Z. Angew. Math. Phys. 61 (2010), 63–72

0044-2275/10/010063-10
published online June 4, 2009
DOI 10.1007/s00033-009-0005-3

Zeitschrift für angewandte Mathematik und Physik ZAMP

Inviscid limit for axisymmetric flows without swirl in a critical Besov space

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Abstract. In this paper, we study the inviscid limit for the 3-D axisymmetric incompressible fluid flows without swirl and prove the convergence rate. We will also prove the uniform persistence of the initial regularity for 3-D axisymmetric Navier–Stokes equations in a critical Besov space.

Mathematics Subject Classification (2000). 35Q30 · 76D05 · 76C05.

Keywords. Axisymmetric flows without swirl · Inviscid limit · Navier–Stokes equations · Euler equations · Besov spaces.

1. Introduction

In this paper, we consider the 3-D Navier–Stokes equations for the viscous incompressible fluid,

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \triangle v + \nabla \pi = 0 \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0. \end{cases} \tag{NS}$$

Here, the vector field $v(x,t) = (v^1, v^2, v^3)(x,t)$ stands for the velocity of the fluid, the scalar function π denotes the pressure and the parameter $\nu > 0$ is the kinematic viscosity.

We will also consider the 3-D Euler equations which are the inviscid case of the system (NS),

$$\begin{cases} \partial_t \bar{v} + \bar{v} \cdot \nabla \bar{v} + \nabla \bar{\pi} = 0 \\ \operatorname{div} \bar{v} = 0 \\ \bar{v}|_{t=0} = v_0. \end{cases} \tag{E}$$

The Navier–Stokes equations and Euler equations have been intensively studied by many authors, and some local results have been obtained, we do not enumerate them here. But the question of global smooth existence is still open and continues to be one of the most outstanding problem in mathematical fluid mechanics. For the convergence for functions spaces of classical solutions of the Navier–Stokes equations to the Euler equations, see for example [9].

Recall that the vorticities $\omega = \nabla \times v$ and $\bar{\omega} = \nabla \times \bar{v}$ satisfies the following equations:

$$\partial_t \omega + v \cdot \nabla \omega - \nu \triangle \omega = \omega \cdot \nabla v \tag{1.1}$$

and

$$\partial_t \bar{\omega} + \bar{v} \cdot \nabla \bar{\omega} = \bar{\omega} \cdot \nabla \bar{v}. \tag{1.2}$$

The main difficulty for establishing global regularity is to understand how the vortex stretching terms $\omega \cdot \nabla v$ and $\bar{\omega} \cdot \nabla \bar{v}$ affect the dynamics of the fluid.

While global existence is not proved for arbitrary initial smooth data, there are partial results in the case of the so-called axisymmetric flows without swirl.

Definition 1.1. We say that a vector field v is *axisymmetric* if it has the form:

$$v(x,t) = v^r(r,z,t)\mathbf{e}_r + v^z(r,z,t)\mathbf{e}_z,$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ is the cylindrical basis of $\mathbb{R}^3$ and the components v^r and v^z do not depend on the angular variable.

The main feature of axisymmetric flows arises in the vorticity which takes the form

$$\omega = (\partial_z v^r - \partial_r v^z)\mathbf{e}_\theta.$$

In this situation, (1.1) and (1.2) become

$$\partial_t \omega + v^r \partial_r \omega + v^z \partial_z \omega - \nu \left(\partial_r^2 \omega + \partial_z^2 \omega + \frac{1}{r}\partial_r \omega - \frac{\omega}{r^2} \right) = \frac{v^r}{r}\omega \tag{1.3}$$

and

$$\partial_t \bar{\omega} + \bar{v}^r \partial_r \bar{\omega} + \bar{v}^z \partial_z \bar{\omega} = \frac{\bar{v}^r}{r}\bar{\omega}. \tag{1.4}$$

