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Energy decay rate of the thermoelastic Bresse system

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Abstract. In this paper, we study the energy decay rate for the thermoelastic Bresse system which describes the motion of a linear planar, shearable thermoelastic beam. If the longitudinal motion and heat transfer are neglected, this model reduces to the well-known thermoelastic Timoshenko beam equations. The system consists of three wave equations and two heat equations coupled in certain pattern. The two wave equations about the longitudinal displacement and shear angle displacement are effectively damped by the dissipation from the two heat equations. Actually, the corresponding energy decays exponentially like the classical one-dimensional thermoelastic system. However, the third wave equation about the vertical displacement is only weakly damped. Thus the decay rate of the energy of the overall system is still unknown. We will show that the exponentially decay rate is preserved when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, only a polynomial type decay rate can be obtained. These results are proved by verifying the frequency domain conditions.

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1. Introduction

In their study on networks of flexible beams, Lagnese, Leugering and Schmidt [9] derived a general model for 3-d nonlinear thermoelastic beams. A special case of this model is a linear planar, shearable thermoelastic beam whose motion is governed by the following system of partial differential equations:

$$\rho h\ddot{w}_1 = (Eh(w_1' - kw_3) - \alpha\theta_1)' - kGh(\phi_2 + w_3' + kw_1), \tag{1.1}$$
$$\rho h\ddot{w}_3 = Gh(\phi_2 + w_3' + kw_1)' + kEh(w_1' - kw_3) - k\alpha\theta_1, \tag{1.2}$$
$$\rho I\ddot{\phi}_2 = EI\phi_2'' - Gh(\phi_2 + w_3' + kw_1) - \alpha\theta_3', \tag{1.3}$$
$$\rho c\dot{\theta}_1 = \theta_1'' - \alpha T_0(\dot{w}_1' - k\dot{w}_3), \tag{1.4}$$
$$\rho c\dot{\theta}_3 = \theta_3'' - \alpha T_0\dot{\phi}_2', \tag{1.5}$$

where w_1, w_3, ϕ_2 are the longitudinal, vertical and shear angle displacements; θ_1, θ_3 are the temperature deviations from the reference temperature T_0 along the longitudinal and vertical directions; $E, G, \rho, I, m, h, k, c$ are positive constants for the elastic and thermal material properties. The dot and prime are used for the partial derivatives with respect to time $t \geq 0$ and spatial location $x \in [0, l]$ respectively.

From this seemingly complicated system (three wave equations coupled with two heat equations), very interesting special cases can be obtained. In particular, the isothermal system is exactly the system obtained by Bresse [4] in 1856. The Bresse system, equations (1.1)-(1.3) with θ_1, θ_3 removed, is more general than the well-known Timoshenko system where the longitudinal displacement w_1 is not considered. If both θ_1 and w_1 are neglected, the Bresse thermoelastic system simplifies to the following Timoshenko thermoelastic system:

$$\rho h \ddot{w}_3 = Gh(\phi_2 + w_3')', \tag{1.6}$$
$$\rho I \ddot{\phi}_2 = EI\phi_2'' - Gh(\phi_2 + w_3') - \alpha\theta_3', \tag{1.7}$$
$$\rho c \dot{\theta}_3 = \theta_3'' - \alpha T_0 \dot{\phi}_2', \tag{1.8}$$

which was studied by Racke and Rivera [13]. For the boundary conditions

$$w_3(t,x) = \phi_2'(t,x) = \theta_3(t,x) = 0, \quad \text{at } x = 0, l \tag{1.9}$$

or

$$w_3(t,x) = \phi_2(t,x) = \theta_3'(t,x) = 0, \quad \text{at } x = 0, l, \tag{1.10}$$

they obtained exponential stability for the thermoelastic Timoshenko system (1.6)–(1.8) when $E = G$. Moreover, they also proved non-exponential stability for the case of boundary condition (1.9) when $E \neq G$ by a method used in [5]. We refer the reader to the references [15], [8] and [2] for the Timoshenko system with other kinds of damping mechanisms such as viscous damping, and viscoelastic damping of Boltzmann type acting on the motion equation of w_3 or ϕ_2. In all three cases, the rotational displacement ϕ_2 of the Timoshenko system is effectively damped due to the thermal energy dissipation in equations (1.7)–(1.8). In fact, the energy associated with this component of motion decays exponentially. The transverse displacement w_3 is only indirectly damped through the coupling, which can be observed from (1.6). The effectiveness of this damping depends on the type of coupling and the wave speeds. When the wave speeds are the same ($E = G$), the indirect damping is actually strong enough to induce exponential stability for the Timoshenko system. But when the wave speeds are different, the Timoshenko system loses the exponential stability. This phenomenon has been observed for partially damped second order evolution equations. We quote [3], [1] for the polynomial energy decay rate by the multiplier technique, [7] for the study of optimal decay rate by spectral compensation, and [16] for the polynomial energy decay rate of hyperbolic-parabolic coupled system by Riesz basis approach.

In this paper, we study the energy decay rate for the thermoelastic Bresse system (1.1)–(1.5) with the boundary conditions

$$w_1'(t,x) = w_3(t,x) = \phi_2'(t,x) = \theta_1(t,x) = \theta_3(t,x) = 0, \text{ for } x = 0, l, \tag{1.11}$$

or

$$w_1(t,x) = w_3(t,x) = \phi_2(t,x) = \theta_1(t,x) = \theta_3(t,x) = 0, \quad \text{for } x = 0, l, \tag{1.12}$$

and initial conditions

$$\begin{aligned} &w_1(0,x) = u_0(x), \;\; \dot{w}_1(0,x) = v_0(x), \;\; \phi_2(0,x) = \phi_0(x), \;\; \dot{\phi}_2(0,x) = \psi_0(x), \\ &w_3(0,x) = w_0(x), \;\; \theta_1(0,x) = \theta_0(x), \;\; \theta_3(0,x) = \xi_0(x), \end{aligned} \tag{1.13}$$

whose total energy is

$$\begin{aligned} \mathcal{E}(t) \;=\; & \frac{1}{2}\int_0^l \Big\{ [Eh(w_1' - kw_3)^2 + Gh(\phi_2 + w_3' + kw_1)^2 + EI(\phi_2')^2] \\ & + \; [\rho h(\dot{w}_1^2 + \dot{w}_3^2) + \rho I \dot{\phi}_2^2] + \frac{\rho c}{T_0}(\theta_1^2 + \theta_3^2) \Big\} dx. \end{aligned} \tag{1.14}$$

