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Exponential stability of an elastic string with local Kelvin–Voigt damping

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Abstract. This paper is devoted to analyzing an elastic string with local Kelvin–Voigt damping. We prove the exponential stability of the system when the material coefficient function near the interface is smooth enough. Our method is based on the frequency method and semigroup theory.

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1. Introduction

In this paper, we study the following elastic wave equation with local Kelvin–Voigt damping.

$$\begin{cases} u_{tt}(t,x) - [u_x(t,x) + \beta(x)u_{xt}(t,x)]_x = 0 & t > 0, \ \ -1 < x < 1, \\ u(t,-1) = u(t,1) = 0 & t \geq 0, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) & -1 \leq x \leq 1. \end{cases} \tag{1.1}$$

The function $\beta(x) \in C^1[-1,1]$ is assumed to be

$$\beta(x) = \begin{cases} a(x), & x \in (0,1], \\ 0, & x \in [-1,0], \end{cases} \tag{1.2}$$

where $a(x) \in C[0,1]$ is nonnegative.

The natural energy of system (1.1) is

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_t(t,x)|^2 + |u_x(t,x)|^2\right] \mathrm{d}x,$$

and it is dissipated according to the following law:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\int_0^1 a(x)|u_{xt}(t,x)|^2 \mathrm{d}x. \tag{1.3}$$

Formula (1.3) shows that the only dissipative mechanism acting on the system is the local Kelvin–Voigt damping $[\beta(x)u_{xt}(t,x)]_x$, which is effective on $(0,1]$. In this paper, we analyze the problem how the dissipative mechanism introduced by the local Kelvin–Voigt damping affects the long time behavior of the energy of system (1.1).

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There are a number of publications concerning the wave equation with local viscous damping (see [1,2,5]). However, only a few results are known for the wave equation with local Kelvin–Voigt damping, which is unbounded in the energy space. Liu and Liu ([6]) proved that the energy of system (1.1) does not decay exponentially if $a(\cdot) \equiv 1$. In this case, local viscoelastic damping reacts strongly on the damped region. However, purely elastic waves are almost completely reflected on the interface because of the discontinuity of material properties at the interface, and then are very weakly dissipated by the viscoelastic damping mechanism. It was proven that the non-exponential stability result also holds for the higher dimensional wave equation with local viscoelastic damping when the material is not continuous at the interface ([12]).

More recently, it was shown that if the material parameter $\beta(\cdot)$ is smooth enough at the interface, the energy of the wave equation with local viscoelastic damping decays exponentially ([7,8]). In [7], the exponential stability of system (1.1) was established for local viscoelastic damping with damping coefficient function $\beta(\cdot) \in C^2[-1,1]$. The higher dimensional case can be found in [8]. Renardy proved that for a special function $\beta(\cdot)$, eigenvalues of the one-dimensional problem are such that the decay rate tends to infinity with frequency ([11]).

In this paper, we study the system with continuous coefficient function $\beta(\cdot)$ which vanishes on the interface and do not belong to $C^2[-1,1]$. By using an equivalent condition of the exponential stability given by Huang [4], we obtain the uniform exponential decay of system (1.1).

This paper is organized as follows. In Sect. 2, we show some preliminary results and state the main result. Section 3 is devoted to the proof of the exponential stability of the system.

2. Preliminaries and main results

We first formulate system (1.1) as an abstract Cauchy problem. Introduce the following Hilbert space:

$$\mathcal{H} = H_0^1(-1,1) \times L^2(-1,1),$$

with norm

$$\|U\|_{\mathcal{H}} = \sqrt{\|u\|^2_{H_0^1(-1,1)} + \|v\|^2_{L^2(-1,1)}}, \qquad \forall\, U = (u,v) \in \mathcal{H}.$$

Define an unbounded operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ by

$$\begin{cases} A(u,v) = (v,\ (u' + \beta v')')\,, \\ D(A) = \left\{(u,v) \in \mathcal{H} \;\middle|\; v \in H_0^1(-1,1),\ (u' + \beta v')' \in L^2(-1,1)\,\right\}. \end{cases} \tag{2.1}$$

Let $U(t) \doteq \big(u(t), u_t(t)\big) \in \mathcal{H}$. Then system (1.1) can be written as an abstract evolution equation on Hilbert space $\mathcal{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t) = AU(t), \quad U(0) = U_0 \in \mathcal{H}. \tag{2.2}$$

The following lemma implies the well-posedness of system (1.1).