For the axisymmetric flows, there are several important works, see for example [3,8,11,12,14]. In particular, for the 3-D axisymmetric Euler equations, Ukhovskii and Yudovich [14] proved the global existence for axisymmetric initial data with finite energy and satisfying in addition $\omega_0 \in L^2 \cap L^\infty$ and $\frac{\omega_0}{r} \in L^2 \cap L^\infty$. In terms of Sobolev regularity these assumptions are satisfied if the velocity v_0 belongs to H^s with $s > \frac{7}{2}$. This is far from the critical regularity of local existence theory $s = \frac{5}{2}$. The optimal result in Sobolev spaces is done by Shirota and Yanagisawa [12] who proved global existence in H^s with $s > \frac{5}{2}$. Recently, in [1] the global well-posedness for the 3-D axisymmetric Euler equations with the initial data lying in the critical Besov spaces $B_{p,1}^{1+\frac{3}{p}}$ with $p \in [1,\infty]$ and satisfying $\frac{\omega_0}{r} \in L^{3,1}$ (Lorentz space) is proved using the geometric properties of axisymmetric vorticity equations skillfully (1.4) and a special decomposition. Making use of completely similar arguments as in [1], we can prove the global well-posedness for the 3-D axisymmetric Navier–Stokes equations in the critical Besov spaces $B_{p,1}^{-1+\frac{3}{p}}$ with $p \in [1,\infty]$.

In this paper, we prove the convergence rate of the inviscid limit of the axisymmetric Navier–Stokes equations (NS) for vanishing viscosity. We prove also the global well-posedness in a critical Besov spaces $B_{2,1}^{\frac{5}{2}}(\mathbb{R}^3)$ and get uniform bounds of the solution on the viscosity. To obtain these results, with the help of similar arguments as in [5], we first make use of twice Fourier localization technique together with Lagrangian coordinates to prove a regularization effect of the vorticity equation (1.3) which allows us to bound the Lipschitz norm of the viscous velocity uniformly on the viscosity ν.

Our main result is the following theorem:

Theorem 1.1. *Let v_0 be an axisymmetric divergence free vector field belonging to $B_{2,1}^{\frac{5}{2}}(\mathbb{R}^3)$. Then, the system* (NS) *has a unique global solution in $\mathcal{C}(\mathbb{R}^+; B_{2,1}^{\frac{5}{2}})$, and the solution v satisfies the following uniform estimate:*

$$\|v(t)\|_{B_{2,1}^{\frac{5}{2}}} \le C_0 \mathrm{e}^{\exp(\exp(\exp C_0 t))}.$$

Moreover, v converges to the Euler solution $\bar{v}$ as the viscosity $\nu \to 0$. More precisely, for all $\nu \in (0,1]$, we have

$$\|v(t) - \bar{v}(t)\|_{L^2} \le C_0 \mathrm{e}^{\exp(\exp C_0 t)}(\nu t),$$

where C_0 is a constant depending only on the initial data but not on the viscosity.

Remark 1.1. In fact, when p, v_0 satisfy one of the following conditions:

1. $p \in (1,3)$ *and* $v_0 \in B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3)$*;*
2. $p = 3$ *and* $v_0 \in B_{3,1}^2(\mathbb{R}^3)$ *satisfying* $\|\frac{\omega_0}{r}\|_{L^{3,1}} < \infty$,

we can obtain the same results. Note that for $p < 3$, the condition $\|\frac{\omega_0}{r}\|_{L^{3,1}} < \infty$ holds automatically from $v_0 \in B_{p,1}^{1+\frac{3}{p}}$ (cf. [1]). In addition, for $p \in (3,\infty]$, $v_0 \in B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3)$ satisfying $\|\frac{\omega_0}{r}\|_{L^{3,1}} < \infty$, we can obtain similar results by similar arguments as in [1,7] where the method used is different from this paper.

The rest of this paper is arranged as follows:

In Sect. 2, we recall the definition and some properties of Besov spaces, and list some useful lemmas. In Sect. 3, we prove a regularization effect of the vorticity equation (1.3). In Sect. 4, we give the proof of Theorem 1.1.

Notation: Throughout the paper, C stands for a constant which may be different in each occurrence. We shall sometimes use the notation $A \lesssim B$ instead of $A \leq CB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

First we recall the Littlewood–Paley Theory. Let (χ, φ) be a couple of smooth radial functions with values in $[0,1]$ such that χ is supported in the ball $\{\xi \in \mathbb{R}^N \big| |\xi| \leq \frac{4}{3}\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^N \big| \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and

$$\chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N;$$

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N \backslash \{0\}.$$

Denoting $\varphi_q(\xi) = \varphi(2^{-q}\xi)$ and $h_q = \mathcal{F}^{-1}\varphi_q$, we define the homogeneous dyadic blocks as

$$\dot{\Delta}_q u := \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y)\,\mathrm{d}\,y, \quad \forall q \in \mathbb{Z}.$$