We first show that $\mathcal{E}(t)$ decays exponentially if assuming $E = G$. However, from the theory of elasticity, E and G denote the Young's modulus and the shear modulus, respectively. These two elastic modulus are not equal since

$$G = \frac{E}{2(1+\nu)}$$

where $\nu \in (0, \frac{1}{2})$ is the Poisson's ratio. Thus, the exponential stability for the case of $E = G$ is only mathematically sound. However, it does provide useful insight to the study of similar models arising from other applications. This result is anticipated once we consider the results in [13]. The thermoelastic Bresse system contains three wave equations, and two of them, (1.1) and (1.3), are effectively damped by the thermal damping from equation (1.4)-(1.5). The third wave equation, (1.2), is indirectly damped through the coupling and a weak thermal damping. If the wave speed of the third equation is the same as the wave speed of the effectively damped wave equation (1.1) or (1.3), then exponential stability of the overall system is expected.

When $E \neq G$, $\mathcal{E}(t)$ does not decay exponentially. However, in this case we are able to obtain a polynomial-type of decay rate. Our results also apply to the thermoelastic Timoshenko system. It is interesting to see that the polynomial decay rate of the system with the boundary conditions (1.11) is faster than the one with boundary conditions (1.12). Our main tools are the frequency domain characterization of exponential decay obtained by Prüss [12] and Huang [6], and of polynomial decay obtained recently by the authors of the present paper [10]. This technique has been successfully applied to many dissipative systems for exponential stability and analyticity of the associated semigroups. Our results in [10] extended this technique further to the systems which are only strongly stable but not exponentially stable. Several examples of such systems and their polynomial decay rate of energy were given in [10]. We believe that this frequency domain method for polynomial decay rate of energy will enjoy success same as its counterpart of exponential decay rate of energy for partially or locally damped distributed systems.

For readers' convenience, we include these two frequency domain conditions here.

Theorem 1.1. ([6], [12]) *A C_0 semigroup $e^{t\mathcal{A}}$ of contractions on a Hilbert space $\mathcal{H}$ is exponentially stable if and only if*

$$\boldsymbol{i}\mathbb{R} \subset \rho(\mathcal{A}), \qquad \sup_{\beta\in\mathbb{R}} \|(\boldsymbol{i}\beta I - \mathcal{A})^{-1}\| < +\infty. \tag{1.15}$$

Theorem 1.2. ([10]) *If a bounded C_0 semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ satisfies*

$$\boldsymbol{i}\mathbb{R} \subset \rho(\mathcal{A}), \qquad \sup_{|\beta|\geq 1} \frac{1}{\beta^j}\|(\boldsymbol{i}\beta I - \mathcal{A})^{-1}\| < +\infty \tag{1.16}$$

for some $j > 0$, then for any positive integer m there exists a constant $C_m > 0$ such that

$$\|e^{t\mathcal{A}}z_0\|_{\mathcal{H}} \leq C_m \left(\frac{\ln t}{t}\right)^{\frac{m}{j}} (\ln t)\|z_0\|_{\mathcal{D}(\mathcal{A}^m)} \tag{1.17}$$

for all $z_0 \in \mathcal{D}(\mathcal{A}^m)$.

This paper is organized as follows. Section 2 is for the semigroup setting of the thermoelastic Bresse system. Section 3 and 4 are devoted to exponential and polynomial decay rate of the system energy, respectively.

2. Semigroup setting

To avoid using subscript for the variables, we denote

$$u = w_1, \;\; w = w_3, \;\; \phi = \phi_2, \;\; v = \dot{w}_1, \;\; y = \dot{w}_3, \;\; \psi = \dot{\phi}_2, \;\; \theta = \theta_1, \;\; \xi = \theta_3.$$

Thus, the state variable vector is $z = (u, w, \phi, v, y, \psi, \theta, \xi)$.

In order to choose the proper state space for the system, we shall find the static solution first. Thus, we consider the static system associated with (1.1)-(1.5)

$$Eh(u' - kw)' - \alpha\theta' - kGh(\phi + w' + ku) = 0, \tag{2.1}$$

$$Gh(\phi + w' + ku)' + kEh(u' - kw) - k\alpha\theta = 0, \tag{2.2}$$

$$EI\phi'' - Gh(\phi + w' + ku) - \alpha\xi' = 0, \tag{2.3}$$

$$\theta'' = 0, \tag{2.4}$$

$$\xi'' = 0. \tag{2.5}$$

Since θ, ξ both vanish at $x = 0$ and l, it follows from the equations (2.4)-(2.5) that $\theta = \xi \equiv 0$. We then multiply (2.1)-(2.3) by u, w, ϕ and integrate from 0 to l, respectively. This yields

$$\int_0^l [Eh(u' - kw)^2 + Gh(\phi + w' + ku)^2 + EI(\phi')^2]dx = 0. \tag{2.6}$$

