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# General decay and blow-up of solutions for a viscoelastic equation with nonlinear boundary damping-source interactions

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**Abstract.** In this paper, a viscoelastic equation with nonlinear boundary damping and source terms of the form

$$u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = a\,|u|^{p-1}\,u, \quad \text{in } \Omega \times (0,\infty),$$
$$u = 0, \text{ on } \Gamma_0 \times (0,\infty),$$
$$\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(s)\mathrm{d}s + h(u_t) = b\,|u|^{k-1}\,u, \quad \text{on } \Gamma_1 \times (0,\infty)$$
$$u(0) = u^0, u_t(0) = u^1, \quad x \in \Omega,$$

is considered in a bounded domain $\Omega$. Under appropriate assumptions imposed on the source and the damping, we establish both existence of solutions and uniform decay rate of the solution energy in terms of the behavior of the nonlinear feedback and the relaxation function $g$, without setting any restrictive growth assumptions on the damping at the origin and weakening the usual assumptions on the relaxation function $g$. Moreover, for certain initial data in the unstable set, the finite time blow-up phenomenon is exhibited.

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## 1. Introduction

It is well known that viscoelastic materials have memory effects. These properties are due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. From the mathematical point of view, their memory effects are modeled by integro-differential equations. Hence, questions related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years.

We study the following viscoelastic problem with a nonlinear boundary dissipation and nonlinear boundary/interior sources:

$$\begin{gathered} u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = a\,|u|^{p-1}\,u, \quad \text{in } \Omega \times (0,\infty), \\ u = 0, \text{ on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial}{\partial \nu}u(s)\mathrm{d}s + h(u_t) = b\,|u|^{k-1}\,u, \quad \text{on } \Gamma_1 \times (0,\infty) \\ u(0) = u^0(x), \quad u_t(0) = u^1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

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where $a > 0,\ b > 0,\ p > 1,\ k > 1$, and $\Omega$ is a bounded domain in $R^n$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, $\Gamma_0$ and $\Gamma_1$ are closed and disjoint with meas $(\Gamma_0) > 0$, and $\nu$ is the unit outward normal to $\Gamma$. The relaxation function $g$ is a positive and uniformly decaying function, $h$ is a function satisfying some conditions given in (A2), and

$$\begin{aligned} &1 \le p \le \frac{n}{n-2},\ n > 2 \text{ and } 1 \le p < \infty, \quad \text{if } n = 2, \\ &1 \le k < \frac{n-1}{n-2},\ n > 2 \text{ and } 1 \le k < \infty, \quad \text{if } n = 2. \end{aligned} \tag{1.2}$$

This problem has been widely studied when the viscoelastic term $g$ is absent in (1.1). Several results have been established. Some of the most important papers are those of Chen [8], Haraux [15], Komornik and Zuazua [17], Lasiecka and Tataru [18] and Nako [32]. Among these works, it is worth noting that the pioneering work of Lasiecka and Tataru [18], in which (1.1) with $g = 0$, was conducted under very weak geometrical conditions on $\Gamma_0$ and $\Gamma_1$. They showed that the energy decays as fast as the solution of an associated differential equation, without imposing that $h$ has a polynomial behavior near zero. However, they did not obtain an explicit decay rate estimate for the energy. Alababu–Boussouira [1] investigated the stabilization of hyperbolic systems by a nonlinear feedback that can be localized on a part of the boundary or locally distributed. Using weight integral inequalities together with convexity arguments, she obtained a semi-explicit formula for the decay rate of the solution energy in terms of the behavior of the nonlinear feedback close to the origin. Recently, Cavalcanti et al. [3] considered (1.1) with $g = 0$. They established the existence, nonexistence and uniform decay of solutions under suitable conditions and relations between the damping and the source terms. Yet, the decay rate is also implicit, as in [18]. Vitillaro [34] considered (1.1) with $g = a = 0,\ b = 1$ and $h(u_t) = |u_t|^{m-1} u_t,\ m \ge 1$. The author proved the local existence of solutions when $m > k$ and global existence when $k \le m$ or the initial data was chosen suitably. We refer the reader to related works [11,19,38] dealing with boundary stabilization.

Conversely, in the presence of the memory term ($g \neq 0$), there are numerous results related to the asymptotic behavior of solutions of viscoelastic systems. For example, the viscoelastic membrane equation

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = 0, \quad \text{in } \Omega \times (0,\ \infty), \\ &u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \quad x \in \Omega, \\ &u(x,t) = 0, \quad x \in \partial\Omega, \quad t \ge 0, \end{aligned} \tag{1.3}$$

is considered in a bounded domain $\Omega \subset R^n$ with smooth boundary, see [4,5,9,10,12,28,33]. A nonlinear case of (1.3) with damping term and force term is

$$\begin{aligned} &u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + h(u_t) = f(u), \quad \text{in } \Omega \times (0,\ \infty), \\ &u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \quad x \in \Omega, \\ &u(x,t) = 0, \quad x \in \partial\Omega, \quad t \ge 0, \end{aligned} \tag{1.4}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary. Problems related to (1.4) have been extensively studied, and several results concerning existence, decay and blow-up have been obtained [6,16,21–26,29,35–37]. In relation to a class of abstract viscoelastic systems, Rivera et al. [30] considered the following

$$u_{tt} + Au - (g * A^\alpha u)(t) = 0, \tag{1.5}$$

where $A$ is positive self-adjoint operator with domain $D(A)$ that is a subset of Hilbert space $H$ and $*$ denotes the convolution product in the variable $t$. They showed that the dissipation given by the memory effect is not strong enough to produce exponential stability with $0 \le \alpha < 1$. Indeed, they obtained that the solutions decay polynomially even if kernel $g$ decays exponentially. Very recently, River et al.

[31] completed the analysis in studying the optimal energy rate for problem (1.5) with $0 \leq \alpha < 1$: that is, they showed that the associated energy to problem (1.5) with $0 \leq \alpha < 1$ is polynomial stable, and subsequently found that the decay rate is optimal.

Problem (1.1) has been considered by Cavalcanti et al. [5] with $a = b = 0$. They showed the global existence and established some uniform decay results under quite restrictive assumptions on both the damping function $h$ and kernel $g$. Later, Cavalcanti et al. [4] generalized the result without imposing a growth condition on $h$ and under a weaker assumption on $g$. Recently, Messaoudi and Mustafa [27] exploited some properties of convex functions [1] and the multiplier method to extend these results. They establish an explicit and general decay rate result without imposing any restrictive growth assumption on damping term $h$ and greatly weakened the assumption on $g$. More recently, Ha [14] studied problem (1.1) with $a = 1$ and $b = 0$. He generalized the result of [4] by applying the method developed by Martinez [20]. In fact, the author proved the existence of solutions and uniform decay rates under greatly weakened assumptions of $g$ and $h$.

Motivated by previous works [3,14,27], it is interesting to investigate the existence of solutions, uniform decay result of solutions and finite time blow-up of solutions to problem (1.1) with two nonlinear source terms (boundary and interior) and without imposing any restrictive growth assumption on the boundary dissipation. The presence of the boundary nonlinear term $|u|^{k-1} u$ brings great difficulty in establishing existence of weak solutions due to the fact that Loptanski condition does not hold for Neumann problem. To overcome this point, we utilize arguments as in Cavalcanti et al. [7] making use of Faedo–Galerkin procedure to study well-posedness of problem (1.1). Then, based on some properties of convex functions and the multiplier method as in Guessmia and Messaoudi's work [13], our next intention is to establish an explicit and general decay rate for equations (1.1) under assumptions on $g$ and $h$, without imposing a specific growth condition on the behavior of $h$ near zero and greatly weakening the usual assumptions on relaxation function $g$. In this way, our results allow a larger class of relaxation functions and improve the results of Messaoudi and Mustafa [27], who considered problem (1.1) in the absence of the boundary/interior source terms. Additionally, two competing nonlinear source terms (boundary and interior) in (1.1) may cause the finite time blow-up of solutions, which was not discussed by Ha [14]. Our last intention is to prove that for certain initial data in the unstable set, there are solutions that blow-up in finite time.

The remainder of this paper is organized as follows. In Sect. 2, we provide assumptions that will be used later, state and prove the existence result Theorem 2.6. In Sect. 3, we prove our stability result that is given in Theorem 3.5. Finally, we prove the blow-up result in Theorem 4.2.

## 2. Preliminary results

In this section, we give assumptions and preliminaries that will be needed throughout the paper. First, we introduce the set

$$H^1_{\Gamma_0} = \left\{ u \in H^1(\Omega) : u|_{\Gamma_0} = 0 \right\},$$

and endow $H^1_{\Gamma_0}$ with the Hilbert structure induced by $H^1(\Omega)$, we have that $H^1_{\Gamma_0}$ is a Hilbert space. For simplicity, we denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{q,\Gamma_1} = \|\cdot\|_{L^q(\Gamma_1)}$, $1 \leq q \leq \infty$. According to (1.2), we have the imbedding: $H^1_{\Gamma_0} \hookrightarrow L^{p+1}(\Omega)$. Let $c_* > 0$ be the optimal constant of Sobolev imbedding which satisfies the inequality

$$\|u\|_{p+1} \leq c_* \|\nabla u\|_2 , \quad \forall u \in H^1_{\Gamma_0}, \tag{2.1}$$

and we use the trace-Sobolev imbedding: $H^1_{\Gamma_0} \hookrightarrow L^{k+1}(\Gamma_1)$, $1 \leq k < \frac{n}{n-2}$. In this case, the imbedding constant is denoted by $B_*$, i.e.,

$$\|u\|_{k+1,\Gamma_1} \leq B_* \|\nabla u\|_2 . \tag{2.2}$$

Next, we state the assumptions for problem (1.1):

**(A1)** $g:[0,\infty)\to(0,\infty)$ is a bounded $C^1$ function satisfying

$$g(0)>0,\ 1-\int_0^\infty g(s)\mathrm{d}s=l>0, \tag{2.3}$$

and there exists a nonincreasing positive differentiable function $\xi$ such that

$$g'(t)\le-\xi(t)g(t), \tag{2.4}$$

for all $t\ge 0$.

**(A2)** $h:R\to R$ is a nondecreasing function with $h(s)s\ge 0$ for all $s\in R$ and there exists a convex and increasing function $H:R^+\to R^+$ of class $C^1(R^+)\cap C^2((0,\infty))$ satisfying $H(0)=0$ and $H$ is linear on $[0,1]$ or $H'(0)=0$ and $H''>0$ on (0,1] such that,

$$\begin{aligned} m_q\left|s\right|^q\le\left|h(s)\right|\le M_q\left|s\right|^q,\ 1\le q<\tfrac{n-1}{n-2}\text{ if }\ |s|\ge 1,\\ h^2(s)\le H^{-1}(sh(s))\text{ if }\ |s|\le 1,\end{aligned} \tag{2.5}$$

where $m_q$ and $M_q$ are positive constants.

**Remark 2.1.** Without loss of generality, we take $a=b=1$ in (1.1) throughout this work.

The energy associated with problem (1.1) is given by

$$E(t)=\frac{1}{2}\left\|u_t\right\|_2^2+J(u(t)),\quad\text{for }u\in H^1_{\Gamma_0}, \tag{2.6}$$

where

$$\begin{aligned}J(u(t))=&\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\left\|\nabla u(t)\right\|^2+\frac{1}{2}(g\circ\nabla u)(t)\\ &-\frac{1}{p+1}\left\|u\right\|_{p+1}^{p+1}-\frac{1}{k+1}\left\|u\right\|_{k+1,\Gamma_1}^{k+1},\end{aligned} \tag{2.7}$$

and

$$(g\circ\nabla u)(t)=\int_0^t g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|_2^2\mathrm{d}s.$$

Next, we define a functional $F$ introduced by Cavalcanti et al. in [3], which helps in establishing desired results. Setting

$$F(x)=\frac{1}{2}x^2-\frac{B_\Omega^{p+1}}{p+1}x^{p+1}-\frac{B_\Gamma^{k+1}}{k+1}x^{k+1},\quad x>0, \tag{2.8}$$

where

$$B_\Omega=\sup_{\substack{u\in H^1_{\Gamma_0},\\ u\ne 0}}\frac{\left\|u\right\|_{p+1}}{\sqrt{l\left\|\nabla u\right\|_2^2}}\text{ and }B_\Gamma=\sup_{\substack{u\in H^1_{\Gamma_0},\\ u\ne 0}}\frac{\left\|u\right\|_{k+1,\Gamma_1}}{\sqrt{l\left\|\nabla u\right\|_2^2}}. \tag{2.9}$$

**Remark 2.2.** (i) As in [3], we can verify that the functional $F$ is increasing in $(0,\lambda_0)$, decreasing in $(\lambda_0,\infty)$, and $F$ has a maximum at $\lambda_0$ with the maximum value

$$\begin{aligned}d&\equiv F(\lambda_0)\\ &=\frac{1}{2}\lambda_0^2-\frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1}-\frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1},\end{aligned} \tag{2.10}$$

where $\lambda_0$ is the first positive zero of the derivative function $F'(x)$.

(ii) From (2.6), (2.7), (2.3), (2.9) and the definition of $F$, we have

$$\begin{aligned} E(t) \geq J(u(t)) &\geq \frac{1}{2} l \left\| \nabla u(t) \right\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ &- \frac{B_\Omega^{p+1}}{p+1} \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^{p+1} \\ &- \frac{B_\Gamma^{k+1}}{k+1} \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^{k+1} \\ &= F\left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right), \; t \geq 0. \end{aligned} \tag{2.11}$$

Now, if one considers

$$l \left\| \nabla u(t) \right\|^2 + (g \circ \nabla u)(t) < \lambda_0^2, \tag{2.12}$$

then, from (2.11), we obtain

$$\begin{aligned} E(t) &\geq F\left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right) \\ &= \frac{1}{2} l \left\| \nabla u(t) \right\|^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{B_\Omega^{p+1}}{p+1} \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^{p+1} \\ &\quad - \frac{B_\Gamma^{k+1}}{k+1} \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^{k+1} \\ &> \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^2 \left( \frac{1}{2} - \frac{B_\Omega^{p+1}}{p+1} \lambda_0^{p-1} - \frac{B_\Gamma^{k+1}}{k+1} \lambda_0^{k-1} \right), \; t \geq 0. \end{aligned}$$

Thus, using the identity

$$1 - B_\Omega^{p+1} \lambda_0^{p-1} - B_\Gamma^{k+1} \lambda_0^{k-1} = 0, \tag{2.13}$$

we have, for $k \geq p$,

$$\begin{aligned} E(t) &\geq F\left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right) \\ &\geq \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^2 \left( \frac{1}{2} - \frac{1}{p+1} + \left( \frac{1}{p+1} - \frac{1}{k+1} \right) B_\Gamma^{k+1} \lambda_0^{k-1} \right) \\ &\geq \frac{p-1}{2(p+1)} \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^2, \end{aligned}$$

and if $p \geq k$, we get

$$\begin{aligned} E(t) &\geq F\left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right) \\ &\geq \frac{k-1}{2(k+1)} \left( \sqrt{l \left\| \nabla u \right\|_2^2 + (g \circ \nabla u)(t)} \right)^2, \end{aligned}$$

where the identity (2.13) is derived because $\lambda_0$ is the first positive zero of the derivative function $F'(x)$. Consequently, we have

$$J(u(t)) \geq 0$$

and

$$l\,\|\nabla u\|_2^2 + (g\circ\nabla u)(t) \le \frac{1}{c_0}F\left(\sqrt{l\,\|\nabla u\|_2^2 + (g\circ\nabla u)(t)}\right) \le \frac{1}{c_0}E(t),\ \text{for } t\ge 0, \tag{2.14}$$

with $c_0 = \begin{cases} \frac{p-1}{2(p+1)}, & \text{if } k\ge p, \\ \frac{k-1}{2(k+1)}, & \text{if } p\ge k. \end{cases}$

Following, we will study the existence of weak solutions to problem (1.1). For this purpose, we use the Faedo–Galerkin procedure and employ the ideas developed in [3,7] to establish the weak solution of problem (1.1). Firstly, we are going to consider regular solutions of the following problem

$$\begin{aligned} u_m''(t) - \Delta u_m(t) + \int_0^t g(t-s)\Delta u_m(s)\mathrm{d}s &= f_{1,m}(u_m),\ \text{in } \Omega\times(0,\infty), \\ u_m &= 0,\ \text{on } \Gamma_0\times(0,\infty), \\ \frac{\partial u_m}{\partial\nu} - \int_0^t g(t-s)\frac{\partial}{\partial\nu}u_m(s)\mathrm{d}s + \frac{1}{m}u_m' + h(u_m') &= f_{2,m}(u_m),\ \text{on } \Gamma_1\times(0,\infty) \\ u(0) = u^0(x),\ u_t(0) &= u^1(x),\ x\in\Omega, \end{aligned} \tag{2.15}$$

where, for each $m\in N,\ f_{i,m},\ i=1,2$, are defined by

$$f_{1,m}(s) = \begin{cases} |s|^{p-1}s, & |s|\le m, \\ |m|^{p-1}m, & s\ge m, \\ |-m|^{p-1}(-m), & s\le -m, \end{cases} \tag{2.16}$$

and

$$f_{2,m}(s) = \begin{cases} |s|^{k-1}s, & |s|\le m, \\ |m|^{k-1}m, & s\ge m, \\ |-m|^{k-1}(-m), & s\le -m. \end{cases} \tag{2.17}$$

Then, a sequence of regular solution of problem (2.15) will be obtained and this sequence will converge to a desired weak solution, as $m$ goes to infinity. However, instead of solving (2.15), we will consider the more general problem given by

$$\begin{aligned} u''(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s &= |u|^{p-1}u,\ \text{in } \Omega\times(0,\infty), \\ u &= 0,\ \text{on } \Gamma_0\times(0,\infty), \\ \frac{\partial u}{\partial\nu} - \int_0^t g(t-s)\frac{\partial}{\partial\nu}u(s)\mathrm{d}s + \alpha u' + h(u') &= |u|^{k-1}u, \text{on } \Gamma_1\times(0,\infty), \\ u(0) = u^0(x),\ u_t(0) &= u^1(x),\ x\in\Omega, \end{aligned} \tag{2.18}$$

where $\alpha>0$ is a positive constant.