We shall also use the following low-frequency cut-off:

$$\dot{S}_q u := \sum_{j\leq q-1} \dot{\Delta}_j u.$$

Definition 2.1. Let $\mathcal{S}'_h$ be the space of tempered distributions u such that

$$\lim_{q\to-\infty} \dot{S}_q u = 0, \quad \text{in } \mathcal{S}'.$$

The formal equality

$$u = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u$$

holds in $\mathcal{S}'_h$ and is called the *homogeneous Littlewood–Paley decomposition*.

Let us now recall the definition of the homogeneous Besov spaces:

Definition 2.2. For $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ and $u \in \mathcal{S}'_h$, we set

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_{q\in\mathbb{Z}} 2^{qsr} \|\dot{\Delta}_q u\|^r_{L^p}\right)^{\frac{1}{r}} \quad \text{if } r < \infty$$

and

$$\|u\|_{\dot{B}^s_{p,\infty}} := \sup_{q\in\mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p}.$$

Now we define the homogeneous Besov spaces as

$$\dot{B}^s_{p,r} := \left\{ u \in \mathcal{S}'_h \,\middle|\, \|u\|_{\dot{B}^s_{p,r}} < \infty \right\}.$$

The above definition does not depend on the choice of the couple (χ, φ). Remark that if $s < \frac{N}{p}$ or $s = \frac{N}{p}$ and $r = 1$, then $\dot{B}^s_{p,r}$ is a Banach space.

We now recall some basic properties of the homogeneous Besov spaces.

Proposition 2.1. *The following properties hold true* (cf. [10,13])*:*

1. *Sobolev embedding: if $p_1 \le p_2$ and $r_1 \le r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-N\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*
2. *Let $\sigma \in \mathbb{R}$, then the operator $(-\triangle)^{\sigma/2}$ is an isomorphism from $\dot{B}^s_{p,r}$ to $\dot{B}^{s-\sigma}_{p,r}$.*
3. *Let $\beta \in (0,1)$, $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, then we have the sharp interpolation result: $\|u\|_{\dot{B}^{\beta s_1 + (1-\beta)s_2}_{p,1}} \lesssim \|u\|^{\beta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\beta}_{\dot{B}^{s_2}_{p,\infty}}$.*

Let $T > 0$ and $\rho \in [1, \infty]$, we denote by $L^\rho_T \dot{B}^s_{p,r}$ the set of all tempered distribution u satisfying

$$\|u\|_{L^\rho_T \dot{B}^s_{p,r}} := \left\| \left(\sum_{q\in\mathbb{Z}} 2^{qsr} \|\dot{\Delta}_q u\|^r_{L^p} \right)^{\frac{1}{r}} \right\|_{L^\rho_T} < \infty.$$

Now we briefly state the definition of the nonhomogeneous Besov spaces. The nonhomogeneous Littlewood–Paley decomposition $(\Delta_q)_{q\ge -1}$ is given by

$$\Delta_q := \dot{\Delta}_q = \varphi(2^{-q}D) \quad \text{if} \quad q \ge 0, \quad \text{and} \quad \Delta_{-1} := \chi(D).$$

And the low-frequency cut-off is defined as $S_q := \sum_{-1\le j\le q-1} \Delta_j$.

Definition 2.3. For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $u \in \mathcal{S}'$, we set

$$\|u\|_{B^s_{p,r}} := \left(\sum_{q\ge -1} 2^{qsr} \|\Delta_q u\|^r_{L^p} \right)^{\frac{1}{r}} \quad \text{if} \quad r < \infty$$

and

$$\|u\|_{B^s_{p,\infty}} := \sup_{q\ge -1} 2^{qs} \|\Delta_q u\|_{L^p}.$$

Then the nonhomogeneous Besov spaces are defined as

$$B^s_{p,r} := \left\{ u \in \mathcal{S}' \,\middle|\, \|u\|_{B^s_{p,r}} < \infty \right\}.$$

Similar to the homogeneous case, we can also define the time–space Besov spaces $L^\rho_T B^s_{p,r}$.

Now we give some useful lemmas.