In the case of boundary condition (1.12), it is clear from (2.6) that $u = w = \phi \equiv 0$. But in the case of boundary condition (1.11)

$$u = -c_1 \cos kx - \frac{c_2}{k}, \quad w = c_1 \sin kx, \quad \phi = c_2, \quad \theta = \xi \equiv 0$$

are nonzero solutions of (2.1)-(2.5) for any constants c_1, c_2 as long as $c_1 \sin kl = 0$. Therefore, we will impose $\int_0^l \phi(x)dx = 0$ to force $\phi \equiv 0$. Furthermore, $u = w \equiv 0$ if $k \neq \frac{n\pi}{l}$. On the other hand, from the dynamical system (1.1) and (1.3), we have

$$\ddot{f}(t) + af(t) = 0$$

with $f(t) = \int_0^l [\phi(x,t) + ku(x,t)]dx$. If the initial condition $f(0) = \dot{f}(0) = 0$ is satisfied, $f(t)$ will be kept at zero for all $t > 0$. Although a translation of state variables can shift the equilibrium state to zero, we are going to deal with this by choosing the state spaces

$$\begin{aligned} \mathcal{H}_1 &= H_*^1 \times H_0^1 \times H_*^1 \times L_*^2 \times L^2 \times L_*^2 \times (L^2)^2, \\ \mathcal{H}_2 &= (H_0^1)^3 \times (L^2)^5, \end{aligned}$$

where

$$H_*^1 = \Big\{ f \in H^1(0,l) \Big| \int_0^l f(x) = 0 \Big\}, \qquad L_*^2 = \Big\{ f \in L^2(0,l) \Big| \int_0^l f(x) = 0 \Big\}. \tag{2.7}$$

Both state spaces are equipped with the inner product which induces the energy norm

$$\begin{aligned} \|z\|_{\mathcal{H}_i}^2 &= Eh\|u' - kw\|^2 + Gh\|\phi + w' + ku\|^2 + EI\|\phi'\|^2 + \rho h\|v\|^2 + \rho h\|y\|^2 \\ &+ \rho I\|\psi\|^2 + \frac{\rho c}{T_0}(\|\theta\|^2 + \|\xi\|^2). \end{aligned} \tag{2.8}$$

Here and after, $\|\cdot\|$ denotes the $L^2(0,l)$ norm.

Define a linear operator $\mathcal{A}_i : \mathcal{H}_i \to \mathcal{H}_i (i = 1,2)$ by

$$\mathcal{A}_i z = \begin{bmatrix} v \\ y \\ \psi \\ \frac{E}{\rho}(u' - kw)' - \frac{\alpha}{\rho h}\theta' - \frac{kG}{\rho}(\phi + w' + ku) \\ \frac{G}{\rho}(\phi + w' + ku)' + \frac{kE}{\rho}(u' - kw) - \frac{k\alpha}{\rho h}\theta \\ \frac{E}{\rho}\phi'' - \frac{Gh}{\rho I}(\phi + w' + ku) - \frac{\alpha}{\rho I}\xi' \\ \frac{1}{\rho c}\theta'' - \frac{\alpha T_0}{\rho c}(v' - ky) \\ \frac{1}{\rho c}\xi'' - \frac{\alpha T_0}{\rho c}\psi' \end{bmatrix} \tag{2.9}$$

with

$$\mathcal{D}(\mathcal{A}_1) = \{ z \in \mathcal{H}_1 \big| w, \theta, \xi \in H_0^1 \cap H^2, u', \phi', y \in H_0^1, v, \psi \in H_*^1 \}, \tag{2.10}$$

$$\mathcal{D}(\mathcal{A}_2) = \{ z \in \mathcal{H}_2 \big| u, w, \phi, \theta, \xi \in H_0^1 \cap H^2, v, y, \psi \in H_0^1 \}. \tag{2.11}$$

Thus, the thermoelastic Bresse beam system is transformed into a first order evolution on the Hilbert space $\mathcal{H}_i$:

$$\dot{z}(t) = \mathcal{A}_i z(t), \quad z(0) = z_0$$

with $i = 1, 2$ corresponding to the boundary conditions (1.11) and (1.12), respectively. Here and after we assume that $k \neq \frac{n\pi}{l}$ for all positive integer n when $i = 1$.

Theorem 2.1. *$\mathcal{A}_i$ generates a C_0 semigroup $S_i(t)$ of contractions on $\mathcal{H}_i$ for $i = 1, 2$.*

Proof. It is clear that $\mathcal{D}(\mathcal{A}_i)$ is dense in $\mathcal{H}_i$. By a straight forward calculation,

$$\mathrm{Re}\langle \mathcal{A}_i z, z\rangle_{\mathcal{H}} = -\frac{1}{T_0}(\|\theta'\|^2 + \|\xi'\|^2) \leq 0. \tag{2.12}$$

Hence, $\mathcal{A}_i$ is dissipative. It is easy to show that

$$\mathcal{A}_i z = F, \qquad \forall\, F \in \mathcal{H}_i \tag{2.13}$$

where $F = (f_1, \cdots, f_8)^T$, has unique solution $z \in \mathcal{D}(\mathcal{A}_i)$. In fact, from the first three equations of (2.13), we get

$$v = f_1, \qquad y = f_2, \qquad \psi = f_3.$$

Substitute them into the last two equation in (2.13) and using the stardard elliptic PDE theory, we have unique solution

$$\theta \in H_0^1 \cap H^2, \qquad \xi \in H_0^1 \cap H^2.$$

Finally, for the unique solvability of solution (u, w, ϕ) to the fourth, fifth and sixth equations in (2.13), we define a bilinear form

$$\begin{aligned} & b((u, w, \phi), (\tilde{u}, \tilde{w}, \tilde{\phi})) \\ &= Eh\langle u' - kw, \tilde{u}' - k\tilde{w}\rangle + Gh\langle \phi + w' + ku, \tilde{\phi} + \tilde{w}' + k\tilde{u}\rangle + EI\langle \phi', \tilde{\phi}'\rangle \end{aligned} \tag{2.14}$$

The conclusion follows from the Lax-Milgram theorem. Thus, $0 \in \rho(\mathcal{A}_i)$. By the resolvent identity, for small $\lambda > 0$ we have $R(\lambda - \mathcal{A}) = \mathcal{H}$ (see Theorem 1.2.4 in [11]), the conclusion now follows from the Lumer-Phillip theorem. □

3. Exponential decay rate: the case of $E = G$

Theorem 3.1. *If $E = G$, then the semigroup $S_i(t)$ is exponentially stable, i.e., there exist constant $M, \epsilon > 0$ independent of $z_0 \in \mathcal{H}_i$ such that*

$$\|S_i(t) z_0\|_{\mathcal{H}_i} \leq M e^{-\epsilon t} \|z_0\|_{\mathcal{H}_i}, \quad t \geq 0,$$

for $i = 1, 2$.