Lemma 2.1. (Liu and Liu [6]) *Assume that function $a(\cdot) \in C[0,1]$ and $a(\cdot) > 0$ on $(0,1]$. Then operator A generates a C_0 semigroup e^{tA} of contraction on $\mathcal{H}$, with $0 \in \rho(A)$, the resolvent set of A.*

To obtain the exponential stability of system (1.1), we assume that function $a(\cdot)$ satisfies the following conditions.

(**A1**) $a(\cdot) \in C^1[0,1]$, $a(0) = a'(0) = 0$, $a(x) > 0$ for all $x \in (0,1]$.

(**A2**) There exists a positive constant a_0 such that $\int_0^x \frac{|a'(s)|^2}{a(s)}\,\mathrm{d}s \le a_0 |a'(x)|$ for all $x \in [0,1]$.

Lemma 2.2. *Let function* $a(\cdot)$ *satisfy* (A1) *and* (A2). *Assume a function* $y(\cdot) \in H^1(0,1)$ *satisfies* $y(1) = 0$. *Then,*

$$\|a'a^{-1/2}y\|_{L^2(0,1)} \leq 2a_0\|a^{1/2}y'\|_{L^2(0,1)},$$

where a_0 *is the positive constant defined in (A2).*

Proof. Set $\Phi(x) \doteq \int_0^x \frac{|a'(s)|^2}{a(s)}\,\mathrm{d}s$ for $x \in [0,1]$. Then we have by (A2) that

$$\frac{|\Phi(x)|^2}{a(x)} \leq a_0^2\frac{|a'(x)|^2}{a(x)} = a_0^2\Phi'(x). \tag{2.3}$$

□

Therefore, from (2.3) and the fact that $y(1) = 0$,

$$\begin{aligned}\int_0^1 \Phi'(x)|y(x)|^2\mathrm{d}x &= -2\mathcal{R}e\int_0^1 \Phi(x)y(x)\overline{y'(x)}\mathrm{d}x\\ &\leq \frac{1}{2a_0^2}\int_0^1 \frac{|\Phi(x)|^2}{a(x)}|y(x)|^2\mathrm{d}x + 2a_0^2\int_0^1 a(x)|y'(x)|^2\mathrm{d}x\\ &\leq \frac{1}{2}\int_0^1 \Phi'(x)|y(x)|^2\mathrm{d}x + 2a_0^2\int_0^1 a(x)|y'(x)|^2\mathrm{d}x.\end{aligned} \tag{2.4}$$

Consequently,

$$\int_0^1 \Phi'(x)|y(x)|^2\mathrm{d}x \leq 4a_0^2\int_0^1 a(x)|y'(x)|^2\mathrm{d}x. \tag{2.5}$$

□

The main result of this paper is as follows.

Theorem 2.1. *Suppose that function* $a(\cdot)$ *satisfies* (A1) *and* (A2). *Then the energy of system* (1.1) *is exponentially stable, i.e., there exist* C, $\delta > 0$ *such that*

$$E(t) \leq Ce^{-\delta t}E(0), \quad \forall t \geq 0.$$

*Remark 2.1***.** It is easy to verify that function $a(x) = x^\alpha$ $(\alpha > 1)$ satisfies $(A1)$ and $(A2)$. Therefore, we extend the exponential stability result in [8], which needed the corresponding coefficient function $\beta(\cdot) \in C^2[-1,1]$. It is natural to expect the uniform exponential stability of (1.1) when $\beta(\cdot) \in C[-1,1]\backslash C^2[-1,1]$. Furthermore, it could be interesting to analyze whether the exponential decay result in [8] obtained by multiplier technique in some particular geometric configuration. These problems are open.

3. Proofs

In this section, we prove Theorem 2.1. The idea is to estimate the energy norm and boundary terms at the interface by the local viscoelastic damping. The difficulty is to deal with the higher order boundary term at the interface so that the energy on subdomain $(-1,\ 0)$ can be controlled by the viscoelastic damping on $(0,\ 1)$.

Our proof is based on the following result [4].

Lemma 3.1. *Let e^{tA} be a C_0 semigroup in Hilbert space $\mathcal{H}$, and there exists a positive constant M such that $\|e^{tA}\|_{\mathcal{L}(\mathcal{H})} \leq M$ $(t \geq 0)$. Then e^{tA} is exponentially stable if and only if $\{\lambda \in \mathbb{C} \mid \mathcal{R}e\lambda = 0\} \in \rho(A)$ and* $\sup\{\|(\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \mid \mathcal{R}e\lambda = 0\} < \infty$.