Lemma 2.2. (cf. [6]) *Let ϕ be a smooth function supported in the shell $\left\{\xi \in \mathbb{R}^N \,\middle|\, R_1 \le |\xi| \le R_2, 0 < R_1 < R_2\right\}$. There exist two positive constants κ and C depending only on ϕ such that for all $1 \le p \le \infty$, $\tau \ge 0$ and $\lambda > 0$, we have*

$$\|\phi(\lambda^{-1}D)\mathrm{e}^{\tau\triangle}u\|_{L^p} \le C\mathrm{e}^{-\kappa\tau\lambda^2} \|\phi(\lambda^{-1}D)u\|_{L^p}.$$

Lemma 2.3. (cf. [4]) *Let v be a vector field belonging to $L^1_{loc}(\mathbb{R}^+; \mathrm{Lip}(\mathbb{R}^N))$. For $q \in \mathbb{Z}$ we set $u_q := \dot{\Delta}_q u$ and denote by ψ_q the flow of the regularized vector field $\dot{S}_{q-1}v$. Then $\forall p \in [1, \infty]$, we have*

$$\|(\triangle u_q) \circ \psi_q - \triangle(u_q \circ \psi_q)\|_{L^p} \le Ce^{CV(t)} V(t) 2^{2q} \|u_q\|_{L^p},$$

where $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} \,\mathrm{d}\tau$ and $C = C(p) > 0$ is a constant.

Lemma 2.4. (cf. [5]) *Let v be a divergence free vector field belonging to* $\mathrm{Lip}(\mathbb{R}^N)$ *and $u \in L^p$, $p \in [1,\infty]$. Then there exists a constant C depending only on N such that, for all $q \geq -1$,*

$$\|[\Delta_q, v\cdot\nabla]u\|_{L^p} \leq C\|\nabla v\|_{L^\infty}\|u\|_{L^p}.$$

Lemma 2.5. (cf. [6,15]) *Assume $N \geq 2$. Let f be a function in Schwartz class and ψ a diffeomorphism of $\mathbb{R}^N$ preserving Lebesgue measure, then we have for all $p \in [1,\infty]$ and for all $j, q \in \mathbb{Z}$,*

$$\|\dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p} \leq C2^{-|j-q|}\|\nabla\psi^{\eta(j,q)}\|_{L^\infty}\|\dot{\Delta}_q f\|_{L^p},$$

with $\eta(j,q) = sign(j-q)$.

3. Regularization effect

In this section, let us prove a regularization effect of the vorticity equation (1.3) which allows us to bound the Lipschitz norm of the velocity uniformly on the viscosity. We will prove it by using twice Fourier localization technique together with Lagrangian coordinates and commutator estimates.

Proposition 3.1. *Let $p \in [1,\infty]$. Let v be an axisymmetric Lipschitz solution of the system (NS) and $\omega = \nabla\times v$ be a solution to (1.3) with initial vorticity $\omega_0 \in L^p$ and $\frac{\omega_0}{r} \in L^{3,1}$. Then for all $q \in \mathbb{N}$, we have*

$$\nu 2^{2q}\int_0^t \|\triangle_q\omega(\tau)\|_{L^p}\,\mathrm{d}\,\tau \leq C\|\omega_0\|_{L^p}\left(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\,\tau\right)\mathrm{e}^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

Proof. Let $\omega_q := \Delta_q\omega$. Applying Δ_q to (1.1) instead of (1.3), we get

$$\partial_t\omega_q + S_{q-1}v\cdot\nabla\omega_q - \nu\triangle\omega_q = f_q$$

with $f_q := \Delta_q(\omega\cdot\nabla v) + (S_{q-1}v - v)\cdot\nabla\omega_q - [\Delta_q, v\cdot\nabla]\omega$.

Let ψ_q be the flow of the regularized vector field $S_{q-1}v$. Denote $\bar{\omega}_q := \omega_q\circ\psi_q$, $\bar{f}_q := f_q\circ\psi_q$. Then we have

$$\partial_t\bar{\omega}_q - \nu\triangle\bar{\omega}_q = \bar{f}_q + \nu G_q \tag{3.1}$$

with $G_q := (\triangle\omega_q)\circ\psi_q - \triangle(\omega_q\circ\psi_q)$.

