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On the integrable rational Abel differential equations

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Abstract. In Cheb-Terrab and Roche (Comput Phys Commun 130(1–2):204–231, 2000) a classification of the Abel equations known as solvable in the literature was presented. In this paper, we show that all the integrable rational Abel differential equations that appear in Cheb-Terrab and Roche (Comput Phys Commun 130(1–2):204–231, 2000) and consequently in Cheb-Terrab and Roche (Eur J Appl Math 14(2):217–229, 2003) can be reduced to a Riccati differential equation or to a first-order linear differential equation through a change with a rational map. The change is given explicitly for each class. Moreover, we have found a unified way to find the rational map from the knowledge of the explicitly first integral.

Mathematics Subject Classification (1991). Primary 34C35 · 34D30.

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1. Introduction and statement of the results

In this work, we study the class of integrable *rational Abel differential equations* of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3, \tag{1}$$

where $f_i(x)$ are rational functions of x. Here *integrable* means that the Abel differential equation has an explicit first integral $H(x, y)$ defined in all $\mathbb{R}^2$ except in a Lebesgue set of zero measure.

Abel equations appear in the reduction of order of many second- and high-order families, and hence are frequently found in the modelling of real problems in varied areas. In [5] it is given a classification according to invariant theory of the integrable Abel differential equations, i.e. of the Abel equations known as solvable in the literature.

The classification of the known integrable cases is derived from the analysis of the works of Abel [1], Apell [2], Liouville [14–16], and Kamke's textbook [13]. During these last 130 years all the integrable rational Abel differential equations that have been found have been reduced by Cheb-Terrab and Roche [5] to four classes depending on one parameter and seven classes formed by a unique equation. All these classes are summarized in Appendix. We remark that the Class 1 of Appendix can be written into the form (1) doing the change $\{X = y,\ Y = 1/(-x - 3y + 3y^2)\}$.

Our main result is the following.

Theorem 1. *Any integrable rational Abel differential equation given in Appendix can be transformed to a Riccati differential equation or to a first-order linear differential equation doing a change with a rational map.*

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The proof will be given in Sect. 2 by providing for each known integrable rational Abel differential equation the explicit rational change to a first-order linear differential equation or to a Riccati equation.

Open problem 1. *Show that any integrable rational Abel differential equation can be transformed to a Riccati differential equation or to a first-order linear differential equation doing a change with a rational or algebraic map.*

Note that Theorem 1 proves the Open Problem 1 restricted to all the known integrable rational Abel differential equations up to now.

It is well known that several different classes of polynomial differential systems can be transformed into Abel differential equations of the form (1) where, in general, the functions $f_i(x)$ are trigonometric instead of rational, see for instance [7–10]. It is also known that many of these classes of Abel differential equations are integrable, for instance the ones coming from quadratic polynomial differential equations having a center, see [17].

Open problem 2. *It is unknown if there exists a non-rational Abel differential equation* (1) *having an explicit first integral which cannot be transformed into a Riccati differential equation or a first-order differential linear equation.*

2. Proof of Theorem 1

The proof of Theorem 1 is inspired from the explicit expression of the first integral. We consider the 2-dimensional system associated to the Abel equation (1) given by

$$\dot{x} = 1, \quad \dot{y} = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3, \tag{2}$$

and its associated vector field

$$\mathcal{X} = \frac{\partial}{\partial x} + (f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3)\frac{\partial}{\partial y}.$$

In [6], generalizing all the integrable classes collected in [5], it was presented a single multi-parameter non-constant invariant class of Abel ordinary differential equations denoted by *Abel Inverse–Abel* (AIA) *type* in the notation of [6]. In addition the class AIA contains a new subclass of *Abel Inverse–Riccati* (AIR) *type* depending on six parameters all of whose members can be systematically transformed into Riccati-type equations. Moreover, this AIR class includes a four-parameter Liouvillian integrable subclass denoted by *Abel Inverse–Linear* (AIL) *subclass* all of whose members can be mapped into first-order linear equations. Additionally this family (AIA) includes new integrable cases constructed from the previously known ones in [5]. Moreover, in [6] it is shown that the integrable equations belonging to family (AIA) constructed from the previously known ones in [5] can all be transformed through an algebraic change to the (AIL) or (AIR) class. These changes are not given explicitly, and they should be found by a composition of several algebraic changes. For instance, for Class 1, it is clear that by this composition an algebraic change exists, but it is not possible to give explicitly the change from the sequence of changes proposed.