Proof. By the Theorem 1.1 in section 1, we need to show

$$\boldsymbol{i}\mathbb{R} \in \rho(\mathcal{A}_i), \tag{3.1}$$

and

$$\limsup_{\beta\to\infty} \|(\boldsymbol{i}\beta I - \mathcal{A}_i)^{-1}\| < \infty. \tag{3.2}$$

We will establish these conditions by contradiction.

For the case of $i = 1$, if (3.2) is false, then there exist a sequence $z_n \in \mathcal{D}(\mathcal{A}_1)$ with $\|z_n\|_{\mathcal{H}_1} = 1$, and a sequence $\beta_n \in \mathbb{R}$ with $\beta_n \to \infty$ such that

$$\lim_{n\to\infty} \|(\boldsymbol{i}\beta_n I - \mathcal{A}_1)z_n\|_{\mathcal{H}_1} = 0, \tag{3.3}$$

i.e., in $L^2(0,l)$ we have the following convergence:

$$\boldsymbol{i}\beta_n(u_n' - kw_n) - (v_n' - ky_n) \to 0, \tag{3.4}$$

$$\boldsymbol{i}\beta_n(\phi_n + w_n' + ku_n) - (\psi_n + y_n' + kv_n) \to 0, \tag{3.5}$$

$$\boldsymbol{i}\beta_n\phi_n' - \psi_n' \to 0, \tag{3.6}$$

$$\boldsymbol{i}\beta_n v_n - \frac{E}{\rho}(u_n' - kw_n)' + \frac{kG}{\rho}(\phi_n + w_n' + ku_n) + \frac{\alpha}{\rho h}\theta_n' \to 0, \tag{3.7}$$

$$\boldsymbol{i}\beta_n y_n - \frac{G}{\rho}(\phi_n + w_n' + ku_n)' - \frac{kE}{\rho}(u_n' - kw_n) + \frac{\alpha k}{\rho h}\theta_n \to 0, \tag{3.8}$$

$$\boldsymbol{i}\beta_n\psi_n - \frac{E}{\rho}\phi_n'' + \frac{Gh}{\rho I}(\phi_n + w_n' + ku_n) + \frac{\alpha}{\rho I}\xi_n' \to 0, \tag{3.9}$$

$$\boldsymbol{i}\beta_n\theta_n - \frac{1}{rc}\theta_n'' + \frac{\alpha T_0}{\rho c}(v_n' - ky_n) \to 0, \tag{3.10}$$

$$\boldsymbol{i}\beta_n\xi_n - \frac{1}{\rho c}\xi_n'' + \frac{\alpha T_0}{\rho c}\psi_n' \to 0. \tag{3.11}$$

Our goal is to derive $\|z_n\|_{\mathcal{H}_1} \to 0$ as a contradiction, which will be proceeded by showing that each term in (2.8) converges to zero. Instead of using the fact $\beta_n \to \infty$, we shall only use its consequence that β_n is bounded away from zero in our proof of condition (3.2). Although this makes our argument somewhat more complicated, it can greatly simplify the proof of condition (3.1).

From (3.3) and (2.12), we obtain

$$\mathrm{Re}\langle(\boldsymbol{i}\beta_n - \mathcal{A}_1)z_n, z_n\rangle_{\mathcal{H}_1} = \frac{1}{T_0}(\|\theta_n'\|^2 + \|\xi_n'\|^2) \to 0. \tag{3.12}$$

Thus, by the Poincare's inequality,

$$\|\theta_n\| \to 0, \quad \|\xi_n\| \to 0. \tag{3.13}$$

Eliminating ψ_n' in (3.11) by (3.6), and taking the inner product with $\frac{\rho c\phi_n'}{\beta_n}$ in

$L^2(0,l)$, we obtain

$$\begin{aligned} &-\frac{1}{\beta_n}\langle \xi_n'', \phi_n' \rangle + \boldsymbol{i}\alpha T_0 \|\phi_n'\|^2 \\ &= -\frac{1}{\beta_n}\xi_n'\bar{\phi}_n'\big|_0^l + \langle \xi_n', \frac{\phi_n''}{\beta_n} \rangle + \boldsymbol{i}\alpha T_0 \|\phi_n'\|^2 \to 0. \end{aligned} \tag{3.14}$$

The boundary terms in (3.14) vanish due to the boundary conditions on ϕ_n. Moreover, $\left\|\frac{\phi_n''}{\beta_n}\right\|$ is bounded which can be easily seen from (3.9). This, combined with (3.12), implies that the second term in (3.14) converges to zero. Hence, we have

$$\|\phi_n'\| \to 0. \tag{3.15}$$

In view of (3.6), we see that $\left\|\frac{\psi_n'}{\beta_n}\right\|$ converges to zero. Since $\int_0^l \psi_n dx = 0$, by the Poincare's inequality we also know that $\left\|\frac{\psi_n}{\beta_n}\right\|$ converges to zero. Hence, the inner product of (3.9) with $\frac{\psi_n}{\beta_n}$ in $L^2(0,l)$ leads to

$$\|\psi_n\| \to 0. \tag{3.16}$$

Similarly, we can eliminate $v_n' - ky_n$ from (3.10) by (3.4), then take the inner product of the resulting equation with $\rho c(u_n' - kw_n)$ in $L^2(0,l)$ to get

$$\begin{aligned} &-\frac{1}{\beta_n}\langle \theta_n'', u_n' - kw_n \rangle + \boldsymbol{i}\alpha T_0 \|u_n' - kw_n\|^2 \\ &= -\frac{1}{\beta_n}\theta_n'(\bar{u}_n' - k\bar{w}_n)\big|_0^l + \left\langle \theta_n', \frac{(u_n' - kw_n)'}{\beta_n} \right\rangle + \boldsymbol{i}\alpha T_0 \|u_n' - kw_n\|^2 \to 0. \end{aligned} \tag{3.17}$$