**Remark 2.3.** Setting

$$f_1(s) = |s|^{p-1}s \text{ and } f_2(s) = |s|^{k-1}s,$$

and defined $f_{1,trunc}$ and $f_{2,trunc}$ as

$$f_{1,trunc}(s)=\begin{cases}|s|^{p-1}s, & |s|\le M,\\ |M|^{p-1}M, & s\ge M,\\ |-M|^{p-1}(-M), & s\le -M,\end{cases}$$

$$f_{2,trunc}(s)=\begin{cases}|s|^{k-1}s, & |s|\le M,\\ |M|^{k-1}M, & s\ge M,\\ |-M|^{k-1}(-M), & s\le -M,\end{cases}$$

where $M$ is a positive constant, so if $F_i(s)=\int_0^s f_i(\tau)\mathrm{d}\tau$ and $F_{i,trunc}(s)=\int_0^s f_{i,trunc}(\tau)\mathrm{d}\tau$ are, respectively, the primitives of $f_i$ and $f_{i,trunc}$, $i=1,2$. We define, for $u\in H^1_{\Gamma_0}$,

$$J_{trunc}(u(t))=\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|^2+\frac{1}{2}(g\circ\nabla u)(t)$$
$$-\int_\Omega F_{1,trunc}(u(t))\mathrm{d}x-\int_{\Gamma_1}F_{2,trunc}(u(t))\mathrm{d}\Gamma,$$

and we can write

$$J(u(t))=\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|^2+\frac{1}{2}(g\circ\nabla u)(t)$$
$$-\int_\Omega F_1(u(t))\mathrm{d}x-\int_{\Gamma_1}F_2(u(t))\mathrm{d}\Gamma.$$

Define

$$E_{trunc}(t)=\frac{1}{2}\|u_t\|_2^2+J_{trunc}(u(t)),$$

considering the energy $E(t)$ defined by (2.6), and noting that $\int_\Omega F_{1,trunc}(u(t))\mathrm{d}x\le\int_\Omega F_1(u(t))\mathrm{d}x$ and $\int_{\Gamma_1}F_{2,trunc}(u(t))\mathrm{d}\Gamma\le\int_{\Gamma_1}F_1(u(t))\mathrm{d}\Gamma$, we deduce that

$$\begin{aligned}E_{trunc}(t)&\ge J_{trunc}(u(t))\\ &\ge\frac{1}{2}\left(l\|\nabla u(t)\|^2+(g\circ\nabla u)(t)\right)-\int_\Omega F_{1,trunc}(u(t))\mathrm{d}x-\int_{\Gamma_1}F_{2,trunc}(u(t))\mathrm{d}\Gamma\\ &\ge\frac{1}{2}\left(l\|\nabla u(t)\|^2+(g\circ\nabla u)(t)\right)-\int_\Omega F_1(u(t))\mathrm{d}x-\int_{\Gamma_1}F_2(u(t))\mathrm{d}\Gamma\\ &\ge\frac{1}{2}l\|\nabla u(t)\|^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{B_\Omega^{p+1}}{p+1}\left(\sqrt{l\|\nabla u\|_2^2+(g\circ\nabla u)(t)}\right)^{p+1}\\ &\quad-\frac{B_\Gamma^{k+1}}{k+1}\left(\sqrt{l\|\nabla u\|_2^2+(g\circ\nabla u)(t)}\right)^{k+1}\\ &=F\left(\sqrt{l\|\nabla u\|_2^2+(g\circ\nabla u)(t)}\right),\end{aligned}$$

which shows that the inequality (2.11) remains true by changing $E_{trunc}(t)$ by $E(t)$ and $J(u(t))$ by $J_{trunc}(u(t))$. Analogously as in deriving (2.14), subject to (2.12), we also obtain

$$J_{trunc}(u(t))\ge 0$$

and

$$l\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)\leq\frac{1}{c_0}F\left(\sqrt{l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)}\right)$$
$$\leq\frac{1}{c_0}E_{trunc}(t),\ t\geq 0.$$

From the above consideration, all the arguments that will be used to prove the existence of regular solutions to problem (2.18) when the initial data is taken in the potential well, can be repeated to the same problem by changing $f_1(s)=|s|^{p-1}s$ and $f_2(s)=|s|^{k-1}s$ by $f_{1,trunc}$ and $f_{2,trunc}$, respectively.

Now, we are ready to state our result.

**Theorem 2.4.** *Let the hypotheses* $(A1)-(A2)$ *and (1.2) hold and* $u^0\in H^1_{\Gamma_0}\cap H^2(\Omega), u^1\in H^1_{\Gamma_0}$ *verifying the compatibility conditions*

$$\frac{\partial u^0}{\partial\nu}+\alpha u^1+h(u^1)=\left|u^0\right|^{k-1}u^0\ \text{ on }\Gamma_1. \tag{2.19}$$

*Assume further that* $l\left\|\nabla u^0\right\|_2^2<\lambda_0^2$ *and* $E(0)<d$. *Then, there exists a unique regular solution* $u$ *of (2.18) satisfying*

$$u\in L^\infty\left([0,\ T);H^1_{\Gamma_0}\cap H^2(\Omega)\right),$$
$$u_t\in L^\infty\left([0,\ T);H^1_{\Gamma_0}\right),$$
$$u_{tt}\in L^\infty\left([0,\ T);L^2(\Omega)\right),$$

*with* $l\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)<\lambda_0^2$, *for* $t>0$.

*Proof.* Let $\{w_m\}_{m\in N}$ be a basis in $H^1_{\Gamma_0}\cap H^2(\Omega)$ and $V_m$ be the space generated by $w_1,\ \ldots,w_m$, $m=1,2,\ldots$. Let us consider

$$u_m(t)=\sum_{i=1}^m r_{im}(t)w_i$$

satisfying the following approximate problem corresponding to (2.18)

$$\begin{gathered}\int_\Omega u_m''(t)w\mathrm{d}x+\int_\Omega\nabla u_m(t)\cdot\nabla w\mathrm{d}x-\int_0^t g(t-\tau)\int_\Omega\nabla u_m(\tau)\cdot\nabla w\mathrm{d}x\mathrm{d}\tau\\ +\alpha\int_{\Gamma_1}u_m'(t)w\mathrm{d}\Gamma+\int_{\Gamma_1}h(u_m'(t))w\mathrm{d}\Gamma\\ =\int_\Omega|u_m(t)|^{p-1}u_m(t)w\mathrm{d}x+\int_{\Gamma_1}|u_m(t)|^{k-1}u_m(t)w\mathrm{d}\Gamma\text{ for }w\in V_m,\\ u_m(0)=u^0\text{ and }u_m'(0)=u^1,\text{ for }m\in N.\end{gathered}\tag{2.20}$$

By standard methods in ordinary differential equations, we prove the existence of solutions to (2.20) on some interval $[0,t_n),\ 0<t_n<T$. In order to extend the solution of (2.20) to the whole interval $[0,T]$, we need the a prior estimate below.

**The First Estimate**

Setting $w = u_m'(t)$ in (2.20), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[&\frac{1}{2}\left\|u_m'(t)\right\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\left\|\nabla u_m(t)\right\|_2^2+\frac{1}{2}(g\circ\nabla u_m)(t)\\ &-\frac{1}{p+1}\left\|u_m\right\|_{p+1}^{p+1}-\frac{1}{k+1}\left\|u_m\right\|_{k+1,\Gamma_1}^{k+1}\Bigg]+\alpha\left\|u_m'(t)\right\|_{2,\Gamma_1}^2+\int_{\Gamma_1}u_m'h(u_m')\mathrm{d}\Gamma\\ &=\frac{1}{2}(g'\circ\nabla u_m)(t)-\frac{1}{2}g(t)\left\|\nabla u_m(t)\right\|_2^2.\end{aligned} \tag{2.21}$$

Then, from (A1) and (A2), we have

$$E_m'(t)=-\alpha\left\|u_m'(t)\right\|_{2,\Gamma_1}^2-\int_{\Gamma_1}u_m'h(u_m')\mathrm{d}\Gamma+\frac{1}{2}(g'\circ\nabla u_m)(t)-\frac{1}{2}g(t)\left\|\nabla u_m(t)\right\|_2^2\le 0.$$

This shows that $E_m(t)$ is a nonincreasing function. For extending the solution to the whole interval, we adapt the idea of Vitillaro [] to our context. Since this result can also be used for existing solutions, for simplicity, we will omit the index $m$. □

**Lemma 2.5.** *Let $u^0\in H_{\Gamma_0}^1\cap H^2(\Omega)$, $u^1\in H_{\Gamma_0}^1$ and the hypotheses (A1)–(A2), (1.2) and (2.19) hold. Assume further that $l\left\|\nabla u^0\right\|_2^2<\lambda_0^2$ and $E(0)<d$. Then, it holds that*

$$l\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)<\lambda_0^2, \tag{2.22}$$

*for all $t\ge 0$.*

*Proof.* Using (2.11) and considering $E(t)$ is a nonincreasing function, we obtain

$$F\left(\sqrt{l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)}\right)\le E(t)\le E(0)<d,\ t\in[0,t_m). \tag{2.23}$$

Further, from Remark 2.2 (i), we observe that $F$ is increasing in $(0,\lambda_0)$, decreasing in $(\lambda_0,\infty)$ and $F(\lambda)\to-\infty$ as $\lambda\to\infty$. Thus, as $E(0)<d$, there exist $\lambda_2'<\lambda_0<\lambda_2$ such that $F(\lambda_2')=F(\lambda_2)=E(0)$, which together with $l\left\|\nabla u^0\right\|_2^2<\lambda_0^2$ infer that

$$F\left(\sqrt{l\left\|\nabla u^0\right\|_2^2}\right)\le E(0)=F(\lambda_2').$$

This implies that $l^{\frac{1}{2}}\left\|\nabla u^0\right\|_2\le\lambda_2'$.

Next, we will prove that

$$\sqrt{l\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)}\le\lambda_2'. \tag{2.24}$$

To establish (2.24), we argue by contradiction. Suppose that (2.24) does not hold, then there exists $t^*\in(0,t_m)$ such that

$$\sqrt{l\left\|\nabla u(t^*)\right\|_2^2+(g\circ\nabla u)(t^*)}>\lambda_2'.$$

Case 1: If $\lambda_2'<\sqrt{l\left\|\nabla u(t^*)\right\|_2^2+(g\circ\nabla u)(t^*)}<\lambda_0$, then

$$F\left(\sqrt{l\left\|\nabla u(t^*)\right\|_2^2+(g\circ\nabla u)(t^*)}\right)>F(\lambda_2')=E(0)\ge E(t^*).$$

This contradicts (2.23).

Case 2: If $\sqrt{l\left\|\nabla u(t^*)\right\|_2^2+(g\diamond\nabla u)(t^*)}\geq\lambda_0$, then by continuity of $\sqrt{l\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)}$, there exists $0<t_1<t^*$ such that

$$\lambda_2'<\sqrt{l\left\|\nabla u(t_1)\right\|_2^2+(g\circ\nabla u)(t_1)}<\lambda_0,$$

then

$$F\left(\sqrt{l\left\|\nabla u(t_1)\right\|_2^2+(g\circ\nabla u)(t_1)}\right)>F(\lambda_2')=E\left(0\right)\geq E(t_1).$$

This is also a contradiction of (2.23). Thus, we have proved the inequality (2.24). This completes the proof of Lemma 2.2. □

We note from (2.1), (2.2) and Lemma 2.2 that

$$\begin{aligned}&\frac{1}{p+1}\left\|u_m\right\|_{p+1}^{p+1}+\frac{1}{k+1}\left\|u_m\right\|_{k+1,\Gamma_1}^{k+1}\\&\quad\leq c_1\left(\left\|\nabla u_m\right\|_2^{p+1}+\left\|\nabla u_m\right\|_2^{k+1}\right)\\&\quad\leq c_1\left(\frac{\lambda_0^{p+1}}{l^{\frac{p+1}{2}}}+\frac{\lambda_0^{k+1}}{l^{\frac{k+1}{2}}}\right),\end{aligned}\tag{2.25}$$

where $c_1=\max\left\{\frac{c_*^{p+1}}{p+1},\frac{B_*^{k+1}}{k+1}\right\}$.

Now, integrating (2.21) over $(0,t)$ and using (A1), (2.22) and (2.25), we conclude that

$$\begin{aligned}&\frac{1}{2}\left\|u_m'(t)\right\|_2^2+\frac{l}{2}\left\|\nabla u_m\right\|_2^2+\frac{1}{2}(g\circ\nabla u_m)(t)\\&\quad+\alpha\int_0^t\left\|u_m'(s)\right\|_{2,\Gamma_1}^2\,\mathrm{d}s+\int_0^t\int_{\Gamma_1}u_m'h(u_m')\mathrm{d}\Gamma\mathrm{d}s\\&\leq E(0)+\frac{\lambda_0^2}{2}+\frac{1}{p+1}\left\|u_m\right\|_{p+1}^{p+1}+\frac{1}{k+1}\left\|u_m\right\|_{k+1,\Gamma_1}^{k+1}\\&\leq E(0)+\lambda_0^2\left(\frac{1}{2}+c_1\left(\frac{\lambda_0^{p-1}}{l^{\frac{p+1}{2}}}+\frac{\lambda_0^{k-1}}{l^{\frac{k+1}{2}}}\right)\right)\\&\equiv L_1,\end{aligned}\tag{2.26}$$

where $L_1$ is a positive constant independent of $m\in N$, $\alpha$ and $t\in(0,T)$.

Additionally, using (2.26) and the growth condition imposed on $h$ given in (A2), we obtain

$$\begin{aligned}&\frac{1}{2}\left\|u_m'(t)\right\|_2^2+\frac{l}{2}\left\|\nabla u_m\right\|_2^2+\frac{1}{2}(g\circ\nabla u_m)(t)+\\&\alpha\int_0^t\left\|u_m'(s)\right\|_{2,\Gamma_1}^2\,\mathrm{d}s+m_q\int_0^t\int_{\Gamma_1}\left|u_m'\right|^{q+1}\mathrm{d}\Gamma\mathrm{d}s-c_ht\\&\leq L_1,\end{aligned}$$

and then

$$\int_0^t\int_{\Gamma_1}\left(\left|h(u_m')\right|^{\frac{q+1}{q}}+\left|u_m'\right|^{q+1}\right)\mathrm{d}\Gamma\mathrm{d}s\leq L(m_q,M_q,L_1,T),\tag{2.27}$$

where $L$ and $c_h$ are some positive constants.

