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An extended similarity theory applied to heated flows in complex geometries

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Abstract. In the traditional similarity theory the influence of temperature- and pressure-dependent fluid properties on the flow field and heat transfer is not described by the basic dimensionless parameters, i.e. Prandtl, Reynolds, Rayleigh, ... number. We present an extended similarity theory that not only takes into account the variable material properties but also can handle small variations in other parameters of the physical model like small changes in the (reference) Prandtl number. The method has general applicability that is suitable for a wide variety of fluid dynamic and heat transfer situations in which variable properties with a strong dependence on temperature and pressure play a significant role. It is especially useful in predicting the behaviour of a certain fluid based on the results for a different one. As an example the Nußelt number of a lid driven heated cavity is determined with fluid properties being temperature dependent.

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1. Introduction

The four physical fluid properties:

- ρ^* (density)
- c_p^* (heat capacity)
- k^* (heat conductivity)
- μ^* (dynamic viscosity)

are in general pressure and temperature dependent. The straight forward way to take these dependences into account requires the insertion of the exact functional formulas into the basic mathematical model equations. Results obtained in this way only hold true for one fluid and one set of boundary conditions including a certain rate of heating. Transfering the results to other fluids or flow and heat transfer regimes in terms of a similarity analysis is not possible. Recent applications of this approach are Kim, Choi [14], Pantokratoras [19], [20], Ali [1], Valueva [22], Maleque, Satter [18], Ben-Mansour, Sahin [3], Hernandez, Zamora [8], Depcik et al. [5], Mahmood et al. [17] and Ligrani, Mahmood [16].

A more general approximate (ad hoc) method is to correct the solutions obtained for constant fluid properties. Classical methods of this type are the property ratio and the reference temperature method.

For example, with the *property ratio method* the result for constant fluid properties, the Nußelt-number Nu_{cp} (cp means constant properties), is corrected in the following way when only temperature dependences are considered:

$$Nu = Nu_{cp} \left(\frac{\rho^*(T_1^*)}{\rho^*(T_2^*)}\right)^{n_\rho} \left(\frac{c_p^*(T_1^*)}{c_p^*(T_2^*)}\right)^{n_{c_p}} \left(\frac{k^*(T_1^*)}{k^*(T_2^*)}\right)^{n_k} \left(\frac{\mu^*(T_1^*)}{\mu^*(T_2^*)}\right)^{n_\mu}.$$

In this formula T_1^*, T_2^* are two characteristic temperatures and $n_\rho, n_{c_p}, n_k, n_\mu$ are empirically determined exponents. The big disadvantage of this approach is that the exponents actually depend on T_1^*, T_2^* though often they are assumed to be constants.

The *reference temperature method* uses a fictitious reference temperature, T_R^*, that in a similar way is empirically determined under the condition that the theoretical results obtained for constant property calculations at this reference temperature represent the real situation for variable fluid properties (see, for example, Benard et al.[2], Jayaraj et al. [13] and Debrestian, Anderson [4]). The reference temperature is denoted in the form $T_R^* = T_1^* + j(T_2^* - T_1^*)$, where T_1^* and T_2^* are characteristic temperatures and j is the empirically determined reference temperature factor. Again, the disadvantage is that j actually depends on T_1^* and T_2^*.

As an alternative an asymptotic method based on an earlier work by the second author, see Herwig, Schäfer [10] and Herwig [9], will be described in detail in the next section. It expands the difference between the constant and the variable property solution in an asymptotic way. This method, called *asymptotic correction method*, uses a perturbation parameter which is a small dimensionless characteristic temperature or pressure difference of the actual flow and heat transfer problem. This method can also be used to convert the empirical property ratio and reference temperature methods into analytical methods as shown in Gersten, Herwig [6].

A main issue of this approach is to present the theory in a dimensionless form. Then, results obtained for one fluid can be systematically transferred to a different fluid. This can be done in terms of an extended similarity theory even when certain parameters of the physical model, like geometry parameters, undergo slight nonsimilar variations. In the last section this method is applied to compute Nußelt numbers for a benchmark example (heated and lid driven cavity) with several different fluids and boundary conditions.

Nomenclature

dimensional quantities

symbol	name	unit
ρ^*	density	kg/m^3
c_p^*	heat capacity	J/(kgK)
k^*	heat conductivity	W/(mK)
μ^*	dynamic viscosity	kg/(ms)
a^*	variable fluid property	-
T^*	temperature	K
p^*	pressure	Pa
ΔT^*	temperature difference	K
L^*	characteristic length	m
$x^*, y*$	cartesian coordinates	m
w^*	characteristic velocity	m/s
$\dot{q}^*$	local heat flux density	W/(m^{2}K)
g^*	gravitational acceleration	m/s^2
B^*	box width	m
H^*	box height	m
T_h^*	hot wall temperature	K
T_c^*	cold wall temperature	K

nondimensional quantities and groups

symbol	name
x, y	cartesian coordinates
Nu	Nußelt number
Re	Reynolds number
Pr	Prandtl number
Gr	Grashof number
Fr	Froude number
Ec	Eckert number
Ma	Mach number
ϵ	temperature difference
θ	temperature
K_a, K_{a2}	K-values, $a = \rho, c_p, k, \mu$
A_a, A_{a2}, A_{ab}	A-values, $a, b = \rho, c_p, k, \mu$
H	box height

Indices

symbol	meaning
R	reference state
cp	constant properties
W	wall
l,r	left/right wall

2. The ACFD-method

With heat transfer applications in mind we explain the method for a situation in which the Nußelt number is sought. We restrict ourselves to Newtonian fluids whose properties $\rho^*, c_p^*, k^*, \mu^*$ are temperature dependent but have a negligible pressure dependence. The asymptotic correction method can also be called ACFD-method which stands for **a**symptotic **c**omputational **f**luid **d**ynamics. It can best be subdivided into four aspects.