All the integrable classes collected in [5], four depending on one parameter, labelled A, B, C and D, and another seven independent of parameters, labelled 1–7 are particular members of the AIL, AIR and AIA. More precisely, the classes A, C, 4 and 5 are subclasses of AIL, the classes B, D, and 2 are subclasses of AIR, and the classes 1, 3, 6, and 7 are subclasses of AIA. It is clear, by the previous paragraph, that all these classes 1–7, and A–D map to a first-order linear equation or to a Riccati equation through an algebraic change. Here, we will prove that it is possible to find explicitly a change with a rational map for each class and in almost all the cases such a map is birational, and consequently Theorem 1 will be proved.

It is important to note that we have found a unified way to find the rational map from the explicit expression of the first integral. It is necessary to separate the transcendental dependence in one variable and the non-transcendental dependence in the other. This unified method to find the map is very important for future works since, up to now, it was not understood how to find such a map when it exists. This observation can allow to detect the integrability of an equation by a perturbative method of construction of its first integral.

The first case of the table given in Appendix is the unique that has a rational first integral and consequently in the expression of the first integral does not appear any transcendental function. In this case, for finding the rational map we shall use Theorem 2 which provides a result of local orbitally linearization with a map in a neighbourhood of a singular point. Theorem 2 is a straightforward generalization of a result given in [11]. We need some previous definitions in order to state Theorem 2.

Let $F = F(x, y)$ and $K = K(x, y)$ be smooth functions. If they satisfy $\mathcal{X}F = KF$, then F is called a *Darboux factor* and K its *cofactor*, see for more details [3,4].

We say that system (2) is *orbitally linearizable with a map* in a neighbourhood U of a singular point if there exist Darboux factors $F_i : U \to \mathbb{R}$ with cofactors $K_i : U \to \mathbb{R}$ for $i = 1, \ldots, m$, and numbers $\alpha_i, \beta_i \in \mathbb{C}$, such that $\sum_{i=1}^{m-1} \alpha_i K_i(x, y) = \lambda h(x, y)$ and $\sum_{i=2}^{m} \beta_i K_i(x, y) = \mu h(x, y)$.

Theorem 2. *Suppose that system (2) is orbitally linearizable with a map in a neighbourhood of a singular point. Then, using the notation of the definition of orbitally linearizable, the change* $(x, y) \mapsto (u, v) = (\prod_{i=1}^{m-1} F_i^{\alpha_i}, \prod_{i=2}^{m} F_i^{\beta_i})$ *brings the system into its orbitally linearizable normal form.*

Class 1 in [5], discovered by Halphen [12] in connection with elliptic functions, is given by $dy/dx = (3y(1 + y) - 4x)/(x(8y - 1))$. This equation can be transformed to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y(y-1) - x}{x(8y-9)}. \tag{3}$$

by the change $\{x = X/4, y = Y - 1\}$ and renaming $\{X \to x, Y \to y\}$. The rational map

$$X = x^3,\ Y = \frac{27x + 4x^2 - 36xy + 8xy^2 - 4y^3 + 4y^4}{(x^2 - 6xy + 2xy^2 + y^4)^3},$$

transforms system (3) into the linear equation $\mathrm{d}Y/\mathrm{d}X = -Y/X$. This change can be obtained using the invariant algebraic curves that appear in the first integral given in the appendix and their respective cofactors and applying Theorem 2 at the singular point $(0, 0)$.

Class 2 in [5], given first by Liouville [15], is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - 2xy^2. \tag{4}$$

The birational map

$$X = x^2 - \frac{1}{y}, \quad Y = x,$$

transforms system (4) into the Riccati equation $\mathrm{d}Y/\mathrm{d}X = Y^2 - X$. In this case the Abel equation has a non-Liouvillian first integral, where the transcendental functions in the first integral are in the variable $x^2 - 1/y$. The change has been obtained by taking $x^2 - 1/y$ as a new variable.