**The Second Estimate**

First of all, we are going to estimate $\|u_m''(0)\|_2$. By taking $t=0$ and $w=u_m''(0)$ in (2.20), we get

$$\|u_m''(0)\|_2^2+\int_\Omega \nabla u^0\cdot\nabla u_m''(0)\mathrm{d}x+\alpha\int_{\Gamma_1} u^1u_m''(0)\mathrm{d}\Gamma+\int_{\Gamma_1} u_m''(0)h(u^1)\mathrm{d}\Gamma$$
$$=\int_\Omega \left|u^0\right|^{p-1}u^0u_m''(0)\mathrm{d}x+\int_{\Gamma_1}\left|u^0\right|^{k-1}u^0u_m''(0)\mathrm{d}\Gamma.$$

Employing Green's formula, (2.19) and Hölder's inequality, we have

$$\|u_m''(0)\|_2\le\left\|\Delta u^0\right\|_2+\left\|u^0\right\|_{2p}^p. \tag{2.28}$$

Next, taking the derivative of (2.20) with respect to $t$ and setting $w=u_m''(t)$ in the resulting expression, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m''(t)\|_2^2+\|\nabla u_m'(t)\|_2^2\right)+\alpha\left\|u_m''(s)\right\|_{2,\Gamma_1}^2+\int_{\Gamma_1} h'(u_m')\left(u_m''\right)^2\mathrm{d}\Gamma$$
$$=g(0)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\nabla u_m(t)\cdot\nabla u_m'(t)\mathrm{d}x-g(0)\left\|\nabla u_m'(t)\right\|_2^2$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g'(t-\tau)\int_\Omega\nabla u_m(\tau)\cdot\nabla u_m'(t)\mathrm{d}x\mathrm{d}\tau\right)-g'(0)\int_\Omega\nabla u_m(t)\cdot\nabla u_m'(t)\mathrm{d}x$$
$$-\int_0^t g''(t-\tau)\int_\Omega\nabla u_m(\tau)\cdot\nabla u_m'(t)\mathrm{d}x\mathrm{d}\tau+p\int_\Omega|u_m|^{p-1}u_m'u_m''(t)\mathrm{d}x$$
$$+k\int_{\Gamma_1}|u_m|^{k-1}u_m'u_m''(t)\mathrm{d}\Gamma. \tag{2.29}$$

We will estimate the terms on the right-hand side of (2.29). By Hölder's inequality, Young's inequality and (A1), we have, for $\varepsilon>0$,

$$-g'(0)\int_\Omega\nabla u_m(t)\cdot\nabla u_m'(t)\mathrm{d}x\le\varepsilon\left\|\nabla u_m(t)\right\|_2^2+\frac{g'(0)^2}{4\varepsilon}\left\|\nabla u_m'(t)\right\|_2^2, \tag{2.30}$$

and

$$\int_0^t g''(t-\tau)\int_\Omega\nabla u_m(\tau)\cdot\nabla u_m'(t)\mathrm{d}x\mathrm{d}\tau\le\left\|\nabla u_m'(t)\right\|_2\int_0^t g''(t-\tau)\left\|\nabla u_m(\tau)\right\|_2\mathrm{d}\tau$$
$$\le\frac{1}{4\varepsilon}\left\|\nabla u_m'(t)\right\|_2^2+\varepsilon\left\|g''\right\|_{L^1}\int_0^t\left|g''(t-\tau)\right|\left\|\nabla u_m(\tau)\right\|_2^2\mathrm{d}\tau. \tag{2.31}$$

Observing that $\frac{p-1}{2p}+\frac{1}{2p}+\frac{1}{2}=1$ and $\frac{k-1}{2k}+\frac{1}{2k}+\frac{1}{2}=1$, and then, from generalized Hölder's inequality, (2.1), (2.2), Young's inequality and (2.22), we obtain

$$\begin{aligned} p\int_{\Omega} |u_m|^{p-1} u_m' u_m''(t)\mathrm{d}x &\le p\,\|u_m\|_{2p}^{p-1}\,\|u_m'\|_{2p}\,\|u_m''\|_2 \\ &\le pc_*^p\,\|\nabla u_m\|_2^{p-1}\,\|\nabla u_m'\|_2\,\|u_m''\|_2 \\ &\le k_1(\varepsilon)\,\|\nabla u_m'\|_2^2+\varepsilon\,\|u_m''\|_2^2, \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} k\int_{\Gamma_1} |u_m|^{k-1} u_m' u_m''(t)\mathrm{d}\Gamma &\le k\,\|u_m\|_{2k,\Gamma_1}^{k-1}\,\|u_m'\|_{2k,\Gamma_1}\,\|u_m''\|_{2,\Gamma_1} \\ &\le kB_*^k\,\|\nabla u_m\|_2^{k-1}\,\|\nabla u_m'\|_2\,\|u_m''\|_{2,\Gamma_1} \\ &\le k_2(\varepsilon)\,\|\nabla u_m'\|_2^2+\varepsilon\,\|u_m''\|_{2,\Gamma_1}^2, \end{aligned} \tag{2.33}$$

where $k_i(\varepsilon)$, $i=1,2$, are positive constants which depends on $\varepsilon$ and the estimate obtained in (2.22). Integrating (2.29) over $(0,t)$ and taking estimates (2.30)–(2.33) into account, we have

$$\begin{aligned} &\frac{1}{2}\left(\|u_m''(t)\|_2^2+\|\nabla u_m'(t)\|_2^2\right)+(\alpha-\varepsilon)\int_0^t \|u_m''(s)\|_{2,\Gamma_1}^2\,\mathrm{d}s+\int_0^t\int_{\Gamma_1} h'(u_m')\,(u_m'')^2\,\mathrm{d}\Gamma\mathrm{d}s \\ &\quad\le \frac{1}{2}\left(\|u_m''(0)\|_2^2+\left\|\nabla u^1\right\|_2^2\right)+g(0)\int_{\Omega}\nabla u_m(t)\cdot\nabla u_m'(t)\mathrm{d}x-g(0)\int_{\Omega}\nabla u^0\cdot\nabla u^1\mathrm{d}x \\ &\qquad+\left(\frac{g'(0)^2+1}{4\varepsilon}+k_1(\varepsilon)+k_2(\varepsilon)-g(0)\right)\int_0^t \|\nabla u_m'\|_2^2\,\mathrm{d}s+\int_0^t g'(t-\tau)\int_{\Omega}\nabla u_m(\tau)\cdot\nabla u_m'(t)\mathrm{d}x\mathrm{d}\tau \\ &\qquad+\left(\varepsilon\,\|g''\|_{L^1}^2+\varepsilon\right)\int_0^t \|\nabla u_m(s)\|_2^2\,\mathrm{d}s+\varepsilon\int_0^t \|u_m''(s)\|_2^2\,\mathrm{d}s. \end{aligned} \tag{2.34}$$

Exploiting Hölder's inequality, Young's inequality and the assumption on $g$ given in (2.4), we observe that

$$g(0)\int_{\Omega}\nabla u_m(t)\cdot\nabla u_m'(t)\mathrm{d}x\le\varepsilon\,\|\nabla u_m'(t)\|_2^2+\frac{g(0)^2}{4\varepsilon}\,\|\nabla u_m(t)\|_2^2$$

and

$$\begin{aligned} &\int_0^t g'(t-\tau)\int_{\Omega}\nabla u_m(\tau)\cdot\nabla u_m'(t)\mathrm{d}x\mathrm{d}\tau \\ &\qquad\le\varepsilon\,\|\nabla u_m'(t)\|_2^2+\frac{\xi(0)\,\|g\|_{L^1}\,\|g\|_{L^\infty}}{4\varepsilon}\,\|\nabla u_m(t)\|_2^2, \end{aligned}$$

then, from (2.34), choosing $\varepsilon$ small enough and combining the estimate (2.22), (2.26), (2.28) and using Gronwall's Lemma, we obtain

$$\|u_m''(t)\|_2^2+\|\nabla u_m'(t)\|_2^2+\int_0^t \|u_m''(s)\|_{2,\Gamma_1}^2\,\mathrm{d}s+\int_0^t\int_{\Gamma_1} h'(u_m')\,(u_m'')^2\,\mathrm{d}\Gamma\mathrm{d}s\le L_2, \tag{2.35}$$

for all $t\in[0,T]$ and $L_2$ is a positive constant independent of $m\in N$.

The estimates (2.22), (2.26), (2.27) and (2.35) permits us to obtain a subsequence of $\{u_m\}$, which we still denote by $\{u_m\}$ and a function $u : \Omega \times (0,\infty) \to R$ satisfying

$$u_m \rightharpoonup u \text{ weak star in } L^\infty(0,T;H^1_{\Gamma_0}), \tag{2.36}$$
$$u'_m \rightharpoonup u' \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \tag{2.37}$$
$$u''_m \rightharpoonup u'' \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{2.38}$$
$$u'_m \rightharpoonup u' \text{ weakly in } L^2(0,T;L^2(\Gamma_1)), \tag{2.39}$$
$$u'_m \rightharpoonup u' \text{ weakly in } L^{q+1}\left((0,T)\times\Gamma_1\right), \tag{2.40}$$
$$h(u'_m) \rightharpoonup \chi \text{ weakly in } L^{\frac{q+1}{q}}\left((0,T)\times\Gamma_1\right). \tag{2.41}$$

In addition, from the assumption (A2) and (2.35), we also notice that

$$\begin{aligned}\int_{\Gamma_1} (h(u'_m))^2\, \mathrm{d}\Gamma &= \int_{|u'_m|\le 1} (h(u'_m))^2\, \mathrm{d}\Gamma + \int_{|u'_m|>1} (h(u'_m))^2\, \mathrm{d}\Gamma\\ &\le c_2 + M_q^2 \left\|u'_m(t)\right\|^{2q}_{2q,\Gamma_1}\\ &\le c_2 + M_q^2 B_*^{2q} \left\|\nabla u'_m(t)\right\|^{2q}_2\\ &\le c_3,\end{aligned}$$

where $c_i,\ i = 2,3$ are some positive constants. Thus, we deduce that

$$h(u'_m) \rightharpoonup \chi \text{ weakly in } L^2\left((0,T)\times\Gamma_1\right). \tag{2.42}$$

Further, noting that $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ and $H^1_{\Gamma_0} \hookrightarrow L^2(\Omega)$ are compact and from Aubin–Lions theorem, we deduce that

$$u_m \to u \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{2.43}$$
$$u'_m \to u' \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{2.44}$$
$$u_m \to u \text{ strongly in } L^2(0,T;L^2(\Gamma_1)), \tag{2.45}$$

and consequently, thanks to Lion's Lemma, we have

$$|u_m|^{p-1} u_m \rightharpoonup |u|^{p-1} u \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{2.46}$$
$$|u_m|^{k-1} u_m \rightharpoonup |u|^{k-1} u \text{ weakly in } L^2(0,T;L^2(\Gamma_1)). \tag{2.47}$$

Multiplying (2.20) by $\theta \in D(0,T)$ and integrating it over $(0,T)$, we get

$$\begin{aligned}&\int_0^T\int_\Omega u''_m v\theta \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \nabla u_m\cdot\nabla v\theta\mathrm{d}x\mathrm{d}t - \int_0^T\int_0^t g(t-\tau)\int_\Omega \nabla u_m(\tau)\cdot\nabla v\theta\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\\ &\quad+\alpha\int_0^T\int_{\Gamma_1} u'_m(t)v\theta\mathrm{d}\Gamma\mathrm{d}t + \int_0^T\int_{\Gamma_1} h(u'_m(t))v\theta\mathrm{d}\Gamma\mathrm{d}t\\ &= \int_0^T\int_\Omega |u_m(t)|^{p-1} u_m(t)v\theta\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Gamma_1} |u_m(t)|^{k-1} u_m(t)v\theta\mathrm{d}\Gamma\mathrm{d}t,\end{aligned} \tag{2.48}$$

for all $\theta \in D(0,T)$ and for all $v \in H^1_{\Gamma_0} \cap H^2(\Omega)$. Convergences (2.36)–(2.47) are sufficient to pass the limit in the approximate problem (2.48). Since $\{w_m\}_{m\in N}$ is a basis in $H^1_{\Gamma_0} \cap H^2(\Omega)$ and $V_m$ is dense in

$H^1_{\Gamma_0} \cap H^2(\Omega)$, after passing to the limit, we obtain

$$\int_0^T \int_\Omega u'' v\theta \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \nabla u \cdot \nabla v\theta \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u(\tau) \cdot \nabla v\theta \mathrm{d}x\mathrm{d}\tau\mathrm{d}t$$
$$+ \alpha \int_0^T \int_{\Gamma_1} u'(t) v\theta \mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_{\Gamma_1} \chi v\theta \mathrm{d}\Gamma\mathrm{d}t$$
$$= \int_0^T \int_\Omega |u(t)|^{p-1} u(t) v\theta \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_1} |u(t)|^{k-1} u(t) v\theta \mathrm{d}\Gamma\mathrm{d}t. \tag{2.49}$$

In particular, let $v\theta \in D((0,T) \times \Omega)$ in (2.49), we obtain

$$u''(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = |u|^{p-1} u \quad \text{in} \quad D'((0,T) \times \Omega).$$

Since $u'', \ |u|^{p-1} u \in L^2([0,T); L^2(\Omega))$, we have

$$\Delta \left( u(t) - \int_0^t g(t-s)u(s)\mathrm{d}s \right) \in L^2(0,T; L^2(\Omega))$$

and therefore

$$u''(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = |u|^{p-1} u \quad \text{in } L^2(0,T; L^2(\Omega)). \tag{2.50}$$

Taking (2.50) into account and making use of the Green's formula, we see that

$$\frac{\partial}{\partial \nu} \left( u - \int_0^t g(t-s)u(s)\mathrm{d}s \right) + \alpha u' + \chi = |u|^{k-1} u \quad \text{in } D'((0,T); H^{-\frac{1}{2}}(\Gamma_1)),$$

and since $|u|^{k-1} u, \ \alpha u', \ \chi \in L^2(0,T; L^2(\Gamma_1))$, we infer

$$\frac{\partial}{\partial \nu} \left( u - \int_0^t g(t-s)u(s)\mathrm{d}s \right) + \alpha u' + \chi = |u|^{k-1} u \text{ in } L^2(0,T; L^2(\Gamma_1)). \tag{2.51}$$

Finally, we want to show that $\chi = h(u')$. Firstly, taking $w = u_m(t)$ in (2.20) and then integrating it over $(0, T)$, we deduce that

$$\begin{aligned}
&\int_0^T \int_\Omega u_m''(t) u_m(t) \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \nabla u_m(t) \cdot \nabla u_m(t) \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u_m(\tau) \cdot \nabla u_m(t) \mathrm{d}x\mathrm{d}\tau\mathrm{d}t \\
&\quad + \alpha \int_0^T \int_{\Gamma_1} u_m'(t) u_m(t) \mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_{\Gamma_1} h(u_m'(t)) u_m(t) \mathrm{d}\Gamma\mathrm{d}t \\
&= \int_0^T \int_\Omega |u_m(t)|^{p+1} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_1} |u_m(t)|^{k+1} \mathrm{d}\Gamma\mathrm{d}t. \qquad (2.52)
\end{aligned}$$

From (2.38), (2.42) and (2.45)–(2.47), we have

$$\begin{aligned}
&\lim_{m\to\infty} \left( \int_0^T \int_\Omega |\nabla u_m(t)|^2 \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u_m(\tau) \cdot \nabla u_m(t) \mathrm{d}x\mathrm{d}\tau\mathrm{d}t \right) \\
&= - \int_0^T \int_\Omega u''(t) u(t) \mathrm{d}x\mathrm{d}t - \alpha \int_0^T \int_{\Gamma_1} u'(t) u(t) \mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_{\Gamma_1} \chi u(t) \mathrm{d}\Gamma\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega |u(t)|^{p+1} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_1} |u(t)|^{k+1} \mathrm{d}\Gamma\mathrm{d}t. \qquad (2.53)
\end{aligned}$$

Employing (2.50), (2.51) and (2.53) and the Green's formula, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \left( \int_0^T \int_\Omega |\nabla u_m(t)|^2 \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u_m(\tau) \cdot \nabla u_m(t) \mathrm{d}x\mathrm{d}\tau\mathrm{d}t \right) \\
&= \int_0^T \int_\Omega |\nabla u(t)|^2 \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u(\tau) \cdot \nabla u(t) \mathrm{d}x\mathrm{d}\tau\mathrm{d}t, \qquad (2.54)
\end{aligned}$$

which implies that

$$\nabla u_m(t) \to \nabla u(t) \text{ strongly in } L^2(0, T; L^2(\Omega)). \qquad (2.55)$$

Secondly, considering $w = u_m'(t)$ in (2.20) and then integrating it over $(0, T)$, we have

$$\int_0^T \int_\Omega u_m''(t) u_m'(t) \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \nabla u_m(t) \cdot \nabla u_m'(t) \mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u_m(\tau) \cdot \nabla u_m'(t) \mathrm{d}x\mathrm{d}\tau\mathrm{d}t$$
$$+\alpha \int_0^T \int_{\Gamma_1} |u_m'(t)|^2 \mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_{\Gamma_1} h(u_m'(t)) u_m'(t) \mathrm{d}\Gamma\mathrm{d}t$$
$$= \int_0^T \int_\Omega |u_m(t)|^{p-1} u_m(t) u_m'(t) \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_1} |u_m(t)|^{k-1} u_m(t) u_m'(t) \mathrm{d}\Gamma\mathrm{d}t.$$

From convergences (2.37), (2.38), (2.44)–(2.47) and (2.55), we deduce that

$$\lim_{m\to\infty} \int_0^T \int_{\Gamma_1} h(u_m'(t)) u_m'(t) \mathrm{d}\Gamma\mathrm{d}t$$
$$= - \int_0^T \int_\Omega u''(t) u'(t) \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \nabla u(t) \cdot \nabla u'(t) \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_0^t g(t-\tau) \int_\Omega \nabla u(\tau) \cdot \nabla u'(t) \mathrm{d}x\mathrm{d}\tau\mathrm{d}t - \alpha \int_0^T \int_{\Gamma_1} |u'(t)|^2 \mathrm{d}\Gamma\mathrm{d}t$$
$$+ \int_0^T \int_\Omega |u(t)|^{p-1} u(t) u'(t) \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_1} |u(t)|^{k-1} u(t) u'(t) \mathrm{d}\Gamma\mathrm{d}t. \quad (2.56)$$

Exploiting (2.50), (2.51), (2.56) and using the Green's formula, we see that

$$\lim_{m\to\infty} \int_0^T \int_{\Gamma_1} h(u_m'(t)) u_m'(t) \mathrm{d}\Gamma\mathrm{d}t = \int_0^T \int_{\Gamma_1} \chi u'(t) \mathrm{d}\Gamma\mathrm{d}t. \quad (2.57)$$

Since $h$ is nondecreasing monotone function, we have, for all $\varphi \in L^{q+1}(\Gamma_1)$,

$$\int_0^T \int_{\Gamma_1} \left(h(u_m'(t)) - h(\varphi)\right)\left(u_m'(t) - \varphi\right) \mathrm{d}\Gamma\mathrm{d}t \geq 0.$$

This yields

$$\int_0^T \int_{\Gamma_1} h(u_m'(t)) \varphi \mathrm{d}\Gamma\mathrm{d}t + \int_0^T \int_{\Gamma_1} h(\varphi)\left(u_m'(t) - \varphi\right) \mathrm{d}\Gamma\mathrm{d}t$$
$$\leq \int_0^T \int_{\Gamma_1} h(u_m'(t)) u_m'(t) \mathrm{d}\Gamma\mathrm{d}t, \quad (2.58)$$

and then

$$\liminf_{m\to\infty}\int_0^T\int_{\Gamma_1} h(u_m'(t))\varphi \mathrm{d}\Gamma\mathrm{d}t + \liminf_{m\to\infty}\int_0^T\int_{\Gamma_1} h(\varphi)\left(u_m'(t)-\varphi\right)\mathrm{d}\Gamma\mathrm{d}t$$
$$\le \liminf_{m\to\infty}\int_0^T\int_{\Gamma_1} h(u_m'(t))u_m'(t)\mathrm{d}\Gamma\mathrm{d}t.$$

Considering the convergence (2.57), (2.40) and (2.41), we obtain

$$\int_0^T\int_{\Gamma_1}\left(\chi-h(\varphi)\right)\left(u'(t)-\varphi\right)\mathrm{d}\Gamma\mathrm{d}t\ge 0,$$

which implies that $\chi = h(u')$.