(1) Taylor series expansion of the fluid properties. All temperature dependent fluid properties $a^* = a^*(T^*)$, where a^* represents ρ^*, c_p^*, μ^* and k^*, are expanded in a Taylor series around a reference temperature T_R^* up to an order $n \geq 1$. The nondimensional representation of this Taylor series expansion reads:

$$\begin{aligned} a &:= \frac{a^*}{a_R^*} = 1 + \varepsilon K_{a1}\theta + \frac{1}{2}\varepsilon^2 K_{a2}\theta^2 + \cdots + \frac{1}{n!}\varepsilon^n K_{an}\theta^n + O(\varepsilon^{n+1}) \\ &= 1 + h_{a1}\theta + \frac{1}{2}h_{a2}\theta^2 + \cdots + \frac{1}{n!}h_{an}\theta^n + O(\varepsilon^{n+1}) \end{aligned} \tag{2.1}$$

with: $a_R^* := a^*(T_R^*)$

$$\varepsilon := \frac{\Delta T_R^*}{T_R^*}$$

$$\begin{aligned}
\theta &:= \frac{T^* - T_R^*}{\Delta T_R^*} \\
K_{a1} &:= \left[\frac{\partial a^*}{\partial T^*}\frac{T^*}{a^*}\right]_R \\
K_{a2} &:= \left[\frac{\partial^2 a^*}{\partial T^{*2}}\frac{T^{*2}}{a^*}\right]_R \\
K_{an} &:= \left[\frac{\partial^n a^*}{\partial T^{*n}}\frac{T^{*n}}{a^*}\right]_R \\
h_{aj} &:= \varepsilon^j K_{aj} = \left[\frac{\partial^j a^*}{\partial T^{*j}}\frac{\Delta T^{*j}}{a^*}\right]_R, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{2.2}$$

In this notation ΔT_R^* is a characteristic dimensional temperature difference of the problem under consideration. The nondimensional temperature derivatives K_{a1}, K_{a2}, …, K_{an} of order 1 to n are material properties and are called *K-values*. By definition, they determine the temperature influenced deviation of the property a^* referred to the reference value a_R^* up to a certain order. The combinations h_{aj} are called *temperature influence factors*.

fluid	air	CO_2	SF_6	NH_3
$\rho^*/(\mathrm{kg/m^3})$	1.188	1.815	6.07	0.707
$10^{-6}\mu^*/(\mathrm{kg/ms})$	18.185	14.69	14.695	9.911
$10^{-3}k^*/(\mathrm{W/mK})$	25.721	16.22	13.17	23.57
$c_p^*/(\mathrm{kJ/kgK})$	1.014	0.846	0.652	2.164
Pr	0.717	0.766	0.727	0.910
$K_{\rho 1}$	-1.000	-1.019	-1.037	-1.046
$K_{\rho 2}$	2.000	2.144	2.037	2.303
$K_{\mu 1}$	0.775	0.969	1.05	1.077
$K_{\mu 2}$	-0.352	-0.155	-0.376	0.390
K_{k1}	0.891	1.441	1.51	1.499
K_{k2}	-0.257	0.761	-0.445	0.699
$K_{c_p 1}$	0.068	0.333	0.672	-0.024
$K_{c_p 2}$	-0.076	0.035	-0.567	1.718

Table 1. K-values for different fluids at $T_R^* = 293$ K and $p_R^* = 1$ bar

As an example K-values K_{a1}, K_{a2} of first and second order at $T_R^* = 293$ K and $p_R^* = 1$ bar are given in table 1 for air, carbon dioxide (CO_2), sulfur hexafluoride (SF_6) and ammonia (NH_3). The K-values for air were taken from [7], those for CO_2, SF_6 and NH_3 were computed from [23]. Density K-values $K_{\rho 1}$, $K_{\rho 2}$ of SF_6 (missing in [23]) are based on data taken from the public online database of the National Institute of Standards and Technology (NIST). All K-values of table 1 will be used in our benchmark example in the last section.

(2) Asymptotic expansion of the difference between the constant and the variable property results. In our benchmark example we want to describe the local Nußelt

Nr.	dimensional influence quantity	notation	unit
1	local heat flux density	$\dot{q}^*_W$	kg/s^3
2	coordinate	y^*	m
3	characteristic length	L^*	m
4	characteristic velocity	w^*	m/s
5	characteristic temperature difference	ΔT^*_R	K
6	reference density	ρ^*_R	kg/m^3
7	reference heat capacity	$c^*_{p,R}$	m^2/(s^2 K)
8	reference heat conductivity	k^*_R	kg m/(s^3 K)
9	reference dynamic viscosity	μ^*_R	kg/(m s)
10	reference speed of sound	c^*_R	m/s
11	gravitational acceleration	g^*	m/s^2
n ×	temperature derivatives of density	$\left[\partial^j \rho^*/\partial T^{*j}\right]_R$	kg/(m^{3}K^j)
n ×	temperature derivatives of heat capacity	$\left[\partial^j c^*_p/\partial T^{*j}\right]_R$	m^2/(s^{2}K^{j+1})
n ×	temperature derivatives of heat conductivity	$\left[\partial^j k^*/\partial T^{*j}\right]_R$	kg m/(s^{3}K^{j+1})
n ×	temperature derivatives of dynamic viscosity	$\left[\partial^j \mu^*/\partial T^{*j}\right]_R$	kg/(m s K^j)

