Z. angew. Math. Phys. 59 (2008) 1081–1110
0044-2275/08/061081-30
DOI 10.1007/s00033-007-7127-2

Zeitschrift für angewandte Mathematik und Physik ZAMP

Covariance in linearized elasticity

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Abstract. In this paper we covariantly obtain all the governing equations of linearized elasticity. Our motivation is to see if one can make a connection between invariance (covariance) properties of the (global) balance of energy in nonlinear elasticity and those of its counterpart in linear elasticity. We start by proving a Green-Naghdi-Rivilin theorem for linearized elasticity. We do this by first linearizing energy balance about a given reference motion and then by postulating its invariance under isometries of the Euclidean ambient space. We also investigate the possibility of covariantly deriving a linearized elasticity theory, without any reference to the local governing equations, e.g. local balance of linear momentum. In particular, we study the consequences of linearizing covariant energy balance and covariance of linearized energy balance. We show that in both cases, covariance gives all the field equations of linearized elasticity.

Mathematics Subject Classification (2000). 74A99, 53Z05.

Keywords. Elasticity, linear elasticity, covariance.

1. Introduction

Linear elasticity is based on the assumption that displacement gradients are small. Balances of linear and angular momenta in linear elasticity have the same forms as those of nonlinear elasticity. Kinematics is described with respect to a reference state and no distinction is made between the manifold describing the material's neutral, "undeformed" state and the spatial manifold (the ambient space) that the material lives in. These results in the Cauchy and the first Piola–Kirchhoff stress tensors being the same. In the constitutive equations, stress and strain tensors are linearly related by a fourth-order tensor of elastic constants. Governing equations of linear elasticity can be obtained by linearizing those of nonlinear elasticity about a reference motion. In the geometric theory, where a body's neutral state is described by a Riemannian manifold[1] and it is assumed that the body deforms in a Riemannian ambient space, one can obtain the governing equations of linear elasticity by a geometric linearization of the governing equations of nonlinear elasticity [7]. Recently, Steigmann [11] studied frame indifference of the governing

[1] Or possibly a more general manifold with torsion, for the case with dislocations.

equations of linear elasticity. His main conclusion is that linearized elasticity is frame-indifferent if it is properly formulated.

It is well known that balance laws in nonlinear elasticity can be obtained by postulating an energy balance and its invariance under time-dependent isometries of the ambient space if the latter is Euclidean [4] and diffeomorphisms of the ambient space if it is Riemannian (covariance) [7, 10, 12]. Now one may ask what the connection between linearized and nonlinear elasticity is in terms of energy balance and its invariance. In this paper, we make this connection in both cases of Euclidean and Riemannian ambient space manifolds.[2] In the case of a Euclidean ambient space, we first linearize the invariance conditions for energy balance around a reference motion. This gives local balance of linear (angular) momentum for the case of invariance under translations (rotations) with constant speed.

We next linearize/quadratize the energy balance around a reference motion (note that working to linear order in the equations of motion corresponds to working to quadratic order in the energy or the Lagrangian). This linearization/quadratization gives two separate equations for the first order and the second order terms, and postulating the invariance of the linearized energy balance under time-dependent rigid translations and rotations of the ambient space gives the equations of linearized elasticity. We also show that assuming that the reference motion is a static equilibrium configuration, the quadratized energy balance is identical to the so-called Power Theorem in linear elasticity. We then obtain the Lagrangian dynamics for the linearized and quadratized versions of the Lagrangian.

In the case of a general Riemannian ambient space and a Riemannian material manifold, we study two things: (i) linearization of the covariance condition for energy balance, and, (ii) covariance of the linearized energy balance. By "linearization of the covariance condition for energy balance" we mean linearization of the difference of energy balance for a nearby motion with respect to a reference motion. We show that this linearization will give the governing equations of linearized elasticity. In the more interesting case, we first linearize energy balance with respect to a reference motion $\mathring{\varphi}_t$ and then postulate the invariance of this linearized energy balance with respect to diffeomorphisms of the ambient space. We also extend the ideas of first variation of "energy" of maps [9] to elasticity, where energy has a more complicated form.

This paper is structured as follows. In §2 we study invariance of linearized energy balance when the ambient space is Euclidean. We show the connection

[2] Note that a Euclidean space is a special Riemannian manifold. It turns out that in this special case, one can obtain all the governing equations by postulating the invariance of energy balance under time-dependent isometries. For a general Riemannian manifold, however, the isometry group may be trivial and hence invariance of energy balance under isometries may be vacuous. What is postulated in this case is invariance of energy balance under arbitrary diffeomorphisms, with a given transformation law for the metric tensor. It turns out that one can obtain all the other balance laws from this approach.

between energy balance in nonlinear and linear elasticity. In §3 we give a brief introduction to geometric elasticity in order to make the paper self-contained. We then review Marsden and Hughes' idea of geometric linearization of elasticity in §4 and present some new results. We also revisit linearization of elasticity using variation of maps and ideas from geometric calculus of variations. In §5 we study different notions of covariance in linearized elasticity. In particular, we covariantly obtain all the governing equations of linearized elasticity. Conclusions are given in §6.

2. Linearized elasticity in Euclidean ambient space

It has long been known that one can obtain all the balance laws of elasticity by postulating balance of energy and its invariance under (time-dependent) rigid translations and rotations of the current configuration with constant speeds [4].[3] Here we are interested in formulating a version of the Green–Naghdi–Rivilin theorem for linearized elasticity.

Let φ_t denote a motion of a body. Energy balance for an arbitrary subbody $\mathcal{U} \subset \mathcal{B}$ is written as

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dv = \int_{\varphi_t(\mathcal{U})} \rho\left(\mathbf{b}\cdot\mathbf{v}+r\right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\mathbf{t}\cdot\mathbf{v}+h\right) da, \quad (2.1)$$

in spatial coordinates, where ρ is the mass density, e, r and h are the internal energy per unit mass, the heat supply per unit mass and the heat flux, respectively, and $\mathbf{v}$, $\mathbf{b}$, and $\mathbf{t}$ are spatial velocity, body force per unit mass, and traction, respectively. In material coordinates energy balance reads

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0\left(E+\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) dV = \int_{\mathcal{U}} \rho_0\left(\mathbf{B}\cdot\mathbf{V}+R\right) dV + \int_{\partial\mathcal{U}} \left(\mathbf{T}\cdot\mathbf{V}+H\right) dA, \quad (2.2)$$

where $E = E(\mathbf{X}, \mathbf{F})$ is the internal energy per unit mass of the undeformed configuration, ρ_0 is the mass per unit undeformed volume, and R, H, $\mathbf{V}$, $\mathbf{B}$ and $\mathbf{T}$ are the material versions of r, h, $\mathbf{v}$, $\mathbf{b}$ and $\mathbf{t}$, respectively. We start with material energy balance as it is written for a fixed domain and makes the calculations simpler.

Let us assume that we are given a reference motion $\mathring{\varphi}_t$. Balance of energy for this fixed motion is written as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0\left(\mathring{E}+\frac{1}{2}\mathring{\mathbf{V}}\cdot\mathring{\mathbf{V}}\right) dV = \int_{\mathcal{U}} \rho_0\left(\mathring{\mathbf{B}}\cdot\mathring{\mathbf{V}}+\mathring{R}\right) dV + \int_{\partial\mathcal{U}} \left(\mathring{\mathbf{T}}\cdot\mathring{\mathbf{V}}+\mathring{H}\right) dA. \quad (2.3)$$

[3] It turns out that one can obtain all the balance laws under the more general case of "isometries with time-dependent speeds" provided that body force is transformed properly. However, invariance under isometries with constant speed turns out to be sufficient as in the original work of Green–Naghdi–Rivilin [4].