**Uniqueness.** Let $u_1$ and $u_2$ be two solutions of problem (2.18). Then, $z = u_1 - u_2$ verifies

$$\begin{aligned}&\int_\Omega z''(t)w\mathrm{d}x + \int_\Omega \nabla z(t)\cdot\nabla w\mathrm{d}x - \int_0^t g(t-\tau)\int_\Omega \nabla z(\tau)\cdot\nabla w\mathrm{d}x\mathrm{d}\tau\\ &\quad +\alpha\int_{\Gamma_1} z'(t)w\mathrm{d}\Gamma + \int_{\Gamma_1}\left(h(u_1')-h(u_2')\right)w\mathrm{d}\Gamma\\ &\quad = \int_\Omega\left(|u_1|^{p-1}u_1-|u_2|^{p-1}u_2\right)w\mathrm{d}x + \int_{\Gamma_1}\left(|u_1|^{k-1}u_1-|u_2|^{k-1}u_2\right)w\mathrm{d}\Gamma,\end{aligned} \tag{2.59}$$

for all $w\in H^1_{\Gamma_0}$. Replacing $w = z'(t)$ in (2.59) and observing that $h$ is monotone, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z'(t)\|_2^2+\|\nabla z(t)\|_2^2\right)+\alpha\|z'(t)\|_{2,\Gamma_1}^2 \le \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g(t-\tau)\int_\Omega\nabla z(\tau)\cdot\nabla z(t)\mathrm{d}x\mathrm{d}\tau\right)\\ &\quad -g(0)\|\nabla z(t)\|_2^2 - \int_0^t g'(t-\tau)\int_\Omega \nabla z(\tau)\cdot\nabla z(t)\mathrm{d}x\mathrm{d}\tau\\ &\quad +\int_\Omega\left(|u_1|^{p-1}u_1-|u_2|^{p-1}u_2\right)z'(t)\mathrm{d}x+\int_{\Gamma_1}\left(|u_1|^{k-1}u_1-|u_2|^{k-1}u_2\right)z'(t)\mathrm{d}\Gamma.\end{aligned} \tag{2.60}$$

Utilizing (2.4), Hölder's inequality and Young's inequality, we have

$$\int_0^t g'(t-\tau)\int_\Omega\nabla z(\tau)\cdot\nabla z(t)\mathrm{d}x\mathrm{d}\tau \le \frac{1}{2}\|\nabla z\|_2^2 + \frac{\|g'\|_{L^1}}{2}\int_0^t g'(t-\tau)\|\nabla z(\tau)\|_2^2\mathrm{d}\tau. \tag{2.61}$$

Using the fact that $\frac{p-1}{2p}+\frac{1}{2p}+\frac{1}{2}=1$ and $\frac{k-1}{2k}+\frac{1}{2k}+\frac{1}{2}=1$ again, and, then, from Hölder's inequality and Young's inequality, we see that

$$\begin{aligned}\int_\Omega & \left(|u_1|^{p-1}u_1-|u_2|^{p-1}u_2\right)z'(t)\mathrm{d}x\\ &\le c_4\int_\Omega\left(|u_1|^{p-1}+|u_2|^{p-1}\right)|z(t)|\,|z'(t)|\,\mathrm{d}x\\ &\le \frac{c_4}{2}\left(\|u_1\|_{2p}^{p-1}+\|u_2\|_{2p}^{p-1}\right)\left(\|\nabla z\|_2^2+\|z'\|_2^2\right),\end{aligned}\tag{2.62}$$

and

$$\begin{aligned}\int_{\Gamma_1} & \left(|u_1|^{k-1}u_1-|u_2|^{k-1}u_2\right)z'(t)\mathrm{d}\Gamma\\ &\le c_5\left(\|u_1\|_{2k.\Gamma_1}^{k-1}+\|u_2\|_{2k,\Gamma_1}^{k-1}\right)\|z\|_{2k,\Gamma_1}\|z'\|_{2,\Gamma_1}\\ &\le \varepsilon\|z'\|_{2,\Gamma_1}^2+\frac{c_5^2}{4\varepsilon}\left(\|u_1\|_{2k.\Gamma_1}^{k-1}+\|u_2\|_{2k,\Gamma_1}^{k-1}\right)^2\|\nabla z\|_2^2,\end{aligned}\tag{2.63}$$

where $\varepsilon>0$ and $c_i,\ i=4,5$ are some positive constants. Integrating (2.60) over $(0,t)$ and taking estimates (2.61)–(2.63), (2.22) and (2.26) into account, we deduce that

$$\begin{aligned}\frac{1}{2}&\left(\|z'(t)\|_2^2+\|\nabla z(t)\|_2^2\right)+(\alpha-\varepsilon)\int_0^t\|z'(s)\|_{2,\Gamma_1}^2\,\mathrm{d}s\\ &\le\frac{c_6}{2}\int_0^t\|z'\|_2^2\,\mathrm{d}s+\left(\frac{\|g'\|_{L^1}^2}{2}+g(0)+\frac{c_5^2}{4\varepsilon}\right)\int_0^t\|\nabla z(s)\|_2^2\,\mathrm{d}s,\end{aligned}$$

where $c_6$ is a positive constant. Thus, choosing $\varepsilon$ sufficiently small and employing Gronwall's lemma, we conclude that

$$\|z'(t)\|_2=\|\nabla z(t)\|_2=0 \text{ for all } t\in[0,T].$$

In order to obtain the existence of weak solutions for problem (1.1), we use standard arguments of density with truncated problem and obtain the next result.

**Theorem 2.6.** *Let the initial data* $\{u^0,\ u^1\}\in H^1_{\Gamma_0}\times L^2(\Omega)$. *Suppose that the hypotheses (A1)-(A2) and (1.2) hold. Assume further that* $l\left\|\nabla u^0\right\|_2^2<\lambda_0^2$ *and* $E(0)<d$. *Then, there exists a weak solution* $u$ *of the problem (1.1) satisfying*

$$u\in C\left([0,\ T);H^1_{\Gamma_0}\right)\cap C^1\left([0,\ T);L^2(\Omega)\right),$$

*with* $l\|\nabla u(t)\|^2+(g\circ\nabla u)(t)<\lambda_0^2$ *for some* $T>0$. *Moreover, we have the following energy relation satisfied*

$$E(t)+\int_0^t\int_{\Gamma_1}u_th(u_t)d\Gamma\,ds+\int_0^t\frac{1}{2}(g'\circ\nabla u)(s)\,ds+\frac{1}{2}\int_0^t g(s)\,\|\nabla u(s)\|_2^2\,ds=E(0).\tag{2.64}$$

*Proof.* Since $u^0\in H^1_{\Gamma_0},\ u^1\in L^2(\Omega)$, and, moreover, $l\left\|\nabla u^0\right\|_2^2<\lambda_0^2$ *and* $E(0)<d$. Then, $l\left\|\nabla u^0\right\|_2^2=\lambda_0^2-\delta_1$ and $E(0)=d-\delta_2$ for some positive numbers $\delta_i,\ i=1,2$. Now, we consider

$$D(-\Delta)=\left\{v\in H^1_{\Gamma_0}\cap H^2(\Omega),\ \frac{\partial v}{\partial\nu}=0\text{ on }\Gamma_1\right\}.$$

Let sequences $\left\{u_m^0, u_m^1\right\}$ in $D(-\Delta)$ and $H^1_{\Gamma_0}(\Omega)$, respectively, such that

$$u_m^0 \to u^0 \text{ in } H^1_{\Gamma_0} \text{ and } u_m^1 \to u^1 \text{ in } L^2(\Omega) \text{ as } m \to \infty. \tag{2.65}$$

So, $\left\{u_m^0, u_m^1\right\}$ satisfy, for all $m \geq m_0$, for some $m_0 \in N$, the compatibility conditions

$$\frac{\partial u_m^0}{\partial \nu} + \frac{1}{m} u_m^1 + h(u_m^1) = f_{1,m}\left(u_m^0\right) \text{ on } \Gamma_1,$$

where $\alpha$ is chosen equal to $\frac{1}{m}$ and $f_{1,m}(s)$ is the sequence of Lipschitz continuous (truncated) functions defined by (2.16). And, moreover, $l\left\|\nabla u_m^0\right\|_2^2 < \lambda_0^2$ *and* $E(u_m^0(0)) < d$. Then, for each $m \in N$, there exists a regular solution $u_m : \Omega \times (0, \infty) \to R$ of (2.15) with initial data $\left\{u_m^0, u_m^1\right\}$. So, we can verify

$$\begin{gathered} u_m''(t) - \Delta u_m(t) + \int_0^t g(t-s)\Delta u_m(s)\mathrm{d}s = f_{1,m}(u_m), \text{ in } \Omega \times (0,\infty), \\ u_m = 0, \text{ on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u_m}{\partial \nu} - \int_0^t g(t-s)\frac{\partial}{\partial \nu} u_m(s)\mathrm{d}s + \frac{1}{m}u_m' + h(u_m') = f_{2,m}(u_m), \text{ on } \Gamma_1 \times (0,\infty), \\ u_m(0) = u_m^0, u_m^1(0) = u_m^1, \ x \in \Omega, \end{gathered} \tag{2.66}$$

where $f_{2,m}(s)$ is the sequence of Lipschitz continuous (truncated) functions defined in (2.17). By the same argument used to prove the estimates given in (2.26) and (2.27), we arrive at

$$l\left\|\nabla u_m(t)\right\|^2 + (g \circ \nabla u_m)(t) < \lambda_0^2, \tag{2.67}$$

$$\begin{aligned} &\frac{1}{2}\left\|u_m'(t)\right\|_2^2 + \frac{l}{2}\left\|\nabla u_m(t)\right\|^2 + \frac{1}{2}(g \circ \nabla u_m)(t) \\ &\quad + \frac{1}{m}\int_0^t \left\|u_m'(s)\right\|_{2,\Gamma_1}^2 \mathrm{d}s + \int_0^t \int_{\Gamma_1} u_m' h(u_m')\mathrm{d}\Gamma\mathrm{d}s \\ &\leq L_1, \end{aligned} \tag{2.68}$$

and

$$\int_0^t \int_{\Gamma_1} \left(\left|h(u_m')\right|^{\frac{q+1}{q}} + \left|u_m'\right|^{q+1}\right)\mathrm{d}\Gamma\mathrm{d}s \leq L(m_q, M_q, L_1, T). \tag{2.69}$$

Similar to deriving (2.25), by (2.1), (2.2) and (2.66), we have

$$\begin{aligned} \int_\Omega F_{1,m}(u_m)\mathrm{d}x + \int_{\Gamma_1} F_{2,m}(u_m)\mathrm{d}\Gamma &\leq \int_\Omega F_1(u_m)\mathrm{d}x + \int_{\Gamma_1} F_2(u_m)\mathrm{d}\Gamma \\ &\leq c_1\left(\frac{\lambda_0^{p+1}}{l^{\frac{p+1}{2}}} + \frac{\lambda_0^{k+1}}{l^{\frac{k+1}{2}}}\right), \end{aligned} \tag{2.70}$$

where $F_{i,m}(s) = \int_0^s f_{i,m}(\tau)\mathrm{d}\tau$ and $F_i(s) = \int_0^s f_i(\tau)\mathrm{d}\tau,\ i = 1, 2$ with $f_1(s) = |s|^{p-1} s$ and $f_2(s) = |s|^{k-1} s$ and $c_1 = \max\left\{\frac{c_*^{p+1}}{p+1}, \frac{B_*^{k+1}}{k+1}\right\}$. Hence, from (2.67)–(2.70), we get

$$\{u_m\} \text{ is bounded in } L^\infty(0,T;H^1_{\Gamma_0}), \tag{2.71}$$
$$\{u'_m\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{2.72}$$
$$\{F_{1,m}(u_m)\} \text{ is bounded in } L^\infty(0,T;L^1(\Omega)), \tag{2.73}$$
$$\{F_{2,m}(u_m)\} \text{ is bounded in } L^\infty(0,T;L^1(\Gamma_1)), \tag{2.74}$$
$$\left\{\frac{1}{\sqrt{m}}u'_m\right\} \text{ is bounded in } L^2(0,T;L^2(\Gamma_1)), \tag{2.75}$$
$$\{u'_m\} \text{ is bounded in } L^{q+1}(0,T;L^{q+1}(\Gamma_1)), \tag{2.76}$$
$$\{h(u'_m)\} \text{ is bounded in } L^{\frac{q+1}{q}}(0,T;L^{\frac{q+1}{q}}(\Gamma_1)). \tag{2.77}$$

Next, we will prove that

$$f_{1,m}(u_m) \to f_1(u) \text{ in } L^2(0,T;L^2(\Omega)), \tag{2.78}$$
$$f_{2,m}(u_m) \to f_2(u) \text{ in } L^2(0,T;L^2(\Gamma_1)). \tag{2.79}$$

Following the similar arguments as in [7], we obtain the proof. Indeed, we have

$$\int_0^T\int_\Omega |f_{1,m}(u_m) - f_1(u)|^2\,\mathrm{d}x\mathrm{d}s$$
$$\le 2\int_0^T\int_\Omega |f_{1,m}(u_m) - f_1(u_m)|^2\,\mathrm{d}x\mathrm{d}s + 2\int_0^T\int_\Omega |f_1(u_m) - f_1(u)|^2\,\mathrm{d}x\mathrm{d}s.$$

Observe that from the definition of truncated sequence given by (2.16) and making use of the Dominated Convergence Theorem, it follows that

$$\int_0^T\int_\Omega |f_1(u_m) - f_1(u)|^2\,\mathrm{d}x\mathrm{d}s \to 0 \text{ as } m \to \infty.$$

So, it remains to show that

$$\int_0^T\int_\Omega |f_{1,m}(u_m) - f_1(u_m)|^2\,\mathrm{d}x\mathrm{d}s \to 0 \text{ as } m \to \infty. \tag{2.80}$$

In fact, from the definition of the truncated sequence given by (2.16), we can write

$$\int_0^T\int_\Omega |f_{1,m}(u_m) - f_1(u_m)|^2\,\mathrm{d}x\mathrm{d}s$$
$$\le 2\left(\int_0^T\int_{\Omega_m} |f_1(u_m)|^2\,\mathrm{d}x\mathrm{d}s + \int_0^T\int_{\Omega_m}\left(|f_1(m)|^2 + |f_1(-m)|^2\right)\mathrm{d}x\mathrm{d}s\right), \tag{2.81}$$

where $\Omega_m = \{x \in \Omega; |u_m(x)| > m\}$. Then, by the embedding $H^1_{\Gamma_0} \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and (2.67), we have for $n > 2$

$$\left(\int_{\Omega_m} m^{\frac{2n}{n-2}} \mathrm{d}x\right)^{\frac{n-2}{2n}} \leq \left(\int_{\Omega_m} |u_m|^{\frac{2n}{n-2}} \, \mathrm{d}x\right)^{\frac{n-2}{2n}} \leq \frac{c_* \lambda_0}{l^{\frac{1}{2}}}.$$

Therefore,

$$\operatorname{meas}(\Omega_m) \leq \frac{c_* \lambda_0}{l^{\frac{1}{2}}} m^{\frac{-2n}{n-2}}. \tag{2.82}$$

Analogously, for $n = 2$, the above inequality remains valid with any exponent for $m$. From the fact that $p < \frac{n}{n-2}$ and (2.82), we infer

$$\int_{\Omega_m} |f_1(u_m)|^2 \, \mathrm{d}x \leq \left(\int_{\Omega_m} |u_m|^{\frac{2n}{n-2}} \, \mathrm{d}x\right)^{\frac{p(n-2)}{n}} (\operatorname{meas}(\Omega_m))^{\frac{n-p(n-2)}{n}} \to 0, \tag{2.83}$$

as $m \to \infty$. Also, we have

$$\int_{\Omega_m} |f_1(m)|^2 \, \mathrm{d}x = \operatorname{meas}(\Omega_m) m^{2p} \leq \frac{c_* \lambda_0}{l^{\frac{1}{2}}} m^{2p - \frac{2n}{n-2}} \to 0, \tag{2.84}$$

as $m \to \infty$. Combining these results in (2.81), (2.83) and (2.84) we obtain (2.80) which proved the desired result (2.78). Similarly, as in [14], we have the result (2.79).