Applying the homogeneous operator $\dot{\Delta}_j$ to (3.1) and using Lemma 2.2 lead to

$$\begin{aligned}\|\dot{\Delta}_j\bar{\omega}_q(t)\|_{L^p} &\lesssim \mathrm{e}^{-\kappa\nu t2^{2j}}\|\dot{\Delta}_j\omega_{0,q}\|_{L^p}\\ &\quad+\int_0^t \mathrm{e}^{-\kappa\nu(t-\tau)2^{2j}}\left(\|\dot{\Delta}_j\bar{f}_q\|_{L^p}+\nu\|\dot{\Delta}_jG_q\|_{L^p}\right)\mathrm{d}\,\tau.\end{aligned} \tag{3.2}$$

Now from Lemma 2.3 we have

$$\|\dot{\Delta}_jG_q(t)\|_{L^p} \leq Ce^{CV(t)}V(t)2^{2q}\|\omega_q\|_{L^p}. \tag{3.3}$$

Since $\|\Delta_q(\omega\cdot\nabla v)\|_p \lesssim \|\nabla v\|_{L^\infty}\|\omega\|_{L^p}$, using the incompressibility of the flow and Lemma 2.4, we deduce that

$$\|\dot{\Delta}_j\bar{f}_q(t)\|_{L^p} \leq C\|\bar{f}_q(t)\|_{L^p} = C\|f_q(t)\|_{L^p} \leq C\|\nabla v\|_{L^\infty}\|\omega\|_{L^p}. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.2), taking the L^1 norm over $[0,t]$ yields

$$\begin{aligned}\|\dot{\Delta}_j\bar{\omega}_q\|_{L^1_tL^p} &\lesssim \nu^{-1}2^{-2j}\|\dot{\Delta}_j\omega_{0,q}\|_{L^p} + \nu^{-1}V(t)2^{-2j}\|\omega\|_{L^\infty_tL^p}\\ &\quad+\mathrm{e}^{CV(t)}V(t)2^{2(q-j)}\|\omega_q\|_{L^1_tL^p}.\end{aligned}$$

Applying the maximum principle to (1.1), we have for all $p \in [1, \infty]$

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\omega(\tau)\|_{L^p} \,\mathrm{d}\,\tau.$$

Gronwall lemma ensures that

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} \mathrm{e}^{\int_0^t \|\nabla v(\tau)\|_{L^\infty} \,\mathrm{d}\,\tau}. \tag{3.5}$$

Thus the above estimate becomes

$$\begin{aligned}\|\dot{\Delta}_j \bar{\omega}_q\|_{L^1_t L^p} \lesssim \nu^{-1} 2^{-2j} \|\dot{\Delta}_j \omega_{0,q}\|_{L^p} + \nu^{-1} \mathrm{e}^{CV(t)} V(t) 2^{-2j} \|\omega_0\|_{L^p} \\ + \mathrm{e}^{CV(t)} V(t) 2^{2(q-j)} \|\omega_q\|_{L^1_t L^p}.\end{aligned}$$

Multiplying both sides by $\nu 2^{2q}$, we get

$$\begin{aligned}\nu 2^{2q} \|\dot{\Delta}_j \bar{\omega}_q\|_{L^1_t L^p} \leq C 2^{2(q-j)} \|\dot{\Delta}_j \omega_{0,q}\|_{L^p} + C e^{CV(t)} V(t) 2^{2(q-j)} \|\omega_0\|_{L^p} \\ + C e^{CV(t)} V(t) 2^{2(q-j)} \nu 2^{2q} \|\omega_q\|_{L^1_t L^p}.\end{aligned} \tag{3.6}$$

Let $M_0 \in \mathbb{N}$ to be fixed hereafter. The incompressibility of the flow shows that

$$\begin{aligned}\nu 2^{2q} \|\omega_q\|_{L^1_t L^p} &= \nu 2^{2q} \|\bar{\omega}_q\|_{L^1_t L^p} \\ &\leq \sum_{|j-q| \leq M_0} \nu 2^{2q} \|\dot{\Delta}_j \bar{\omega}_q\|_{L^1_t L^p} + \sum_{|j-q| > M_0} \nu 2^{2q} \|\dot{\Delta}_j \bar{\omega}_q\|_{L^1_t L^p}.\end{aligned} \tag{3.7}$$