Class 3 in [5], was also found by Liouville [16], and is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3}{4x^2} - y^2. \tag{5}$$

Doing first the change $\{z = x - 1/y,\ x = x\}$ and later the change $\{u = z^2 - 1/(2x),\ z = z\}$ we obtain a Riccati equation $\mathrm{d}u/\mathrm{d}z = -u + z^2$. In short the birational map

$$X = \frac{2x - 4x^2y - y^2 + 2x^3y^2}{2xy^2}, \quad Y = \frac{xy - 1}{y},$$

transforms system (5) into the Riccati equation $\mathrm{d}Y/\mathrm{d}X = Y^2 - X$. In this case the Abel equation has also a non-Liouvillian first integral and the change has been obtained in such a way that transcendental dependence in the first integral will be only in one variable and the non-transcendental dependence in the other.

Class 4 in [5] is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - \frac{(x+1)\,y^2}{x}. \tag{6}$$

The change $z = x - 1/y$ transforms system (6) into the Riccati equation $\mathrm{d}x/\mathrm{d}z = x(x-z)/z$ which has a Liouville first integral because it has the algebraic particular solution $x = 0$. Hence, there exists a change that transforms system (6) into a linear equation. In fact the birational map

$$X = \frac{1-2y-xy}{2(-1+xy)}, \quad Y = \frac{2+x}{x},$$

transforms system (6) into the linear equation $\mathrm{d}Y/\mathrm{d}X = 4(X+Y)/(1+2X)^2$.

Class 5 in [5], is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(2x+3)(x+1)y^3}{2x^5} + \frac{(5x+8)\,y^2}{2x^3}. \tag{7}$$

The change $z = (x+1)/(x\sqrt{A})$, where $A = 4/y - 10/x - 6/x^2 - 4$ transforms system (7) into the Riccati equation

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{x(1+x+6z^2+4xz^2)}{z(1+4z^2)},$$

which has a Liouville first integral because it has the algebraic particular solution $x = 0$. Hence there exists a change that transforms system (7) into a linear equation. More specifically the birational map

$$X = \frac{2x^2 - xy - y}{2x^2 + xy + x^2 y}, \quad Y = \frac{1}{x},$$

transforms system (7) into the linear equation $\mathrm{d}Y/\mathrm{d}X = (X+Y)/(2X(X-1)(3X-2))$.

Class 6 in [5] is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^3}{x^2(x-1)^2} + \frac{(1-x-x^2)y^2}{x^2(x-1)^2}. \tag{8}$$

The change $z = (y + x^2 - x)/(xy(x-1))$ transforms system (8) into the Riccati equation $\mathrm{d}x/\mathrm{d}z = -x + x^2 - 1/z$, which has a Liouville first integral because it has the algebraic particular solution $x = -1/z$. Hence, there exists a change that transforms system (8) into a linear equation. More specifically, the birational map

$$X = \frac{x - x^2 - y + 2xy - 2x^2 y}{2(-x + x^2 + y)}, \quad Y = \frac{-1 + x - y + 2xy}{2(-1 + x + y)},$$

transforms system (8) into the linear equation $\mathrm{d}Y/\mathrm{d}X = 4(X+Y)/(1+2X)^2$.

Class 7 in [5] is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4x^4+5x^2+1)y^3}{2x^3} + y^2 + \frac{(1-4x^2)y}{2x(x^2+1)}. \tag{9}$$

The change $z = (x - 2yx^4 - 3yx^2 - y)/(x(x + yx^2 + y))$ transforms system (9) into the Riccati equation

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{1+xz+x^2}{2+2z^2},$$

which has a Liouville first integral because it has the algebraic particular solution $x = -1/z$. Hence there exists a change that transforms system (9) into a linear equation. More specifically, the birational map

$$X = \frac{2(x^2 + xy + x^3y)}{y - x + 3x^2y + 2x^4y}, \quad Y = \frac{y - x + x^2y}{(x + x^3)y},$$

transforms system (9) into the linear equation $\mathrm{d}Y/\mathrm{d}X = -2(X + Y)/(X(4 + X^2)$.