Taking the above estimates into account, there exists a sequence, which we still denote by $\{u_m\}$, such that

$$u_m \rightharpoonup u \text{ weak star in } L^\infty(0, T; H^1_{\Gamma_0}), \tag{2.85}$$

$$u'_m \rightharpoonup u' \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \tag{2.86}$$

$$u'_m \rightharpoonup u' \text{ weakly in } L^{q+1}(0, T; L^{q+1}(\Gamma_1)), \tag{2.87}$$

$$h(u'_m) \rightharpoonup \chi \text{ weakly in } L^{\frac{q+1}{q}}(0, T; L^{\frac{q+1}{q}}(\Gamma_1)), \tag{2.88}$$

for some $\chi \in L^{\frac{q+1}{q}}(0, T; L^{\frac{q+1}{q}}(\Gamma_1))$. Defining $z_{m,l} = u_m - u_l, \ m, l \in N$, from (2.66), it holds that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\left\|z'_{m,l}(t)\right\|_2^2 + \|\nabla z_{m,l}(t)\|_2^2\right) + \frac{1}{m} \|u'_m(t)\|^2_{2,\Gamma_1} - \frac{1}{m} \int_{\Gamma_1} u'_m u'_l \mathrm{d}\Gamma \\
&\quad - \frac{1}{l} \int_{\Gamma_1} u'_m u'_l \mathrm{d}\Gamma + \frac{1}{l} \|u'_l(t)\|^2_{2,\Gamma_1} + \int_{\Gamma_1} (h(u'_m) - h(u'_l))(u'_m - u'_l) \, \mathrm{d}\Gamma \\
&\leq \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^t g(t-\tau) \int_\Omega \nabla z_{m,l}(\tau) \cdot \nabla z_{m,l}(t) \mathrm{d}x \mathrm{d}\tau\right) - g(0) \|\nabla z_{m,l}(t)\|_2^2 \\
&\quad - \int_0^t g'(t-\tau) \int_\Omega \nabla z_{m,l}(\tau) \cdot \nabla z_{m,l}(t) \mathrm{d}x \mathrm{d}\tau + \int_\Omega (f_{1,m}(u_m) - f_{1,l}(u_l)) \, z'_{m,l}(t) \mathrm{d}x \\
&\quad + \int_{\Gamma_1} (f_{2,m}(u_m) - f_{2,l}(u_l)) \, z'_{m,l}(t) \mathrm{d}\Gamma.
\end{aligned}$$

Employing Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|z_{m,l}'(t)\right\|_2^2+\|\nabla z_{m,l}(t)\|_2^2\right)+\int\limits_{\Gamma_1}\left(h(u_m')-h(u_l')\right)\left(u_m'-u_l'\right)\mathrm{d}\Gamma\\&\le\left(\frac{1}{m}+\frac{1}{l}\right)\int\limits_{\Gamma_1}\left(\left|u_m'(t)\right|_2^2+\left|u_l'(t)\right|_2^2\right)\mathrm{d}\Gamma\\&+\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_0^t g(t-\tau)\int\limits_\Omega\nabla z_{m,l}(\tau)\cdot\nabla z_{m,l}(t)\mathrm{d}x\mathrm{d}\tau\right)\\&-g(0)\left\|\nabla z_{m,l}(t)\right\|_2^2-\int\limits_0^t g'(t-\tau)\int\limits_\Omega\nabla z_{m,l}(\tau)\cdot\nabla z_{m,l}(t)\mathrm{d}x\mathrm{d}\tau\\&+\int\limits_\Omega\left(f_{1,m}(u_m)-f_{1,l}(u_l)\right)z_{m,l}'(t)\mathrm{d}x+\int\limits_{\Gamma_1}\left(f_{2,m}(u_m)-f_{2,l}(u_l)\right)z_{m,l}'(t)\mathrm{d}\Gamma.\end{aligned}\tag{2.89}$$

Integrating (2.89) over $(0,t)$ and applying the similar estimate as in deriving (2.35), we obtain

$$\begin{aligned}&\frac{1}{2}\left(\left\|z_{m,l}'(t)\right\|_2^2+\|\nabla z_{m,l}(t)\|_2^2\right)+\int\limits_0^t\int\limits_{\Gamma_1}\left(h(u_m')-h(u_l')\right)\left(u_m'-u_l'\right)\mathrm{d}\Gamma\mathrm{d}s\\&\le\left\|u_m^1-u_l^1\right\|_2^2+\left\|\nabla u_m^0-\nabla u_l^0\right\|_2^2+\left(\frac{1}{m}+\frac{1}{l}\right)\int\limits_0^t\int\limits_{\Gamma_1}\left(\left|u_m'(t)\right|_2^2+\left|u_l'(t)\right|_2^2\right)\mathrm{d}\Gamma\mathrm{d}s\\&+c_7\int\limits_0^t\int\limits_\Omega\left|\nabla u_m-\nabla u_l\right|^2\mathrm{d}x\mathrm{d}s+\int\limits_0^t\int\limits_\Omega\left(f_{1,m}(u_m)-f_{1,l}(u_l)\right)z_{m,l}'(t)\mathrm{d}x\mathrm{d}s\\&+\int\limits_0^t\int\limits_{\Gamma_1}\left(f_{2,m}(u_m)-f_{2,l}(u_l)\right)z_{m,l}'(t)\mathrm{d}\Gamma\mathrm{d}s,\end{aligned}\tag{2.90}$$

where $c_7$ is some positive constant.

Convergences (2.65), (2.78), (2.79) and (2.85) imply the convergence to zero (when $m,l\to\infty$) of the terms on the right-hand side of (2.90). Therefore, we deduce that

$$u_m\to u\text{ in }C^0(0,T;H_{\Gamma_0}^1)\cap C^1(0,T;L^2(\Omega))\tag{2.91}$$

and

$$\lim_{m,l\to\infty}\int\limits_0^t\int\limits_{\Gamma_1}\left(h(u_m')-h(u_l')\right)\left(u_m'-u_l'\right)\mathrm{d}\Gamma\mathrm{d}s=0.\tag{2.92}$$

From (2.87), (2.88) and (2.92), we also obtain

$$\lim_{m\to\infty}\int\limits_0^t\int\limits_{\Gamma_1}\left(h(u_m')u_m'-h(u_l')u'-\chi u_m'\right)\mathrm{d}\Gamma\mathrm{d}s+\lim_{l\to\infty}\int\limits_0^t\int\limits_{\Gamma_1}h(u_l')u_l'\mathrm{d}\Gamma\mathrm{d}s=0.$$

Consequently, again using (2.87), (2.88) and changing $m$ to $l$, we see that

$$2\lim_{m\to\infty}\int_0^t\int_{\Gamma_1} h(u'_m)u'_m \mathrm{d}\Gamma\mathrm{d}s = 2\int_0^t\int_{\Gamma_1}\chi u' \mathrm{d}\Gamma\mathrm{d}s. \tag{2.93}$$

By (2.92) combined with (2.87), (2.88) and the monotonicity of $h$, we get that $\chi = h(u')$.

The above convergences (2.78), (2.79), (2.85)–(2.88) allow us to pass the limit in (2.64) in order to obtain

$$\begin{gathered} u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = |u|^{p-1}u, \text{ in } D'(\Omega\times(0,T)), \\ u = 0, \text{ on } \Gamma_0\times(0,\infty), \\ \frac{\partial u}{\partial\nu} - \int_0^t g(t-s)\frac{\partial}{\partial\nu}u(s)\mathrm{d}s + h(u_t) = |u|^{k-1}u, \text{ in } L^{\frac{q+1}{q}}(0,T;L^{\frac{q+1}{q}}(\Gamma_1)) \\ u(0) = u_0, u_t(0) = u_1, \ x\in\Omega. \end{gathered}$$

Finally, for the last assertion in the theorem, we first derive the energy identity for the approximate solutions $u_m$,

$$\begin{aligned} E(t) &+ \frac{1}{m}\int_0^t\int_{\Gamma_1} |u'_m|^2 \mathrm{d}\Gamma\mathrm{d}s + \int_0^t\int_{\Gamma_1} u'_m h(u'_m)\mathrm{d}\Gamma\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}(g'\circ\nabla u_m)(s)\mathrm{d}s + \frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|_2^2\mathrm{d}s = E(0), \end{aligned}$$

and then, due to the monotonicity of $h$ and (2.91)–(2.92), we pass to the limit. Therefore, we complete the proof. □

## 3. Uniform decay

In this section, we prove decay rate estimates for regular solutions of the following problem

$$\begin{gathered} u''(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = |u|^{p-1}u, \text{ in } \Omega\times(0,\infty), \\ u = 0, \text{ on } \Gamma_0\times(0,\infty), \\ \frac{\partial u}{\partial\nu} - \int_0^t g(t-s)\frac{\partial}{\partial\nu}u(s)\mathrm{d}s + \alpha u' + h(u') = |u|^{k-1}u, \text{ on } \Gamma_1\times(0,\infty), \\ u(0) = u^0, u_t(0) = u^1, \ x\in\Omega, \end{gathered} \tag{3.1}$$

where $\alpha > 0$ is a positive constant. Further, we observe that the same result remains true when one considers truncated Lipschitz functions instead of $f_1(s) = |s|^{p-1}s$ and $f_2(s) = |s|^{k-1}s$. Based on this fact and considering the density arguments used in Sect. 2, we also can extend our result to weak solutions of problem (1.1). We consider $h$ satisfies (2.5) with $q = 1$ i.e.,

$$\alpha_1|s| \le |h(s)| \le \alpha_2|s| \text{ for all } |s| \ge 1. \tag{3.2}$$

Adopting the proof of [27], we still have the following result.

**Lemma 3.1.** *Let $u$ be the solution of (3.1), then, under assumptions (A1)–(A2), $E(t)$ is a nonincreasing function on $[0,T)$ and*

$$E'(t) = -\alpha\|u_t\|_{2,\Gamma_1}^2 - \int_{\Gamma_1} u_t h(u_t)d\Gamma + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0. \tag{3.3}$$

**Theorem 3.2.** *Let $u^0 \in H^1_{\Gamma_0} \cap H^2(\Omega), u^1 \in H^1_{\Gamma_0}$ and (A1)–(A2) and (1.2) hold. Assume further that $l\left\|\nabla u^0\right\|_2^2 < \lambda_0^2$ and $E(0) < d$, then the problem (3.1) admits a global solution.*

*Proof.* It follows from (2.14) and (2.11) that

$$\begin{aligned}\frac{1}{2}\|u_t\|_2^2 + c_0\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right) &\le \frac{1}{2}\|u_t\|_2^2 + F\left(\sqrt{l\|\nabla u\|_2^2 + (g\circ\nabla u)(t)}\right)\\ &\le \frac{1}{2}\|u_t\|_2^2 + J(u(t))\\ &= E(t) < E(0) < d.\end{aligned}\tag{3.4}$$

Thus, we establish the boundedness of $u_t$ in $L^2(\Omega)$ and the boundedness of $u$ in $H^1_{\Gamma_0}$. Moreover, from (2.1) and (2.2), we also obtain

$$\begin{aligned}\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1} &\le c_*^{p+1}\|\nabla u\|_2^{p+1} + B_*^{k+1}\|\nabla u\|_2^{k+1}\\ &\le \frac{1}{l}\left(c_*^{p+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{p-1}{2}} + B_*^{k+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{k-1}{2}}\right) l\|\nabla u\|_2^2\\ &\le L\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right),\end{aligned}$$

which implies that the boundedness of $u$ in $L^{p+1}(\Omega)$ and in $L^{k+1}(\Gamma_1)$ with $L = \frac{1}{l}\left(c_*^{p+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{p-1}{2}} + B_*^{k+1}\left(\frac{E(0)}{lc_0}\right)^{\frac{k-1}{2}}\right)$. Hence, it must have $T = \infty$. □

Now, we shall investigate the asymptotic behavior of the energy function $E(t)$. First, we define some functionals and establish several lemmas. Let

$$G(t) = ME(t) + \varepsilon\Phi(t) + \Psi(t),\tag{3.5}$$

where

$$\Phi(t) = \int_\Omega u_t u\,\mathrm{d}x,\tag{3.6}$$

$$\Psi(t) = \int_\Omega u_t \int_0^t g(t-s)\left(u(s) - u(t)\right)\mathrm{d}s\mathrm{d}x,\tag{3.7}$$

and $M$, $\varepsilon$ are some positive constants to be be specified later.

**Lemma 3.3.** *There exist two positive constants $\beta_1$ and $\beta_2$ such that the relation*

$$\beta_1 E(t) \le G(t) \le \beta_2 E(t)\tag{3.8}$$

*holds, for $\varepsilon > 0$ small enough while $M > 0$ is large enough.*

*Proof.* By Hölder's inequality, Young's inequality and (2.1), we deduce that

$$|\Phi(t)| \le \frac{1}{2}\|u_t\|_2^2 + \frac{c_*^2}{2}\|\nabla u\|_2^2,\tag{3.9}$$

and

$$\begin{aligned}|\Psi(t)| &\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)\left(u(t) - u(s)\right)\mathrm{d}s\right)^2\mathrm{d}x\\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{c_*^2(1-l)}{2}\left(g\circ\nabla u\right)(t),\end{aligned}\tag{3.10}$$

where the last inequality is obtained due to $\int_0^t g(s)\mathrm{d}s \leq \int_0^\infty g(s)\mathrm{d}s = 1-l$. Hence, it follows from (3.5),(3.9) and (3.10) that

$$\begin{aligned} G(t) &= ME(t) + \varepsilon\Phi(t) + \Psi(t) \\ &\leq ME(t) + c_1\left\|u_t\right\|^2 + c_2\left\|\nabla u\right\|_2^2 + c_3\left(g\circ\nabla u\right)(t) \end{aligned}$$

and

$$G(t) \geq ME(t) - c_4\left(\|u_t\|_2^2 + \|\nabla u\|_2^2 + \left(g\circ\nabla u\right)(t)\right),$$

where $c_1 = \frac{\varepsilon+1}{2}$, $c_2 = \frac{\varepsilon c_*^2}{2}$, $c_3 = \frac{c_*^2(1-l)}{2}$, and $c_4 = \max(c_1, c_2, c_3)$. Thus, selecting $\varepsilon > 0$ small enough and $M$ sufficiently large, there exist two positive constants $\beta_1$ and $\beta_2$ such that

$$\beta_1 E(t) \leq G(t) \leq \beta_2 E(t).$$

□

**Lemma 3.4.** *Let (A1)–(A2) and (1.2) hold, then, for any $t_0 > 0$, the functional $G(t)$ verifies, along solution of (3.1),*

$$G'(t) \leq -\alpha_1 E(t) + \alpha_2\left(g\circ\nabla u\right)(t) + \alpha_3\int_{\Gamma_1} h^2(u_t)d\Gamma,$$

*where $\alpha_i$, $i = 1, 2, 3$ are some positive constants independent of $\alpha$.*

*Proof.* In the following, we estimate the derivative of $G(t)$. From (3.6) and (3.1), we have

$$\begin{aligned} \Phi'(t) = \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x \\ - \int_{\Gamma_1} h(u_t)u\mathrm{d}\Gamma - \alpha\int_{\Gamma_1} u_t u\mathrm{d}\Gamma + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}. \end{aligned} \tag{3.11}$$

Employing Hölder's inequality, Young's inequality, (2.2) and (2.3), the third and fourth terms on the right-hand side of (3.11) can be estimated as follows, for $\eta$, $\delta > 0$,

$$\begin{aligned} &\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x \\ &\quad\leq \left[\frac{1}{2} + \frac{1}{2}(1+\eta)(1-l)^2\right]\|\nabla u\|_2^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\left(g\circ\nabla u\right)(t) \end{aligned} \tag{3.12}$$

and

$$\left|\int_{\Gamma_1} h(u_t)u\mathrm{d}\Gamma\right| \le \delta \|u\|_{2,\Gamma_1}^2 + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma$$
$$\le \delta B_*^2 \|\nabla u\|_2^2 + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma. \tag{3.13}$$