Now consider a smooth variation of this motion $\varphi_{t,s}$ parametrized by s, such that $\varphi_{t,0} = \mathring{\varphi}_t$. For each value of s, the energy balance is of the form (2.2). Now let us assume that for any value of s, energy balance is invariant under a time-dependent rigid translation of the deformed configuration, of the form $\xi_t(\mathbf{x}) = \mathbf{x} + (t - t_0)\mathbf{w}$. This would give the following two relations for $s = 0$ and $s \neq 0$ [12]

$$\int_{\mathcal{U}} \frac{\partial \rho_0}{\partial t} \left(\mathbf{w} \cdot \mathring{\mathbf{V}} + \frac{1}{2} \mathbf{w} \cdot \mathbf{w} \right) dV = \int_{\mathcal{U}} \rho_0 (\mathring{\mathbf{B}} - \mathring{\mathbf{A}}) \cdot \mathbf{w} dV + \int_{\partial \mathcal{U}} \mathring{\mathbf{T}} \cdot \mathbf{w} dA, \tag{2.4}$$

$$\int_{\mathcal{U}} \frac{\partial \rho_0}{\partial t} \left(\mathbf{w} \cdot \mathbf{V} + \frac{1}{2} \mathbf{w} \cdot \mathbf{w} \right) dV = \int_{\mathcal{U}} \rho_0 (\mathbf{B} - \mathbf{A}) \cdot \mathbf{w} dV + \int_{\partial \mathcal{U}} \mathbf{T} \cdot \mathbf{w} dA, \tag{2.5}$$

where $\mathring{\mathbf{A}}$ and $\mathbf{A}$ are the material accelerations for motions $\mathring{\varphi}_t$ and $\varphi_{t,s}$, respectively. Arbitrariness of $\mathbf{w}$ gives conservation of mass $\frac{\partial \rho_0}{\partial t} = 0$ and subtracting the above two relations then gives

$$\int_{\mathcal{U}} \rho_0 (\mathbf{A} - \mathring{\mathbf{A}}) \cdot \mathbf{w} dV = \int_{\mathcal{U}} \rho_0 (\mathbf{B} - \mathring{\mathbf{B}}) \cdot \mathbf{w} dV + \int_{\partial \mathcal{U}} (\mathbf{T} - \mathring{\mathbf{T}}) \cdot \mathbf{w} dA. \tag{2.6}$$

We will next linearize this equation in s. Let us start by defining the linearization and the quadratization around a motion $\mathring{\varphi}$ of a tensor field $\mathbf{H}_s(\varphi_{t,s})$ that depends on s through $\varphi_{t,s}$ and possibly also explicitly. These definitions are not really essential at this point since they are given in terms of ordinary derivatives, but their generalization in the following sections will be less trivial.

Definition 2.1. *Given a smooth tensor field $\mathbf{H} = \mathbf{H}_s$ parametrized by s, one can write*

$$\mathbf{H}_s(\varphi_{t,s}) = \mathbf{H}_0(\mathring{\varphi}_t) + \mathcal{L}(\mathbf{H})\ s + \frac{1}{2} \mathcal{Q}(\mathbf{H})\ s^2 + o(s^2), \tag{2.7}$$

where the linearization $\mathcal{L}(\mathbf{H})$ and quadratization $\mathcal{Q}(\mathbf{H})$ of $\mathbf{H}$ are defined as

$$\mathcal{L}(\mathbf{H}) = \mathcal{L}(\mathbf{H}; \mathring{\varphi}_t) = \frac{d}{ds}\Big|_{s=0} \mathbf{H}_s(\varphi_{t,s}), \tag{2.8}$$

$$\mathcal{Q}(\mathbf{H}) = \mathcal{Q}(\mathbf{H}; \mathring{\varphi}_t) = \frac{d^2}{ds^2}\Big|_{s=0} \mathbf{H}_s(\varphi_{t,s}). \tag{2.9}$$

Let $\mathbf{U}(t)$ denote the time-dependent vector field whose integral curves at time t are given by $c_t(\mathbf{x}) = \varphi_{t,s}$, i.e., $U^i = \frac{\partial \varphi^i_{t,s}}{\partial s}\Big|_{s=0}$. $\mathbf{U}$ represents the linearized displacement from the reference motion. It can be shown that

$$\mathcal{L}(\mathbf{V}) = \dot{\mathbf{U}}, \quad \mathcal{L}(\mathbf{A}) = \ddot{\mathbf{U}}. \tag{2.10}$$

Now, the linearization of the invariance equation (2.6) (i.e., the "linearized invariance") reads

$$\int_{\mathcal{U}} \rho_0 \ddot{\mathbf{U}} \cdot \mathbf{w} dV = \int_{\mathcal{U}} \rho_0 \mathcal{L}(\mathbf{B}) \cdot \mathbf{w} dV + \int_{\partial \mathcal{U}} \mathcal{L}(\mathbf{T}) \cdot \mathbf{w} dA. \tag{2.11}$$

Since $\mathcal{L}(\mathbf{T}) = \operatorname{Div}\mathcal{L}(\mathbf{P}) \cdot \hat{\mathbf{N}}$, where $\mathbf{P}$ is the stress and $\hat{\mathbf{N}}$ is the unit normal vector to $\partial\mathcal{U}$ at $\mathbf{X} \in \partial\mathcal{U}$, arbitrariness of $\mathbf{w}$ and $\mathcal{U}$ will imply

$$\rho_0 \ddot{\mathbf{U}} = \rho_0 \mathcal{L}(\mathbf{B}) + \operatorname{Div}\mathcal{L}(\mathbf{P}), \tag{2.12}$$

which is nothing but linearization of the local balance of linear momentum. Similarly, assuming invariance of energy balance under rotations with constant velocity and linearizing, one gets the linearization of the balance of angular momentum.

We will next expand the balance of energy (2.2) to second order in the deviation from a reference motion $\overset{\circ}{\varphi}$, and postulate the invariance of the linear term under isometries of the Euclidean ambient space. This will be equivalent to the linearization of invariance derived above, since linearization and the calculation of the change due to a rigid translation/rotation commute with each other.