Again, the boundary terms in (3.17) vanish. Since $\left\|\frac{(u_n' - kw_n)'}{\beta_n}\right\|$ is bounded due to (3.7), the second term in (3.17) converge to zero. Hence,

$$\|u_n' - kw_n\| \to 0. \tag{3.18}$$

This further leads to $\frac{1}{\beta_n}\|v_n' - ky_n\| \to 0$. Therefore, $\frac{1}{\beta_n}\|v_n'\|$ is bounded. From the $L^2(0,l)$ inner product of (3.9) and $\frac{v_n}{\beta_n}$ we obtain

$$\left\langle \phi_n + w_n' + ku_n, \frac{v_n}{\beta_n} \right\rangle \to 0.$$

Now, we take the inner product of (3.7) with $\frac{v_n}{\beta_n}$ in $L^2(0,l)$, and integrate by part to the second term in the resulting expression. It is easy to see from there that the second, third and last terms in the expression all converge to zero. We then obtain

$$\|v_n\| \to 0. \tag{3.19}$$

So far, we have not used (3.5) and (3.8), which is certainly needed in the last part of our proof, i.e., showing

$$\|\phi_n + w_n' + ku_n\| \to 0, \quad \|y_n\| \to 0. \tag{3.20}$$

However, the above strategy of using the heat equations (3.10)-(3.11) to get the dissipation of the elastic part of the energy does not work anymore since (3.8) is only weakly damped. Hence, we will proceed with a different approach which relies on the assumption of the same wave speed ($E = G$). For the Bresse system, this makes all three wave equations to have the same wave speed. But we actually only need a pair of them, one effectively damped and the other not, satisfying this requirement. In the following we only use the equations (1.2) and (1.3) to get (3.20).

Taking the inner product of (3.9) with $\phi_n + w'_n + ku_n$ and (3.8) with ϕ'_n in $L^2(0,l)$, respectively, we have

$$\langle \boldsymbol{i}\beta_n\psi_n, \phi_n + w'_n + ku_n\rangle - \frac{E}{\rho}\langle \phi''_n, \phi_n + w'_n + ku_n\rangle + \frac{Gh}{\rho I}\|\phi_n + w'_n + ku_n\|^2 \to 0, \tag{3.21}$$

$$\langle \boldsymbol{i}\beta_n y_n, \phi'_n\rangle - \frac{G}{\rho}(\phi_n + w'_n + ku_n)\overline{\phi}'_n\Big|_0^l + \frac{G}{\rho}\langle \phi_n + w'_n + ku_n, \phi''_n\rangle \to 0. \tag{3.22}$$

The boundary terms in (3.22) vanish. Moreover, the first term in (3.22) can be written as the following,

$$\begin{aligned}\langle \boldsymbol{i}\beta_n y_n, \phi'_n\rangle &= -\langle y_n, \boldsymbol{i}\beta_n\phi'_n\rangle\\ &= -\langle y_n, \psi'_n\rangle + o(1)\\ &= \langle y'_n, \psi_n\rangle + o(1)\\ &= \langle \psi_n + y'_n + kv_n, \psi_n\rangle + o(1)\\ &= -\langle \phi_n + w'_n + ku_n, \boldsymbol{i}\beta_n\psi_n\rangle + o(1),\end{aligned}$$

Therefore, the real part of the sum of (3.21) and (3.22) yields the first part of (3.20). The inner product of (3.8) with $\frac{y_n}{\beta_n}$ gives us the second part of (3.20).

For the case of $i = 2$, the above arguments also apply. The only difference is that this time we have to estimate the boundary terms in (3.14), (3.17) and (3.22). First, we have

$$\left|\frac{1}{\beta_n}\xi'_n(x)\phi'_n(x)\right| \le C\|\xi'_n\|^{\frac12}\left\|\frac{\xi''_n}{\beta_n}\right\|^{\frac12}\|\phi'_n\|^{\frac12}\left\|\frac{\phi''_n}{\beta_n}\right\|^{\frac12}$$

and

$$\left|\frac{1}{\beta_n}\theta'_n(x)(u'_n(x) - kw_n(x))\right| \le C\|\theta'_n\|^{\frac12}\left\|\frac{\theta''_n}{\beta_n}\right\|^{\frac12}\|u'_n - kw_n\|^{\frac12}\left\|\frac{(u'_n - kw_n)'}{\beta_n}\right\|^{\frac12}$$

for $x = 0, l$ and some constant $C > 0$. Since $\left\|\frac{\xi''_n}{\beta_n}\right\|$, $\left\|\frac{\phi''_n}{\beta_n}\right\|$, $\left\|\frac{\theta''_n}{\beta_n}\right\|$ and $\left\|\frac{(u'_n - kw_n)'}{\beta_n}\right\|$ are all bounded, the boundary terms in (3.14) and (3.17) converge to zero due to (3.12).

We now claim that the boundary terms in (3.22) also converge to zero. In fact, taking the inner product of (3.9) with $2\rho x\phi'_n$ and (3.8) with $2\rho x(\phi_n + w'_n + ku_n)$

in $L^2(0,l)$, respectively, then integrating by parts we obtain

$$\rho\|\psi_n\|^2 + E\|\phi_n'\|^2 - El|\phi_n'(l)|^2 \to 0, \tag{3.23}$$

$$\rho\|y_n\|^2 + G\|\phi_n + w_n' + ku_n\|^2 - Gl|w_n'(l)|^2 \to 0. \tag{3.24}$$

In view of (3.15)-(3.16) and $\|z_n\|_{\mathcal{H}_2} = 1$, we know that $|\phi_n'(l)|$ converges to zero and $|w_n'(l)|$ is uniformly bounded in n. Similarly, repeating above process with $2\rho(x-l)\phi_n'$ and $2\rho(x-l)(\phi_n + w_n' + ku_n)$ we have the same estimate for $|\phi_n'(0)|$ and $|w_n'(0)|$. Moreover, ϕ_n and u_n already vanish on the boundary due to the boundary conditions. Hence, the claim is proved.

