Z. Angew. Math. Phys. 62 (2011), 1065–1082
© 2011 Springer Basel AG
0044-2275/11/061065-18
published online July 7, 2011
DOI 10.1007/s00033-011-0145-0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Existence and asymptotic stability of a viscoelastic wave equation with a delay

Mokhtar Kirane and Belkacem Said-Houari

Abstract. In this paper, we consider the viscoelastic wave equation with a delay term in internal feedbacks; namely, we investigate the following problem

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0$$

together with initial conditions and boundary conditions of Dirichlet type. Here $(x,t) \in \Omega \times (0,\infty), g$ is a positive real valued decreasing function and μ_1, μ_2 are positive constants. Under an hypothesis between the weight of the delay term in the feedback and the weight of the term without delay, using the Faedo–Galerkin approximations together with some energy estimates, we prove the global existence of the solutions. Under the same assumptions, general decay results of the energy are established via suitable Lyapunov functionals.

Mathematics Subject Classification (2000). 35L05 · 35L15 · 35L70 · 93D15.

Keywords. Global existence · General decay · Relaxation function · Delay feedbacks.

1. Introduction

In this paper, we consider the following linear viscoelastic wave equation with a linear damping and a delay term

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0 \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau) \end{cases} \tag{1}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N \geq 1), \mu_1,\ \mu_2$ are positive constants, $\tau > 0$ represents the time delay and u_0 , u_1, f_0 are given functions belonging to suitable spaces.

The purpose of this paper is to study the existence and the asymptotic stability of problem (1), with a delay term appearing in the control term in the first equation.

Introducing the delay term $\mu_2 u_t(x,t-\tau)$ makes the problem different from those considered in the literature.

In recent years, PDEs with time delay effects have become an active area of research; see for example [2,23] and references therein. In [9], the authors showed that a small delay in a boundary control is a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary; see for instance [17,18,25] and references therein. In [17], the authors examined

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a system of wave equation with a linear boundary damping term with a delay. Namely, they considered the following system

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega, \\ u(x,t-\tau) = g_0(x,t-\tau) & x \in \Omega, t \in (0,\tau), \end{cases} \tag{2}$$

and proved under the assumption

$$\mu_2 < \mu_1 \tag{3}$$

that the energy is exponentially stable. On the contrary, if (3) does not hold, they found a sequence of delays for which the corresponding solution of (2) is unstable. The main approach used in [17] is an observability inequality combined with a Carleman estimate. The same results were showed if both the damping and the delay act inside the domain. Let us also mention the result by Xu et al. [25], where they proved the same result as in [17] for the one space dimension by adopting the spectral analysis approach. According to the previous results, the decay rate of solutions depends on the delay, and as it was shown in [19], the decay rate decreases whenever the delay increases. In other words, the decay is slower when τ becomes larger.

The case of time-varying delay in the wave equation has been studied recently by Nicaise et al. [20] in one space dimension. In that work, an exponential stability result was given under the condition

$$\mu_2 \le \sqrt{1-d}\,\mu_1 \tag{4}$$

where d is a constant such that

$$\tau'(t) \le d < 1, \quad \forall t > 0. \tag{5}$$

In the absence of the delay term, that is for $\mu_2 = 0$, problems similar to (1) in bounded domains or in the whole N-dimensional space have been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions, have been established by several authors over the past three decades. See in this regard [3,4,6,13–15,22] and references therein.

Here, we recall some results regarding the viscoelastic wave equation.

The single viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(x,s)\,\mathrm{d}s + h(u_t) = f(u), \tag{6}$$

in $\Omega \times (0,\infty)$, subjected to initial conditions and boundary conditions of Dirichlet type has been considered by Cavalcanti et al. [6] in the case where $f = 0$ and $h(u_t) = a(x)u_t$. More precisely, they studied the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(x,s)\,\mathrm{d}s + a(x)u_t = 0 \tag{7}$$

in $\Omega \times (0,\infty)$, where $a : \Omega \to \mathbb{R}^+$ is a function, which may be null on a part of the domain Ω. By assuming $a(x) \ge a_0$ on $\omega \subset \Omega$ and

$$-\zeta_1 g(t) \le g'(t) \le -\zeta_2 g(t), \quad \forall t \ge 0$$

the authors showed an exponential decay result under some geometric restrictions on the subset ω. The result in [6] has been improved by Berrimi and Messaoudi [4], who showed the same result as in [6], under weaker conditions on both a and g. In [3], more general abstract version of Eq. (6) has been considered and an uniform stability result has been obtained. In fact, the decay rates obtained in [3] are in line with the ones obtained in [4] for Eq. (6).

Using the piecewise multipliers method, Cavalcanti and Oquendo [8] showed some stability results for a more general problem than the one considered in [6]. More precisely, they investigated the following problem

$$u_{tt} - k_0 \Delta u + \int_0^t div[a(x) g(t-s) \Delta u(x,s)] \mathrm{d}s + b(x) h(u_t) + f(u) = 0, \tag{8}$$

and proved that, under the same conditions on the function g and for $a(x)+b(x) \geq \rho > 0$, an exponential stability result if g decays exponentially and h is linear, and a polynomial stability result for g decaying polynomially and h nonlinear.

Fabrizio and Polidoro [10] treated the problem (7) with $a(x) = a_0$ and showed that the solution of the problem (7) decays exponentially only if the relaxation kernel g does. That is to say the presence of the memory term may prevent the exponential decay due to the linear frictional damping term. We mention also the paper of Messaoudi [14] in which the author considered a problem related to (6) and proved a general decay result. In fact, his result allows a large class of relaxation functions and improves earlier results in which only the exponential and polynomial rates were established.

Cavalcanti et al. [5] investigated the following problem

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(x,s) \, \mathrm{d}s - \gamma \Delta u_t = 0, \qquad \rho > 0, \tag{9}$$

in $\Omega \times (0, \infty)$. For $\gamma \geq 0$, they showed a global existence result. Furthermore, they obtained an exponential decay result for $\gamma > 0$ provided that the function g decays exponentially. Using the potential well theory, Tatar and Messaoudi [15] extended the result in [5] to a situation where a source term of the form $|u|^{p-2} u$ is present in Eq. (9).