Class A in [5] is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\alpha x + \frac{1}{x} + \frac{1}{x^3}\right) y^3 + y^2. \tag{10}$$

The rational map $\{z = x^3/(y + x),\ u = -yx^2/(y + x)\}$ transforms system (10) into the linear equation

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{u - u^2 + \alpha u^3 + 2z}{u - u^2 + \alpha u^3}.$$

This map has been obtained, as in the Class 3, in such a way that the transcendental dependence in the first integral will be only in one variable and the non-transcendental dependence in the other. Following the sequence of transformations indicated in [6] we can also construct the rational map

$$X = \frac{x^2}{2}, \quad Y = \frac{x^2 y}{2(x + y)},$$

which transforms system (10) into the linear equation $\mathrm{d}Y/\mathrm{d}X = 2(X + Y)/(X(1 - 2X + 4\alpha X^2))$. Both maps are not birational maps and it seems it does not exist a birational one for this Class A.

Class B in [5], discovered first by Liouville [16], is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x^2 - \alpha^2)y^3 + 2(x + 1)y^2. \tag{11}$$

The birational map $\{z = x^2 + 1/y,\ x = x\}$ transforms system (11) into the Riccati equation $\mathrm{d}x/\mathrm{d}z = (x^2 - z)/(2(z - \alpha^2))$. In this case the Abel equation has a non-Liouvillian first integral, where the transcendental functions are in the variable $x^2 + 1/y$. The change has been obtained taking $x^2 + 1/y$ as a new variable. Following the sequence of transformations indicated in [6] we can also construct the birational map

$$X = -\frac{1}{x^2 + y}, \quad Y = x,$$

which transforms system (11) into the Riccati equation $\mathrm{d}Y/\mathrm{d}X = (X + Y^2)/(2(\alpha^2 + X))$.

Class C in [5] is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha(1 - x^2)y^3}{2x} + (\alpha - 1)y^2 - \frac{\alpha y}{2x}. \tag{12}$$

The birational map $\{z = (1 - xy)/y,\ x = x\}$ transforms system (12) into the Riccati equation $\mathrm{d}x/\mathrm{d}z = 2x(x + z)/(\alpha(z^2 - 1))$. This Riccati equation has a Liouville first integral because it has the algebraic particular solution $x = 0$. Hence there exists a change that transforms system (12) into a linear equation. In fact, the birational map

$$X = \frac{\sqrt{\alpha}\, y}{xy - 1}, \quad Y = \frac{\sqrt{\alpha}}{x},$$

transforms system (12) into the linear equation $dY/dX = -2(X + Y)/(X(X^2 - \alpha))$.

Class D in [5], related with Appell's work [2], is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^3}{x} - \frac{(\alpha + x^2)y^2}{x^2}. \tag{13}$$

The change $\{z = x - \alpha/x - 1/y,\ x = x\}$ transforms system (13) into the Riccati equation $\mathrm{d}x/\mathrm{d}z = \alpha + xz + x^2$. In this case the Abel equation has a non–Liouvillian first integral, where the transcendental functions are in the variable $x - \alpha/x - 1/y$. The change has been obtained taking $x - \alpha/x - 1/y$ as a new variable. Following the sequence of transformations indicated in [6] we arrive to the same birational map

$$X = \frac{x + \alpha y - x^2 y}{xy}, \quad Y = x,$$

which transforms system (13) into the Riccati equation $\mathrm{d}Y/\mathrm{d}X = -\alpha + XY + Y^2$.