Table 2. List of dimensional quantities influencing the problem of a lid driven heated cavity (benchmark problem)

number as a function of all nondimensional groups of the problem including the nondimensional temperature influence factors h_{aj}, $a = \rho, c_p, k, \mu$, $j = 1, \ldots, n$. In general, one has to carry out a complete formal dimensional analysis of the problem under consideration. Again, as an example, table 2 contains all quantities that influence our benchmark problem. In the last four rows of table 2 the index j runs from 1 to n. Thus there are $11 + 4n$ influence quantities and 4 dimensions (kg,m,s,K) which according to the Buckingham-Π-theorem results in $7 + 4n$ nondimensional groups given in table 3. Therefore

$$\begin{aligned} Nu &= Nu(Pr, Re, Fr, Ec, Ma, h, y, Geom) && (2.3)\\ h &:= (h_{\rho 1}, \ldots, h_{\rho n}, h_{c_p 1}, \ldots, h_{c_p n}, h_{k1}, \ldots, h_{kn}, h_{\mu 1}, \ldots, h_{\mu n}) && (2.4)\\ &= (h_1, \ldots, h_m)\\ m &:= 4n. \end{aligned}$$

In (2.3) *Geom* stands for further unspecified nondimensional geometry parameters. The aim is now to decouple the fluid property influences represented by h from the remaining parameters. For this reason the right hand side of equation (2.3) is formally expanded in a Taylor series with respect to the multidimensional variable h around $h = 0$:

$$Nu(Pr, Re, Fr, Ec, Ma, h, y, Geom) = \sum_\alpha C_\alpha \, h^\alpha \qquad (2.5)$$

$$\text{with:} \quad C_\alpha = C_\alpha(Pr, Re, Fr, Ec, Ma, y, Geom) := \frac{1}{\alpha!}\left[\frac{\partial^{|\alpha|} Nu}{\partial h^\alpha}\right]_{h=0}. \qquad (2.6)$$

The summation is taken over all multiindices

$$\begin{aligned} \alpha &= (\alpha_{\rho 1}, \ldots, \alpha_{\rho n}, \alpha_{c_p 1}, \ldots, \alpha_{c_p n}, \alpha_{k1}, \ldots, \alpha_{kn}, \alpha_{\mu 1}, \ldots, \alpha_{\mu n})\\ &= (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \end{aligned}$$

for which we use the common abbreviations

$$\begin{aligned} h^\alpha &:= h_1^{\alpha_1} \cdots h_m^{\alpha_m} \\ \alpha! &:= \alpha_1 \cdots \alpha_m \\ |\alpha| &:= \alpha_1 + \cdots + \alpha_m. \end{aligned}$$

Nr.	nondimensional group	name
1	$Nu = \dot{q}_W^* L^*/(\Delta T_R^* k_R^*)$	Nußelt number
2	$y = y^*/L^*$	nondimensional coordinate
3	$Re = \rho_R^* w^* L^*/\mu_R^*$	Reynolds number
4	$Pr = \mu_R^* c_{p,R}^*/k_R^*$	Prandtl number
5	$Fr = w^*/\sqrt{g^* L^*}$	Froude number
6	$Ec = w^{*2}/(c_{p,R}^* \Delta T_R^*)$	Eckert number
7	$Ma = w^*/c_R^*$	Mach number
n×	$h_{\rho j} = \frac{\Delta T_R^{*j}}{\rho_R^*}\left[\frac{\partial^j \rho^*}{\partial T^{*j}}\right]_R$	density temperature influence factors
n×	$h_{c_p j} = \frac{\Delta T_R^{*j}}{c_{p,R}^*}\left[\frac{\partial^j c_p^*}{\partial T^{*j}}\right]_R$	heat capacity temperature influence factors
n×	$h_{kj} = \frac{\Delta T_R^{*j}}{k_R^*}\left[\frac{\partial^j k^*}{\partial T^{*j}}\right]_R$	heat conductivity temperature influence factors
n×	$h_{\mu j} = \frac{\Delta T_R^{*j}}{\mu_R^*}\left[\frac{\partial^j \mu^*}{\partial T^{*j}}\right]_R$	dynamic viscosity temperature influence factors

Table 3. Nondimensional groups of the benchmark problem

Notice that the first summand of (2.5) for $\alpha = 0$ by definition (2.6) is the constant property solution:

$$\begin{aligned} C_0(Pr, Re, Fr, Ec, Ma, y, Geom) &= Nu(Pr, Re, Fr, Ec, Ma, h = 0, y, Geom) \\ &= Nu_{cp}(Pr, Re, Fr, Ec, Ma, y, Geom). \end{aligned}$$

Division of (2.5) by Nu_{cp} gives

$$\frac{Nu}{Nu_{cp}} = \frac{Nu(Pr, Re, Fr, Ec, Ma, h, y, Geom)}{Nu_{cp}(Pr, Re, Fr, Ec, Ma, y, Geom)} = \sum_\alpha A_\alpha h^\alpha \tag{2.7}$$

$$\begin{aligned} A_\alpha &= A_\alpha(Pr, Re, Fr, Ec, Ma, y, Geom) \\ &:= \frac{C_\alpha(Pr, Re, Fr, Ec, Ma, y, Geom)}{Nu_{cp}(Pr, Re, Fr, Ec, Ma, y, Geom)} = \frac{1}{\alpha! Nu_{cp}}\left[\frac{\partial^{|\alpha|} Nu}{\partial h^\alpha}\right]_{h=0}. \end{aligned} \tag{2.8}$$

The numbers A_α are called *A-values*. They still depend on Pr, Re, Fr, Ec, Ma, y, $Geom$ as indicated in (2.8) but not on the variable material properties of the fluid.