Recently, Messaoudi and Tatar [16] have studied (9) with ($\gamma = 0$) and showed that the viscoelastic damping term is strong enough to stabilize the system. We mention also the paper [7], in which the authors investigated a problem similar to (6) with a nonlinear feedback acting on the boundary of the domain Ω and showed uniform decay rates of the energy without imposing any restrictive growth assumption on the damping term. We refer also to [1,11] for some results on the asymptotic stability and global nonexistence results of the wave equation with boundary dissipation of the memory type. A wave equation with acoustic and memory boundary conditions on a part of the boundary of the domain Ω has been also investigated recently in [24], where the existence and uniqueness of global solution have been proved.

In the present work, we are concerned with problem (1). Our goal here is twofold:

First, using the Faedo–Galerkin approximations together with some energy estimates, and under some restriction on the parameters μ_1 and μ_2, the system is showed to be well-posed.

Second, under the hypothesis $\mu_2 \leq \mu_1$ between the weight of the delay term in the feedback and the weight of the term without delay, we prove a general decay of the total energy of our problem. Our method of proof uses some ideas developed in [17] for the wave equation with delay and some estimates of the viscoelastic wave equation, enabling us to obtain suitable Lyapunov functionals, from which are derived the desired results. We recall that for $\mu_1 = \mu_2$, Nicaise and Pignotti showed in [17] that some instabilities may occur. Here, due to the presence of the viscoelastic damping, we prove that our solution is still asymptotically stable even if $\mu_1 = \mu_2$.

The paper is organized as follows: in the next section, we fix notations and, for the convenience of the reader, we recall without proofs some useful lemmas. In Sect. 3, we will prove the well-posedness of the solution. In the section 4, we will show a general decay of the energy defined by (30) provided that the weight of the delay is less than the weight of the damping. We will also prove the same decay result even if the weight of the delay is equal to the weight of the damping.

2. Preliminaries

In this section, we present some material that shall be used in order to prove our main result.

Let us first introduce the following notations:

$$(\phi * \psi)(t) := \int_0^t \phi(t-\tau)\psi(\tau)\,\mathrm{d}\tau$$

$$(\phi \diamond \psi)(t) := \int_0^t \phi(t-\tau)\left|\psi(t)-\psi(\tau)\right|\mathrm{d}\tau$$

$$(\phi \circ \psi)(t) := \int_0^t \phi(t-\tau)\int_\Omega \left|\psi(t)-\psi(\tau)\right|^2 \mathrm{d}x\mathrm{d}\tau.$$

The following lemma was introduced in [21]; it will be used later in order to define the new modified functional energy of problem (1).

Lemma 2.1. *For any function $\phi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,T)$, we have*

$$(\phi * \psi)(t)\,\psi_t(t) = -\frac{1}{2}\phi(t)\left|\psi(t)\right|^2 + \frac{1}{2}(\phi' \diamond \psi)(t) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)\,\mathrm{d}\tau\right)\left|\psi(t)\right|^2\right\}.$$

Lemma 2.2. *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\right)^2 \leq (1-l)\,C_*^2\,(g \circ \nabla u)(t), \tag{10}$$

where C_* is the Poincaré constant and l is given in (G1).

For the proof of Lemma 2.2, we refer to [14].

For the relaxation function g, we assume

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0$$

(G2) There exists a positive nonincreasing differentiable function $\zeta(t)$ such that

$$g'(t) \leq -\zeta(t)\,g(t), \qquad \forall t \geq 0, \tag{11}$$

and

$$\int_0^{+\infty} \zeta(t)\,\mathrm{d}t = +\infty.$$

3. Well-posedness of the problem

In this section, we will prove the global existence and the uniqueness of the solution of problem (1). We will first transform the problem (1) into the problem (14) below, by adding a new unknown, and then, we use the Faedo–Galerkin approximations together with some energy estimates, to prove the existence of the unique solution of problem (14).

In order to prove the existence of a unique solution of problem (1), we introduce as in [18], the new variable

$$z\left(x,\rho,t\right)=u_t\left(x,t-\tau\rho\right),\ x\in\Omega,\ \rho\in\left(0,1\right),\quad t>0. \tag{12}$$

Then, we have

$$\tau z_t\left(x,\rho,t\right)+z_\rho\left(x,\rho,t\right)=0,\ \text{in }\Omega\times\left(0,1\right)\times\left(0,+\infty\right). \tag{13}$$

Therefore, problem (1) is equivalent to:

$$\begin{cases} u_{tt}(x,t)-\Delta u(x,t)+\int\limits_0^t g(t-s)\Delta u(x,s)\mathrm{d}s+\mu_1 u_t(x,t)+\mu_2 z(x,1,t)=0, & x\in\Omega,\ t>0\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, & x\in\Omega,\rho\in\left(0,1\right),\ t>0\\ u(x,t)=0, & x\in\partial\Omega,\ t>0\\ z(x,0,t)=u_t(x,t) & x\in\Omega,\ t>0\\ u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x) & x\in\Omega\\ z(x,\rho,0)=f_0(x,t-\tau) & x\in\Omega,\ t\in(0,\tau)\end{cases} \tag{14}$$

The first natural question is the existence of solutions of the problem (14). In this section, we will give a sufficient condition that guarantees that this problem is well-posed.

First, let ξ be a positive constant such that

$$\tau\mu_2<\xi<\tau\left(2\mu_1-\mu_2\right). \tag{15}$$

The existence result reads as follows:

Theorem 3.1. *Assume that $\mu_1\leq\mu_2$. Then given $u_0\in H_0^1(\Omega)$, $u_1\in L^2(\Omega)$, $f_0\in L^2\left(\Omega\times(0,1)\right)$ and $T>0$, there exists a unique weak solution (u,z) of the problem (14) on $(0,T)$ such that*

$$\begin{aligned} u&\in C\left([0,T],H_0^1(\Omega)\right)\cap C^1\left([0,T],L^2(\Omega)\right),\\ u_t&\in L^2\left(0,T;H_0^1(\Omega)\right)\cap L^2\left((0,T)\times\Omega\right).\end{aligned}$$

Proof. We divide the proof of Theorem 3.1 in two steps: the construction of approximations and then thanks to certain energy estimates, we pass to the limit.