Let us begin by defining $\delta E = E - \overset{\circ}{E}$, $\delta\mathbf{V} = \mathbf{V} - \overset{\circ}{\mathbf{V}}$, $\delta\mathbf{B} = \mathbf{B} - \overset{\circ}{\mathbf{B}}$, etc. Then, balance of energy for the perturbed motion is written as

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left[\overset{\circ}{E} + \delta E + \frac{1}{2}(\overset{\circ}{\mathbf{V}} + \delta\mathbf{V}) \cdot (\overset{\circ}{\mathbf{V}} + \delta\mathbf{V})\right] dV \\ = \int_{\mathcal{U}} \rho_0 \left[(\overset{\circ}{\mathbf{B}} + \delta\mathbf{B}) \cdot (\overset{\circ}{\mathbf{V}} + \delta\mathbf{V}) + \overset{\circ}{R} + \delta R\right] dV \\ + \int_{\partial\mathcal{U}} \left[(\overset{\circ}{\mathbf{T}} + \delta\mathbf{T}) \cdot (\overset{\circ}{\mathbf{V}} + \delta\mathbf{V}) + \overset{\circ}{H} + \delta H\right] dA.\end{aligned} \tag{2.13}$$

Subtracting (2.3) from (2.13) yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left[\delta E + \overset{\circ}{\mathbf{V}} \cdot \delta\mathbf{V} + \frac{1}{2}\delta\mathbf{V} \cdot \delta\mathbf{V}\right] dV \\ = \int_{\mathcal{U}} \rho_0 \left[\overset{\circ}{\mathbf{B}} \cdot \delta\mathbf{V} + \overset{\circ}{\mathbf{V}} \cdot \delta\mathbf{B} + \delta\mathbf{B} \cdot \delta\mathbf{V} + \delta R\right] dV \\ + \int_{\partial\mathcal{U}} \left[\overset{\circ}{\mathbf{T}} \cdot \delta\mathbf{V} + \delta\mathbf{T} \cdot \overset{\circ}{\mathbf{V}} + \delta\mathbf{T} \cdot \delta\mathbf{V} + \delta H\right] dA.\end{aligned} \tag{2.14}$$

As above, let $\mathbf{U}$ be the vector field given by $U^i = \frac{\partial \varphi^i_{t,s}}{\partial s}\big|_{s=0}$, and let $\mathbf{W}$ be given by $W^i = \frac{\partial^2 \varphi^i_{t,s}}{\partial s^2}\big|_{s=0}$. Working up to second order in s, we have

$$\delta\mathbf{V} = s\dot{\mathbf{U}} + \frac{s^2}{2}\dot{\mathbf{W}} + o(s^2), \tag{2.15}$$

$$\delta\mathbf{A} = s\ddot{\mathbf{U}} + \frac{s^2}{2}\ddot{\mathbf{W}} + o(s^2), \tag{2.16}$$

$$\delta\mathbf{F} = s\boldsymbol{\nabla}\mathbf{U} + \frac{s^2}{2}\boldsymbol{\nabla}\mathbf{W} + o(s^2), \tag{2.17}$$

where $\mathbf{F} = T\varphi_t = \boldsymbol{\nabla}\varphi_t$ is the deformation gradient. $E = E(\mathbf{X}, \mathbf{F})$ gives

$$\begin{aligned}
\delta E &= s \frac{d}{ds}\Big|_{s=0} E + \frac{s^2}{2} \frac{d}{ds}\Big|_{s=0} \frac{dE}{ds} + o(s^2) && (2.18)\\
&= s \left(\frac{\partial E}{\partial \mathbf{F}} \frac{d\mathbf{F}}{ds} \right)_{s=0} + \frac{s^2}{2} \frac{d}{ds}\Big|_{s=0} \left(\frac{\partial E}{\partial \mathbf{F}} \frac{d\mathbf{F}}{ds} \right) + o(s^2) && (2.19)\\
&= s \overset{\circ}{\mathbf{P}} \cdot \boldsymbol{\nabla}\mathbf{U} + \frac{s^2}{2} \left(\boldsymbol{\nabla}\mathbf{U} \cdot \overset{\circ}{\mathsf{C}} \cdot \mathbf{U} + \overset{\circ}{\mathbf{P}} \cdot \boldsymbol{\nabla}\mathbf{W} \right) + o(s^2), && (2.20)
\end{aligned}$$

where $\overset{\circ}{\mathsf{C}} = \left(\frac{\partial^2 E}{\partial \mathbf{F} \partial \mathbf{F}} \right)_{s=0}$ is the elasticity tensor and $\overset{\circ}{\mathbf{P}}$ is the stress tensor, evaluated at the reference motion. The body force $\mathbf{B}$ and the traction $\mathbf{T}$ can also be linearized/quadratized as follows.

$$\begin{aligned}
\delta \mathbf{B} &= s\mathcal{L}(\mathbf{B}) + \frac{s^2}{2} \mathcal{Q}(\mathbf{B}), && (2.21)\\
\delta \mathbf{T} &= s\mathcal{L}(\mathbf{T}) + \frac{s^2}{2} \mathcal{Q}(\mathbf{T}), && (2.22)
\end{aligned}$$

where $\mathcal{L}(\mathbf{T}) = \operatorname{Div} \mathcal{L}(\mathbf{P}) \cdot \hat{\mathbf{N}}$ can be further expanded as

$$\mathcal{L}(\mathbf{T}) = \operatorname{Div} \mathcal{L}(\mathbf{P}) \cdot \hat{\mathbf{N}} = \operatorname{Div} \left(\frac{\partial \mathbf{P}}{\partial \mathbf{F}}\Big|_{s=0} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \cdot \hat{\mathbf{N}} = \operatorname{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \cdot \hat{\mathbf{N}}. \quad (2.23)$$

Now, (2.14) can be expanded to read

$$\begin{aligned}
&\left\{ \frac{d}{dt} \int_{\mathcal{U}} \left(\overset{\circ}{\mathbf{P}} \cdot \boldsymbol{\nabla}\mathbf{U} + \rho_0 \overset{\circ}{\mathbf{V}} \cdot \dot{\mathbf{U}} \right) dV - \int_{\mathcal{U}} \rho_0 \left[\overset{\circ}{\mathbf{B}} \cdot \dot{\mathbf{U}} + \overset{\circ}{\mathbf{V}} \cdot \mathcal{L}(\mathbf{B}) + \mathcal{L}(\mathbf{B}) \cdot \dot{\mathbf{U}} + \mathcal{L}(R) \right] dV \right.\\
&\quad \left. - \int_{\partial\mathcal{U}} \left[\overset{\circ}{\mathbf{T}} \cdot \dot{\mathbf{U}} + \mathcal{L}(\mathbf{T}) \cdot \overset{\circ}{\mathbf{V}} + \mathcal{L}(H) \right] dA \right\} s\\
&\quad + \left\{ \frac{d}{dt} \int_{\mathcal{U}} \left[\frac{1}{2} \boldsymbol{\nabla}\mathbf{U} \cdot \overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} + \frac{1}{2} \overset{\circ}{\mathbf{P}} \cdot \boldsymbol{\nabla}\mathbf{W} + \frac{1}{2} \rho_0 \dot{\mathbf{U}} \cdot \dot{\mathbf{U}} + \frac{1}{2} \rho_0 \overset{\circ}{\mathbf{V}} \cdot \dot{\mathbf{W}} \right] dV \right.\\
&\qquad - \int_{\mathcal{U}} \rho_0 \left[\mathcal{L}(\mathbf{B}) \cdot \dot{\mathbf{U}} + \frac{1}{2} \left(\overset{\circ}{\mathbf{B}} \cdot \dot{\mathbf{W}} + \overset{\circ}{\mathbf{V}} \cdot \mathcal{Q}(\mathbf{B}) \right) + \frac{1}{2} \mathcal{Q}(R) \right] dV\\
&\qquad \left. - \int_{\partial\mathcal{U}} \left[\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \cdot \dot{\mathbf{U}} + \frac{1}{2} \overset{\circ}{\mathbf{T}} \cdot \dot{\mathbf{W}} + \frac{1}{2} \mathcal{Q}(\mathbf{T}) \cdot \overset{\circ}{\mathbf{V}} + \frac{1}{2} \mathcal{Q}(H) \right] dA \right\} s^2\\
&\qquad + o(s^2) = 0. && (2.24)
\end{aligned}$$

We call (2.24) the ***perturbed energy balance***. As s is arbitrary in (2.24), the coefficients of s and s^2 must be zero separately, i.e.