Proof of Theorem 2.1. By Lemma 3.1, it suffices to show that there exists $r > 0$ such that

$$\inf\left\{\|\mathrm{i}\omega U - AU\|_{\mathcal{H}} \,\middle|\, \|U\|_{\mathcal{H}} = 1, \omega \in \mathbb{R}\right\} \geq r. \tag{3.1}$$

We suppose that (3.1) fails. Then there exist a sequence of real numbers $\omega_n \neq 0$ and a sequence of functions $U_n = (u_n, v_n) \in \mathcal{D}(A)$ with $\|U_n\|_{\mathcal{H}} = 1$ such that

$$\|\mathrm{i}\omega_n U_n - AU_n\|_{\mathcal{H}} \to 0. \tag{3.2}$$

Define

$$y_{1,n} \doteq u_n\chi_{[0,1]}, \quad y_{2,n} \doteq v_n\chi_{[0,1]}, \quad z_{1,n} \doteq u_n\chi_{[-1,0]}, \quad z_{2,n} \doteq v_n\chi_{[-1,0]}.$$

Then, by (3.2),

$$f_{1,n} \doteq \mathrm{i}\omega_n y_{1,n} - y_{2,n} \to 0 \quad \text{in} \quad H^1(0,1), \tag{3.3}$$

$$f_{2,n} \doteq \mathrm{i}\omega_n y_{2,n} - T_n' \to 0 \quad \text{in} \quad L^2(0,1), \tag{3.4}$$

$$g_{1,n} \doteq \mathrm{i}\omega_n z_{1,n} - z_{2,n} \to 0 \quad \text{in} \quad H^1(-1,0), \tag{3.5}$$

$$g_{2,n} \doteq \mathrm{i}\omega_n z_{2,n} - z_{1,n}'' \to 0 \quad \text{in} \quad L^2(-1,0), \tag{3.6}$$

where

$$T_n = y_{1,n}' + a y_{2,n}'.$$

It follows from (3.2) that $\lim\limits_{n\to\infty} \mathcal{R}e\big((\mathrm{i}\omega_n I - A)U_n,\, U_n\big)_{\mathcal{H}} = 0$. Thus, a direct computation gives that

$$\lim_{n\to\infty} \|a^{1/2} y_{2,n}'\|_{L^2(0,1)} = 0. \tag{3.7}$$

We shall estimate $\|U_n\|_{\mathcal{H}}$ by the dissipative term $\|a^{1/2} y_{2,n}'\|_{L^2(0,1)}$, then reach the contradiction to the hypothesis $\|U_n\|_{\mathcal{H}} = 1$. The proof is divided into four steps.

Step 1. We multiply (3.4) by qT_n, where $q \in C^2[0,1]$ is an arbitrary real-value function,

$$\int_0^1 (\mathrm{i}\omega_n y_{2,n} - T_n') q\overline{T_n}\mathrm{d}x \to 0. \tag{3.8}$$

By (3.3), we have

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{R}e \int_0^1 \mathrm{i}\omega_n y_{2,n} q\left(\overline{y_{1,n}' + a y_{2,n}'}\right)\mathrm{d}x \\ &= \frac{1}{2}\lim_{n\to\infty}\left(q(0)|y_{2,n}(0)|^2 + \int_0^1 q'|y_{2,n}|^2\mathrm{d}x\right) - \mathcal{I}m \lim_{n\to\infty} \omega_n \int_0^1 q a y_{2,n}\overline{y_{2,n}'}\mathrm{d}x.\end{aligned} \tag{3.9}$$

Furthermore, a direct computation gives

$$-\mathcal{R}e \int_0^1 T_n' q T_n \mathrm{d}x = -\frac{1}{2}\left(q(1)|T_n(1)|^2 - q(0)|T_n(0)|^2\right) + \frac{1}{2}\int_0^1 q'|T_n|^2\mathrm{d}x. \tag{3.10}$$

Thus, replacing (3.9) and (3.10) into (3.8) yields

$$\frac{1}{2}\int_0^1 q'|y_{2,n}|^2\mathrm{d}x + \frac{1}{2}\int_0^1 q'|T_n|^2\mathrm{d}x - \omega_n \mathcal{I}m \int_0^1 qay_{2,n}\overline{y'_{2,n}}\mathrm{d}x$$
$$+\frac{1}{2}\left[q(0)|y_{2,n}(0)|^2 - q(1)|T_n(1)|^2 + q(0)|T_n(0)|^2\right] \to 0. \quad (3.11)$$

In the next steps, we will estimate the last four terms on the left hand side of (3.11).