Step 1 : Faedo–Galerkin approximation.

We construct approximations of the solution (u,z) by the Faedo–Galerkin method as follows. For every $n\geq 1$, let $W_n=\operatorname{span}\{w_1,\ldots,w_n\}$, be a Hilbertian basis of the space $H_0^1(\Omega)$.

Now, we define for $1\leq j\leq n$ the sequence $\varphi_j(\rho,x)$ as follows:

$$\varphi_j(x,0)=w_j\left(x\right).$$

Then, we may extend $\varphi_j(x,0)$ by $\varphi_j(x,\rho)$ over $L^2\left(\Omega\times[0,1]\right)$ and denote $V_n=\operatorname{span}\{\varphi_1,\ldots,\varphi_n\}$,

We choose two sequences (u_{0n}) and (u_{1n}) in W_n and a sequence (z_{0n}) in V_n such that $u_{0n}\to u_0$ strongly in $H_0^1(\Omega)$, $u_{1n}\to u_1$ strongly in $L^2(\Omega)$ and $z_{0n}\to f_0$ strongly in $L^2(\Omega\times(0,1))$.

We define now the approximations:

$$u_n(t,x)=\sum_{j=1}^n g_{jn}(t)w_j\left(x\right),\qquad\text{and}\qquad z_n(t,x,\rho)=\sum_{j=1}^n h_{jn}(t)\varphi_j(x,\rho) \tag{16}$$

where $(u_n(t), z_n(t))$ are solutions to the finite dimensional Cauchy problem (written in normal form):

$$\begin{cases} \int_\Omega u_{ttn}(t) w_j \, \mathrm{d}x + \int_\Omega \nabla u_n \nabla w_j - \int_0^t g(t-s) \int_\Omega \nabla u_n(s) \nabla w_j \, \mathrm{d}x \mathrm{d}s \\ + \int_\Omega \left(\mu_1 u_{nt}(t,x) + \mu_2 z_n(x,1,t)\right) w_j \, \mathrm{d}x = 0, \\ z_n(x,0,t) = u_{tn}(x,t) \\ (u_n(0), u_{tn}(0)) = (u_{0n}, u_{1n}) \end{cases} \tag{17}$$

and

$$\begin{cases} \int_\Omega \left(\tau z_{nt}(x,\rho,t) + z_{n\rho}(x,\rho,t) \varphi_j\right) \mathrm{d}x = 0 \\ z_n(\rho,0) = z_{0n}. \end{cases} \tag{18}$$

According to the standard theory of ordinary differential equations, the finite dimensional problem (17), (18) has solution $(g_{jn}(t), h_{jn}(t))_{j=1,n}$ defined on $[0, t_n)$. The a priori estimates that follow imply that in fact $t_n = T$.

Step 2 : Energy estimates.

Multiplying Eq. (17) by $g'_{jn}(t)$ integrating over $(0,t)$, using integration by parts and Lemma 2.1 we get, for every $n \geq 1$,

$$\begin{aligned} &\frac{1}{2}\left[\left(1 - \int_0^t g(s)\,\mathrm{d}s\right) \|\nabla u_n(t)\|_2^2 + \|u_{tn}(t)\|_2^2 + (g \circ \nabla u_n)(t)\right] + \mu_1 \int_0^t \|u_{tn}(s)\|_2^2 \,\mathrm{d}s \\ &\quad + \mu_2 \int_0^t \int_\Omega z_n(x,1,s) u_{tn}(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t g(s) \|\nabla u_n(s)\|_2^2 \,\mathrm{d}s - \frac{1}{2}\int_0^t (g' \circ \nabla u_n)(s)\,\mathrm{d}s \\ &= \frac{1}{2}\left[\|\nabla u_0\|_2^2 + \|u_1\|_2^2\right]. \end{aligned} \tag{19}$$

Let $\xi > 0$ to be chosen later. Multiplying Eq. (18) by $(\xi/\tau) h'_{jn}(t)$ integrating over $(0,t) \times (0,1)$, we obtain:

$$\begin{aligned} &\frac{\xi}{2} \int_\Omega \int_0^1 z_n^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x + \frac{\xi}{\tau} \int_0^t \int_\Omega \int_0^1 z_{n\rho} z_n(x,\rho,s)\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \frac{\xi}{2} \|z_{0n}\|^2_{L^2(\Omega\times(0,1))}. \end{aligned} \tag{20}$$

Now, to handle the last term in the left-hand side of (20), we remark that we have:

$$\begin{aligned} \int_0^t \int_\Omega \int_0^1 z_{n\rho} z_n(x,\rho,s)\,\mathrm{d}\rho\,\mathrm{d}x\mathrm{d}s &= \frac{1}{2} \int_0^t \int_\Omega \int_0^1 \frac{\partial}{\partial\rho} z_n^2(x,\rho,s)\,\mathrm{d}\rho\,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{1}{2} \int_0^t \int_\Omega \left(z_n^2(x,1,s) - z_n^2(x,0,s)\right) \mathrm{d}x\mathrm{d}s. \end{aligned} \tag{21}$$

Summing up the identities (19) and (20) and taking into account (21), we get:

$$\begin{aligned}
&\mathscr{E}_n(t) + \left(\mu_1 - \frac{\xi}{2\tau}\right) \int_0^t \|u_{tn}(s)\|_2^2 \, ds + \frac{\xi}{2\tau} \int_0^t \int_\Omega z_n^2(\sigma, 1, s) \, d\sigma \, ds \\
&\quad + \mu_2 \int_0^t \int_\Omega z_n(x, 1, s) u_{tn}(s, x) \, dx \, ds \\
&\quad + \frac{1}{2} \int_0^t g(s) \|\nabla u_n(s)\|_2^2 \, ds - \frac{1}{2} \int_0^t (g' \circ \nabla u_n)(s) \, ds \\
&= \mathscr{E}_n(0) \,, \qquad (22)
\end{aligned}$$

where

$$\mathscr{E}_n(t) = \frac{1}{2} \left[\left(1 - \int_0^t g(s) \, ds\right) \|\nabla u_n(t)\|_2^2 + \|u_{tn}(t)\|_2^2 + (g \circ \nabla u_n)(t) \right] + \frac{\xi}{2} \|z_n\|_{L^2(\Omega \times (0,1))}^2 . \qquad (23)$$