To prove the condition (3.1) we again use a contradiction argument. We already know that $0 \in \rho(\mathcal{A}_1)$. Assuming $\beta \neq 0$ and $\boldsymbol{i}\beta \in \sigma(\mathcal{A}_1)$, there exists a sequence $z_n \in \mathcal{D}(\mathcal{A}_1)$ with $\|z_n\|_{\mathcal{H}_1} = 1$ for all n such that

$$\lim_{n\to\infty} \|(\boldsymbol{i}\beta I - \mathcal{A}_1)\, z_n\|_{\mathcal{H}_1} = 0.$$

Let's recall the statement we made following (3.11). Since we did not use the fact $\beta_n \to \infty$ to verify condition (3.2), a repetition of the above argument with β_n replaced by β leads to the same contradiction. The proof is thus achieved.

Remark 3.1. When $E = G$, all three wave equations (1.1)-(1.3) have the same wave speed. However, in the proof of (3.20), we only used the fact that equations (1.2) and (1.3) have the same wave speed. Actually our method works as long as there is a pair of wave equations (one of them is effectively damped and the other is not) with the same wave speed. To illustrate this point, we include another proof of (3.20) which only requires equations (1.1) and (1.2) to have the same wave speed.

For the case $i = 1$, taking the inner product of (3.7) with $\phi_n + w_n' + ku_n$ and (3.8) with $u_n' - kw_n$ in $L^2(0,l)$, respectively, we have

$$\langle \boldsymbol{i}\beta_n v_n, \phi_n + w_n' + ku_n\rangle - \frac{E}{\rho}\langle (u_n' - kw_n)', \phi_n + w_n' + ku_n\rangle + \frac{kG}{\rho}\|\phi_n + w_n' + ku_n\|^2 \to 0, \tag{3.25}$$

$$\langle \boldsymbol{i}\beta_n y_n, u_n' - kw_n\rangle - \frac{G}{\rho}(\phi_n + w_n' + ku_n)(\overline{u_n' - kw_n})\Big|_0^l + \frac{G}{\rho}\langle \phi_n + w_n' + ku_n, (u_n' - kw_n)'\rangle \to 0. \tag{3.26}$$

The boundary terms in (3.26) vanish again. Moreover, the first term in (3.26) can be written as the following,

$$\begin{aligned}\langle \boldsymbol{i}\beta_n y_n, u_n' - kw_n\rangle &= -\langle y_n, \boldsymbol{i}\beta_n(u_n' - kw_n)\rangle \\ &= -\langle y_n, v_n' - ky_n\rangle + o(1) \\ &= \langle y_n', v_n\rangle + k\|y_n\|^2 + o(1) \\ &= \langle \boldsymbol{i}\beta_n(\phi_n + w_n' + ku_n), v_n\rangle + k\|y_n\|^2 + o(1), \\ &= -\langle \phi_n + w_n' + ku_n, \boldsymbol{i}\beta_n v_n\rangle + k\|y_n\|^2 + o(1),\end{aligned}$$

Therefore, the real part of the sum of (3.25) and (3.26) leads to (3.20).

For the case $i = 2$, a similar argument also shows that the boundary terms in (3.26) converge to zero.

4. Polynomial decay rate: the case of $E \neq G$

From the proof of Theorem 4.1 in [10], we can see that the thermoelastic Bresse system (1.1)-(1.5) with the boundary condition (1.11) is not exponentially stable when $E \neq G$. The idea is to find a sequence of $\lambda_n \in \mathbb{R}$ with $|\lambda_n| \to \infty$ and a sequence $z_n \in \mathcal{D}(\mathcal{A}_1)$ with $\|z_n\|_{\mathcal{H}_1} = 1$ such that $\|(\boldsymbol{i}\lambda_n I - \mathcal{A}_1)z_n\|_{\mathcal{H}_1} \to 0$. In the case of boundary condition (1.11), this approach worked well due to the fact that all eigenmodes are separable, i.e., the system operator can be decomposed to a block-diagonal form according to the frequency when the state variables are expanded into Fourier series. However, in the case of boundary condition (1.12), this approach has no success in the literature to our knowledge. A complicated technique developed in [14] is promising. We will not dive into the details here. Our main results are the following polynomial-type decay rate estimations.

Theorem 4.1. *If $E \neq G$, then there exists a constant $C_m > 0$ independent of $z_0 \in \mathcal{D}(\mathcal{A}_i^m)$ such that*

$$\|S_i(t)z_0\|_{\mathcal{H}_i} \leq C_m \left(\frac{\ln t}{t}\right)^{\frac{m}{j}} (\ln t)\|z_0\|_{\mathcal{D}(\mathcal{A}_i^m)}, \qquad m = 1, 2, \cdots$$

with

$$j = \begin{cases} 4 & i = 1, \\ 8 & i = 2. \end{cases}$$

Remark 4.1. It is interesting to see that the polynomial decay rate depends on the boundary conditions. Although we can't guarantee that our estimate of the decay rate is optimal since we are only verifying sufficient conditions, the reader will see from the following proof that the best l is chosen in order to get a contradiction. However, we do not have a physical explanation why case one has a faster decay rate than case two.

Proof. By Theorem 1.2 in section 1, we need to show

$$\boldsymbol{i}\mathbb{R} \in \rho(\mathcal{A}_i), \tag{4.1}$$

and

$$\lim_{\beta\to\infty}\frac{1}{\beta^j}\|(\boldsymbol{i}\beta I-\mathcal{A}_i)^{-1}\|<\infty. \tag{4.2}$$