As for the fifth term, we consider, without loss of generality that $\alpha \le 1$. Thus, for $\delta > 0$, we obtain

$$\left|\alpha\int_{\Gamma_1} u_t u\mathrm{d}\Gamma\right| \le \delta B_*^2 \|\nabla u\|_2^2 + \frac{\alpha}{4\delta}\|u_t\|_{2,\Gamma_1}^2. \tag{3.14}$$

A substitution of (3.12)–(3.14) into (3.11) yields

$$\Phi'(t) \le \|u_t\|_2^2 - \left(\frac{1}{2} - \frac{1}{2}(1+\eta)(1-l)^2 - 2\delta B_*^2\right)\|\nabla u\|_2^2$$
$$+\frac{1}{2}(1+\frac{1}{\eta})(1-l)\left(g\circ\nabla u\right)(t) + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma$$
$$+\frac{\alpha}{4\delta}\|u_t\|_{2,\Gamma_1}^2 + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}.$$

Letting $\eta = \frac{l}{1-l} > 0$ and $\delta = \frac{l}{8B_*^2}$ in above inequality, we obtain

$$\Phi'(t) \le -\frac{l}{4}\|\nabla u\|_2^2 + \|u_t\|_2^2 + \frac{1-l}{2l}\left(g\circ\nabla u\right)(t) + \frac{2B_*^2}{l}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma$$
$$+\frac{2\alpha B_*^2}{l}\|u_t\|_{2,\Gamma_1}^2 + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}. \tag{3.15}$$

Next, we estimate $\Psi'(t)$. Taking the derivative of $\Psi(t)$ in (3.7) and using (3.1), we obtain

$$\Psi'(t) = \int_\Omega \nabla u(t)\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)\mathrm{d}s\mathrm{d}x$$
$$-\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\mathrm{d}s\right)\mathrm{d}x$$
$$+\int_{\Gamma_1} h(u_t)\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}\Gamma + \alpha\int_{\Gamma_1} u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}\Gamma$$
$$-\int_{\Gamma_1} |u|^{k-1}u\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}\Gamma$$
$$-\int_\Omega |u|^{p-1}u\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}x$$
$$-\int_\Omega u_t\int_0^t g'(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2. \tag{3.16}$$

Similar to deriving (3.15), in what follows we will estimate the right-hand side of (3.16). Using Young's inequality, Hölder's inequality, and (2.3), for $\delta > 0$, we have

$$\begin{aligned}
&\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\mathrm{d}s\mathrm{d}x\right| \\
&\quad\le \delta\left\|\nabla u\right\|_2^2+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\mathrm{d}s\right)^2\mathrm{d}x \\
&\quad\le \delta\left\|\nabla u\right\|_2^2+\frac{1-l}{4\delta}\left(g\circ\nabla u\right)(t),
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
&\left|\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\mathrm{d}s\right)\mathrm{d}x\right| \\
&\quad\le 2\delta\left(1-l\right)^2\left\|\nabla u\right\|_2^2+\left(2\delta+\frac{1}{4\delta}\right)(1-l)\left(g\circ\nabla u\right)(t).
\end{aligned}\tag{3.18}$$

Utilizing Hölder's inequality, Young's inequality and (2.2) and noting that $\alpha \le 1$, the third term and fourth term on the right-hand side of (3.16) can be estimated as

$$\begin{aligned}
&\left|\int_{\Gamma_1} h(u_t)\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}\Gamma\right| \\
&\quad\le \frac{1}{2}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma+\frac{(1-l)B_*^2}{2}\left(g\circ\nabla u\right)(t),
\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}
&\left|\alpha\int_{\Gamma_1} u_t\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}\Gamma\right| \\
&\quad\le \frac{\alpha}{2}\left\|u_t\right\|_{2,\Gamma_1}^2+\frac{(1-l)B_*^2}{2}\left(g\circ\nabla u\right)(t)
\end{aligned}\tag{3.20}$$

As for the fifth and sixth terms on the right-hand side of (3.16), using Hölder's inequality, Young's inequality, (2.1)–(2.3) and (3.4), we obtain

$$\begin{aligned}
&\int_{\Gamma_1}\left|u\right|^{k-1}u\int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}\Gamma \\
&\quad\le \delta\left\|u\right\|_{2k,\Gamma_1}^{2k}+\frac{(1-l)B_*^2}{4\delta}\left(g\circ\nabla u\right)(t) \\
&\quad\le \delta B_*^{2k}\left(\frac{E(0)}{lc_0}\right)^{k-1}\left\|\nabla u\right\|_2^2+\frac{(1-l)B_*^2}{4\delta}\left(g\circ\nabla u\right)(t),
\end{aligned}\tag{3.21}$$

and

$$\left|\int_\Omega |u|^{p-1} u \int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}x\right|$$
$$\le \delta \|u\|_{2p}^{2p} + \frac{(1-l)c_*^2}{4\delta}\left(g\circ\nabla u\right)(t)$$
$$\le \delta c_*^{2p}\left(\frac{E(0)}{lc_0}\right)^{p-1}\|\nabla u\|_2^2 + \frac{(1-l)c_*^2}{4\delta}\left(g\circ\nabla u\right)(t). \tag{3.22}$$

Exploiting Hölder's inequality, Young's inequality and (A1) to estimate the seventh term, we have

$$\left|\int_\Omega u_t \int_0^t g'(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}x\right|$$
$$\le \delta\|u_t\|_2^2 - \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t). \tag{3.23}$$

Then, combining these estimates (3.17)–(3.23), (3.16) becomes

$$\Psi'(t) \le -\left(\int_0^t g(s)\mathrm{d}s - \delta\right)\|u_t\|_2^2 + c_5\delta\|\nabla u\|_2^2 + c_6\left(g\circ\nabla u\right)(t)$$
$$+\frac{\alpha}{2}\|u_t\|_{2,\Gamma_1}^2 - \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t) + \frac{1}{2}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma, \tag{3.24}$$

where $c_5 = 1+2(1-l)^2 + c_*^{2p}\left(\frac{E(0)}{lc_0}\right)^{p-1} + B_*^{2k}\left(\frac{E(0)}{lc_0}\right)^{k-1}$ and $c_6 = (1-l)\left(\frac{1}{2\delta}+2\delta+B_*^2+\frac{c_*^2+B_*^2}{4\delta}\right)$. Hence, we conclude from (3.5), (3.3), (3.15), and (3.24) that

$$G'(t) = ME'(t) + \varepsilon\Phi'(t) + \Psi'(t)$$
$$\le -\left(\frac{M}{2} - \frac{g(0)c_*^2}{4\delta}\right)\left(-g'\circ\nabla u\right)(t) - \left(g_0-\delta-\varepsilon\right)\|u_t\|_2^2 - \alpha\left(M - \frac{2B_*^2}{l} - \frac{1}{2}\right)\|u_t\|_{2,\Gamma_1}^2$$
$$+\left(c_5\delta - \frac{\varepsilon l}{4}\right)\|\nabla u\|_2^2 + \left(c_6 + \frac{(1-l)\varepsilon}{2l}\right)\left(g\circ\nabla u\right)(t)$$
$$+\left(\frac{1}{2}+\frac{c_*^2\varepsilon}{l}\right)\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma + \varepsilon\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right),$$

where we have used the fact that for any $t_0 > 0$,

$$\int_0^t g(s)\mathrm{d}s \ge \int_0^{t_0} g(s)\mathrm{d}s = g_0, \quad \forall\, t \ge t_0,$$

because $g$ is positive and continuous with $g(0) > 0$. At this point, we choose $\varepsilon > 0$ small enough so that Lemma 3.3 holds and $\varepsilon < \frac{g_0}{2}$. Once $\varepsilon$ is fixed, we choose $\delta$ to satisfy

$$\delta < \min\left\{\frac{\varepsilon l}{8c_5}, \frac{g_0}{4}\right\},$$

and then pick $M$ sufficiently large such that

$$M > \max\left\{\frac{g(0)c_*^2}{2\delta}, \frac{2B_*^2}{l} + \frac{1}{2}\right\}.$$

Hence, for all $t \geq t_0$, we arrive at

$$G'(t) \leq -\frac{\varepsilon l}{8}\|\nabla u\|_2^2 - \frac{g_0}{4}\|u_t\|_2^2 + c_7 (g \circ \nabla u)(t) + c_8 \int_{\Gamma_1} h^2(u_t) d\Gamma$$
$$+\varepsilon \left( \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1} \right),$$

which yields

$$G'(t) \leq -\alpha_1 E(t) + \alpha_2 (g \circ \nabla u)(t) + \alpha_3 \int_{\Gamma_1} h^2(u_t) d\Gamma, \tag{3.25}$$

or

$$E(t) \leq -\beta_3 G'(t) + \beta_4 (g \circ \nabla u)(t) + \beta_5 \int_{\Gamma_1} h^2(u_t) d\Gamma,$$

where $c_i,\ i = 7, 8,\ \alpha_j,\ j = 1, 2, 3$ and $\beta_k,\ k = 3, 4, 5$ are all positive constants independent of $\alpha$. □

Before stating our main result, we need to recall that if $\phi$ is a proper convex function from $R$ to $R \cup \{\infty\}$, then its convex conjugate $\phi^*$ is defined as

$$\phi^*(y) = \sup_{x \in R} \{xy - \phi(x)\}. \tag{3.26}$$

Now, we are ready to prove our main results by adopting and modifying the arguments in [13,14]. We consider the following partition of $\Gamma_1$

$$\Gamma_1^+ = \{x \in \Gamma_1 \,|\, |u_t| > 1\}, \Gamma_1^- = \{x \in \Gamma_1 \,|\, |u_t| \leq 1\}.$$

**Theorem 3.5.** *Let $u^0 \in H^1_{\Gamma_0} \cap H^2(\Omega), u^1 \in H^1_{\Gamma_0}$ be given and $(A1) - (A2)$ and (1.2) hold. Suppose further that $l \left\|\nabla u^0\right\|_2^2 < \lambda_0^2$ and $E(0) < d$. Then, for each $t_0 > 0$ and $k_1,\ k_2$ and $\varepsilon_0$ are positive constants, the solution energy of (3.1) satisfies*

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_0^t \xi(s) ds\right),\ t \geq t_0, \tag{3.27}$$

*where*

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds \tag{3.28}$$

*and*

$$H_2(t) = \begin{cases} t, & \text{if } H \text{ is linear on } [0,1], \\ tH'(\varepsilon_0 t), & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on} (0,1]. \end{cases} \tag{3.29}$$

*Proof.* Let $u^0 \in H^1_{\Gamma_0} \cap H^2(\Omega), u^1 \in H^1_{\Gamma_0}$ such that $l \left\|\nabla u^0\right\|_2^2 < \lambda_0^2$ and $E(0) < d$, then the global existence of solution $u$ of (3.1) is guaranteed directly by Theorem 3.2. Next, we consider the following two cases: (i) $H$ is linear on $[0, 1]$ and (ii) $H'(0) = 0$ and $H'' > 0$ on $(0, 1]$.

Case 1: $H$ is linear on $[0, 1]$. In this case, there exists $\alpha_1' > 0$ such that $|h(s)| \leq \alpha_1' |s|$, for all $s \in R$. By (3.3), we have

$$\int_{\Gamma_1} h^2(u_t) d\Gamma \leq \alpha_1' \int_{\Gamma_1} u_t h(u_t) d\Gamma \leq -\alpha_1' E'(t),$$

which together with (3.25) implies that

$$(G(t)+c_9E(t))' \le -\alpha_1 H_2(E(t)) + \alpha_2 (g\circ\nabla u)(t), \tag{3.30}$$

where $H_2(s)=s$ and $c_9=\alpha_1'\alpha_3$.

Case 2: $H'(0)=0$ and $H''>0$ on $(0,1]$.

In this case, we first estimate $\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma$ on the right-hand side of (3.25). Given (3.2), noting that $H^{-1}$ is concave and increasing and using Jensen's inequality and (3.3), we deduce that

$$\begin{aligned}\int_{\Gamma_1} h^2(u_t)\mathrm{d}\Gamma &= \int_{\Gamma_1^+} h^2(u_t)\mathrm{d}\Gamma + \int_{\Gamma_1^-} h^2(u_t)\mathrm{d}\Gamma\\ &\le M_q\int_{\Gamma_1^+} u_t h(u_t)\mathrm{d}\Gamma + \int_{\Gamma_1^-} h^2(u_t)\mathrm{d}\Gamma\\ &\le -M_qE'(t) + \int_{\Gamma_1^-} H^{-1}(u_t h(u_t))\,\mathrm{d}\Gamma\\ &\le -M_qE'(t) + \frac{1}{c_{10}}H^{-1}\left(c_{10}\int_{\Gamma_1^-} u_t h(u_t)\mathrm{d}\Gamma\right)\\ &\le -M_qE'(t) + \frac{1}{c_{10}}H^{-1}\left(-c_{10}E'(t)\right),\end{aligned}$$

where $c_{10}=\frac{1}{vol(\Gamma_1^-)}$. Hence, (3.25) becomes

$$F_1(t)' \le -\alpha_1 E(t) + c_{11}H^{-1}\left(-c_{10}E'(t)\right) + \alpha_2 (g\circ\nabla u)(t),\ \forall t\ge t_0, \tag{3.31}$$

where $c_{11}=\frac{\alpha_3}{c_{10}}$ and

$$F_1(t) = G(t) + M_q\alpha_3 E(t). \tag{3.32}$$

Now, we define

$$F_2(t) = H'(\varepsilon_0 E(t))F_1(t) + \beta E(t), \tag{3.33}$$

where $\varepsilon_0>0$ and $\beta>0$ to be determined later. Then, using $E'(t)\le 0,\ H''(t)\ge 0$, and (3.31), we obtain

$$\begin{aligned}F_2'(t) &= \varepsilon_0 E'(t)H''(\varepsilon_0E(t))F_1(t) + H'(\varepsilon_0E(t))F_1'(t) + \beta E'(t)\\ &\le -\alpha_1 H'(\varepsilon_0E(t))E(t) + \alpha_2 H'(\varepsilon_0E(t))\,(g\circ\nabla u)(t)\\ &\quad + c_{11}H'(\varepsilon_0E(t))H^{-1}\left(-c_{10}E'(t)\right) + \beta E'(t).\end{aligned} \tag{3.34}$$

Let $H^*$ denote the Legendre transform of $H$ defined by (3.26), then (see [2])

$$H^*(s) = s\left(H'\right)^{-1}(s) - H\left[\left(H'\right)^{-1}(s)\right],\ \text{if } s\in R^+ \tag{3.35}$$

and $H^*$ satisfies the following inequality

$$AB \le H^*(A) + H(B),\ \text{for } A,\ B\ge 0. \tag{3.36}$$

Further, using (3.35) and noting that $H'(0)=0,\ \left(H'\right)^{-1}$ is increasing and $H$ is also increasing yield

$$H^*(s) \le s\left(H'\right)^{-1}(s),\ s\ge 0. \tag{3.37}$$

Taking $H'(\varepsilon_0 E(t)) = A$ and $H^{-1}(-c_{10}E'(t)) = B$ in (3.34), applying (3.37) and (3.36), noting that $0 \le H'(\varepsilon_0 E(t)) \le H'(\varepsilon_0 E(0))$ due to $H'$ is increasing, we obtain

$$\begin{aligned} F_2'(t) &\le -\alpha_1 H'(\varepsilon_0 E(t))E(t) + c_{11}H^*\left(H'(\varepsilon_0 E(t))\right) + c_{13}\left(g \circ \nabla u\right)(t) + \left(\beta - c_{12}\right)E'(t) \\ &\le -\left(\alpha_1 - c_{11}\varepsilon_0\right)H'(\varepsilon_0 E(t))E(t) + c_{13}\left(g \circ \nabla u\right)(t) + \left(\beta - c_{12}\right)E'(t). \end{aligned}$$

Thus, choosing $0 < \varepsilon_0 < \frac{\alpha_1}{c_{11}}$, $\beta > c_{12}$ and using $E'(t) \le 0$ by (3.3), we obtain

$$\begin{aligned} F_2'(t) &\le -c_{14}H'(\varepsilon_0 E(t))E(t) + c_{13}\left(g \circ \nabla u\right)(t) \\ &= -c_{14}H_2(E(t)) + c_{13}\left(g \circ \nabla u\right)(t), \end{aligned} \tag{3.38}$$

where $H_2(s) = sH'(\varepsilon_0 s)$, $c_{12} = c_{10}c_{11}$ , $c_{13} = \alpha_2 \cdot H'(\varepsilon_0 E(0)) > 0$ and $c_{14}$ is a positive constant.