At this point, we have to distinguish the following two cases:

Case 1: We suppose that $\mu_2 < \mu_1$. Let us choose then ξ that satisfies inequality (15). Using Young's inequality, (22) leads to:

$$\begin{aligned}
&\mathscr{E}_n(t) + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right) \int_0^t \|u_{tn}(s)\|_2^2 \, ds + \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right) \int_0^t \int_\Omega z_n^2(x, 1, s) \, dx ds \\
&+ \frac{1}{2} \int_0^t g(s) \|\nabla u_n(s)\|_2^2 \, ds - \frac{1}{2} \int_0^t (g' \circ \nabla u_n)(s) \, ds \le \mathscr{E}_n(0) .
\end{aligned}$$

Consequently, using (15), we can find two positive constants c_1 and c_2 such that:

$$\begin{aligned}
&\mathscr{E}_n(t) + c_1 \int_0^t \|u_{tn}(s)\|_2^2 \, ds + c_2 \int_0^t \int_\Omega z_n^2(x, 1, s) \, dx \, ds \\
&+ \frac{1}{2} \int_0^t g(s) \|\nabla u_n(s)\|_2^2 \, ds - \frac{1}{2} \int_0^t (g' \circ \nabla u_n)(s) \, ds \le \mathscr{E}_n(0) \,. \qquad (24)
\end{aligned}$$

Case 2: We suppose that $\mu_2 = \mu_1 = \mu$ and choose then $\xi = \tau\mu$. Whereupon, inequality (24) takes the form

$$\mathscr{E}_n(t) + \frac{1}{2} \int_0^t g(s) \|\nabla u_n(s)\|_2^2 \, ds - \frac{1}{2} \int_0^t (g' \circ \nabla u_n)(s) \, ds \le \mathscr{E}_n(0). \qquad (25)$$

Now, in both cases and since the sequences $(u_{n0})_{n\in\mathbb{N}}, (u_{n1})_{n\in\mathbb{N}}$ and $(z_{0n})_{n\in\mathbb{N}}$ converge, and using (G1) and (G2), we can find a positive constant C independent of n such that

$$\mathscr{E}_n(t) \le C \,. \qquad (26)$$

Therefore, using the fact that $1 - \int_0^t g(s) ds \ge l$, the last estimate (26) together with (23) give us, for all $n \in \mathbb{N}$, $t_n = T$; we deduce

$$(u_n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)), \qquad (27)$$

$$(u_{tn})_{n\in\mathbb{N}} \text{ is bounded in } L^{\infty}(0,T;L^2(\Omega)), \tag{28}$$

and

$$(z_n)_{n\in\mathbb{N}} \text{ is bounded in } L^{\infty}(0,T;L^2(\Omega\times(0,1))). \tag{29}$$

Consequently, we may conclude that:

$$\begin{aligned} u_n &\rightharpoonup u \text{ weak}^* && \text{in } L^{\infty}(0,T;H_0^1(\Omega)),\\ u_{tn} &\rightharpoonup u_t \text{ weak}^* && \text{in } L^{\infty}(0,T;L^2(\Omega)),\\ z_n &\rightharpoonup z \text{ weak}^* && \text{in } L^{\infty}(0,T;L^2(\Omega\times(0,1))). \end{aligned}$$

From (27), (28) and (29), we have $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^{\infty}\left(0,T;H_0^1(\Omega)\right)$. Then, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$. Since $(u_{tn})_{n\in\mathbb{N}}$ is bounded in $L^{\infty}\left(0,T;L^2(\Omega)\right)$, $(u_{tn})_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;L^2(\Omega)\right)$. Consequently $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1\left(0,T;H^1(\Omega)\right)$.

Since the embedding $H^1\left(0,T;H^1(\Omega)\right)\hookrightarrow L^2\left(0,T;L^2(\Omega)\right)$ is compact, using Aubin–Lions theorem [12], we can extract a subsequence $(u_\mu)_{\mu\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$u_\mu \to u \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right).$$

Therefore,

$$u_\mu \to u \text{ strongly and a.e on } (0,T)\times\Omega.$$

The proof now can be completed arguing as in [12, Théorème 3.1] □

4. Asymptotic behavior

In this section, we show, using the energy method and suitable Lyapunov functionals that under the hypothesis $\mu_2 \le \mu_1$, the energy of the solution of problem (1) decreases exponentially as t tends to infinity. We will discuss two case, the case where $\mu_2 < \mu_1$ and the case $\mu_2 = \mu_1$. We will separate the two cases since the proofs are slightly different.

4.1. Exponential stability for $\mu_2 < \mu_1$

In this subsection, we will show that under the assumption $\mu_2 < \mu_1$, the solution of problem (1) decays to the trivial steady state. To achieve our goal, we will use the energy method combined with the choice of a suitable Lyapunov functional.

For a positive constant ξ satisfying the inequality (15), we define the functional energy of problem (14) as

$$\begin{aligned} E(t) = E(t,z,u) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2\\ &\quad+\frac{1}{2}(g\circ\nabla u)(t) + \frac{\xi}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x. \end{aligned} \tag{30}$$

Our goal now is to prove that the above energy $E(t)$ is a decreasing function along the trajectories. More precisely, we have the following result:

Lemma 4.1. *Suppose that* (G1) *and* (G2) *hold and let* (u, z) *be a solution of the problem* (14). *Then, the energy functional defined by* (30) *is a nonincreasing function, that is there exists a positive constant* C *such that*

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq -C\left(\int_\Omega u_t^2(x,t)\,\mathrm{d}x + \int_\Omega z^2(x,1,t)\,\mathrm{d}x\right) + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2$$
$$\leq 0, \qquad \forall t \geq 0 \tag{31}$$