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{U}} \left[\overset{\circ}{\mathbf{P}} \cdot \boldsymbol{\nabla}\mathbf{U} + \rho_0 \overset{\circ}{\mathbf{V}} \cdot \dot{\mathbf{U}} \right] dV &= \int_{\mathcal{U}} \rho_0 \left[\overset{\circ}{\mathbf{B}} \cdot \dot{\mathbf{U}} + \overset{\circ}{\mathbf{V}} \cdot \mathcal{L}(\mathbf{B}) + \mathcal{L}(\mathbf{B}) \cdot \dot{\mathbf{U}} + \mathcal{L}(R) \right] dV\\
&\quad + \int_{\partial\mathcal{U}} \left[\overset{\circ}{\mathbf{T}} \cdot \dot{\mathbf{U}} + \mathcal{L}(\mathbf{T}) \cdot \overset{\circ}{\mathbf{V}} + \mathcal{L}(H) \right] dA. && (2.25)
\end{aligned}$$

And[4]

$$\frac{d}{dt}\int_{\mathcal{U}}\left[\frac{1}{2}\boldsymbol{\nabla}\mathbf{U}\cdot\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}+\frac{1}{2}\overset{\circ}{\mathbf{P}}\cdot\boldsymbol{\nabla}\mathbf{W}+\frac{1}{2}\rho_0\dot{\mathbf{U}}\cdot\dot{\mathbf{U}}+\frac{1}{2}\rho_0\overset{\circ}{\mathbf{V}}\cdot\dot{\mathbf{W}}\right]dV$$
$$=\int_{\mathcal{U}}\rho_0\left[\mathcal{L}(\mathbf{B})\cdot\dot{\mathbf{U}}+\frac{1}{2}\left(\overset{\circ}{\mathbf{B}}\cdot\dot{\mathbf{W}}+\overset{\circ}{\mathbf{V}}\cdot\mathcal{Q}(\mathbf{B})\right)+\frac{1}{2}\mathcal{Q}(R)\right]dV$$
$$+\int_{\partial\mathcal{U}}\left[\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\cdot\dot{\mathbf{U}}+\frac{1}{2}\overset{\circ}{\mathbf{T}}\cdot\dot{\mathbf{W}}+\frac{1}{2}\mathcal{Q}(\mathbf{T})\cdot\overset{\circ}{\mathbf{V}}+\frac{1}{2}\mathcal{Q}(H)\right]dA. \tag{2.26}$$

We call (2.25) the ***linearized energy balance*** and (2.26) the ***quadratized energy balance***.

In classical linear elasticity, the initial motion is a stress-free static configuration, i.e. $\overset{\circ}{\mathbf{V}}=\mathbf{0}$, $\overset{\circ}{\mathbf{P}}=\mathbf{0}$, and $\overset{\circ}{\mathbf{B}}=\mathbf{0}$ and it is assumed that there are no heat sources and fluxes, i.e., $\delta R=\delta H=0$. In this case, the linearized energy balance becomes trivial, and the quadratized energy balance reads

$$\frac{d}{dt}\int_{\mathcal{U}}\left[\frac{1}{2}\boldsymbol{\nabla}\mathbf{U}\cdot\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}+\frac{1}{2}\rho_0\dot{\mathbf{U}}\cdot\dot{\mathbf{U}}\right]dV=\int_{\mathcal{U}}\rho_0\mathcal{L}(\mathbf{B})\cdot\dot{\mathbf{U}}dV+\int_{\partial\mathcal{U}}\mathcal{L}(\mathbf{T})\cdot\dot{\mathbf{U}}dA. \tag{2.27}$$

This is the so-called ***Power Theorem*** in classical linear elasticity [3]. In other words, under the conditions of classical linear elasticity the linearized energy balance is vacuous since it is related to the linearization of a function at its minimum, and one needs to look at the quadratized energy balance.

We can now consider the invariance of the energy balance under isometries (translations and rotations) of the Euclidean ambient space.

2.1. Invariance of the linearized energy balance under isometries of the Euclidean ambient space

Let us first consider a rigid translation of the deformed configuration of the following form

$$\mathbf{x}'=\xi_t(\mathbf{x})=\mathbf{x}+(t-t_0)\mathbf{w}, \tag{2.28}$$

where $\mathbf{w}$ is a constant vector. Under this change of frame we have

$$\varphi_t'=\xi_t\circ\varphi_t,\qquad \overset{\circ}{\varphi}{}_t'=\xi_t\circ\overset{\circ}{\varphi}_t\ . \tag{2.29}$$

Therefore, at $t=t_0$

$$\mathbf{V}'=\mathbf{V}+\mathbf{w},\qquad \overset{\circ}{\mathbf{V}}{}'=\overset{\circ}{\mathbf{V}}+\mathbf{w}. \tag{2.30}$$

Also

$$\mathbf{U}'=\mathbf{U},\qquad \dot{\mathbf{U}}'=\dot{\mathbf{U}}. \tag{2.31}$$

[4] As the second variation of motion $\mathbf{W}$ is not common in the elasticity literature, we will comment on its dynamics in §2.2.

Linearized balance of energy in the new frame at $t = t_0$ is written as[5]

$$\int_{\mathcal{U}} \rho_0 \left(\dot{\overline{\overset{\circ}{\mathbf{P}}}} \cdot \nabla\mathbf{U} + \overset{\circ}{\mathbf{P}} \cdot \nabla\dot{\mathbf{U}} + (\overset{\circ}{\mathbf{V}} + \mathbf{w}) \cdot \ddot{\mathbf{U}} + \overset{\circ}{\mathsf{C}} \cdot \dot{\mathbf{U}} \right) dV$$
$$= \int_{\mathcal{U}} \rho_0 \left[\overset{\circ}{\mathbf{B}} \cdot \dot{\mathbf{U}} + (\overset{\circ}{\mathbf{V}} + \mathbf{w}) \cdot \mathcal{L}(\mathbf{B}) + \mathcal{L}(R) \right] dV$$
$$+ \int_{\partial\mathcal{U}} \left[\left(\overset{\circ}{\mathbf{P}} \cdot \dot{\mathbf{U}} + \overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U} \cdot (\overset{\circ}{\mathbf{V}} + \mathbf{w}) \right) \cdot \hat{\mathbf{N}} + \mathcal{L}(H) \right] dA, \qquad (2.33)$$

where we used the conservation of mass $\dot{\rho}_0 = 0$. Subtracting (2.25) from (2.33) yields

$$\int_{\mathcal{U}} \rho_0 \ddot{\mathbf{U}} \cdot \mathbf{w}\; dV = \int_{\mathcal{U}} \rho_0 \mathcal{L}(\mathbf{B}) \cdot \mathbf{w}\; dV + \int_{\mathcal{U}} \mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U} \right) \cdot \mathbf{w}\; dV. \qquad (2.34)$$

Because $\mathcal{U}$ and $\mathbf{w}$ are arbitrary, we conclude that

$$\mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U} \right) + \rho_0 \mathcal{L}(\mathbf{B}) = \rho_0 \ddot{\mathbf{U}}. \qquad (2.35)$$