In order to derive an asymptotic expansion in ε (recalling that in c.f. (2.2), $h_{aj} = \varepsilon^j K_{aj}$) we now reorder the sum (2.7) by collecting terms with equal powers of ε^i, $i \in \mathbb{N}_0$. Up to second order this expansion has the form

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon[K_{\rho 1} A_\rho + K_{\mu 1} A_\mu + K_{k1} A_k + K_{c_p 1} A_{c_p}] + \varepsilon^2[K_{\rho 2} A_{\rho 2} + K_{\mu 2} A_{\mu 2}$$

$$
\begin{aligned}
&+K_{k2}A_{k2} + K_{c_p2}A_{c_p2} + K_{\rho1}^2 A_{\rho\rho} + K_{\mu1}^2 A_{\mu\mu} + K_{k1}^2 A_{kk} + K_{c_p1}^2 A_{c_pc_p} \\
&+K_{\rho1}(K_{\mu1}A_{\rho\mu} + K_{k1}A_{\rho k} + K_{c_p1}A_{\rho c_p}) + K_{\mu1}(K_{k1}A_{\mu k} + K_{c_p1}A_{\mu c_p}) \\
&+K_{k1}K_{c_p1}A_{kc_p}] + O(\varepsilon^3) \qquad (2.9)\\
= \; & 1 + \varepsilon A_1 + \varepsilon^2 A_2 + O(\varepsilon^3) \qquad (2.10)\\
A_1 := \; & K_{\rho1}A_\rho + K_{\mu1}A_\mu + K_{k1}A_k + K_{c_p1}A_{c_p} \qquad (2.11)\\
A_2 := \; & K_{\rho2}A_{\rho2} + K_{\mu2}A_{\mu2} + K_{k2}A_{k2} + K_{c_p2}A_{c_p2} \qquad (2.12)\\
&+K_{\rho1}^2 A_{\rho\rho} + K_{\mu1}^2 A_{\mu\mu} + K_{k1}^2 A_{kk} + K_{c_p1}^2 A_{c_pc_p} \\
&+K_{\rho1}(K_{\mu1}A_{\rho\mu} + K_{k1}A_{\rho k} + K_{c_p1}A_{\rho c_p}) + K_{\mu1}(K_{k1}A_{\mu k} + K_{c_p1}A_{\mu c_p}) \\
&+K_{c_p1}K_{k1}A_{c_pk}
\end{aligned}
$$

$$A_a := \frac{1}{Nu_{cp}} \left[\frac{\partial Nu}{\partial h_{a1}}\right]_{h=0} \qquad (2.13)$$

$$A_{aa} := \frac{1}{2Nu_{cp}} \left[\frac{\partial^2 Nu}{\partial h_{a1}^2}\right]_{h=0} \qquad (2.14)$$

$$A_{a2} := \frac{1}{Nu_{cp}} \left[\frac{\partial Nu}{\partial h_{a2}}\right]_{h=0} \qquad (2.15)$$

$$A_{ab} := \frac{1}{Nu_{cp}} \left[\frac{\partial^2 Nu}{\partial h_{a1}\partial h_{b1}}\right]_{h=0} \qquad (2.16)$$

where $a, b \in \{\rho, c_p, k, \mu\}$, $a \neq b$.

The big advantage of this asymptotic approach is: Once the A-values are known, Nu/Nu_{cp} can be computed for

- selected fluids (K-value variation)
- selected small driving temperature differences (ε variation)

provided that the nondimensional groups $Pr, Re, Fr, Ec, Ma, Geom$ remain unchanged.

Figure 1 sketches the process of the ACFD-method that converts the results from one fluid to a second one: The left branch represents the computation of the A-values plus the constant property solution. These computations require several solutions for $h \to 0$, for h see (2.4). They are carried out numerically via CFD-simulations for certain fictitious fluids. This procedure will be described in detail in step 3. The right branch of the picture illustrates the application for a certain fluid (fixed K-values) and certain small temperature differences ($\varepsilon_1, \varepsilon_2, \varepsilon_3$). Going from the left to the right branch is only possible if $Pr, Re, Fr, Ec, Ma, Geom$ are kept the same. Later on, in step 4, we will extend the theory allowing for small variations in these parameters since for example keeping the reference Prandtl number constant strongly restricts the number of fluids which might be interchanged.

(3) Computation of the A-values. For complex flow and heat transfer problems, an analytic determination of the A-values is not possible. Therefore, commercial

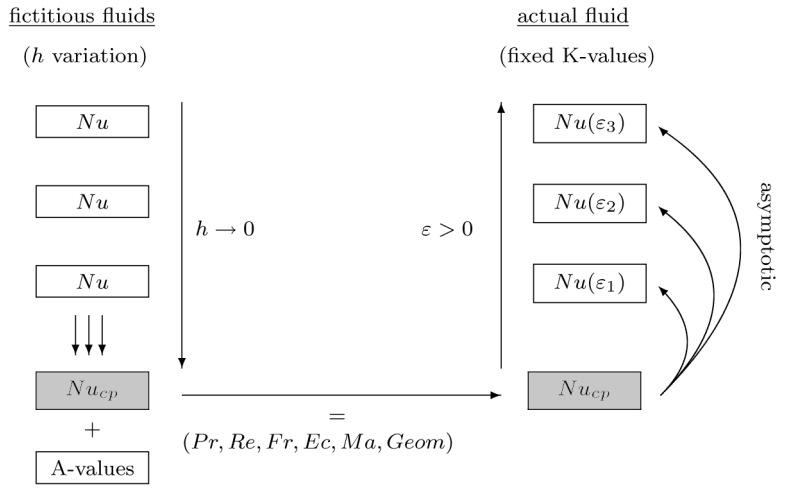

Figure 1. The ACFD-method for converting results from one fluid to a second one

CFD-codes like FLUENT can be used to determine them numerically. This is why the approach is called ACFD-method: **a**symptotic **c**omputational **f**luid **d**ynamics (see [10]). In the numerical determination of the A-values the partial derivatives on the right-hand sides of (2.13),..., (2.16) are formally approximated by appropriate difference quotients. There are many ways to do this. We chose the following central difference schemes to approximate the derivatives:

$$A_a = \frac{1}{Nu_{cp}} \lim_{s\to 0} \left\{ \frac{Nu(h_{a1}=s) - Nu(h_{a1}=-s)}{2s} \right\}_0 \tag{2.17}$$

$$A_{aa} = \frac{1}{2Nu_{cp}} \lim_{s\to 0} \left\{ \frac{Nu(h_{a1}=s) - 2Nu_{cp} + Nu(h_{a1}=-s)}{s^2} \right\}_0 \tag{2.18}$$

$$A_{a2} = \frac{1}{Nu_{cp}} \lim_{s\to 0} \left\{ \frac{Nu(h_{a2}=s) - Nu(h_{a2}=-s)}{2s} \right\}_0 \tag{2.19}$$

$$A_{ab} = \frac{1}{Nu_{cp}} \lim_{s\to 0} \frac{1}{4s^2} \{Nu(h_{a1}=h_{b1}=s) + Nu(h_{a1}=h_{b1}=-s) - Nu(h_{a1}=-h_{b1}=s) - Nu(h_{a1}=-h_{b1}=-s)\}_0 . \tag{2.20}$$

Here the limiting process $\lim_{s\to 0}$ ... means that s must be small enough to avoid a higher order influence, but not too small since otherwise truncation errors appear. The index 0 at the right curly brackets indicates that all other variables h_{c1}, h_{c2} are set to zero. Each of the eight terms

- $Nu(h_{a1}=s)_0$, $Nu(h_{a1}=-s)_0$
- $Nu(h_{a2}=s)_0$, $Nu(h_{a2}=-s)_0$

• $Nu(h_{a1} = h_{b1} = s)_0$, $Nu(h_{a1} = h_{b1} = -s)_0$, $Nu(h_{a1} = -h_{b1} = s)_0$, $Nu(h_{a1} = -h_{b1} = -s)_0$

corresponds to exactly one CFD-simulation. Hence, the calculation of (2.17), (2.18) and (2.19) needs two CFD-simulations for a fixed small value of s, while the determination of A_{ab} according to (2.20) needs four CFD-simulations. With one further CFD-calculation the constant property solution Nu_{cp} is determined. Details of the strategy to determine the A-values are:

- Since the method holds for all Newtonian fluids it also holds for arbitrarily chosen fictitious fluids for which only one property is temperature dependent (in some arbitrary way) with all other properties being constant. Such fluids serve as „dummy fluids“ from which the general A-values can be determined.
- A CFD-simulation for $Nu(h_{a1} = s)_0$ might be done with a dummy fluid whose K_{a1}-value equals s/ε where ε corresponds to a freely chosen temperature difference. All other K-values of this dummy fluid are set to zero.
- In the same way a CFD-simulation for $Nu(h_{a1} = -s)_0$ may use a dummy fluid whose K_{a1}-value equals $-s/\varepsilon$ and whose other K-values are zero again.
- The two CFD-simulations for $Nu(h_{a2} = \pm s)_0$ use dummy fluids with $K_{a2} = \pm s/\varepsilon^2$.
- Finally, the four CFD-simulations for $Nu(h_{a1} = \pm h_{b1} = \pm s)_0$ use dummy fluids with $K_{a1} = \pm s/\varepsilon$ and $\pm K_{b1} = \pm s/\varepsilon$.

CFD-programs in general work with dimensional variables. This requires some retranslations we want to mention here for the readers convenience. In FLUENT, for example, it is possible to define fictitious fluids with dimensional polynomial temperature dependent properties a^* like:

$$a^*(T^*) = a_0^* + a_1^* T^* + a_2^* T^{*2}. \tag{2.21}$$

Then a fictitious fluid behaves according to prescribed values of h_{a1} and h_{a2} if

$$a_0^* := a_R^* \left[1 - \frac{h_{a1} T_R^*}{\Delta T^*} + \frac{h_{a2} T_R^{*2}}{2(\Delta T^*)^2} \right] \tag{2.22}$$

$$a_1^* := a_R^* \left[\frac{h_{a1}}{\Delta T^*} - \frac{h_{a2} T_R^*}{(\Delta T^*)^2} \right] \tag{2.23}$$

$$a_2^* := \frac{a_R^* h_{a2}}{2(\Delta T^*)^2} \tag{2.24}$$

where ΔT^* is the dimensional characteristic temperature difference and T_R^* is the dimensional reference temperature.

Finally, all computed A-values are stored in databases as arrays of the form $(y, A(y))$ which represent curves with respect to a local coordinate y, i.e. along a certain line in the flow field. Take, for example, figure 3 in which all first order A-values have been computed for the benchmark problem.

(4) Variation of reference parameters. Since the reference Prandtl number is a property of the fluid the Pr-dependence of the A-values restricts the number of

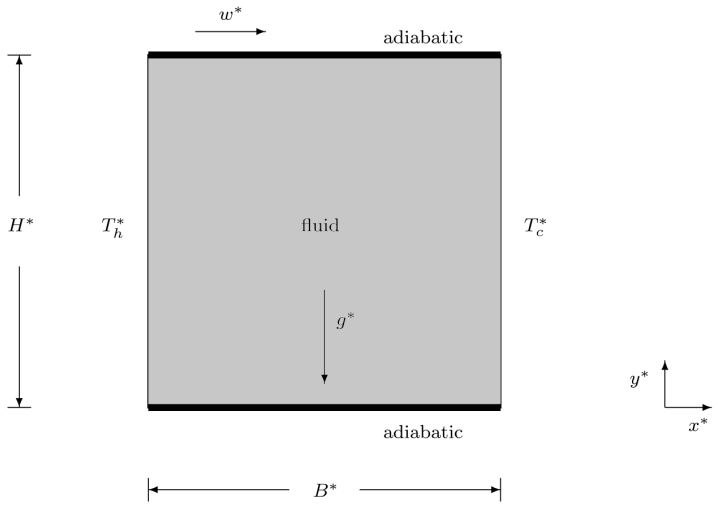

Figure 2. Two-dimensional heated and driven cavity (HDC)