Proof. Multiplying the first equation in (14) by u_t, integrating over Ω and using integration by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\|u_t\|_2^2 + \|\nabla u\|_2^2\right) + \mu_1\|u_t\|_2^2 + \mu_2\int_\Omega z(x,1,t)u_t(x,t)\mathrm{d}x$$
$$= \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(\tau)\,\mathrm{d}x\mathrm{d}\tau. \tag{32}$$

Now, using Lemma 2.1, the term in the right-hand side of (32) can be rewritten as follows

$$\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(\tau)\,\mathrm{d}x\mathrm{d}\tau + \frac{1}{2}g(t)\|\nabla u\|_2^2$$
$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t g(\tau)\|\nabla u\|_2^2\,\mathrm{d}\tau - (g\circ\nabla u)(t)\right] + \frac{1}{2}(g'\circ\nabla u)(t). \tag{33}$$

Consequently, equality (32) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left\{\|u_t(t)\|_2^2 + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)\right\}$$
$$= -\mu_1\|u_t\|_2^2 - \mu_2\int_\Omega z(x,1,t)u_t(x,t)\mathrm{d}x - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t). \tag{34}$$

We multiply the second equation in (14) by ξz and integrate the result over $\Omega\times(0,1)$, to obtain:

$$\frac{\xi}{\tau}\int_\Omega\int_0^1 z_t z(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x = -\frac{\xi}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\,\mathrm{d}\rho\,\mathrm{d}x$$
$$= -\frac{\xi}{2\tau}\int_\Omega\left(z^2(x,1,t) - z^2(x,0,t)\right)\mathrm{d}x\,. \tag{35}$$

From (34), (35), using the equation (12) and Young's inequality, we obtain

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|u_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x$$
$$-\frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t).$$

Then, using (15) our conclusion holds. $\square$

Our stability result reads as follows:

Theorem 4.2. *Let u be the solution of (1). Assume that $\mu_2 < \mu_1$ and g satisfies* (G1) *and* (G2). *Then, there exist two positive constants K and λ such that the energy of problem (1) satisfies*

$$E(t) \leq K e^{-\lambda \int_0^t \zeta(s)\mathrm{d}s}, \qquad \forall t \geq 0. \tag{36}$$

The proof of Theorem 4.2 will be done through several Lemmas. We construct a functional $\mathscr{L}(t)$, equivalent to the energy $E(t)$, satisfying

$$\frac{\mathrm{d}\mathscr{L}(t)}{\mathrm{d}t} \leq -\Lambda \mathscr{L}(t), \qquad \forall t \geq 0,$$

where Λ is a positive constant. In order to construct such functional, let us first define the following

$$\Psi(t) := \int_\Omega u_t u \mathrm{d}x. \tag{37}$$

Then, we have the following estimate.

Lemma 4.3. *Let (u, z) be the solution of (14), then for any $\delta_1 > 0$, we have*

$$\begin{aligned}\frac{\Psi(t)}{\mathrm{d}t} \leq & \left(1 + \frac{\mu_1}{4\delta_1}\right) \|u_t\|_2^2 - \left(\frac{l}{2} - \delta_1 C_*^2 (\mu_1 + \mu_2)\right) \|\nabla u\|_2^2 \\ & + \frac{1}{4\delta_1} \int_\Omega z^2(x,1,t)\mathrm{d}x + \frac{(1-l)}{2} (g \circ \nabla u)(t).\end{aligned} \tag{38}$$

Proof. Using the first equation in (14), a direct computation leads to the following identity

$$\Psi'(t) = \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u(t) . \int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x - \mu_1 \int_\Omega u_t u \mathrm{d}x - \mu_2 \int_\Omega z(x,1,t) u \mathrm{d}x. \tag{39}$$

Now, the third term in the right-hand side of (39) can be estimated as follows:

$$\begin{aligned}\int_\Omega \nabla u(t) . \int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x &\leq \frac{1}{2}\int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)^2 \mathrm{d}x \\ &\leq \frac{1}{2}\int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|\mathrm{d}s\right)^2 \mathrm{d}x.\end{aligned}$$

Using the estimate (10) in Lemma 2.2, Young's inequality and the fact that $\int_0^t g(s)\mathrm{d}s \leq \int_0^\infty g(s)\mathrm{d}s = 1-l$, We get for any $\eta > 0$, (see relation (20) in [14])

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|\mathrm{d}s\right)^2 \mathrm{d}x \\ &\leq (1+\eta)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)|\mathrm{d}s\right)^2 \mathrm{d}x + \left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|\mathrm{d}s\right)^2 \mathrm{d}x \\ &\leq \left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t) + (1+\eta)(1-l)^2 \int_\Omega |\nabla u(t)|^2 \mathrm{d}x.\end{aligned} \tag{40}$$

Consequently, we arrive at

$$\int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s\mathrm{d}x \le \frac{1}{2}\left(1+(1+\eta)(1-l)^2\right)\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t). \tag{41}$$

Next, Young's inequality and Poincaré's inequality imply that, for any $\delta_1 > 0$

$$\int_\Omega u_t u\,\mathrm{d}x \le \delta_1 C_*^2 \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{4\delta_1}\int_\Omega u_t^2\mathrm{d}x, \tag{42}$$

and

$$\int_\Omega z(x,1,t)u\,\mathrm{d}x \le \delta_1 C_*^2 \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)\mathrm{d}x. \tag{43}$$

By inserting the estimates (41), (42) and (43) into (39) and choosing $\eta = l/(1-l)$, then (38) holds. □

Now, let us introduce the following functional

$$I(t) := \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x. \tag{44}$$

Differentiating (44) with respect to t and using the second equation in (14), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\right) &= -\frac{1}{\tau}\int_\Omega \int_0^1 e^{-2\tau\rho} z z_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\ &= -\int_\Omega \int_0^1 \rho e^{-2\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{1}{2\tau}\int_\Omega \int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x.\end{aligned}$$

Then, using an integration by parts, the above formula leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \le -\rho I(t) + \frac{1}{2\tau}\int_\Omega u_t^2(x,t)\mathrm{d}x - \frac{c}{2\tau}\int_\Omega z^2(x,1,t)\mathrm{d}x \tag{45}$$

where c is a positive constant.