As $\dot{\Delta}_j \omega_{0,q} = 0$ for $|j-q| > 1$, by (3.6) we get

$$\begin{aligned}\sum_{|j-q| \leq M_0} \nu 2^{2q} \|\dot{\Delta}_j \bar{\omega}_q\|_{L^1_t L^p} \leq C \|\omega_0\|_{L^p} + C e^{CV(t)} V(t) 2^{2M_0} \|\omega_0\|_{L^p} \\ + C e^{CV(t)} V(t) 2^{2M_0} \nu 2^{2q} \|\omega_q\|_{L^1_t L^p}.\end{aligned} \tag{3.8}$$

On the other hand, Lemma 2.5 implies that

$$\begin{aligned}\sum_{|j-q| > M_0} \nu 2^{2q} \|\dot{\Delta}_j \bar{\omega}_q\|_{L^1_t L^p} &\leq C \sum_{|j-q| > M_0} 2^{-|j-q|} \mathrm{e}^{CV(t)} \nu 2^{2q} \|\omega_q\|_{L^1_t L^p} \\ &\leq C e^{CV(t)} 2^{-M_0} \nu 2^{2q} \|\omega_q\|_{L^1_t L^p}.\end{aligned} \tag{3.9}$$

Plugging (3.8) and (3.9) into (3.7), we have

$$\begin{aligned}\nu 2^{2q} \|\omega_q\|_{L^1_t L^p} \leq C \|\omega_0\|_{L^p} + C e^{CV(t)} V(t) 2^{2M_0} \|\omega_0\|_{L^p} \\ + \left(C e^{CV(t)} V(t) 2^{2M_0} + C e^{CV(t)} 2^{-M_0} \right) \nu 2^{2q} \|\omega_q\|_{L^1_t L^p}.\end{aligned}$$

Choose M_0 and t such that

$$C e^{CV(t)} V(t) 2^{2M_0} + C e^{CV(t)} 2^{-M_0} \leq \frac{1}{2} \quad \text{and} \quad V(t) \leq C_1,$$

where C_1 is a small absolute constant. Thus the above estimate becomes

$$\nu 2^{2q} \|\omega_q\|_{L^1_t L^p} \leq C \|\omega_0\|_{L^p}. \tag{3.10}$$

Now for arbitrary time T, we split $[0, T]$ into m subintervals like as $[0, T_1]$, $[T_1, T_2]$ and so on, such that

$$\int_{T_i}^{T_{i+1}} \|\nabla v(t)\|_{L^\infty} \,\mathrm{d}\,t \simeq C_1. \tag{3.11}$$

Similar arguments as in deriving (3.10) lead to

$$\nu 2^{2q}\|\omega_q\|_{L^1(T_i,T_{i+1};L^p)} \le C\|\omega(T_i)\|_{L^p}. \tag{3.12}$$

Now we prove a new maximum principle for the axisymmetric equation (1.3). By (1.3), we have for all $p \in [1,\infty]$

$$\|\omega(t)\|_{L^p} \le \|\omega_0\|_{L^p} + \int_0^t \left\|\frac{v^r}{r}(\tau)\right\|_{L^\infty} \|\omega(\tau)\|_{L^p}\,\mathrm{d}\,\tau.$$

It has been shown in [1] that

$$\left\|\frac{v^r}{r}\right\|_{L^\infty} \le C\left\|\frac{\omega}{r}\right\|_{L^{3,1}}. \tag{3.13}$$

On the other hand, the following estimate holds (see [14])

$$\left\|\frac{\omega}{r}\right\|_{L^p} \le \left\|\frac{\omega_0}{r}\right\|_{L^p}, \quad \forall p > 1.$$

Thus by interpolation, the inequality (3.13) reduces to

$$\left\|\frac{v^r}{r}\right\|_{L^\infty} \le C\left\|\frac{\omega_0}{r}\right\|_{L^{3,1}}.$$

This together with Gronwall lemma ensures that

$$\|\omega(t)\|_{L^p} \le \|\omega_0\|_{L^p}\mathrm{e}^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

Plugging this into (3.12) yields

$$\nu 2^{2q}\|\omega_q\|_{L^1(T_i,T_{i+1};L^p)} \le C\|\omega_0\|_{L^p}\mathrm{e}^{CT\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

Noting that $mC_1 \simeq 1 + V(T)$, we end up with

$$\nu 2^{2q}\|\omega_q\|_{L^1_T L^p} \le Cm\|\omega_0\|_{L^p}\mathrm{e}^{CT\|\frac{\omega_0}{r}\|_{L^{3,1}}} \le C\|\omega_0\|_{L^p}(1+V(T))\mathrm{e}^{CT\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

Thus the proof is completed. □

Remark 3.1. Note that the vorticity equation (1.3) is different from the 2-D case and our proof is substantially different from the one in [5].