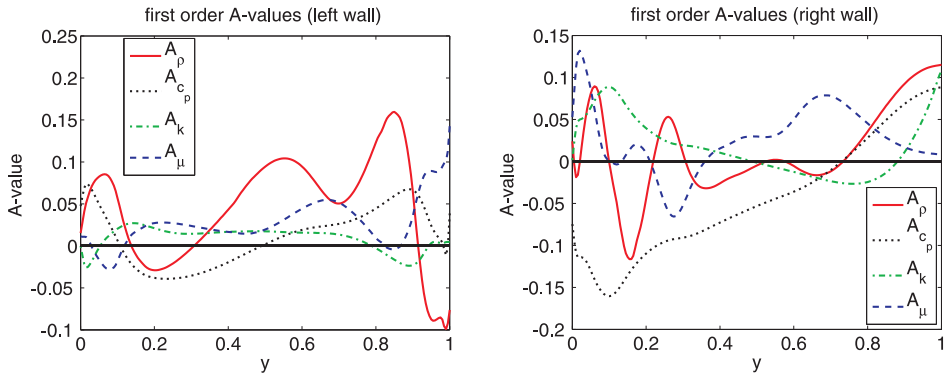

Figure 3. First order A-values for the benchmark problem (HDC) at both side walls

fluids which might be interchanged. Many gases have a Prandtl number around 0.7 however, so that only small deviations in Pr will have to be taken into account. The obvious approach is a second Taylor series expansion of the A-values with respect to the Prandtl number around the original reference Prandtl number, now called Pr_R. The A-values then read:

$$A_a(Pr, y) = A_a(Pr_R, y) + \left[\frac{\partial A_a(Pr, y)}{\partial Pr}\right]_R \Delta Pr + O((\Delta Pr)^2) \qquad (2.25)$$

$$A_{a2}(Pr, y) = A_{a2}(Pr_R, y) + \left[\frac{\partial A_{a2}(Pr, y)}{\partial Pr}\right]_R \Delta Pr + O((\Delta Pr)^2) \qquad (2.26)$$

$$A_{ab}(Pr,y) = A_{ab}(Pr_R,y) + \left[\frac{\partial A_{ab}(Pr,y)}{\partial Pr}\right]_R \Delta Pr + O((\Delta Pr)^2) \quad (2.27)$$

with $\Delta Pr := Pr - Pr_R$ and $a, b \in \{\rho, c_p, k, \mu\}$. All other nondimensional groups $Re, Fr, Ec, Ma, Geom$ are considered to remain constant. They also might be expanded in Taylor series, so that small variations in the Reynolds number or in the geometry can be handled, but we will not deal with such variations in this study.

3. A benchmark-case: the lid driven heated cavity

In this section the ACFD-method is applied to a benchmark case for mixed convection, the heated and driven cavity (HDC): A two-dimensional box, see figure 2, of hight H^*[m], width B^*[m] filled with a Newtonian fluid whose properties ρ^*, c_p^*, k^*, μ^* are all temperature dependent. The left wall of the box has a constant temperature T_h^* [K] (h stands for hot) and the right wall has a lower constant temperature T_c^* [K] (c stands for cold). Bottom and top of the box are adiabatic walls. The top wall is moving with a constant velocity w^* [m/s]. We choose

$$\begin{aligned} T_R^* &:= \frac{T_h^* + T_c^*}{2} = 293K \\ \Delta T_R^* &:= T_h^* - T_c^* \\ L^* &:= B^* = 10^{-2}m \end{aligned}$$

in the nondimensional groups given in table 3. Here $H := H^*/B^*$ is a geometric nondimensional group that replaces the term $Geom$ in (2.3). We examine the local Nußelt number

$$Nu = Nu(Pr, Re, Fr, Ec, Ma, h, y, H)$$

at each position y on the left (heated) or the right (cooled) wall for certain test cases in order to show the advantages of the ACFD-method.

Measurements of Koseff and Street [15] showed that the flow in a rectangular lid driven cavity with a span-wise aspect ratio (the ratio of cavity span to width) ranging from 1 to 3 starts to become turbulent for Reynolds numbers between 6000 and 8000. At $Re = 10000$ the entire flow is turbulent. For a detailed review with respect to lid driven cavity flows see Shankar and Deshpande [21]. In our CFD-simulations $Re = 50000$ to ensure that the flow is entirely turbulent. FLUENT 6.3 with a RNG k-ϵ turbulence model and a steady-state 2D solver was used. This approach is in accordance with FLUENT-simulations done by Isaev et al. [11], [12].