3. Appendix

Class	Integrable rational Abel differential equations with a first integral
1	$\mathrm{d}y/\mathrm{d}x = \frac{3\,y^2-3\,y-x}{x(8\,y-9)},\quad H(x,y) = \frac{x^3\left(4\,x^2+\left(8\,y^2-36\,y+27\right)x+4\,y^4-4\,y^3\right)}{\left(x^2+2\,x\left(y^2-3\,y\right)+y^4\right)^3}.$
2	$\mathrm{d}y/\mathrm{d}x = -2\,y^2x + y^3,\quad H(x,y) = \frac{x\mathrm{Ai}\left(x^2-\frac{1}{y}\right)+\mathrm{Ai}\left(1,x^2-\frac{1}{y}\right)}{x\mathrm{Bi}\left(x^2-\frac{1}{y}\right)+\mathrm{Bi}\left(1,x^2-\frac{1}{y}\right)}.$
3	$\mathrm{d}y/\mathrm{d}x = \frac{y^3}{4\,x^2} - y^2,\quad H(x,y) = \frac{\left(x-\frac{1}{y}\right)\mathrm{Ai}\left(\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)+\mathrm{Ai}\left(1,\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)}{\left(x-\frac{1}{y}\right)\mathrm{Bi}\left(\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)+\mathrm{Bi}\left(1,\left(x-\frac{1}{y}\right)^2-\frac{1}{2x}\right)}.$
4	$\mathrm{d}y/\mathrm{d}x = y^3 - \frac{x+1}{x}y^2,\quad H(x,y) = \frac{1}{x}\mathrm{e}^{\frac{1}{y}-x} - \mathrm{Ei}(1, x - \frac{1}{y}).$
5	$\mathrm{d}y/\mathrm{d}x = -\frac{(2\,x+3)(x+1)y^3}{2\,x^5} + \frac{(5\,x+8)y^2}{2\,x^3},\quad H(x,y) = \frac{\sqrt{A}}{\sqrt[4]{4\,\frac{(x+1)^2}{x^2\,A}+1}} + \int^{2\,\frac{x+1}{x\sqrt{A}}}\left(z^2+1\right)^{-5/4}\mathrm{d}z = 0,$ $A = \frac{4}{y} - \frac{10}{x} - \frac{6}{x^2} - 4.$
6	$\mathrm{d}y/\mathrm{d}x = -\frac{y^3}{x^2(x-1)^2} + \frac{\left(1-x-x^2\right)y^2}{x^2(x-1)^2},\quad H(x,y) = -\mathrm{Ei}\left(1,\,\frac{y+x^2-x}{xy(x-1)}\right) + \frac{(x-1)\,y\,\varepsilon^{\frac{x-y-x^2}{xy(x-1)}}}{x-1+y}.$
7	$\mathrm{d}y/\mathrm{d}x = \frac{\left(4\,x^4+5\,x^2+1\right)y^3}{2\,x^3} + y^2 + \frac{\left(1-4\,x^2\right)y}{2\,x\left(x^2+1\right)}\quad H(x,y) = 2\,\frac{x+A}{\sqrt[4]{A^2+1}(Ax-1)} + \int^{A}\left(z^2+1\right)^{-5/4}\mathrm{d}z,$ $A = \frac{x - 2\,yx^4 - 3\,yx^2 - y}{x\,(x + yx^2 + y)}.$
A	$\mathrm{d}y/\mathrm{d}x = \left(\alpha\,x + \frac{1}{x} + \frac{1}{x^3}\right)y^3 + y^2,\quad H(x,y) = \frac{x^3}{y+x}\,\exp\left(\int^{\frac{-yx^2}{y+x}}\frac{2\,\mathrm{d}z}{z^2-z-\alpha z^3}\right) - \int^{\frac{-yx^2}{y+x}}\exp\left(\int\frac{2\,\mathrm{d}z}{z^2-z-\alpha z^3}\right)\mathrm{d}z.$
B	$\mathrm{d}y/\mathrm{d}x = 2\,\left(x^2-\alpha^2\right)y^3 + 2\,(x+1)\,y^2,\quad H(x,y) = \frac{(\alpha+x)\mathrm{K}\left(\alpha,-\sqrt{x^2+\frac{1}{y}-\alpha^2}\right)+\sqrt{x^2+\frac{1}{y}-\alpha^2}\;\mathrm{K}\left(1+\alpha,-\sqrt{x^2+\frac{1}{y}-\alpha^2}\right)}{(\alpha+x)\mathrm{I}\left(\alpha,-\sqrt{x^2+\frac{1}{y}-\alpha^2}\right)-\sqrt{x^2+\frac{1}{y}-\alpha^2}\;\mathrm{I}\left(1+\alpha,-\sqrt{x^2+\frac{1}{y}-\alpha^2}\right)}.$
C	$\mathrm{d}y/\mathrm{d}x = \frac{\alpha\left(1-x^2\right)y^3}{2\,x} + (\alpha-1)\,y^2 - \frac{\alpha y}{2\,x},\quad H(x,y) = \frac{\alpha}{x}\left(1 - \left(\frac{1}{y} - x\right)^2\right)^{1/\alpha} - 2\int^{\frac{1-xy}{y}}\left(1-z^2\right)^{\frac{1-\alpha}{\alpha}}\mathrm{d}z.$
D	$\mathrm{d}y/\mathrm{d}x = -\frac{y^3}{x} - \frac{\left(\alpha+x^2\right)y^2}{x^2},\quad H(x,y) = \frac{(\alpha+1)\mathrm{M}\left(-\frac{\alpha}{2}-\frac{3}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)+\left(\frac{x}{y}-x^2\right)\mathrm{M}\left(-\frac{\alpha}{2}+\frac{1}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)}{\left(\alpha^2+\alpha\right)\mathrm{W}\left(-\frac{\alpha}{2}-\frac{3}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)+2\left(\frac{x}{y}-x^2\right)\mathrm{W}\left(-\frac{\alpha}{2}+\frac{1}{4},\frac{1}{4},\frac{1}{2}\left(x-\frac{\alpha}{x}-\frac{1}{y}\right)^2\right)}.$