Let

$$G_1(t) = \begin{cases} G(t) + c_9 E(t), & \text{if } H \text{ is linear on } [0,1], \\ F_2(t), & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0,1]. \end{cases}$$

Then, by Lemma 3.3 and the definition of $F_2$ by (3.32)–(3.33), there exist $\beta_1',\ \beta_2' > 0$ such that

$$\beta_2' E(t) \le G_1(t) \le \beta_1' E(t) \tag{3.39}$$

and from (3.30) and (3.38), we have

$$G_1'(t) \le -c_{15}H_2(E(t)) + c_{16}\left(g \circ \nabla u\right)(t),\ t \ge t_0, \tag{3.40}$$

where $c_{15}$ and $c_{16}$ denote some positive constants. Additionally, using (3.39) and $\xi(t) \le \xi(0)$ by (A1), we see that

$$\xi(t)G_1(t) + 2c_{16}E(t) \le l_1 E(t),\ t \ge t_0, \tag{3.41}$$

where $l_1 = \beta_1'\xi(0) + 2c_{16} > 0$. Now, we define

$$F_3(t) = \varepsilon\left(\xi(t)G_1(t) + 2c_{16}E(t)\right),\ 0 < \varepsilon < \frac{1}{l_1}, \tag{3.42}$$

which is equivalent to $E(t)$ by (3.39). Using (3.40), (2.4) and (3.3), we arrive at

$$\begin{aligned} F_3'(t) &= \varepsilon\left(\xi'(t)G_1(t) + \xi(t)G_1'(t) + 2c_{16}E'(t)\right) \\ &\le -c_{15}\varepsilon\xi(t)H_2(E(t)) + c_{16}\varepsilon\xi(t)\left(g \circ \nabla u\right)(t) + 2c_{16}\varepsilon E'(t) \\ &\le -c_{15}\varepsilon\xi(t)H_2(E(t)) - c_{16}\varepsilon\left(g' \circ \nabla u\right)(t) + 2c_{16}\varepsilon E'(t) \\ &\le -c_{15}\varepsilon\xi(t)H_2(E(t)). \end{aligned}$$

Exploiting the fact that $H_2$ is increasing, using (3.41) and noting $0 < \varepsilon < \frac{1}{l_1}$ by (3.42), we obtain

$$\begin{aligned} F_3'(t) &\le -c_{15}\varepsilon\xi(t)H_2\left(\frac{1}{l_1}\left(\xi(t)G_1(t) + 2c_{16}E(t)\right)\right) \\ &\le -c_{15}\varepsilon\xi(t)H_2\left(\varepsilon\left(\xi(t)G_1(t) + 2c_{16}E(t)\right)\right) \\ &= -c_{15}\varepsilon\xi(t)H_2\left(F_3(t)\right). \end{aligned}$$

Given that $H_1'(t) = -\frac{1}{H_2(t)}$ by (3.28), we have

$$F_3'(t)H_1'(F_3(t)) \ge c_{15}\varepsilon\xi(t),\ t \ge t_0.$$

Integrating this over $(t_0, t)$ and noting that $H_1^{-1}$ is decreasing on (0,1], we deduce that

$$F_3(t) \le H_1^{-1}\left(H_1(F_3(t_0)) + c_{15}\varepsilon \int_0^t \xi(s)\mathrm{d}s - c_{15}\varepsilon \int_0^{t_0} \xi(s)\mathrm{d}s\right)$$

$$\le H_1^{-1}\left(c_{15}\varepsilon \int_0^t \xi(s)\mathrm{d}s\right),$$

where we require $\varepsilon > 0$ sufficiently small so that $H_1(F_3(t_0)) - c_{15}\varepsilon \int_0^{t_0} \xi(s)\mathrm{d}s > 0$. Consequently, the equivalent relation between of $F_3$ and $E$ yields

$$E(t) \le k_2 H_1^{-1}\left(k_1 \int_0^t \xi(s)\mathrm{d}s\right), \ t \ge t_0,$$

where $k_1$ and $k_2$ are positive constants. Hence, we complete the proof. □

**Remark 3.6.** From the definition of $H_2$ by (3.29), we deduce that $\lim_{t\to 0} H_1(t) = \infty$. Thus, if $\int_0^\infty \xi(t)\mathrm{d}t = \infty$, we have the strong stability of (3.1): that is,

$$\lim_{t\to\infty} E(t) = 0.$$

## 4. Blow-up

In this section, we investigate the blow-up properties for problem (1.1). To state our results, we make extra assumptions on $g$ and $h$:

(A3) $h$ is monotone, continuous and satisfies

$$m_q |s|^{q+1} \le h(s)s \le M_q |s|^{q+1}, \text{ for all } s \in R, \tag{4.1}$$

where $m_q$ and $M_q$ are positive constants with $q > \frac{k}{r-k}$ and $r = \frac{2(n-1)}{n-2}$.

(B3)

$$\int_0^\infty g(s)\mathrm{d}s < \frac{1}{1 + \frac{1}{\left(\theta(1-\alpha)^2 + 2\alpha(1-\alpha)\right)(\theta-2)}}, \tag{4.2}$$

where $0 < \alpha < 1$ is a fixed number and $\theta$ is a positive constant given in (4.11).

To prove our result, the following lemma is needed.

**Lemma 4.1.** *Suppose that (A1) and (1.2) hold and assume further that* $l\|\nabla u_0\|_2^2 > \lambda_0^2$ *and* $E(0) < \alpha d$, *then there exists* $\lambda_2 > \lambda_0$ *such that, for all* $t \in [0, T)$,

$$l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \ge \lambda_2^2 \tag{4.3}$$

*and*

$$\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1} \ge \frac{1}{c_0'}\left(\frac{B_\Omega^{p+1}}{p+1}\lambda_2^{p+1} + \frac{B_\Gamma^{k+1}}{k+1}\lambda_2^{k+1}\right), \tag{4.4}$$

*where* $c_0' = \max(\frac{1}{p+1}, \frac{1}{k+1})$.

*Proof.* Considering that $E(t)$ is nonincreasing by (2.64), we get

$$E(t) \leq E(0) < \alpha d,$$

for all $t > 0$. Applying arguments similar to those used in deriving (2.22), we can obtain (4.3). Indeed, from the properties of $F$ given in Remark 2.2 (i), there exist $\lambda_2' < \lambda_0 < \lambda_2$ such that $F(\lambda_2') = F(\lambda_2) = E(0)$. Then, as $F(0) = 0$, $l\left\|\nabla u^0\right\|_2^2 > \lambda_0^2$ and the continuity in time of $F\left(\sqrt{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}\right)$ we deduce that

$$l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \geq \lambda_2^2, \ \forall t \in [0, T).$$

In addition, from the definition of $E(t)$ by (2.6), using (2.3), (4.3) and the definition of $F$ by (2.8), we see that

$$\begin{aligned} \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1} &\geq \frac{1}{c_0'}\left(\frac{1}{2}\left(l\|\nabla u(t)\|^2 + (g \circ \nabla u)(t)\right) - E(0)\right) \\ &\geq \frac{1}{c_0'}\left(\frac{1}{2}\lambda_2^2 - F(\lambda_2)\right) \\ &= \frac{1}{c_0'}\left(\frac{B_\Omega^{p+1}}{p+1}\lambda_2^{p+1} + \frac{B_\Gamma^{k+1}}{k+1}\lambda_2^{k+1}\right). \end{aligned}$$

Hence, we complete the proof. □

Following the approach developed by Cavalcanti et al. in [3], we are ready to state and prove the blow-up result.

**Theorem 4.2.** *Let (A1), (A3), (A4) and (1.2) hold and $k > q$. For any fixed number $0 < \alpha < 1$, suppose that $u^0 \in H_{\Gamma_0}^1(\Omega), u^1 \in L^2(\Omega)$ with $l\left\|\nabla u^0\right\|_2^2 > \lambda_0^2$, $E(0) < \alpha d$. Assume that either one of the following conditions holds:*

*(i)* $E(0) < 0$,

*(ii)* $E(0) \geq 0$, $E(0) < \frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$ *if $p > k$ or* $E(0) < \frac{\alpha\lambda_0^2(p-1)}{2(p+1)}$ *if $k > p$,*

*(iii)* $E(0) \geq 0$, $E(0) \geq \frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$ *if $p > k$ and the difference $p - k$ is small enough or* $E(0) \geq \frac{\alpha\lambda_0^2(p-1)}{2(p+1)}$ *if $k > p$ and the difference $k - p$ is small enough.*

*Then, the solution $u$ of (1.1) blows up in finite time, i.e., there exists $T^* < \infty$ such that*

$$\lim_{T \to T^{*-}} \left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right) = \infty. \tag{4.5}$$

*Proof.* By contradiction, we suppose that the solution of problem (1.1) is global. Set

$$H(t) = E_2 - E(t), \tag{4.6}$$

where

$$E(0) < E_2 < \alpha d. \tag{4.7}$$

By (2.64), we see that $H'(t) \geq 0$. Thus, we obtain

$$H(t) \geq H(0) = E_2 - E(0) > 0. \tag{4.8}$$

Define

$$Z(t) = H^{1-\gamma}(t) + \varepsilon \int_\Omega u_t u \mathrm{d}x, \tag{4.9}$$

where $0<\varepsilon<1$ to be determined later and

$$0<\gamma<\min\left\{\frac{1}{q+1}-\frac{1}{k+1},\frac{1}{2}-\frac{1}{p+1}\right\}.$$

Next, we prove that there exist positive constants $c_i,\ i=1-5$ such that the following inequality holds:

$$Z'(t)\geq(1-\gamma)H^{-\gamma}(t)H'(t)+c_1\varepsilon\int\limits_{\Omega}u_t^2\mathrm{d}x+c_2\varepsilon\left(l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)\right)$$
$$+c_3\varepsilon\left\|u\right\|_{p+1}^{p+1}+c_4\varepsilon\left\|u\right\|_{k+1,\Gamma_1}^{k+1}+c_5\varepsilon H(t)-\varepsilon\int\limits_{\Gamma_1}h\left(u_t\right)u\mathrm{d}\Gamma. \tag{4.10}$$

Taking the derivative of (4.9) and using equation (1.1), we obtain

$$\begin{aligned}Z'(t)&=(1-\gamma)H^{-\gamma}(t)H'(t)+\varepsilon\int\limits_{\Omega}u_t^2\mathrm{d}x+\varepsilon\int\limits_{\Omega}u_{tt}u\mathrm{d}x\\&=(1-\gamma)H^{-\gamma}(t)H'(t)+\varepsilon\int\limits_{\Omega}u_t^2\mathrm{d}x-\varepsilon\left\|\nabla u\right\|_2^2\\&\quad+\varepsilon\left(\left\|u\right\|_{p+1}^{p+1}+\left\|u\right\|_{k+1,\Gamma_1}^{k+1}\right)-\varepsilon\int\limits_{\Gamma_1}h\left(u_t\right)u\mathrm{d}\Gamma\\&\quad+\varepsilon\int\limits_{\Omega}\nabla u(t)\int\limits_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Exploiting Hölder's inequality and Young's inequality, for $\eta>0$,

$$\begin{aligned}&\int\limits_{\Omega}\nabla u(t)\int\limits_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x\\&\quad=\int\limits_{\Omega}\int\limits_0^t g(t-s)\nabla u(t)\cdot\left(\nabla u(s)-\nabla u(t)\right)\mathrm{d}s\mathrm{d}x+\int\limits_0^t g(t-s)\mathrm{d}s\left\|\nabla u(t)\right\|_2^2\\&\quad\geq-\eta(g\circ\nabla u)(t)+\left(1-\frac{1}{4\eta}\right)\int\limits_0^t g(s)\mathrm{d}s\left\|\nabla u(t)\right\|_2^2.\end{aligned}$$

Thus,

$$Z'(t)\geq(1-\gamma)H^{-\gamma}(t)H'(t)+\varepsilon\int\limits_{\Omega}u_t^2\mathrm{d}x+\varepsilon\left(-1-\left(\frac{1}{4\eta}-1\right)\int\limits_0^t g(s)\mathrm{d}s\right)\left\|\nabla u\right\|_2^2$$
$$-\varepsilon\eta(g\circ\nabla u)(t)+\varepsilon\left(\left\|u\right\|_{p+1}^{p+1}+\left\|u\right\|_{k+1,\Gamma}^{k+1}\right)-\varepsilon\int\limits_{\Gamma_1}h\left(u_t\right)u\mathrm{d}\Gamma.$$

Adding the term $\theta\left(H(t)-E_2+E(t)\right)$ and using the definition of $E(t)$ by (2.6), we deduce that

$$\begin{aligned}Z'(t) \geq\ & (1-\gamma)H^{-\gamma}(t)H'(t)+\varepsilon\left(1+\frac{\theta}{2}\right)\int_{\Omega} u_t^2\mathrm{d}x\\&+\varepsilon\left(\frac{\theta}{2}-1-\left(\frac{\theta}{2}+\frac{1}{4\eta}-1\right)\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\\&+\varepsilon\left(\frac{\theta}{2}-\eta\right)(g\circ\nabla u)(t)+\varepsilon\left(1-\frac{\theta}{p+1}\right)\|u\|_{p+1}^{p+1}\\&+\varepsilon\left(1-\frac{\theta}{k+1}\right)\|u\|_{k+1,\Gamma_1}^{k+1}+\varepsilon\theta H-\varepsilon\theta E_2-\varepsilon\int_{\Gamma_1} h\left(u_t\right)u\mathrm{d}\Gamma,\end{aligned}$$

where $\theta$ is considered as follows

$$\begin{array}{ll}\text{If } p>k,\ \text{take } \theta=p+1-\varepsilon_1, & \text{with } 0<p-k<\varepsilon_1<p-1;\\ \text{if } k>p,\ \text{take } \theta=k+1-\varepsilon_1, & \text{with } 0<k-p<\varepsilon_1<k-1.\end{array} \tag{4.11}$$

To obtain (4.10), we take $\eta$ to satisfy

$$\frac{1-l}{2(1-\alpha)l(\theta-2)}<\eta<\frac{\theta(1-\alpha)}{2}+\alpha$$

which is possible because of (4.2). Then, using the fact that $l\left\|\nabla u(t)\right\|_2^2+(g\circ\nabla u)(t)\geq\lambda_2^2$ by (4.3) to get

$$\begin{aligned}&\left(\frac{\theta}{2}-1-\left(\frac{\theta}{2}+\frac{1}{4\eta}-1\right)\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\\&\quad+\left(\frac{\theta}{2}-\eta\right)(g\circ\nabla u)(t)-\theta E_2\\&\geq\alpha\left(\frac{\theta}{2}-1\right)\left(l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)\right)-\theta E_2\\&=\alpha\left(\frac{\theta}{2}-1\right)\frac{\lambda_2^2-\lambda_0^2}{\lambda_2^2}\left(l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)\right)\\&\quad+\alpha\left(\frac{\theta}{2}-1\right)\frac{\lambda_0^2}{\lambda_2^2}\left(l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)\right)-\theta E_2\\&\geq c_6\left(l\left\|\nabla u\right\|_2^2+(g\circ\nabla u)(t)\right)+K(\theta),\end{aligned}$$

where

$$c_6=\alpha\left(\frac{\theta}{2}-1\right)\frac{\lambda_2^2-\lambda_0^2}{\lambda_2^2}>0$$

via (4.11) and

$$K(\theta)=\alpha\left(\frac{\theta}{2}-1\right)\lambda_0^2-\theta E_2. \tag{4.12}$$

Similar to the situation with [3], we would like to obtain $K(\theta)$ is also positive. In fact, in the case $p > k$, we have

$$\begin{aligned} K(\theta) &= \alpha \left( \frac{\theta}{2} - 1 \right) \lambda_0^2 - \theta E_2 \\ &= \left( \frac{-\alpha \lambda_0^2}{2} + E_2 \right) \varepsilon_1 + \frac{\alpha (p-1) \lambda_0^2}{2} - (p+1) E_2, \end{aligned}$$

because $\theta = p + 1 - \varepsilon_1$. Further, from $E_2 < \alpha d$ and the definition of $d$ by (2.10), we observe that

$$\begin{aligned} \frac{-\alpha \lambda_0^2}{2} + E_2 &< \alpha \left( \frac{-\lambda_0^2}{2} + d \right) \\ &= \alpha \left( \frac{-\lambda_0^2}{2} + \frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1} \lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1} \lambda_0^{k+1} \right) < 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{\alpha (p-1) \lambda_0^2}{2} - (p+1) E_2 \\ & > \alpha \left( \frac{(p-1) \lambda_0^2}{2} - (p+1) d \right) \\ & = \alpha \left( \frac{(p-1) \lambda_0^2}{2} - (p+1) \left( \frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1} \lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1} \lambda_0^{k+1} \right) \right) \\ & \geq \alpha \left( -\lambda_0^2 + B_\Omega^{p+1} \lambda_0^{p+1} + B_\Gamma^{k+1} \lambda_0^{k+1} \right) \\ & = \alpha \lambda_0^2 \left( -1 + B_\Omega^{p+1} \lambda_0^{p-1} + B_\Gamma^{k+1} \lambda_0^{k-1} \right) = 0, \end{aligned}$$

where we used the identity (2.13). Similarly, if $k > p$, we get

$$\begin{aligned} K(\theta) &= \alpha \left( \frac{\theta}{2} - 1 \right) \lambda_0^2 - \theta E_2 \\ &= \left( \frac{-\alpha \lambda_0^2}{2} + E_2 \right) \varepsilon_1 + \frac{\alpha (k-1) \lambda_0^2}{2} - (k+1) E_2, \end{aligned}$$

with

$$\frac{\alpha (k-1) \lambda_0^2}{2} - (k+1) E_2 \geq 0.$$

Hence, from above arguments, we note that for $p > k$,

$$K(\theta) > 0 \text{ if and only if } 0 < \varepsilon_1 < H_p, \tag{4.13}$$

and in case $k > p$,

$$K(\theta) > 0 \text{ if and only if } 0 < \varepsilon_1 < H_k, \tag{4.14}$$

where

$$H_p = \frac{(p+1) E_2 - \frac{\alpha (p-1) \lambda_0^2}{2}}{E_2 - \frac{\alpha \lambda_0^2}{2}} \text{ and } H_k = \frac{(k+1) E_2 - \frac{\alpha (k-1) \lambda_0^2}{2}}{E_2 - \frac{\alpha \lambda_0^2}{2}}.$$

To derive the inequality (4.10), we choose $\varepsilon_1$ small enough such that

$$1 - \frac{p + 1 - \varepsilon_1}{k+1} > 0, \ \text{if } p > k,$$

and

$$1-\frac{k+1-\varepsilon_1}{p+1}>0,\ \text{if } k>p.$$

This implies that $\varepsilon_1>p-k$, if $p>k$ (respectively $\varepsilon_1>k-p$, if $k>p$). However, as we have already considered $\varepsilon_1<p-1$, if $p>k$ (respectively $\varepsilon_1<k-1$, if $k>p$) in (4.11). Thus, we consider

$$p-k<\varepsilon_1<p-1,\ \text{if } p>k \text{ (respectively } k-p<\varepsilon_1<k-1,\ \text{if } k>p).$$

Using this fact and (4.13)–(4.14), we require $\varepsilon_1$ such that

$$\begin{aligned}&p-k<\varepsilon_1<\min\{p-1,H_p\},\ \text{if } p>k\\&(\text{respectively } k-p<\varepsilon_1<\min\{k-1,H_k\},\ \text{if } k>p).\end{aligned}\tag{4.15}$$

Now, we consider the following cases to obtain desired inequality (4.10).