Let us now consider a rigid rotation of the deformed configuration with constant angular velocity, i.e.

$$\mathbf{x}' = e^{\mathbf{\Omega}(t-t_0)} \mathbf{x}, \qquad (2.36)$$

where $\mathbf{\Omega}^{\mathsf{T}} = -\mathbf{\Omega}$. Therefore, at $t = t_0$

$$\mathbf{V}' = \mathbf{V} + \mathbf{\Omega}\mathbf{x}, \qquad \overset{\circ}{\mathbf{V}}{}' = \overset{\circ}{\mathbf{V}} + \mathbf{\Omega} \overset{\circ}{\mathbf{x}}. \qquad (2.37)$$

This means that

$$\mathbf{U}' = \mathbf{U}, \qquad \dot{\mathbf{U}}' = \dot{\mathbf{U}}. \qquad (2.38)$$

Subtracting balance of energy for $\overset{\circ}{\varphi}_t$ from that of $\overset{\circ}{\varphi}{}'_t$ and using balance of linear momentum for the perturbed motion, we obtain

$$\int_{\mathcal{U}} \left(\overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U} \right) : \mathbf{\Omega}\; dV = 0. \qquad (2.39)$$

Because $\mathcal{U}$ and $\mathbf{w}$ are arbitrary, we conclude that

$$\left(\overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U} \right)^{\mathsf{T}} = \overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U}. \qquad (2.40)$$

Therefore, we have proven the following proposition.

Proposition 2.2. *Invariance of the linearized balance of energy under time-dependent rigid translations and rotations of the Euclidean ambient space with constant velocity is equivalent to the linearized balances of linear and angular momenta.*

[5] Note that we assume that

$$\left(\mathbf{A}' - \mathbf{B}' \right)_{t=t_0} = \mathbf{A} - \mathbf{B}. \qquad (2.32)$$

2.2. Lagrangian field theory of linearized elasticity

Similar ideas can be used in obtaining equations of linear elasticity in the framework of Lagrangian mechanics. The starting point in the Lagrangian field theory of elasticity is a Lagrangian density $\mathfrak{L} = \mathfrak{L}(\mathbf{X}, t, \varphi, \dot{\varphi}, \mathbf{F})$. ***Hamilton's principle of least action*** states that for the equilibrium configuration the first variation of the action integral vanishes, i.e., $\delta S = 0$, where

$$S = \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathfrak{L} dV dt. \tag{2.41}$$

Given the motions $\overset{\circ}{\varphi}$ and φ_s, where $\varphi_0 = \overset{\circ}{\varphi}$, we can write

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \overset{\circ}{\mathfrak{L}} \, dV dt = 0 \quad \text{and} \quad \delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathfrak{L}(s) dV dt = 0 \qquad \forall \ s \in \ I. \tag{2.42}$$

Or

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \left[\mathfrak{L}(s) - \overset{\circ}{\mathfrak{L}} \right] \, dV dt = 0 \qquad \forall \ s \in \ I. \tag{2.43}$$

Assuming that $\mathfrak{L} = \mathfrak{T}(\dot{\varphi}) - \mathfrak{V}(\mathbf{X}, t, \varphi, \mathbf{F})$, where $\mathfrak{T}$ and $\mathfrak{V}$ are kinetic energy and potential energy, respectively, we can write

$$\mathfrak{L}(s) - \overset{\circ}{\mathfrak{L}} = \mathcal{L}(\mathfrak{L}) s + \mathcal{Q}(\mathfrak{L}) \frac{1}{2} s^2 + o(s^2), \tag{2.44}$$

where

$$\mathcal{L}(\mathfrak{L}) = \frac{d}{ds}\Big|_{s=0} \mathfrak{L}(s) = \frac{\partial \mathfrak{L}}{\partial \overset{\circ}{\varphi}} \cdot \mathbf{U} + \frac{\partial \mathfrak{L}}{\partial \dot{\overset{\circ}{\varphi}}} \cdot \dot{\mathbf{U}} + \frac{\partial \mathfrak{L}}{\partial \overset{\circ}{\mathbf{F}}} \cdot \boldsymbol{\nabla} \mathbf{U}, \tag{2.45}$$

$$\begin{aligned} \mathcal{Q}(\mathfrak{L}) = \frac{d^2}{ds^2}\Big|_{s=0} \mathfrak{L}(s) = & \frac{\partial^2 \mathfrak{L}}{\partial \overset{\circ}{\varphi} \, \partial \overset{\circ}{\varphi}} \cdot \mathbf{U} \otimes \mathbf{U} + 2 \frac{\partial^2 \mathfrak{L}}{\partial \overset{\circ}{\varphi} \, \partial \overset{\circ}{\mathbf{F}}} \cdot \mathbf{U} \otimes \boldsymbol{\nabla} \mathbf{U} \\ & + \frac{\partial^2 \mathfrak{L}}{\partial \dot{\overset{\circ}{\varphi}} \, \partial \dot{\overset{\circ}{\varphi}}} \cdot \dot{\mathbf{U}} \otimes \dot{\mathbf{U}} + \frac{\partial^2 \mathfrak{L}}{\partial \overset{\circ}{\mathbf{F}} \, \partial \overset{\circ}{\mathbf{F}}} \cdot \boldsymbol{\nabla} \mathbf{U} \otimes \boldsymbol{\nabla} \mathbf{U} \\ & + \frac{\partial \mathfrak{L}}{\partial \overset{\circ}{\varphi}} \cdot \mathbf{W} + \frac{\partial \mathfrak{L}}{\partial \dot{\overset{\circ}{\varphi}}} \cdot \dot{\mathbf{W}} + \frac{\partial \mathfrak{L}}{\partial \overset{\circ}{\mathbf{F}}} \cdot \boldsymbol{\nabla} \mathbf{W}. \end{aligned} \tag{2.46}$$

The action principle is now written as

$$\left(\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L}(\mathfrak{L}) \ dV dt \right) \ s + \left(\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{Q}(\mathfrak{L}) \ dV dt \right) \ \frac{1}{2} s^2 + o(s^2) = 0. \tag{2.47}$$

Because s is arbitrary this means that the coefficients of s and s^2 should be zero independently, i.e.

