))_x+gh(h+b)_x=-\alpha hu,$$

and then

$$\big(h(u+\sigma_x)^2\big)_t+\big(hu(u+\sigma_x)^2\big)_x+2gh(h+b)_x\sigma_x+2gh(h+b)_xu=-2\alpha hu(u+\sigma_x).$$

We integrate this over $(0,t)\times\mathbb{R}$ and use the calculation in (2.2) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\big(h(u+\sigma_x)^2+g(h+b-\bar{h})^2\big)\,\mathrm{d}x+2g\varepsilon\int_0^t\int_{-\infty}^{\infty}\frac{h+\delta}{h}|h_x|^2\,\mathrm{d}x\mathrm{d}t+2g\int_0^t\int h\sigma_xb_x\,\mathrm{d}x\mathrm{d}t\\ =\int_{-\infty}^{\infty}\big(h_0(u_0+\sigma_{0,x})^2+g(h_0+b-\bar{h})\big)\,\mathrm{d}x-2\alpha\int_0^t\int hu(u+\sigma_x)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Moreover, the terms on the left can be estimated as

$$\begin{aligned}&\left|\int_0^t\int_{-\infty}^{\infty}b_xh\sigma_x\,\mathrm{d}x\mathrm{d}t\right|\\ &\quad\le\int_0^t\int_{-\infty}^{\infty}\big(h(\sigma_x)^2+(h+b-\bar{h})|b_x|^2+|b-\bar{h}||b_x|^2\big)\,\mathrm{d}x\mathrm{d}t\\ &\quad\le\int_0^t\int_{-\infty}^{\infty}\big(h(\sigma_x)^2+(h+b-\bar{h})^2\big)\,\mathrm{d}x\mathrm{d}t+\|b_x\|^4_{L^4(\mathbb{R})}+\|b-\bar{h}\|_{L^\infty(\mathbb{R})}\|b_x\|^2_{L^2(\mathbb{R})},\end{aligned}$$

and

$$\left|\int_0^t \int hu(u+\sigma_x)\,\mathrm{d}x\mathrm{d}t\right| \le 2\int_0^t \int h(u+\sigma_x)^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^t \int h|u|^2\,\mathrm{d}x\mathrm{d}t.$$

Then, (6.5) follows by the assumptions on the initial data and $b(x)$ and (6.4).

With the uniform estimates (6.4) and (6.5), the other steps for the uniform apriori estimates are the same, and the functional framework of the inviscid limit remains unchanged as in Sects. 2–5.

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References

1. Bouchut, F.: Nonlinear Stability of Finite Volume Methods for Hyperbolic Conservation Laws and Well-Balanced Schemes for Sources. Birkhüser Verlag, Basel (2004)
2. Bresch, D., Desjardins, B.: Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. Commun. Math. Phys. **238**, 211–223 (2003)
3. Chen, G.-Q.: Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III). Acta Math. Sci. **6B**, 75–120 (1986) (in English)
4. Chen, G.-Q.: Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III). Acta Math. Sci. **8A**, 243–276 (1988) (in Chinese)
5. Chen, G.-Q., Perepelitsa, M.: Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow. Commun. Pure Appl. Math. **63**, 1469–1504 (2010)
6. Chen, G.-Q., LeFloch, Ph.G.: Compressible Euler equations with general pressure law. Arch. Ration. Mech. Anal. **153**, 221–259 (2000)
7. Chen, G.-Q., LeFloch, Ph.G.: Existence theory for the isentropic Euler equations. Arch. Ration. Mech. Anal. **166**, 81–98 (2003)
8. Dafermos, C.M.: Hyperbolic Conservation Laws in Continuum Physics. Springer, Berlin (2000)
9. Ding, X., Chen, G.-Q., Luo, P.: Convergence of the Lax-Friedrichs scheme for the isentropic gas dynamics (I)–(II). Acta Math. Sci. **5B**, 483–500, 501–540 (1985) (in English)
10. Ding, X., Chen, G.-Q., Luo, P.: Convergence of the Lax-Friedrichs scheme for the isentropic gas dynamics (I). Acta Math. Sci. **7A**, 467–480 (1987) (in Chinese)
11. Ding, X., Chen, G.-Q., Luo, P.: Convergence of the Lax-Friedrichs scheme for the isentropic gas dynamics (II). Acta Math. Sci. **8A**, 61–94 (1988) (in Chinese)
12. Ding, X., Chen, G.-Q., Luo, P.: Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics. Commun. Math. Phys. **121**, 63–84 (1989)
13. DiPerna, R.J.: Convergence of the viscosity method for isentropic gas dynamics. Commun. Math. Phys. **91**, 1–30 (1983)
14. Gerbeau, J.-F., Perthame, B.: Derivation of viscous Saint-Venant system for laminar shallow water: numerical validation. Discrete Continuous Dyn. Syst. **1B**, 89–102 (2001)
15. Hoff, D.: Global solutions of the equations of one-dimensional, compressible flow with large data and forces, and with differing end states. Z. Angew. Math. Phys. **49**, 774–785 (1998)
16. Hugoniot, H.: Sur la propagation du movement dans les corps et epécialement dans les gaz parfaits. J. Ecole Polytechnique **58**, 1–125 (1889)
17. LeFloch, Ph.G., Westdickenberg, M.: Finite energy solutions to the isentropic Euler equations with geometric effects. J. Math. Pures Appl. **88**, 386–429 (2007)
18. Lions, P.-L., Perthame, B., Tadmor, E.: Kinetic formulation of the isentropic gas dynamics and p-systems. Commun. Math. Phys. **163**, 415–431 (1994)
19. Lions, P.-L., Perthame, B., Souganidis, P.E.: Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. Commun. Pure Appl. Math. **49**, 599–638 (1996)

20. Mascia, C.: A dive into shallow water. Riv. Mat. Univ. Parma. **1**, 77–149 (2010)
21. Mellet, A., Vasseur, A.: On the barotropic compressible Navier-Stokes equations. Commun. Partial Differ. Equ. **32**, 431–452 (2007)
22. Rankine, W.J.M.: On the thermodynamic theory of waves of finite longitudinal disturbance. Philos. Trans. R. Soc. Lond. **1960**, 277–288 (1870)
23. Rayleigh, L.(Strutt, J.W.): Aerial plane waves of finite amplitude. Proc. R. Soc. Lond. **84A**, 247–284 (1910)
24. Stokes, G.G.: On a difficulty in the theory of sound. Philos. Mag. **33**, 349–356 (1848)
25. Whitham, G.B.: Linear and Nonlinear Waves, Reprint of the 1974 original. Wiley, New York (1999)

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