(1) If $E(0)<0$, then, we can choose $E_2$ such that $E(0)<E_2<0<\alpha d$. However, in this case, we note for $p>k$ that

$$E_2<0 \text{ if and only if } p-1<H_p$$

and in case $k>p$,

$$E_2<0 \text{ if and only if } k-1<H_k.$$

Hence, when $E(0)<0$, we take $\varepsilon_1>0$ small enough satisfying $p-k<\varepsilon_1<p-1<H_p$, if $p>k$ (respectively $k-p<\varepsilon_1<k-1<H_k$, if $k>p$) to obtain inequality (4.10).

(2) When $0\le E(0)<\alpha d$ and $p>k$, we note that if $p-k<H_p$, then $E_2<\frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$, so we have two possibilities: (i) $E(0)<\frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$ (ii) $E(0)\ge\frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$.

In the first case, it is sufficient to show that $\frac{\lambda_0^2(k-1)}{2(k+1)}<d$. However, this is already proved in [3]. Thus, we conclude that when $0\le E(0)<\frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$, we can choose $E_2$ such that

$$0<E(0)<E_2<\frac{\alpha\lambda_0^2(k-1)}{2(k+1)}<\alpha d.$$

Moreover, using above inequality and taking (4.15) into account, we can choose $\varepsilon_1$ such that

$$p-k<\varepsilon_1<\min\left\{p-1,\ H_p=\frac{(p+1)E_2-\frac{\alpha(p-1)\lambda_0^2}{2}}{E_2-\frac{\alpha\lambda_0^2}{2}}\right\}=H_p.\tag{4.16}$$

Finally, when $E(0)\ge\frac{\alpha\lambda_0^2(k-1)}{2(k+1)}$, it does not seem possible to find $\varepsilon_1$ verifying the inequality (4.16). Hence, in this case we are forced to obtain the difference $p-k$ sufficiently small. Now, returning to (4.11) and (4.12), we note that

$$\lim_{\varepsilon_1\to 0}\theta=\theta^*,$$

where

$$\theta^*=p+1,\ \text{if } p>k \text{ or } \theta^*=k+1,\ \text{if } k>p.$$

Letting $\varepsilon_1 \to 0$ in (4.12), using $E_2 < \alpha d$ by (4.7) and the definition of $d$ by (2.10), we deduce that

$$\begin{aligned}
\lim_{\varepsilon_1 \to 0} K(\theta) &= \alpha\left(\frac{\theta^*}{2} - 1\right)\lambda_0^2 - \theta^* E_2 \\
&> \alpha\left(\frac{\theta^*}{2} - 1\right)\lambda_0^2 - \alpha\theta^* d \\
&= \alpha\lambda_0^2\left(-1 + \frac{\theta^*}{p+1}B_\Omega^{p+1}\lambda_0^{p-1} + \frac{\theta^*}{k+1}B_\Gamma^{k+1}\lambda_0^{k-1}\right).
\end{aligned}$$

For the case $p > k$, substituting $\theta^* = p+1$ and using the identity (2.13), we see that

$$\begin{aligned}
\lim_{\varepsilon_1 \to 0} K(\theta) &> \alpha\lambda_0^2\left(-1 + \frac{\theta^*}{p+1}B_\Omega^{p+1}\lambda_0^{p-1} + \frac{\theta^*}{k+1}B_\Gamma^{k+1}\lambda_0^{k-1}\right) \\
&> \alpha\lambda_0^2\left(-1 + B_\Omega^{p+1}\lambda_0^{p-1} + B_\Gamma^{k+1}\lambda_0^{k-1}\right) = 0,
\end{aligned}$$

which proves the desired inequality (4.10). The analysis is analogous, for the case $k > p$, so we omit the detailed proof.

Under the above arguments, we have the desired inequality (4.10): that is, there exist positive constants $c_i,\ i = 1-5$, such that

$$\begin{aligned}
Z'(t) \geq\ & (1-\gamma)H^{-\gamma}(t)H'(t) + \varepsilon c_1 \int_\Omega u_t^2 \mathrm{d}x \\
& + \varepsilon c_2\left(l\left\|\nabla u\right\|_2^2 + (g \circ \nabla u)(t)\right) + \varepsilon c_3 \left\|u\right\|_{p+1}^{p+1} \\
& + \varepsilon c_4 \left\|u\right\|_{k+1,\Gamma}^{k+1} + \varepsilon c_5 H - \varepsilon \int_{\Gamma_1} h\left(u_t\right) u \mathrm{d}\Gamma.
\end{aligned} \tag{4.17}$$

Using (A3), Hölder's inequality, $k > q$ and Young's inequality, the last term on the right-hand side of (4.17) can be estimated as, for $\delta_1 > 0$,

$$\begin{aligned}
\int_{\Gamma_1} h\left(u_t\right) u \mathrm{d}\Gamma &\leq M_q \left\|u_t\right\|_{q+1,\Gamma_1}^q \left\|u\right\|_{q+1,\Gamma_1} \\
&\leq c_6 \left\|u_t\right\|_{q+1,\Gamma_1}^q \left\|u\right\|_{k+1,\Gamma_1} \\
&\leq c_6 \left\|u_t\right\|_{q+1,\Gamma_1}^q \left(\left\|u\right\|_{k+1,\Gamma_1}^{k+1} + \left\|u\right\|_{p+1}^{p+1}\right)^{\frac{1}{q+1}} \left(\left\|u\right\|_{k+1,\Gamma_1}^{k+1} + \left\|u\right\|_{p+1}^{p+1}\right)^{-\gamma_1} \\
&\leq \left(c_7\left(\delta_1\right)\left\|u_t\right\|_{q+1,\Gamma_1}^{q+1} + \delta_1\left(\left\|u\right\|_{k+1,\Gamma_1}^{k+1} + \left\|u\right\|_{p+1}^{p+1}\right)\right)\left(\left\|u\right\|_{k+1,\Gamma_1}^{k+1} + \left\|u\right\|_{p+1}^{p+1}\right)^{-\gamma_1},
\end{aligned} \tag{4.18}$$

where $c_6 = M_q vol\left(\Gamma_1\right)^{\frac{k-q}{(k+1)(q+1)}}$, $\gamma_1 = \frac{1}{q+1} - \frac{1}{k+1} > 0$ and $c_7\left(\delta_1\right) > 0$ is a constant. Moreover, from (4.6), the definition of $E(t)$ by (2.6), (4.3), (4.7) and the definition of $d$ by (2.10), we have

$$\begin{aligned}
H(t) &= E_2 - E(t) \\
&\leq E_2 - \frac{1}{2}l\left\|\nabla u(t)\right\|^2 - \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p+1}\left\|u\right\|_{p+1}^{p+1} + \frac{1}{k+1}\left\|u\right\|_{k+1,\Gamma_1}^{k+1} \\
&< \alpha d - \frac{1}{2}\lambda_2^2 + \frac{1}{p+1}\left\|u\right\|_{p+1}^{p+1} + \frac{1}{k+1}\left\|u\right\|_{k+1,\Gamma_1}^{k+1} \\
&= \frac{\alpha\lambda_0^2 - \lambda_2^2}{2} - \frac{\alpha B_\Omega^{p+1}}{p+1}\lambda_2^{p+1} - \frac{\alpha B_\Gamma^{k+1}}{k+1}\lambda_2^{k+1} + \frac{1}{p+1}\left\|u\right\|_{p+1}^{p+1} + \frac{1}{k+1}\left\|u\right\|_{k+1,\Gamma_1}^{k+1}
\end{aligned}$$

$$\leq \frac{1}{p+1}\|u\|_{p+1}^{p+1}+\frac{1}{k+1}\|u\|_{k+1,\Gamma_1}^{k+1}$$
$$\leq c_0'\left(\|u\|_{p+1}^{p+1}+\|u\|_{k+1,\Gamma_1}^{k+1}\right), \tag{4.19}$$

which together with (4.18) implies that

$$\int_{\Gamma_1} h(u_t)\,u\mathrm{d}\Gamma \leq c_0^{\gamma_1}\left[c_7(\delta_1)\|u_t\|_{q+1,\Gamma_1}^{q+1}+\delta_1\left(\|u\|_{k+1,\Gamma_1}^{k+1}+\|u\|_{p+1}^{p+1}\right)\right]H(t)^{-\gamma_1}, \tag{4.20}$$

where $c_0'=\max\left(\frac{1}{p+1},\frac{1}{k+1}\right)$. In addition, from (2.94) and (4.1), we have

$$H'(t)\geq \int_{\Gamma_1} h(u_t)\,u_t\mathrm{d}\Gamma \geq m_q\|u_t\|_{q+1,\Gamma_1}^{q+1}.$$

Substituting the above inequality into (4.20), noting that $H$ is increasing by (4.8) and letting $0<\gamma<\gamma_1$, we obtain

$$\begin{aligned}\int_{\Gamma_1} h(u_t)\,u\mathrm{d}\Gamma &\leq c_0^{\gamma_1}\left[c_8(\delta_1)H'(t)+\delta_1\left(\|u\|_{k+1,\Gamma_1}^{k+1}+\|u\|_{p+1}^{p+1}\right)\right]H(t)^{-\gamma_1}\\ &\leq c_0^{\gamma_1}\Big[c_8(\delta_1)H'(t)H(t)^{-\gamma}H(0)^{\gamma-\gamma_1}\\ &\quad+\delta_1\left(\|u\|_{k+1,\Gamma_1}^{k+1}+\|u\|_{p+1}^{p+1}+l\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right)H(0)^{-\gamma_1}\Big],\end{aligned}$$

where $c_8(\delta_1)=\frac{c_7(\delta_1)}{m_q}$. Thus,

$$\begin{aligned}Z'(t)\geq&\left(1-\gamma-\varepsilon c_0^{\gamma_1}c_8(\delta_1)H(0)^{\gamma-\gamma_1}\right)H^{-\gamma}(t)H'(t)+\epsilon c_1\|u_t\|_2^2\\ &+\varepsilon\left(c_2-\delta_1H(0)^{-\gamma_1}\right)\left(l\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right)\\ &+\varepsilon\left(c_3-\delta_1H(0)^{-\gamma_1}\right)\|u\|_{p+1}^{p+1}+\varepsilon\left(c_4-\delta_1H(0)^{-\gamma_1}\right)\|u\|_{k+1,\Gamma_1}^{k+1}+\varepsilon c_5H(t).\end{aligned}$$

At this point, choosing $\delta_1>0$ small enough and $\varepsilon>0$ small enough such that

$$\begin{aligned}&c_2-\delta_1H(0)^{-\gamma_1}>0,\\ &c_3-\delta_1H(0)^{-\gamma_1}>0,\\ &c_4-\delta_1H(0)^{-\gamma_1}>0,\\ &1-\gamma-\varepsilon c_0^{\gamma_1}c_8(\delta_1)H(0)^{\gamma-\gamma_1}>0,\end{aligned}$$

and

$$H^{1-\gamma}(0)+\varepsilon\int_\Omega u_0u_1\mathrm{d}x>0.$$

Thus,

$$Z'(t)\geq\varepsilon c_9\left(\|u_t\|_2^2+l\|\nabla u\|_2^2+(g\circ\nabla u)(t)+\|u\|_{p+1}^{p+1}+\|u\|_{k+1,\Gamma_1}^{k+1}+H\right) \tag{4.21}$$

and

$$Z(0)=H^{1-\gamma}(0)+\varepsilon\int_\Omega u_0u_1\mathrm{d}x>0,$$

where $c_9$ is a positive constant. Consequently,

$$Z(t)\geq Z(0)>0,\ \forall t\geq 0.$$

Now set $\theta_1 = \frac{1}{1-\gamma}$. As $\gamma < \gamma_1 < 1$, it is evident that $1 < \theta_1 < \frac{1}{1-\gamma_1}$. Applying Young's inequality and Hölder's inequality in (4.9), we see that

$$Z(t)^{\theta_1} \leq 2^{\theta_1 - 1}\left[H(t) + \left(\varepsilon \int_\Omega u_t u \mathrm{d}x\right)^{\theta_1}\right]. \tag{4.22}$$

Further, using Hölder's inequality and Young's inequality, for $p > 1$, we have

$$\left(\int_\Omega u_t u \mathrm{d}x\right)^{\theta_1} \leq c_{10} \|u_t\|_2^{\theta_1} \|u\|_{p+1}^{\theta_1} \leq c_{11}\left(\|u\|_{p+1}^{\theta_1\beta_1} + \|u_t\|_2^{\theta_1\beta_2}\right),$$

where $c_{10} = (\mathrm{vol}(\Omega))^{\frac{\theta_1(p-1)}{2(p+1)}}$, $\frac{1}{\beta_1} + \frac{1}{\beta_2} = 1$, and $c_{11} = c_{11}(c_{10}, \beta_1, \beta_2) > 0$. Taking $\theta_1\beta_2 = 2$ to get $\theta_1\beta_1 = \frac{2}{1-2\gamma} \leq p+1$. Then, taking (4.19) into consideration, we deduce that

$$\begin{aligned}\|u\|_{p+1}^{\theta_1\beta_1} &\leq \left[\frac{c_0}{H(0)}\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right)\right]^{\frac{\theta_1\beta_1}{p+1}}\left(\frac{c_0}{H(0)}\right)^{-\frac{\theta_1\beta_1}{p+1}} \\ &\leq c_{12}\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right),\end{aligned}$$

where $c_{12} = \left(\frac{c_0}{H(0)}\right)^{1-\frac{\theta_1\beta_1}{p+1}}$. Hence, (4.22) becomes

$$Z(t)^{\theta_1} \leq c_{13}\left[H(t) + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1} + \|u_t\|_2^2\right], \tag{4.23}$$

here, $c_{13}$ is some positive constant. Combining (4.21) and (4.23) together, we obtain

$$Z'(t) \geq c_{14} Z(t)^{\theta_1},\ t \geq 0, \tag{4.24}$$

here, $c_{14} = \frac{c_9\varepsilon}{c_{13}}$. An integration of (4.24) over $(0,t)$ yields

$$Z(t) \geq \left(Z(0)^{1-\theta_1} - c_{14}(\theta_1 - 1)t\right)^{-\frac{1}{\theta_1 - 1}}.$$

As $Z(0) > 0$, (4.24) shows that $Z$ becomes infinite in a finite time

$$T \leq T^* = \frac{Z(0)^{1-\theta_1}}{c_{14}(\theta_1 - 1)}.$$

Moreover, in view of the inequality induced by (2.6) and (2.64), we have

$$\begin{aligned}&\frac{1}{2}\|u_t(t)\|_2^2 + \left(l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right) \\ &\quad \leq E(0) + \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{1}{k+1}\|u\|_{k+1,\Gamma_1}^{k+1},\end{aligned}$$

which together with (4.23) and (4.19) implies that

$$\lim_{T \to T^{*-}}\left(\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma_1}^{k+1}\right) = \infty.$$

Thus, we obtain (4.5). Hence, the proof is completed. □

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