Proof of Theorem 4.2. Let us define the Lyapunov functional

$$\mathscr{L}(t) := E(t) + \varepsilon\Psi(t) + \varepsilon I(t),$$

where ε is a positive real number which will be chosen later. It is straightforward to see that for $\varepsilon > 0$, $\mathscr{L}(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 depending on ε such that for all $t \ge 0$

$$\beta_1 E(t) \le \mathscr{L}(t) \le \beta_2 E(t)\,. \tag{46}$$

Using the estimates (31), (38) and (45), we may write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \leq &-\left(C-\varepsilon\left(1+\frac{\mu_1}{4\delta_1}\right)-\frac{\varepsilon}{2\tau}\right)\|u_t\|_2^2-\varepsilon\left(\frac{l}{2}-\delta_1 C_*^2(\mu_1+\mu_2)\right)\|\nabla u\|_2^2\\ &+\frac{(1-l)}{2}(g\circ\nabla u)(t)-\left(C+\frac{\varepsilon c}{2\tau}-\frac{\varepsilon}{4\delta_1}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x\\ &+\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u\|_2^2-\varepsilon\rho I(t).\end{aligned}$$

By choosing δ_1 and ε small enough, we can find two positive constants γ_1 and γ_2 such that

$$\mathscr{L}'(t) \leq -\gamma_1 E(t)+\gamma_2(g\circ\nabla u)(t), \qquad \forall t\geq 0. \tag{47}$$

By multiplying (47) by $\zeta(t)$, we arrive at

$$\zeta(t)\mathscr{L}'(t) \leq -\gamma_1\zeta(t)E(t)+\gamma_2\zeta(t)(g\circ\nabla u)(t), \qquad \forall t\geq 0.$$

Recalling (G2) and using (31), we get

$$\begin{aligned}\zeta(t)\mathscr{L}'(t) &\leq -\gamma_1\zeta(t)E(t)-\gamma_2(g'\circ\nabla u)(t)\\ &\leq -\gamma_1\zeta(t)E(t)-2\gamma_2 E'(t), \qquad \forall t\geq 0.\end{aligned}$$

That is

$$(\zeta(t)\mathscr{L}(t)+2\gamma_2 E(t))'-\zeta'(t)\mathscr{L}(t)\leq -\gamma_1\zeta(t)E(t), \qquad \forall t\geq 0.$$

Using the fact that $\zeta'(t)\leq 0$, $\forall\, t\geq 0$ and letting

$$\mathscr{F}(t)=\zeta(t)\mathscr{L}(t)+2\gamma_2 E(t)\sim E(t) \tag{48}$$

we obtain

$$\mathscr{F}'(t)\leq -\gamma_1\zeta(t)E(t)\leq -\gamma_3\zeta(t)\mathscr{F}(t), \qquad \forall t\geq 0 \tag{49}$$

A simple integration of (49) over $(0,t)$ leads to

$$\mathscr{F}(t)\leq \mathscr{F}(0)e^{-\gamma_3\int_0^t\zeta(s)\mathrm{d}s}, \qquad \forall t\geq 0. \tag{50}$$

A combination of (48) and (50) leads to (36). The proof of Theorem 4.2 is thus completed. □

4.2. Exponential stability for $\mu_1=\mu_2$

In this subsection, we assume that $\mu_1=\mu_2=\mu$. As we will see, we cannot directly perform the same proof as for the case where $\mu_2<\mu_1$. We point out here that in the absence of the viscoelastic damping, that is for $g=0$, Nicaise and Pignotti have proved recently in [17] that for $\mu_1=\mu_2$ some instabilities may occur. Here and due to the presence of the viscoelastic term, we show that the solution is still exponentially stable even for $\mu_1=\mu_2$.

Theorem 4.4. *Let u be the solution of* (1). *Assume that $\mu_1=\mu_2$ and g satisfies* (G1) *and* (G2). *Then for any $t_0>0$, there exist two positive constants $\hat{K}$ and $\hat{\lambda}$ such that the solution of problem* (1) *satisfies*

$$E(t)\leq \hat{K}e^{-\hat{\lambda}\int_{t_0}^t\zeta(s)\,ds}, \qquad \forall t\geq t_0. \tag{51}$$

If $\mu_1=\mu_2=\mu$, then we can choose $\xi=\tau\mu$ in (15) and Lemma 4.1 takes the form

Lemma 4.5. *Suppose that* (G1) *and* (G2) *hold and let* (u,z) *be a solution of the problem* (14)*. Then, the energy functional defined by* (30) *is a nonincreasing function and it satisfies*

$$\frac{dE(t)}{dt} \leq \frac{1}{2}\left(g' \circ \nabla u\right)(t) - \frac{1}{2} g(t) \left\|\nabla u\right\|_2^2 \leq 0, \qquad \forall t \geq 0. \tag{52}$$

The proof of Lemma 4.5 is an immediate consequence of Lemma 4.1, by choosing $\xi = \tau\mu$.
Now, let us introduce the functional:

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-s)\left(u(t) - u(s)\right) \mathrm{d}s\mathrm{d}x. \tag{53}$$

We start with

Lemma 4.6. *Let* (u,z) *be the solution of (14), then we have the estimate*

$$\begin{aligned} \frac{d\chi(t)}{dt} \leq{}& \left(\delta + 2\delta(1-l)^2\right)\left\|\nabla u\right\|_2^2 + \left(\delta_2(1+\mu) - \int_0^t g(s)\,ds\right)\left\|u_t\right\|_2^2 \\ &+ \left(\frac{1-l}{2\delta} + 2\delta(1-l) + \frac{\mu C_*^2}{4\delta_2} + \frac{\mu C_*^2}{4\delta_4}\right)\left(g \circ \nabla u\right)(t) \\ &+ \mu\delta_4 \int_\Omega z^2(x,1,t)\,dx - \frac{g(0)}{4\delta_2} C_*^2 \left(g' \circ \nabla u\right)(t), \end{aligned} \tag{54}$$

where ε_2*,* δ*,* δ_2 *and* δ_4 *are arbitrary positive constants.*

Proof. Differentiate (53) with respect to t, to get using the first equation in (14)

$$\begin{aligned} \chi'(t) ={}& \int_\Omega \nabla u(t) . \left(\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) \mathrm{d}s\right) \mathrm{d}x \\ &- \int_\Omega \left(\int_0^t g(t-s) \nabla u(s)\,\mathrm{d}s\right) . \left(\int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)\mathrm{d}s\right)\mathrm{d}x \\ &- \int_\Omega u_t \int_0^t g'(t-s)\left(u(t) - u(s)\right)\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\,\mathrm{d}s\right)\left\|u_t\right\|_2^2 \\ &- \mu_1 \int_\Omega u_t \int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}x - \mu_2 \int_\Omega z(x,1,t) \int_0^t g(t-s)\left(u(t)-u(s)\right)\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{55}$$