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L}(\mathfrak{L}) \ dV dt = 0, \quad \delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{Q}(\mathfrak{L}) \ dV dt = 0. \tag{2.48}$$

We call these the linearized and quadratized action principles, respectively. Assuming that $\delta\mathbf{U} = \mathbf{0}$ at the boundary points, the linearized action principle can be simplified to read

$$\int_{t_0}^{t_1}\int_{\mathcal{B}}\left[\frac{\partial\mathfrak{L}}{\partial\,\mathring{\varphi}} - \frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{\mathring{\varphi}}}\right) - \mathrm{Div}\left(\frac{\partial\mathfrak{L}}{\partial\,\mathring{\mathbf{F}}}\right)\right]\cdot\delta\mathbf{U}\ dV dt = 0. \tag{2.49}$$

This gives nothing but the Euler–Lagrange (EL) equations for the reference motion $\mathring{\varphi}$. For the quadratized action principle, there are two independent variations $\delta\mathbf{U}$ and $\delta\mathbf{W}$. Note that $\mathbf{W}$ appears in the quadratized action linearly and hence it can be easily shown that its variation reproduces the EL equations for the reference motion $\mathring{\varphi}$. The $\mathbf{U}$-variation can be simplified to read

$$\begin{aligned}\delta_{\mathbf{U}}\int_{t_0}^{t_1}\int_{\mathcal{B}}\mathcal{Q}(\mathfrak{L})dVdt = 2\int_{t_0}^{t_1}\int_{\mathcal{B}}\Bigg[&\frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\varphi}\,\partial\,\mathring{\varphi}}\cdot\mathbf{U} + \frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\varphi}\,\partial\,\mathring{\mathbf{F}}}\cdot\boldsymbol{\nabla}\mathbf{U} - \mathrm{Div}\left(\frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\varphi}\,\partial\,\mathring{\mathbf{F}}}\cdot\mathbf{U}\right)\\ &-\frac{d}{dt}\left(\frac{\partial^2\mathfrak{L}}{\partial\dot{\mathring{\varphi}}\ \partial\dot{\mathring{\varphi}}}\cdot\dot{\mathbf{U}}\right) - \mathrm{Div}\left(\frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\mathbf{F}}\,\partial\,\mathring{\mathbf{F}}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)\Bigg]\cdot\delta\mathbf{U}\ dVdt = 0.\end{aligned} \tag{2.50}$$

Therefore

$$\frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\varphi}\,\partial\,\mathring{\varphi}}\cdot\mathbf{U} + \frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\varphi}\,\partial\,\mathring{\mathbf{F}}}\cdot\boldsymbol{\nabla}\mathbf{U} - \frac{\partial^2\mathfrak{L}}{\partial\dot{\mathring{\varphi}}\ \partial\dot{\mathring{\varphi}}}\cdot\ddot{\mathbf{U}} - \mathrm{Div}\left(\frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\varphi}\,\partial\,\mathring{\mathbf{F}}}\cdot\mathbf{U} + \frac{\partial^2\mathfrak{L}}{\partial\,\mathring{\mathbf{F}}\,\partial\,\mathring{\mathbf{F}}}\cdot\boldsymbol{\nabla}\mathbf{U}\right) = \mathbf{0}, \tag{2.51}$$

which is nothing but the linearization of the EL equations about $\mathring{\varphi}$. Therefore, as it was expected, the quadratized action principle gives the linearized EL equations.

3. Geometric elasticity

In this section, in order to make the paper self-contained, we review some notation from the geometric approach to elasticity. Refer to [7] for more details and also [1] and [8].

For a smooth n-manifold M, the tangent space to M at a point $p \in M$ is denoted T_pM and the whole tangent bundle is denoted TM. We denote by $\mathcal{B}$ a reference manifold for our body and by $\mathcal{S}$ the space in which the body moves. We assume that $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with metrics $\mathbf{G}$ and $\mathbf{g}$, respectively. Local coordinates on $\mathcal{B}$ are denoted by $\{X^A\}$ and those on $\mathcal{S}$ by $\{x^a\}$.

A ***deformation*** of the body is a C^1 embedding $\varphi : \mathcal{B} \to \mathcal{S}$. The tangent map of φ is denoted $\mathbf{F} = T\varphi : T\mathcal{B} \to T\mathcal{S}$, which is often called the deformation gradient. In local charts on $\mathcal{B}$ and $\mathcal{S}$, the tangent map of φ is given by the Jacobian matrix of partial derivatives of the components of φ, as

$$\mathbf{F} = T\varphi : T\mathcal{B} \to T\mathcal{S}, \quad T\varphi(\mathbf{X},\mathbf{Y}) = (\varphi(\mathbf{X}), \mathbf{D}\varphi(\mathbf{X})\cdot\mathbf{Y}). \tag{3.1}$$

If $F : \mathcal{B} \to \mathbb{R}$ is a C^1 scalar function, $\mathbf{X} \in \mathcal{B}$ and $\mathbf{V_X} \in T_\mathbf{X}\mathcal{B}$, then $\mathbf{V_X}[F]$ denotes the derivative of F at $\mathbf{X}$ in the direction of $\mathbf{V_X}$, i.e., $\mathbf{V_X}[F] = \mathbf{D}F(\mathbf{X}) \cdot \mathbf{V}$. In local coordinates $\{X^A\}$ on $\mathcal{B}$

$$\mathbf{V_X}[F] = \frac{\partial F}{\partial X^A} V^A. \tag{3.2}$$

For $f : \mathcal{S} \to \mathbb{R}$, the ***pull-back*** of f by φ is defined by

$$\varphi^* f = f \circ \varphi. \tag{3.3}$$

If $F : \mathcal{B} \to \mathbb{R}$, the ***push-forward*** of F by φ is defined by

$$\varphi_* F = F \circ \varphi^{-1}. \tag{3.4}$$

If $\mathbf{Y}$ is a vector field on $\mathcal{B}$, then $\varphi_*\mathbf{Y} = T\varphi \cdot \mathbf{Y} \circ \varphi^{-1}$, or using the $\mathbf{F}$ notation, $\varphi_*\mathbf{Y} = \mathbf{F} \cdot \mathbf{Y} \circ \varphi^{-1}$ is a vector field on $\varphi(\mathcal{B})$ called the ***push-forward*** of $\mathbf{Y}$ by φ. Similarly, if $\mathbf{y}$ is a vector field on $\varphi(\mathcal{B}) \subset \mathcal{S}$, then $\varphi^*\mathbf{y} = T(\varphi^{-1}) \cdot \mathbf{y} \circ \varphi$ is a vector field on $\mathcal{B}$ and is called the pull-back of $\mathbf{y}$ by φ.

The cotangent bundle of a manifold M is denoted T^*M and the fiber at a point $p \in M$ (the vector space of one-forms at p) is denoted by T_p^*M. If β is a one-form on $\mathcal{S}$, i.e., a section of the cotangent bundle $T^*\mathcal{S}$, then the one-form on $\mathcal{B}$ defined as

$$(\varphi^*\beta)_\mathbf{X} \cdot \mathbf{V_X} = \beta_{\varphi(\mathbf{X})} \cdot (T\varphi \cdot \mathbf{V_X}) = \beta_{\varphi(\mathbf{X})} \cdot (\mathbf{F} \cdot \mathbf{V_X}) \tag{3.5}$$

for $\mathbf{X} \in \mathcal{B}$ and $\mathbf{V_X} \in T_\mathbf{X}\mathcal{B}$, is called the ***pull-back*** of β by φ. Likewise, the ***push-forward*** of a one-form α on $\mathcal{B}$ is the one form on $\varphi(\mathcal{B})$ defined by $\varphi_*\alpha = (\varphi^{-1})^*\alpha$.

We can associate a vector field $\beta^\sharp$ to a one-form β on a Riemannian manifold M through the equation

$$\langle \beta_\mathbf{x}, \mathbf{v_x} \rangle = \langle\!\langle \beta^\sharp_\mathbf{x}, \mathbf{v_x} \rangle\!\rangle_\mathbf{x}, \tag{3.6}$$

where $\langle\,,\rangle$ denotes the natural pairing between the one form $\beta_\mathbf{x} \in T^*_\mathbf{x}M$ and the vector $\mathbf{v_x} \in T_\mathbf{x}M$ and where $\langle\!\langle \beta^\sharp_\mathbf{x}, \mathbf{v_x} \rangle\!\rangle_\mathbf{x}$ denotes the inner product between $\beta^\sharp_\mathbf{x} \in T_\mathbf{x}M$ and $\mathbf{v_x} \in T_\mathbf{x}M$ induced by the metric $\mathbf{g}$. In coordinates, the components of $\beta^\sharp$ are given by $\beta^a = g^{ab}\beta_b$.