Remark 3.2. The method used in the proof of Proposition 3.1 is necessary only for $p = \infty$. The proof can be done in an easier way for $p < \infty$ by using the refined Bernstein estimate in [2].

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We will divide the proof into three parts.

4.1. Convergence rate

In this subsection, we give the estimate of the convergence rate.

The following global estimates for 3-D axisymmetric Euler equations have been obtained in [1]:

$$\|\bar{v}(t)\|_{B^{\frac{5}{2}}_{2,1}} \le C_0\mathrm{e}^{\exp(\exp C_0 t)}, \quad \|\nabla\bar{v}(t)\|_{L^\infty} \le C_0\mathrm{e}^{\exp C_0 t}, \tag{4.1}$$

with C_0 a constant depending only on the initial data. We set

$$\tilde{v} = v - \bar{v}, \quad \tilde{\pi} = \pi - \bar{\pi}.$$

Then $(\tilde{v},\tilde{\pi})$ satisfies the following

$$\begin{cases}\partial_t\tilde{v}+v\cdot\nabla\tilde{v}-\nu\triangle\tilde{v}+\nabla\tilde{\pi}=\nu\triangle\bar{v}-\tilde{v}\cdot\nabla\bar{v},\\ \tilde{v}|_{t=0}=0.\end{cases}$$

Taking the L^2 inner product and using the incompressibility of the flow, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\tilde{v}\|_{L^2}^2+\nu\|\nabla\tilde{v}\|_{L^2}^2\le\nu\|\triangle\bar{v}\|_{L^2}\|\tilde{v}\|_{L^2}+\|\nabla\bar{v}\|_{L^\infty}\|\tilde{v}\|_{L^2}^2.$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\|\tilde{v}\|_{L^2}\le\nu\|\triangle\bar{v}\|_{L^2}+\|\nabla\bar{v}\|_{L^\infty}\|\tilde{v}\|_{L^2}.$$

By Gronwall lemma, we get

$$\|\tilde{v}\|_{L^2}\le\nu\mathrm{e}^{\int_0^t\|\nabla\bar{v}(\tau)\|_{L^\infty}\,\mathrm{d}\,\tau}\int_0^t\|\triangle\bar{v}(\tau)\|_{L^2}\,\mathrm{d}\,\tau.$$

Using (4.1), we can obtain

$$\begin{aligned}\|\tilde{v}\|_{L^2}&\le\nu\mathrm{e}^{\exp(\exp C_0t)}\int_0^t\|\bar{v}(\tau)\|_{B^2_{2,1}}\,\mathrm{d}\,\tau\\&\le\nu\mathrm{e}^{\exp(\exp C_0t)}\int_0^t\|\bar{v}(\tau)\|_{B^{\frac{5}{2}}_{2,1}}\,\mathrm{d}\,\tau\\&\le\nu\mathrm{e}^{\exp(\exp C_0t)}\int_0^tC_0\mathrm{e}^{\exp(\exp C_0\tau)}\,\mathrm{d}\,\tau\\&\le C_0\mathrm{e}^{\exp(\exp C_0t)}(\nu t).\end{aligned}\tag{4.2}$$

This yields the desired result.

Remark 4.1. Similar arguments show that the same result holds true for L^3:

$$\|v-\bar{v}\|_{L^3}\le C_0\mathrm{e}^{\exp(\exp C_0t)}(\nu t).$$

4.2. Uniform Lipschitz estimate of the velocity

In this subsection, we prove the uniform Lipschitz estimate of the velocity.