Similarly as in (38), we estimate the right-hand side terms of (55) as follows:
First, using Young's inequality and (10), we obtain for any $\delta > 0$,

$$\begin{aligned} &\left|\int_\Omega \nabla u(t) . \left(\int_0^t g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}x\right| \\ &\quad \leq \delta \left\|\nabla u\right\|_2^2 + \frac{1-l}{4\delta}\left(g \circ \nabla u\right)(t). \end{aligned} \tag{56}$$

Also, the second term can be estimated as follows (see [14])

$$\int_\Omega \left(\int_0^t g(t-\tau)\nabla u(\tau)\,\mathrm{d}\tau\right)\cdot\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}x$$
$$\leq \left(2\delta+\frac{1}{4\delta}\right)(1-l)(g\circ\nabla u)(t)+2\delta(1-l)^2\|\nabla u\|_2^2. \tag{57}$$

Concerning the third term, we have for $\delta_2>0$

$$\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\mathrm{d}x \leq \delta_2\|u_t\|_2^2-\frac{g(0)}{4\delta_2}C_*^2(g'\circ\nabla u)(t). \tag{58}$$

The fifth term can be estimated as follows:

$$\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\mathrm{d}x \leq \delta_2\|u_t\|_2^2+\frac{C_*^2}{4\delta_2}(g\circ\nabla u)(t). \tag{59}$$

For the sixth term, we have

$$\int_\Omega z(x,1,t)\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\mathrm{d}x \leq \delta_4\int_\Omega z^2(x,1,t)\mathrm{d}x+\frac{C_*^2}{4\delta_4}(g\circ\nabla u)(t),\ \ \delta_4>0. \tag{60}$$

Inserting the above estimates (56)–(60) into (55), the assertion of the Lemma 4.6 is established. □

Proof of Theorem 4.4. As in the proof of Theorem 4.2, we define the following Lyapunov function $\hat{\mathscr{L}}$ as:

$$\hat{\mathscr{L}}(t):=NE(t)+\epsilon_1\Psi(t)+\chi(t)+\epsilon_3 I(t) \tag{61}$$

where N, ϵ_1 and ϵ_3 are positive real numbers which will be chosen later.

Since the function g is positive, continuous and $g(0)>0$, then for any $t\geq t_0>0$, we have

$$\int_0^t g(s)\,\mathrm{d}s \geq \int_0^{t_0} g(s)\,\mathrm{d}s = g_0.$$

Now, using (38), (52) and (54), we get

$$\begin{aligned}\frac{d\hat{\mathscr{L}}(t)}{\mathrm{d}t} \leq{}& \left\{\epsilon_1\left(1+\frac{\mu}{4\delta_1}\right)+(\delta_2(1+\mu)-g_0)+\frac{\epsilon_3}{2\tau}\right\}\|u_t\|_2^2-\epsilon_3 I(t)\\ &+\left\{\delta\left(1+2(1-l)^2\right)-\epsilon_1\left(\frac{l}{2}-2\mu\delta_1C_*^2\right)\right\}\|\nabla u\|_2^2\\ &+\left(\frac{N}{2}-\frac{g(0)}{4\delta_2}C_*^2\right)(g'\circ\nabla u)(t)+\left(\frac{\epsilon_1}{4\delta_1}+\mu\delta_4-\frac{\epsilon_3 c}{2\tau}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x\\ &+\left\{\epsilon_1\frac{(1-l)}{2}+\left(\frac{1-l}{2\delta}+2\delta(1-l)+\frac{\mu C_*^2}{4\delta_2}+\frac{\mu C_*^2}{4\delta_4}\right)\right\}(g\circ\nabla u)(t).\end{aligned} \tag{62}$$

Now, we have to choose our constants in (62) very carefully.

First, let us take δ_1 small enough such that

$$2\mu\delta_1C_*^2\leq\frac{l}{4}.$$

Then, we select δ_2 small enough so that

$$\delta_2 (1 + \mu) \leq \frac{g_0}{2}.$$

After that, we pick ϵ_3 so small that

$$\frac{\epsilon_3}{2\tau} \leq \frac{g_0}{8}.$$

Once ϵ_3 is fixed, then we choose δ_4 small so that

$$\mu\delta_4 \leq \frac{\epsilon_3 c}{4\tau}.$$

Further, we take ϵ_1 small such that

$$\epsilon_1 < \min \left(1/\left(1 + \frac{\mu}{4\delta_1}\right) \frac{g_0}{8}, \frac{\delta_1 \epsilon_3 c}{2\tau} \right).$$

Also, let us take δ small so that

$$\delta \left(1 + 2(1 - l)^2\right) < \frac{\epsilon_1 l}{8}.$$

Finally, we choose N large enough such that

$$\frac{N}{4} > \frac{g(0)}{4\delta_2} C_*^2.$$

Consequently, there exist two positive constants $\hat{\gamma}_1$ and $\hat{\gamma}_2$ such that

$$\frac{d\hat{\mathscr{L}}(t)}{\mathrm{d}t} \leq -\hat{\gamma}_1 E(t) + \hat{\gamma}_2 (g \circ \nabla u)(t), \qquad \forall t \geq t_0. \tag{63}$$

The remaining part of the proof of inequality (51) can be finished, following the same steps as in the proof of Theorem 4.2; we omit the details.