A type $\binom{m}{n}$-tensor at $\mathbf{X} \in \mathcal{B}$ is a multilinear map

$$\mathbf{T} : \underbrace{T^*_\mathbf{X}\mathcal{B} \times ... \times T^*_\mathbf{X}\mathcal{B}}_{m \text{ copies}} \times \underbrace{T_\mathbf{X}\mathcal{B} \times ... \times T_\mathbf{X}\mathcal{B}}_{n \text{ copies}} \to \mathbb{R}. \tag{3.7}$$

$\mathbf{T}$ is said to be contravariant of order m and covariant of order n. In a local coordinate chart

$$\mathbf{T}(\alpha^1, ..., \alpha^m, \mathbf{V}_1, ..., \mathbf{V}_n) = T^{\;i_1 \dots i_m}{}_{j_1 \dots j_n} \alpha^1_{i_1} ... \alpha^m_{i_m} V_1^{j_1} ... V_n^{j_n}, \tag{3.8}$$

where $\alpha^k \in T^*_\mathbf{X}\mathcal{B}$ and $\mathbf{V}^k \in T_\mathbf{X}\mathcal{B}$.

Suppose $\varphi : \mathcal{B} \to \mathcal{S}$ is a regular map and $\mathbf{T}$ is a tensor of type $\binom{m}{n}$. Push-forward of $\mathbf{T}$ by φ is denoted $\varphi_*\mathbf{T}$ and is a $\binom{m}{n}$-tensor on $\varphi(\mathcal{B})$ defined by

$$(\varphi_*\mathbf{T})(\mathbf{x})(\alpha^1, ..., \alpha^m, \mathbf{v}_1, ..., \mathbf{v}_n) = \mathbf{T}(\mathbf{X})(\varphi^*\alpha^1, ..., \varphi^*\alpha^m, \varphi^*\mathbf{v}_1, ..., \varphi^*\mathbf{v}_n), \quad (3.9)$$

where $\alpha^k \in T^*_{\mathbf{x}}\mathcal{S}, \mathbf{v}_k \in T_{\mathbf{x}}\mathcal{S}, \mathbf{X} = \varphi^{-1}(\mathbf{x}), \varphi^*(\alpha^k) \cdot \mathbf{v}_l = \alpha^k \cdot (T\varphi \cdot \mathbf{v}_l)$ and $\varphi^*(\mathbf{v}_l) = T(\varphi^{-1})\mathbf{v}_l$. Similarly, pull-back of a tensor $\mathbf{t}$ defined on $\varphi(\mathcal{B})$ is given by $\varphi^*\mathbf{t} = (\varphi^{-1})_*\mathbf{t}$.

A ***two-point tensor*** $\mathbf{T}$ of type $\begin{pmatrix} m & r \\ n & s \end{pmatrix}$ at $\mathbf{X} \in \mathcal{B}$ over a map $\varphi : \mathcal{B} \to \mathcal{S}$ is a multilinear map

$$\begin{aligned} T : &\underbrace{T^*_{\mathbf{X}}\mathcal{B} \times ... \times T^*_{\mathbf{X}}\mathcal{B}}_{m \text{ copies}} \times \underbrace{T_{\mathbf{X}}\mathcal{B} \times ... \times T_{\mathbf{X}}\mathcal{B}}_{n \text{ copies}} \\ &\times \underbrace{T^*_{\mathbf{x}}\mathcal{S} \times ... \times T^*_{\mathbf{x}}\mathcal{S}}_{r \text{ copies}} \times \underbrace{T_{\mathbf{x}}\mathcal{S} \times ... \times T_{\mathbf{x}}\mathcal{S}}_{s \text{ copies}} \to \mathbb{R}, \end{aligned} \quad (3.10)$$

where $\mathbf{x} = \varphi(\mathbf{X})$.

Let $\mathbf{w} : \mathcal{U} \to T\mathcal{S}$ be a vector field, where $\mathcal{U} \subset \mathcal{S}$ is open. A curve $\mathbf{c} : I \to \mathcal{S}$, where I is an open interval, is an ***integral curve*** of $\mathbf{w}$ if

$$\frac{d\mathbf{c}}{dt}(r) = \mathbf{w}(\mathbf{c}(r)) \qquad \forall\ r \in I. \quad (3.11)$$

If $\mathbf{w}$ depends on the time variable explicitly, i.e., $\mathbf{w} : \mathcal{U} \times (-\epsilon, \epsilon) \to T\mathcal{S}$, an integral curve is defined by

$$\frac{d\mathbf{c}}{dt} = \mathbf{w}(\mathbf{c}(t), t). \quad (3.12)$$

Let $\mathbf{w} : \mathcal{S} \times I \to T\mathcal{S}$ be a vector field. The collection of maps $F_{t,s}$ such that for each s and $\mathbf{x}$, $t \mapsto F_{t,s}(\mathbf{x})$ is an integral curve of $\mathbf{w}$ and $F_{s,s}(\mathbf{x}) = \mathbf{x}$ is called the flow of $\mathbf{w}$. Let $\mathbf{w}$ be a C^1 vector field on $\mathcal{S}$, $F_{t,s}$ its flow, and $\mathbf{t}$ a C^1 tensor field on $\mathcal{S}$. The ***Lie derivative*** of $\mathbf{t}$ with respect to $\mathbf{w}$ is defined by

$$\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{dt}\left(F^*_{t,s}\mathbf{t}_t\right)\Big|_{s=t}. \quad (3.13)$$

If we hold t fixed in $\mathbf{t}$ then we denote

$$\mathfrak{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{dt}\left(F^*_{t,s}\mathbf{t}_s\right)\Big|_{s=t}, \quad (3.14)$$

which is called the ***autonomous Lie derivative***. Therefore

$$\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{\partial}{\partial t}\mathbf{t} + \mathfrak{L}_{\mathbf{w}}\mathbf{t}. \quad (3.15)$$

Let $\mathbf{y}$ be a vector field on $\mathcal{S}$ and $\varphi : \mathcal{B} \to \mathcal{S}$ a regular and orientation preserving C^1 map. The ***Piola transform*** of $\mathbf{y}$ is defined as

$$\mathbf{Y} = J\varphi^*\mathbf{y}, \quad (3.16)$$

where J is the Jacobian of φ. If $\mathbf{Y}$ is the Piola transform of $\mathbf{y}$, then the ***Piola identity*** holds:

$$\operatorname{Div}\mathbf{Y} = J(\operatorname{div}\mathbf{y}) \circ \varphi. \tag{3.17}$$