Condition (4.1) has been verified in last section. We first establish condition (4.2) for $i=1$ by contradiction. If (4.2) is false, then there exist a sequence $z_n\in\mathcal{D}(\mathcal{A}_1)$ with $\|z_n\|_{\mathcal{H}_1}=1$, and a sequence $\beta_n\in\mathbb{R}$ with $\beta_n\to\infty$ such that

$$\lim_{n\to\infty}\beta_n^4\|(\boldsymbol{i}\beta_n I-\mathcal{A}_1)z_n\|_{\mathcal{H}_1}=0, \tag{4.3}$$

i.e., in $L^2(0,l)$ we have the following convergence:

$$\beta_n^4[\boldsymbol{i}\beta_n(u_n'-kw_n)-(v_n'-ky_n)]\to 0, \tag{4.4}$$

$$\beta_n^4[\boldsymbol{i}\beta_n(\phi_n+w_n'+ku_n)-(\psi_n+y_n'+kv_n)]\to 0, \tag{4.5}$$

$$\beta_n^4[\boldsymbol{i}\beta_n\phi_n'-\psi_n']\to 0, \tag{4.6}$$

$$\beta_n^4\Big[\boldsymbol{i}\beta_n v_n-\frac{E}{\rho}(u_n'-kw_n)'+\frac{kG}{\rho}(\phi_n+w_n'+ku_n)+\frac{\alpha}{\rho h}\theta_n'\Big]\to 0, \tag{4.7}$$

$$\beta_n^4\Big[\boldsymbol{i}\beta_n y_n-\frac{G}{\rho}(\phi_n+w_n'+ku_n)'-\frac{kE}{\rho}(u_n'-kw_n)+\frac{\alpha k}{\rho h}\theta_n\Big]\to 0, \tag{4.8}$$

$$\beta_n^4\Big[\boldsymbol{i}\beta_n\psi_n-\frac{E}{\rho}\phi_n''+\frac{Gh}{\rho I}(\phi_n+w_n'+ku_n)+\frac{\alpha}{\rho I}\xi_n'\Big]\to 0, \tag{4.9}$$

$$\beta_n^4\Big[\boldsymbol{i}\beta_n\theta_n-\frac{1}{\rho c}\theta_n''+\frac{\alpha T_0}{\rho c}(v_n'-ky_n)\Big]\to 0, \tag{4.10}$$

$$\beta_n^4\Big[\boldsymbol{i}\beta_n\xi_n-\frac{1}{mc}\xi_n''+\frac{\alpha T_0}{mc}\psi_n'\Big]\to 0. \tag{4.11}$$

Our goal is to derive $\|z_n\|_{\mathcal{H}_1}\to 0$ as a contradiction. From (4.3) and (2.12), we obtain

$$\mathrm{Re}\langle\beta_n^4(\boldsymbol{i}\beta_n-\mathcal{A})z_n,z_n\rangle_{\mathcal{H}}=\frac{1}{T_0}(\|\beta_n^2\theta_n'\|^2+\|\beta_n^2\xi_n'\|^2)\to 0. \tag{4.12}$$

Thus, by the Poincare's inequality,

$$\|\beta_n^2\theta_n\|\to 0,\quad \|\beta_n^2\xi_n\|\to 0. \tag{4.13}$$

Dividing (4.11) by β_n^3 and replacing $\beta_n\psi_n'$ by $\boldsymbol{i}\beta_n^2\phi_n'$ in view of (4.6) leads to

$$-\beta_n\xi_n''+\boldsymbol{i}\alpha T_0\beta_n^2\phi_n'\to 0 \text{ in } L^2(0,l).$$

Now we take the inner product of the above with ϕ_n' in $L^2(0,l)$ to get

$$\begin{aligned}&-\beta_n\langle\xi_n'',\phi_n'\rangle+\alpha T_0\boldsymbol{i}\|\beta_n\phi_n'\|^2\\ &=\Big\langle\beta_n^2\xi_n',\frac{\phi_n''}{\beta_n}\Big\rangle+\alpha T_0\boldsymbol{i}\|\beta_n\phi_n'\|^2\to 0.\end{aligned} \tag{4.14}$$

This implies that

$$\|\beta_n\phi_n'\|\to 0 \tag{4.15}$$

because of (4.12) and the uniform boundedness of $\|\frac{\phi_n''}{\beta_n}\|$ in n. Thus, $\|\psi_n'\|$ also converges to zero, which leads to

$$\|\psi_n\| \to 0. \tag{4.16}$$

Repeating the above process to (4.10) and (4.4) also gives

$$\|\beta_n(u_n' - kw_n)\| \to 0, \tag{4.17}$$

which further leads to $\|v_n' - ky_n\| \to 0$. Therefore, $\|v_n'\|$ is uniformly bounded in n. From the $L^2(0,l)$ inner product of (4.9) and $\frac{v_n}{\beta_n^5}$ we obtain

$$\left\langle \phi_n + w_n' + ku_n, \frac{v_n}{\beta_n} \right\rangle \to 0.$$

Now, we take the inner product of (4.7) with $\frac{v_n}{\beta_n}$ in $L^2(0,l)$, and integrate by part to the second term in this expression. It is easy to see from there that the second, third and last terms in that expression all converge to zero. We then obtain

$$\|v_n\| \to 0. \tag{4.18}$$

Let $f_n(x) = \displaystyle\int_0^x [\phi_n(p) + w_n'(p) + ku_n(p)]dp$ so that $f_n(0) = 0$ and $f_n' = \phi_n + w_n' + ku_n$. Note that we also have $f_n(l) = 0$ since $\phi_n, u_n \in H_*^1$. We can rewrite the $L^2(0,l)$ inner product of (4.9) and $\frac{\rho I f_n'}{\beta_n^4}$ as

$$-\boldsymbol{i}\rho I\langle \psi_n', \beta_n f_n\rangle + EI\left\langle \beta_n\phi_n', \frac{f_n''}{\beta_n}\right\rangle + Gh\|f_n'\|^2 \to 0. \tag{4.19}$$

It follows from (4.5) and (4.8) that $\|\beta_n f_n\|$ and $\frac{1}{\beta_n}\|f_n''\|$ are both uniformly bounded in n. Thus, the first two terms on the left hand side of (4.19) converge to zero which implies that

$$\|f_n'\| = \|\phi_n + w_n' + ku_n\| \to 0. \tag{4.20}$$

Finally, we take the inner product of (4.8) and $\frac{y_n}{\beta_n}$ in $L^2(0,l)$ to get

$$\|y_n\| \to 0. \tag{4.21}$$

Now every term in $\|z_n\|_{\mathcal{H}_1}$ has converged to zero, which is a contradiction. Thus, the proof for the case of $i = 1$ is complete.