Step 2. In this step, we estimate the third term in (3.11). Multiplying (3.4) by $\mathrm{i}\omega_n ay_{2,n}$,

$$\lim_{n\to\infty}\|\omega_n a^{1/2}y_{2,n}\|^2_{L^2(0,1)} = -\lim_{n\to\infty}\left[\mathrm{i}\omega_n(T'_n, ay_{2,n})_{L^2(0,1)} + \mathrm{i}\omega_n(f_{2,n}, ay_{2,n})_{L^2(0,1)}\right]. \quad (3.12)$$

It is clear from (3.4) that

$$\lim_{n\to\infty}|-\mathrm{i}\omega_n(f_{2,n}, ay_{2,n})_{L^2(0,1)}| \le \frac{1}{4}\lim_{n\to\infty}\|\omega_n a^{1/2}y_{2,n}\|^2_{L^2(0,1)}. \quad (3.13)$$

Moreover,

$$\mathcal{R}e\left[-\mathrm{i}\omega_n\int_0^1 T'_n a\overline{y_{2,n}}\mathrm{d}x\right] = \mathcal{R}e\left[\mathrm{i}\omega_n\int_0^1 (a'y'_{1,n}\overline{y_{2,n}} + aa'y'_{2,n}\overline{y_{2,n}} + ay'_{1,n}\overline{y'_{2,n}})\mathrm{d}x\right]. \quad (3.14)$$

By (3.3), (3.7) and Lemma 2.2,

$$\lim_{n\to\infty}\left|\mathrm{i}\omega_n\int_0^1 a'y'_{1,n}\overline{y_{2,n}}\mathrm{d}x\right| \le \lim_{n\to\infty}\|a^{1/2}y'_{2,n}\|_{L^2(0,1)}\|a'a^{-1/2}y_{2,n}\|_{L^2(0,1)} = 0. \quad (3.15)$$

Furthermore, there exists a positive constant C such that

$$\lim_{n\to\infty}\left|\mathrm{i}\omega_n\int_0^1 aa'y'_{2,n}\overline{y_{2,n}}\mathrm{d}x\right| \le C\lim_{n\to\infty}\|a^{1/2}y'_{2,n}\|^2_{L^2(0,1)} + \frac{1}{4}\lim_{n\to\infty}\|\omega_n a^{1/2}y_{2,n}\|^2_{L^2(0,1)}$$
$$= \frac{1}{4}\lim_{n\to\infty}\|\omega_n a^{1/2}y_{2,n}\|^2_{L^2(0,1)}. \quad (3.16)$$

From (3.3) and (3.7),

$$\lim_{n\to\infty}\mathrm{i}\omega_n\int_0^1 ay'_{1,n}\overline{y'_{2,n}}\mathrm{d}x = \lim_{n\to\infty}\int_0^1 a|y'_{2,n}|^2\mathrm{d}x = 0. \quad (3.17)$$

Therefore, replacing (3.15)–(3.17) into (3.14), we have that

$$\lim_{n\to\infty}\mathcal{R}e\left[-\mathrm{i}\omega_n\int_0^1 T'_n a\overline{y_{2,n}}\mathrm{d}x\right] \le \frac{1}{4}\lim_{n\to\infty}\|\omega_n a^{1/2}y_{2,n}\|^2_{L^2(0,1)}. \quad (3.18)$$

Now, combining (3.12), (3.13) and (3.18) yields

$$\lim_{n\to\infty}\|\omega_n a^{1/2}y_{2,n}\|^2_{L^2(0,1)} = 0. \quad (3.19)$$

Consequently,

$$\left|\omega_n\mathcal{I}m\int_0^1 qay_{2,n}\overline{y'_{2,n}}\mathrm{d}x\right| \le C\|\omega_n a^{1/2}y_{2,n}\|_{L^2(0,1)}\|a^{1/2}y'_{2,n}\|_{L^2(0,1)} \to 0, \quad C>0. \quad (3.20)$$

Thus, combining (3.11) with (3.20), we have

$$\frac{1}{2}\int_0^1 q'|y_{2,n}|^2\mathrm{d}x + \frac{1}{2}\int_0^1 q'|T_n|^2\mathrm{d}x + \frac{1}{2}\left[(q(0)|y_{2,n}(0)|^2 - q(1)|T_n(1)|^2 + q(0)|T_n(0)|^2\right] \to 0. \tag{3.21}$$