The influence of Fr, Ec, Ma on the Nußelt number is neglected in all considered test cases since forced convection dominates (no Fr influence), viscous dissipation is small compared to heat transfer effects (no Ec influence) and Mach numbers are close to zero. Therefore $Nu = Nu(Pr, Re, h, y, H)$ is left. The focus of this study is on effects of variable properties. Nevertheless the constant property result

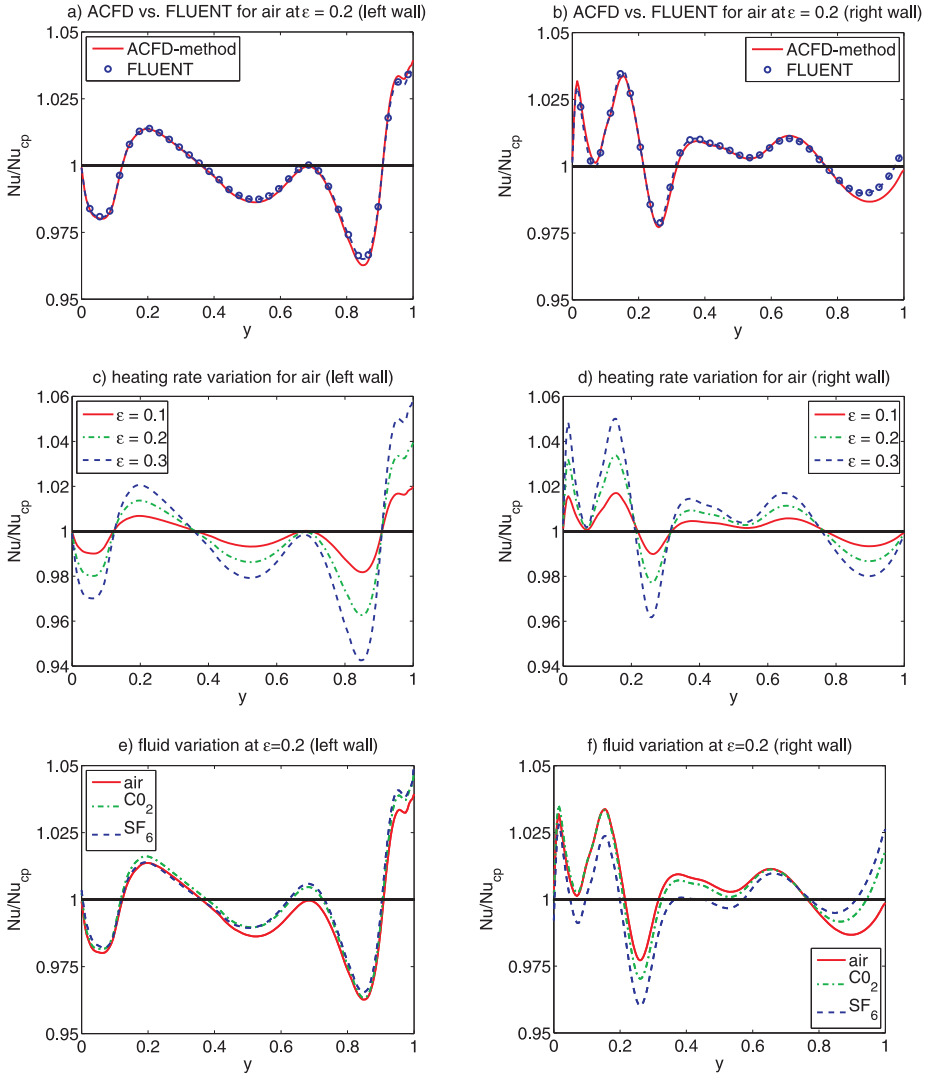

Figure 4. Nu/Nu_{cp} for a heated and lid driven cavity (HDC), $Re = 50000$

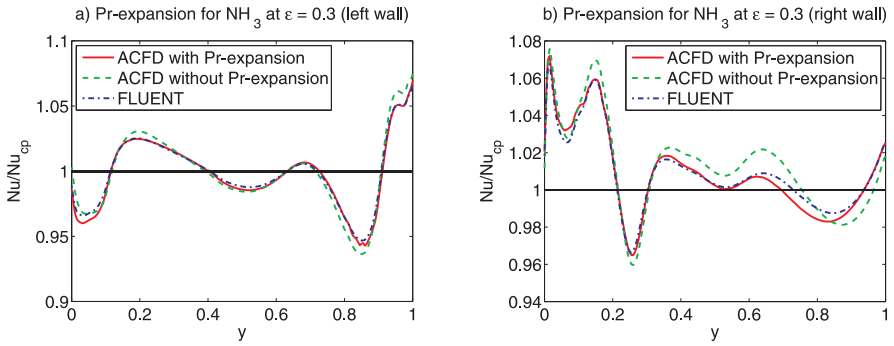

Figure 5. Results based on a Taylor series expansion of the A-values with respect to the Prandtl number compared to those without expansion and to full numerical solutions, $Re = 50000$

should also be given. It is shown in the appendix of this paper.

The example of how to treat variable property effects according to our asymptotic method is presented in five consecutive steps:

1. First all A-values $A(Pr, Re, y, H)$ used in (2.9) are computed for

$$\begin{aligned} Pr_R &:= 0.717 \\ Re_R &:= 50000 \\ H_R &:= 1. \end{aligned}$$

The Prandtl number is that of air at $T_R^* = 293$ K and $p_R^* = 1$ bar. All first order A-values $A_\rho, A_{c_p}, A_k, A_\mu$ for both side walls are shown in figure 3.

2. Next, Nu/Nu_{cp} according to (2.9) is evaluated for air (K-values from table 1) at a fixed temperature difference $T_h^* - T_c^* = 60$K ($\varepsilon = 0.2$). Also Nu/N_{cp} was determined from two FLUENT-solutions, one for constant properties and one with variable properties for air and a heating rate corresponding to $\varepsilon = 0.2$. Both results are shown in figure 4 a), b) and compare quite well.

3. Now we fix the fluid (air) and vary the temperature difference $T_h^* - T_c^* = 30$K, 60K, 90K which corresponds to $\varepsilon = 0.1, 0.2, 0.3$. Formula (2.9) immediately supplies the ACFD-solutions given in figure 4 c), d).