Integrable rational Abel differential equations presented in [5].

References

1. Abel, N.H.: Oeuvres Complètes II. In Lie, S., Sylow, L. (Eds.) Christiana (1881)
2. Appell, P.: Sur les invariants de quelques équations différentielles. J. Math. **5**, 361–423 (1889)
3. Chavarriga, J., Giacomini, H., Giné, J., Llibre, J.: On the integrability of two-dimensional flows. J. Differ. Equ. **157**(1), 163–182 (1999)
4. Chavarriga, J., Giacomini, H., Giné, J., Llibre, J.: Darboux integrability and the inverse integrating factor. J. Differ. Equ. **194**(1), 116–139 (2003)
5. Cheb-Terrab, E.S., Roche, A.D.: Abel ODE's: equivalence and integrable classes. Comput. Phys. Commun. **130**(1–2), 204–231 (2000)

6. Cheb-Terrab, E.S., Roche, A.D.: An Abel ordinary differential equation class generalizing known integrable classes. Eur. J. Appl. Math. **14**(2), 217–229 (2003)
7. Giné, J., Llibre, J.: Integrability and algebraic limit cycles for polynomial differential systems with homogeneous nonlinearities. J. Differ. Equ. **197**, 147–161 (2004)
8. Giné, J., Llibre, J.: Darboux integrability and limit cycles for a class of polynomial differential systems. Differential equations with symbolic computation. Trends Math., Birkhäuser, Basel, pp. 55–65 (2005)
9. Giné, J., Llibre, J.: A family of isochronous foci with Darboux first integral. Pac. J. Math. **218**, 343–355 (2005)
10. Giné, J., Llibre, J.: Integrability, degenerate centers, and limit cycles for a class of polynomial differential systems. Comput. Math. Appl. **51**, 1453–1462 (2006)
11. Giné, J., Maza, S.: Orbital linearization in the quadratic Lotka–Volterra systems around singular points via Lie symmetries. Universitat de Lleida (2007, preprint)
12. Halphen, G.: Sur la multiplication des fonctions elliptiques. Comptes Rendus Des Séances de l'Académie Des Sciences **88**, 414–417 (1879)
13. Kamke, E.: Differentialgleichungen "losungsmethoden und losungen", Col. Mathematik und ihre anwendungen, vol. **18**, Akademische Verlagsgesellschaft Becker und Erler Kom-Ges., Leipzig (1943)
14. Liouville, R.: Sur certaines équations différentielles du premier ordre. Comptes Rendus Des Séances de l'Académie Des Sciences **103**, 476–479 (1886)
15. Liouville, R.: Sur une classe d' équations différentielles du premier ordre et sur les formations invariantes qui s'y rapportent. Comptes Rendus Des Séances de l'Académie Des Sciences **105**, 460–463 (1887)
16. Liouville, R.: Sur une équation différentielle du premier ordre. Acta Math. **26**, 55–78 (1902)
17. Lunkevich, V.A., Sibirskii, K.S.: Integrals of a general quadratic differential system in cases of a center. Differ. Equ. **18**, 563–568 (1982)

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