Let M_1 be a positive integer which will be fixed later. By the triangle inequality we have

$$\begin{aligned}\int_0^t\|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\,\tau\le\int_0^t\|S_{M_1}\nabla\bar{v}(\tau)\|_{L^\infty}\,\mathrm{d}\,\tau+\int_0^t\|S_{M_1}\nabla(v-\bar{v})(\tau)\|_{L^\infty}\,\mathrm{d}\tau\\+\int_0^t\|(\mathrm{Id}-S_{M_1})\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau.\end{aligned}\tag{4.3}$$

By (4.1), we have

$$\int_0^t\|S_{M_1}\nabla\bar{v}(\tau)\|_{L^\infty}\,\mathrm{d}\,\tau\le C\int_0^t\|\nabla\bar{v}(\tau)\|_{L^\infty}\,\mathrm{d}\,\tau\le C_0\mathrm{e}^{\exp C_0t}.\tag{4.4}$$

On the other hand, it is easy to show that by Bernstein lemma (noticing Remark 4.1),

$$\int_0^t \|S_{M_1}\nabla(v-\bar v)(\tau)\|_{L^\infty}\,\mathrm{d}\tau \le C2^{2M_1}\int_0^t \|(v-\bar v)(\tau)\|_{L^3}\,\mathrm{d}\tau$$
$$\le C_0\nu t 2^{2M_1}\mathrm{e}^{\exp(\exp C_0 t)}. \tag{4.5}$$

For the last term in (4.3), we make use of Proposition 3.1 to get

$$\int_0^t \|(\mathrm{Id}-S_{M_1})\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau \le C\sum_{q\ge M_1}\int_0^t \|\Delta_q\omega\|_{L^\infty}\,\mathrm{d}\tau$$
$$\le C\frac{\|\omega_0\|_{L^\infty}}{\nu 2^{2M_1}}\left(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right)\mathrm{e}^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}. \tag{4.6}$$

Plugging (4.4), (4.5) and (4.6) into (4.3), we obtain that

$$\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau \le C_0\left(1+\nu t 2^{2M_1}\right)\mathrm{e}^{\exp(\exp C_0 t)}$$
$$+C\frac{\|\omega_0\|_{L^\infty}}{\nu 2^{2M_1}}\left(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right)\mathrm{e}^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

Choosing M_1 such that

$$C\frac{\|\omega_0\|_{L^\infty}}{\nu 2^{2M_1}}\mathrm{e}^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}\simeq\frac{1}{2},$$

which leads to

$$\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau \le C_0(1+t\|\omega_0\|_{L^\infty}\mathrm{e}^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}})\mathrm{e}^{\exp(\exp C_0 t)} \le C_0\mathrm{e}^{\exp(\exp C_0 t)}. \tag{4.7}$$

4.3. Uniform persistence of the initial regularity

By similar arguments as in Proposition A.2 in [1], we can obtain

$$\mathrm{e}^{-CV(t)}\|\omega(t)\|_{B^{\frac{3}{2}}_{2,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{2}}_{2,1}} + \int_0^t \|\nabla v(\tau)\|_{L^\infty}\mathrm{e}^{-CV(\tau)}\|\omega(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\,\mathrm{d}\tau.$$

Applying Gronwall lemma to the above inequality, we get

$$\|\omega(t)\|_{B^{\frac{3}{2}}_{2,1}} \le \|\omega_0\|_{B^{\frac{3}{2}}_{2,1}}\mathrm{e}^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau}. \tag{4.8}$$

Now for the velocity, we obtain that by (4.8) and the maximum principle,

$$\|v(t)\|_{B^{\frac{5}{2}}_{2,1}} \le \|\Delta_{-1}v(t)\|_{L^2} + \sum_{q\in\mathbb{N}} 2^{\frac{5}{2}q}\|\Delta_q v(t)\|_{L^2}$$
$$\le C\|v(t)\|_{L^2} + \|\omega(t)\|_{B^{\frac{3}{2}}_{2,1}}$$
$$\le C\|v_0\|_{L^2} + C_0\mathrm{e}^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau}$$
$$\le C\|v_0\|_{B^{\frac{5}{2}}_{2,1}} + C_0\mathrm{e}^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau}.$$

Plugging (4.7) into the above inequality, we can get

$$\|v(t)\|_{B^{\frac{5}{2}}_{2,1}} \leq C_0 e^{\exp(\exp(\exp C_0 t))}.$$

Therefore, the proof is completed.

Acknowledgments

The author would like to thank Prof. Changxing Miao, Sahbi Keraani and Xiaoming Wang for helpful comments and suggestions. The author is also deeply grateful to the referees for their valuable advice.

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(Received: October 2, 2008)