4. We now fix the temperature difference $T_h^* - T_c^* = 60$K ($\varepsilon = 0.2$) and vary the fluid by switching from air to carbon dioxide (CO_2) and sulfur hexafluoride (SF_6). Since these gases have Prandtl-numbers around 0.7 at the reference state (cf. table 1) the A-values computed for $Pr = 0.717$ are appropriate as a first approximation.
Using the K-values of CO_2 and SF_6 (cf. table 1) in (2.9) yields the curves shown in figure 4 e), f).

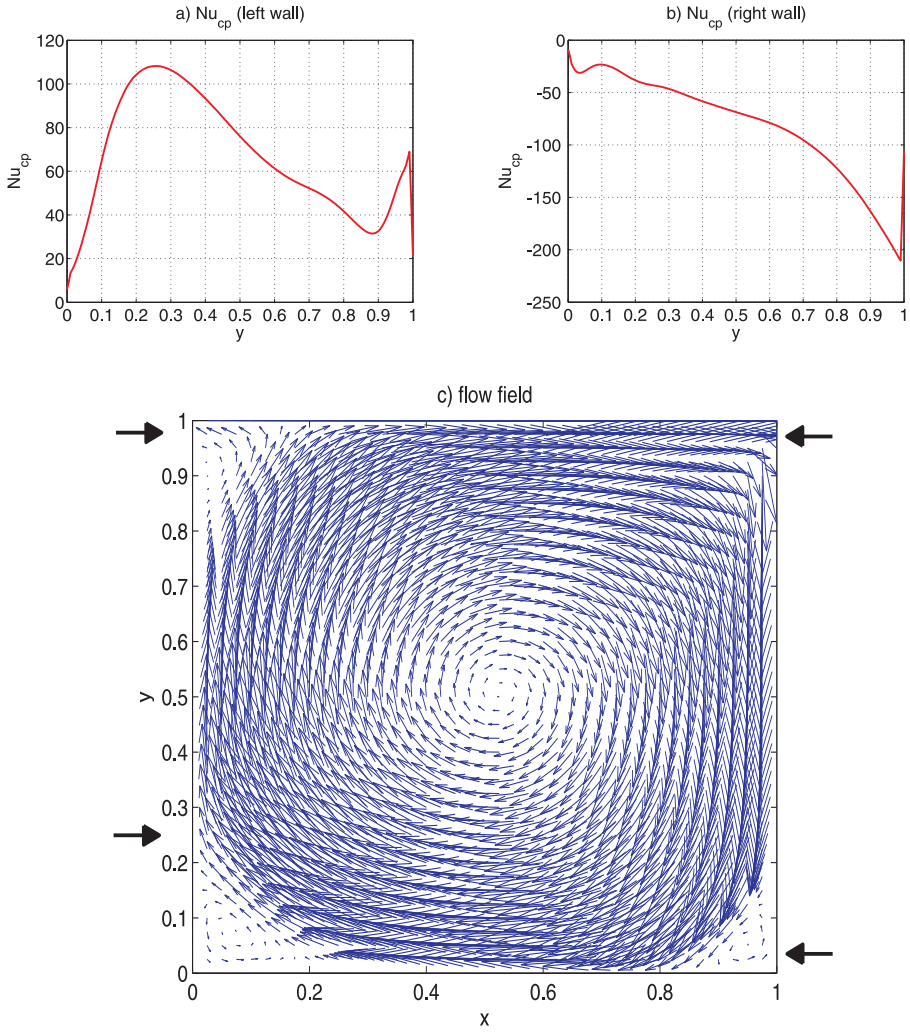

Figure 6. Local Nußelt number distribution along the side walls and the flow field of the HDC for constant fluid properties at $Pr = 0.717$ and $Re = 50000$

5. Finally the A-value expansion with respect to the Prandtl number is determined for the reference Prandtl number $Pr_R = 0.717$ as described in aspect (4) of section 2 (see (2.25), 2.26) (2.27)). Figure 5 compares the ACFD-solution for NH_3 at $\varepsilon = 0.3$ with these corrected A-values to the results without A-value-Pr-expansion. For comparison figure 5 includes the Nu/Nu_{cp}-results from two FLUENT-simulation for NH_3 under the given boundary conditions to assess the quality of both ACFD-solutions.

4. Conclusions

An asymptotic method was presented that includes variable property effects in similarity considerations with respect to complex flows. This is a clear strategy for a numerical approach to a complex problem. However, also experimental results that were gained with a certain fluid have to be treated likewise when they should be used in corresponding situations but with different fluids. It is always the „detour“ over the constant property results that is required. Then, however, experiments have to be accompanied by numerical solutions from which the constant property results can be deduced. One then gets the information whether variable property effects are crucial or can be neglected (within the error band of the experiments). In cases in which they are non-negligible their influence can be quantified by the asymptotic method of this study. The numerical results for this purpose need not be of highest quality, however, since systematic errors will probably drop out when corresponding results for two different fluids are determined on the basis of one common (not high quality) numerical solution of the problem under consideration.

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Appendix

Figure 6 a),b) shows the Nußelt number for constant fluid properties along the left and right side walls of the lid driven heated cavity. On both walls maxima in the Nußelt number distribution occur where the flow “hits the surface” as indicated by arrows in figure 6 c) of the flow field.

At the left (heated) wall this is the case in the region $0.2 < y < 0.3$ where fluid from the rotating primary vortex approaches the wall. Also fluid from the the secondary vortex in the left upper corner approaches the wall almost perpendicularly near $y = 1$ (before it turns due to the non-slip condition) which causes a strong increase of the Nußelt number.

In the upper right corner the fluid impinges on the right (cooled) wall with high velocity (close to that of the moving upper wall) which results in a Nußelt number maximum near $y = 1$. The Nußelt number then decreases monotonically in the region where the flow is parallel to the right wall like it does with a wall jet. The secondary vortex in the right lower corner causes a small increase of the Nußelt number at y near 0 only since it is rather weak.

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