Similarly, for the case of $i = 2$, if condition (4.2) is false, we have

$$\beta_n^8[\boldsymbol{i}\beta_n(u_n' - kw_n) - (v_n' - ky_n)] \to 0, \tag{4.22}$$

$$\beta_n^8[\boldsymbol{i}\beta_n(\phi_n + w_n' + ku_n) - (\psi_n + y_n' + kv_n)] \to 0, \tag{4.23}$$

$$\beta_n^8[\boldsymbol{i}\beta_n\phi_n' - \psi_n'] \to 0, \tag{4.24}$$

$$\beta_n^8\Big[\boldsymbol{i}\beta_n v_n - \frac{E}{\rho}(u_n' - kw_n)' + \frac{kG}{\rho}(\phi_n + w_n' + ku_n) + \frac{\alpha}{\rho h}\theta_n'\Big] \to 0, \tag{4.25}$$

$$\beta_n^8\Big[\boldsymbol{i}\beta_n y_n - \frac{G}{\rho}(\phi_n + w_n' + ku_n)' - \frac{kE}{\rho}(u_n' - kw_n) + \frac{\alpha k}{\rho h}\theta_n\Big] \to 0, \tag{4.26}$$

$$\beta_n^8\Big[\boldsymbol{i}\beta_n\psi_n - \frac{E}{\rho}\phi_n'' + \frac{Gh}{\rho I}(\phi_n + w_n' + ku_n) + \frac{\alpha}{\rho I}\xi_n'\Big] \to 0, \tag{4.27}$$

$$\beta_n^8\Big[\boldsymbol{i}\beta_n\theta_n - \frac{1}{\rho c}\theta_n'' + \frac{\alpha T_0}{\rho c}(v_n' - ky_n)\Big] \to 0, \tag{4.28}$$

$$\beta_n^8\Big[\boldsymbol{i}\beta_n\xi_n - \frac{1}{\rho c}\xi_n'' + \frac{\alpha T_0}{\rho c}\psi_n'\Big] \to 0. \tag{4.29}$$

and

$$\mathrm{Re}\langle\beta_n^8(\boldsymbol{i}\beta_n - \mathcal{A}_2)z_n, z_n\rangle_{\mathcal{H}} = \frac{1}{T_0}(\|\beta_n^4\theta_n'\|^2 + \|\beta_n^4\xi_n'\|^2) \to 0. \tag{4.30}$$

Thus, by the Poincare's inequality,

$$\|\beta_n^4\theta_n\| \to 0, \quad \|\beta_n^4\xi_n\| \to 0. \tag{4.31}$$

Dividing (4.29) by β_n^7 and replacing $\beta_n\psi_n'$ by $\boldsymbol{i}\beta_n^2\phi_n'$ in view of (4.24) leads to

$$-\beta_n\xi_n'' + \boldsymbol{i}\alpha T_0\beta_n^2\phi_n' \to 0 \ \text{ in } \ L^2(0,l).$$

Now we take the inner product of the above with ϕ_n' in $L^2(0,l)$ to get

$$\begin{aligned} &-\beta_n\langle\xi_n'', \phi_n'\rangle + \alpha T_0\boldsymbol{i}\|\beta_n\phi_n'\|^2 \\ &= -\beta_n\xi_n'\bar{\phi}_n'\big|_0^l + \langle\beta_n^2\xi_n', \frac{\phi_n''}{\beta_n}\rangle + \alpha T_0\boldsymbol{i}\|\beta_n\phi_n'\|^2 \to 0. \end{aligned} \tag{4.32}$$

Observing that (4.32) has extra boundary terms than (4.14), in order to get $\|\beta_n\phi_n'\| \to 0$ we must show that the boundary terms in (4.32) converges to zero. From the following sharp estimate

$$|\beta_n\xi_n'(x)\bar{\phi}_n'(x)| \le C\|\beta_n^4\xi_n'\|^{\frac12}\left\|\frac{\xi_n''}{\beta_n}\right\|^{\frac12}\|\phi_n'\|^{\frac12}\left\|\frac{\phi_n''}{\beta_n}\right\|^{\frac12},$$

we see that at least $j = 8$ is needed for obtaining (4.15)-(4.18) again. Next, we take the inner product of (4.27) with $\frac{\rho I f_n'}{\beta_n^8}$ in $L^2(0,l)$ to get

$$\boldsymbol{i}\rho I\beta_n\psi_n(l)\bar{f}_n(l) - \boldsymbol{i}\rho I\langle\psi_n', \beta_n f_n\rangle - EI\phi_n'\bar{w}_n'\big|_0^l + EI\Big\langle\beta_n\phi_n', \frac{f_n''}{\beta_n}\Big\rangle + Gh\|f_n'\|^2 \to 0, \tag{4.33}$$

where f_n has been defined before. But with the boundary condition (1.12), $f_n(l)$ does not equal to zero anymore. Comparing with (4.19), if we can show the boundary terms in (4.33) converge to zero, then the same argument for obtaining (4.20)-(4.21) is also valid here. Since

$$|\beta_n \psi_n(l)\bar{f}_n(l)| \le C\|\beta_n f_n\|^{\frac{1}{2}}\|f_n'\|^{\frac{1}{2}}\|\beta_n\psi_n\|^{\frac{1}{2}}\|\psi_n'\|,$$

and $\|\beta_n\psi_n\|$ is uniformly bounded in n which can be seen from the $L^2(0,l)$ inner product of (4.27) and $\frac{\psi_n}{\beta_n^7}$, the first boundary term in (4.33) does converge to zero. In order to show

$$\left|\phi_n' \bar{w}_n'\Big|_0^l\right| \to 0,$$

we divide (4.26) and (4.27) by β_n^8. Then by the same multiplier technique in deriving (3.23) and (3.24), we again obtain that $w_n'(x)$ is bounded and $\phi_n'(x)$ converges to zero for $x = 0$ and l. □

Remark 4.2. The thermoelastic Timoshenko system and thermoelastic Bresse system both belong to an abstract linear system consisting of coupled conservative hyperbolic equations and dissipative parabolic equations. Their difference is that the latter has more state variables and corresponding equations. But, their basic structure is the same. Therefore, it is plausible to study such an abstract system to obtain general results.

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