Step 3. In this step, we shall estimate the sequence $\{|T_n(1)|\}_{n=1}^{\infty}$. Multiplying (3.4) by $ay_{1,n}$ and using (3.3), we have

$$\lim_{n\to\infty}\left[-\int_0^1 a|y_{2,n}|^2\mathrm{d}x + \int_0^1 (y'_{1,n} + ay'_{2,n})\overline{(a'y_{1,n} + ay'_{1,n})}\mathrm{d}x\right] = 0. \tag{3.22}$$

We now estimate the last integral on the left hand side of (3.22). First, it follows from (3.3) and (3.7) that

$$\lim_{n\to\infty}\|a^{1/2}y'_{1,n}\|_{L^2(0,1)} = \lim_{n\to\infty}\|a^{1/2}T_n\|_{L^2(0,1)} = 0. \tag{3.23}$$

By using (3.23) and Lemma 2.2,

$$\left|\int_0^1 a'y'_{1,n}\overline{y_{1,n}}\mathrm{d}x\right| \le \|a^{1/2}y'_{1,n}\|_{L^2(0,1)}\|a'a^{-1/2}y_{1,n}\|_{L^2(0,1)} \to 0. \tag{3.24}$$

Furthermore, from (3.7), (3.23), we have

$$\left|\int_0^1 aa'y'_{2,n}\overline{y_{1,n}}\mathrm{d}x\right| \le C\|a^{1/2}y'_{2,n}\|_{L^2(0,1)}\|y_{1,n}\|_{L^2(0,1)} \to 0, \tag{3.25}$$

and

$$\left|\int_0^1 (y'_{1,n} + ay'_{2,n})a\overline{y'_{1,n}}\mathrm{d}x\right| \le \|a^{1/2}y'_{1,n}\|^2_{L^2(0,1)} + \|a^{1/2}y'_{1,n}\|_{L^2(0,1)}\|a^{1/2}y'_{2,n}\|_{L^2(0,1)} \to 0. \tag{3.26}$$

Hence, replacing (3.24)–(3.26) into (3.22),

$$\lim_{n\to\infty}\|a^{1/2}y_{2,n}\|_{L^2(0,1)} = 0. \tag{3.27}$$

Therefore, by taking $q = \int_0^x a(s)\mathrm{d}s$ in (3.21) and using (3.23), (3.27) and (A1), we have

$$\lim_{n\to\infty}|T_n(1)| = 0. \tag{3.28}$$

Combining (3.21) with (3.28) yields

$$\frac{1}{2}\int_0^1 q'(|y_{2,n}|^2 + |T_n|^2)\mathrm{d}x + \frac{1}{2}\left[q(0)|y_{2,n}(0)|^2 + q(0)|T_n(0)|^2\right] \to 0. \tag{3.29}$$

Step 4. By setting $q = x$ in (3.29), we obtain

$$\lim_{n\to\infty}\|y_{2,n}\|_{L^2(0,1)} = \lim_{n\to\infty}\|T_n\|_{L^2(0,1)} = 0. \tag{3.30}$$

Thus, we can deduce from (3.7) and (3.30) that

$$\lim_{n\to\infty}\|y'_{1,n}\|_{L^2(0,1)} = 0. \tag{3.31}$$

Furthermore, taking $q = 1 - x$ in (3.29), we have from (3.30) that

$$\lim_{n\to\infty}y_{2,n}(0) = \lim_{n\to\infty}T_n(0) = 0. \tag{3.32}$$

On the other hand, we take the inner product of (3.6) with $(1+x)z'_{1,n}$ on $L^2(-1,0)$ and use (3.5),

$$-|z_{2,n}(0)|^2+\int_{-1}^{0}|z_{2,n}|^2\mathrm{d}x-|z'_{1,n}(0)|^2+\int_{-1}^{0}|z'_{1,n}|^2\mathrm{d}x\to 0. \tag{3.33}$$

Therefore, by (3.32), (3.33) and the fact that $z_{2,n}(0)=y_{2,n}(0)$, $z'_{1,n}(0)=y'_{1,n}(0)=T_n(0)$, we obtain

$$\lim_{n\to\infty}\|z'_{1,n}\|_{L^2(-1,0)}=\lim_{n\to\infty}\|z_{2,n}\|_{L^2(-1,0)}=0. \tag{3.34}$$

In summary, we have from (3.30), (3.31) and (3.34) that $\lim_{n\to\infty}\|U_n\|_{\mathcal{H}}=0$, and then reach the contradiction. The proof is completed. □

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