The last step in the proof of Theorem 4.4 is to show that $\hat{\mathscr{L}}(t)$ and $E(t)$ are equivalent. So, we have the following lemma.

Lemma 4.7. *There exist two positive constants $\hat{\beta}_1$ and $\hat{\beta}_2$ depending on N, ϵ_1 and ϵ_3, such that*

$$\hat{\beta}_1 E(t) \leq \hat{\mathscr{L}}(t) \leq \hat{\beta}_2 E(t), \qquad \forall t \geq 0. \tag{64}$$

Proof. We consider the functional

$$H(t) = \epsilon_1 \Psi(t) + \chi(t) + \epsilon_3 I(t)$$

and show that

$$|H(t)| \leq CE(t), \qquad C > 0. \tag{65}$$

Using Young's inequality, Poincaré's inequality and Lemma 2.2, we obtain

$$\begin{aligned} |\chi(t)| &= \left| \int_\Omega u_t \int_0^t g(t-s)(u(t) - u(s))\,\mathrm{d}s\mathrm{d}x \right| \\ &\leq \frac{1}{2} \int_\Omega u_t^2 \mathrm{d}x + \frac{1}{2} \int_\Omega \left(\int_0^t g(t-s)(u(t) - u(s))\,\mathrm{d}s \right)^2 \mathrm{d}x \\ &\leq \frac{1}{2} \int_\Omega u_t^2 \mathrm{d}x + \frac{1}{2}(1 - l) C_*^2 (g \circ \nabla u)(t). \end{aligned} \tag{66}$$

Similarly, we have

$$|\epsilon_1 \Psi(t) + \epsilon_3 I(t)| = \left| \epsilon_1 \int_\Omega u_t u \mathrm{d}x \right| + \left| \epsilon_3 \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x \right|$$
$$\leq \frac{\epsilon_1}{2} \int_\Omega u_t^2 \mathrm{d}x + \frac{\epsilon_1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x + \epsilon_3 \hat{c} \int_\Omega \int_0^1 z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x. \tag{67}$$

Using $1 - \int_0^t g(s)\,\mathrm{d}s \geq l$, (30), (66) and (67), we get (65) for some positive constant C. Now, it is obvious that from (61), (65) and choosing N large enough, our result is proved. □

5. Examples and concluding remarks

In this section, we give some examples to illustrate our results and we conclude with few remarks, pointing out some open problems and future directions worth pursuing.

Example 5.1. *Let*

$$g(t) = a_1 e^{-b(1+t)^\nu}, \quad \textit{with } a,\, b,\, \nu > 0,$$

It is then clear that (11) *holds for* $\zeta(t) = b\nu(1+t)^{\min(0,\nu-1)}$. *Consequently, applying* (36), *we obtain the following exponential decay*

$$E(t) \leq K e^{-\lambda b (1+t)^{\min(1,\nu)}}.$$

Example 5.2. *If*

$$g(t) = a e^{-b[\ln(1+t)]^\nu} \quad \textit{with } a,\, b > 0,\ \ \nu > 1.$$

Then for

$$\zeta(t) = \frac{b\nu \left(\ln(1+t)\right)^{\nu-1}}{1+t},$$

the inequality (36) *gives*

$$E(t) \leq K e^{-\lambda b (\ln(1+t))^\nu}.$$

Example 5.3. *If*

$$g(t) = \frac{a}{(2+t)^\nu \left(\ln(2+t)\right)^b},$$

where

$$a > 0 \ \textit{and} \ \begin{cases} \nu > 1 & \textit{and} \quad b \in \mathbb{R} \\ & \textit{or} \\ \nu = 1 & \textit{and} \quad b > 1 \end{cases}.$$

Then for

$$\zeta(t) = \frac{\nu\left(\ln(2+t)\right) + b}{(2+t)\left(\ln(2+t)\right)^b},$$

we obtain from (36)

$$E(t) \leq \frac{K}{\left[(2+t)^{\nu_1} \left(\ln(2+t)\right)^b\right]^\lambda}.$$

Remark 5.4. *The above examples clearly illustrate the effect of the behavior of the function g on the decay rate of the total energy of our problem. For example, the exponential decay of the relaxation function g is a sufficient condition to the exponential decay of the solution, whereas for $g = 0$ and $\mu_1 > \mu_2$, the solution of problem* (1) *decays exponentially. See* [17] *for more details.*

5.1. Open problems

There are many interesting questions to be investigated in connection with the problem we have addressed here. We mention here some of them.

- It is clear that the presence of the linear damping term $\mu_1 u_t$ in the first equation of problem (1) plays a decisive role in our proofs. Thus, the problem of whether the stability properties we have proved here are preserved when $\mu_1 = 0$ is open.
- As we have mentioned above, for the wave equation, some instability results have been shown in [17] in the case $\mu_2 \geq \mu_1$. It would be interesting to study the case $\mu_2 > \mu_1$ in our problem (1).
- It would be interesting to investigate the viscoelastic wave equation with a time delay in the boundary condition and a velocity term in the equation. Concerning the wave equation, Datko et *al* [9] treated the one-dimensional problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, & 0 < x < 1, \ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -ku(1,t-\tau), & t > 0, \end{cases} \tag{68}$$

  and show that if the positive constants a and k satisfy

$$k\frac{e^{2a}+1}{e^{2a}-1} < 1,$$

  then the delayed feedback system (68) is stable for all sufficiently small delays.
- We have considered here general decay of the relaxation function g, but the best decay rate that we obtained is exponential. What happens when the function g decays faster than exponentially?

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Mokhtar Kirane
Mathématiques
Image et Applications Pole Sciences et Technologie
Université de la Rochelle
La Rochelle
France
e-mail: mokhtar.kirane@univ-lr.fr

Belkacem Said-Houari
Department of Mathematics and Statistics
University of Konstanz
78457 Konstanz
Germany
e-mail: saidhouarib@yahoo.fr

Belkacem Said-Houari (Current address)
Division of Mathematical and Computer Sciences and Engineering
4700 King Abdullah University of Science and Technology (KAUST)
Thuwal 23955-6900
Saudi Arabia
e-mail: saidhouarib@yahoo.fr

(Received: June 16, 2010; revised: March 23, 2011)