A p-form on a manifold M is a skew-symmetric $\binom{0}{p}$-tensor. The space of p-forms on M is denoted by $\Omega^p(M)$. If $\varphi : M \to N$ is a regular and orientation preserving C^1 map and $\alpha \in \Omega^p(\varphi(M))$, then

$$\int_{\varphi(M)} \alpha = \int_M \varphi^* \alpha. \tag{3.18}$$

Let $\pi : E \to \mathcal{S}$ be a vector bundle over a manifold $\mathcal{S}$ and let $\mathcal{E}(\mathcal{S})$ be the space of smooth sections of E and $\mathcal{X}(\mathcal{S})$ the space of vector fields on $\mathcal{S}$. A ***connection*** on E is a map $\boldsymbol{\nabla} : \mathcal{X}(\mathcal{S}) \times \mathcal{E}(\mathcal{S}) \to \mathcal{E}(\mathcal{S})$ such that $\forall\, f, f_1, f_2 \in C^\infty(\mathcal{S})$, $\forall\, a_1, a_2 \in \mathbb{R}$

$$i) \quad \boldsymbol{\nabla}_{f_1\mathbf{X}_1 + f_2\mathbf{X}_2}\mathbf{Y} = f_1\boldsymbol{\nabla}_{\mathbf{X}_1}\mathbf{Y} + f_2\boldsymbol{\nabla}_{\mathbf{X}_2}\mathbf{Y}, \tag{3.19}$$

$$ii) \quad \boldsymbol{\nabla}_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\boldsymbol{\nabla}_{\mathbf{X}}(\mathbf{Y}_1) + a_2\boldsymbol{\nabla}_{\mathbf{X}}(\mathbf{Y}_2), \tag{3.20}$$

$$iii) \quad \boldsymbol{\nabla}_{\mathbf{X}}(f\mathbf{Y}) = f\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}. \tag{3.21}$$

A ***linear connection*** on $\mathcal{S}$ is a connection on $T\mathcal{S}$, i.e., $\boldsymbol{\nabla} : \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S}) \to \mathcal{X}(\mathcal{S})$. In a local chart

$$\boldsymbol{\nabla}_{\partial_i}\partial_j = \gamma_{ij}^k \partial_k, \tag{3.22}$$

where γ_{ij}^k are Christoffel symbols of the connection and $\partial_i = \frac{\partial}{\partial x^i}$. A linear connection is said to be compatible with the metric of the manifold if

$$\boldsymbol{\nabla}_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle = \langle\!\langle \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle + \langle\!\langle \mathbf{Y}, \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle. \tag{3.23}$$

It can be shown that $\boldsymbol{\nabla}$ is compatible with $\mathbf{g}$ if and only if $\boldsymbol{\nabla}\mathbf{g} = \mathbf{0}$. ***Torsion*** of a connection is defined as

$$\mathfrak{T}(\mathbf{X}, \mathbf{Y}) = \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y} - \boldsymbol{\nabla}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}], \tag{3.24}$$

where

$$[\mathbf{X}, \mathbf{Y}](F) = \mathbf{X}(\mathbf{Y}(F)) - \mathbf{Y}(\mathbf{X}(F)) \qquad \forall\, F \in C^\infty(\mathcal{S}), \tag{3.25}$$

is the ***commutator*** of $\mathbf{X}$ and $\mathbf{Y}$. $\boldsymbol{\nabla}$ is symmetric if it is torsion-free, i.e.

$$\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y} - \boldsymbol{\nabla}_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]. \tag{3.26}$$

It can be shown that on any Riemannian manifold $(\mathcal{S}, \mathbf{g})$ there is a unique linear connection $\boldsymbol{\nabla}$ that is compatible with $\mathbf{g}$ and is torsion-free with the following Christoffel symbols

$$\gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right). \tag{3.27}$$

This is the ***Fundamental Lemma of Riemannian Geometry*** [6] and this connection is called the ***Levi-Civita connection***.

Curvature tensor $\boldsymbol{\mathcal{R}}$ of a Riemannian manifold $(\mathcal{S},\mathbf{g})$ is a $\begin{pmatrix}1\\3\end{pmatrix}$-tensor $\boldsymbol{\mathcal{R}}$: $T^*_{\mathbf{x}}\mathcal{S}\times T_{\mathbf{x}}\mathcal{S}\times T_{\mathbf{x}}\mathcal{S}\times T_{\mathbf{x}}\mathcal{S}\to\mathbb{R}$ defined as

$$\boldsymbol{\mathcal{R}}(\alpha,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)=\alpha\left(\boldsymbol{\nabla}_{\mathbf{w}_1}\boldsymbol{\nabla}_{\mathbf{w}_2}\mathbf{w}_3-\boldsymbol{\nabla}_{\mathbf{w}_2}\boldsymbol{\nabla}_{\mathbf{w}_1}\mathbf{w}_3-\boldsymbol{\nabla}_{[\mathbf{w}_1,\mathbf{w}_2]}\mathbf{w}_3\right) \tag{3.28}$$

for $\alpha\in T^*_{\mathbf{x}}S,\ \mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\in T_{\mathbf{x}}S$. In a coordinate chart $\{x^a\}$

$$\mathcal{R}^a{}_{bcd}=\frac{\partial\gamma^a_{bd}}{\partial x^c}-\frac{\partial\gamma^a_{bc}}{\partial x^d}+\gamma^a_{ce}\gamma^e_{bd}-\gamma^a_{de}\gamma^e_{bc}. \tag{3.29}$$

Let us next review a few of the basic notions of geometric continuum mechanics.

A ***body*** $\mathcal{B}$ is identified with a Riemannian manifold $\mathcal{B}$ and a ***configuration*** of $\mathcal{B}$ is a mapping $\varphi:\mathcal{B}\to\mathcal{S}$, where $\mathcal{S}$ is another Riemannian manifold. The set of all configurations of $\mathcal{B}$ is denoted $\mathcal{C}$. A ***motion*** is a curve $c:\mathbb{R}\to\mathcal{C};t\mapsto\varphi_t$ in $\mathcal{C}$.

For a fixed t, $\varphi_t(\mathbf{X})=\varphi(\mathbf{X},t)$ and for a fixed $\mathbf{X}$, $\varphi_{\mathbf{X}}(t)=\varphi(\mathbf{X},t)$, where $\mathbf{X}$ is position of material points in the undeformed configuration $\mathcal{B}$. The ***material velocity*** is the map $\mathbf{V}_t:\mathcal{B}\to\mathbb{R}^3$ given by

$$\mathbf{V}_t(\mathbf{X})=\mathbf{V}(\mathbf{X},t)=\frac{\partial\varphi(\mathbf{X},t)}{\partial t}=\frac{d}{dt}\varphi_{\mathbf{X}}(t). \tag{3.30}$$

Similarly, the ***material acceleration*** is defined by

$$\mathbf{A}_t(\mathbf{X})=\mathbf{A}(\mathbf{X},t)=\frac{\partial\mathbf{V}(\mathbf{X},t)}{\partial t}=\frac{d}{dt}\mathbf{V}_{\mathbf{X}}(t). \tag{3.31}$$

In components

$$A^a=\frac{\partial V^a}{\partial t}+\gamma^a_{bc}V^bV^c, \tag{3.32}$$

where γ^a_{bc} is the Christoffel symbol of the local coordinate chart $\{x^a\}$.

Here it is assumed that φ_t is invertible and regular. The ***spatial velocity*** of a regular motion φ_t is defined as

$$\mathbf{v}_t:\varphi_t(\mathcal{B})\to\mathbb{R}^3,\qquad \mathbf{v}_t=\mathbf{V}_t\circ\varphi_t^{-1}, \tag{3.33}$$

and the ***spatial acceleration*** $\mathbf{a}_t$ is defined as

$$\mathbf{a}=\dot{\mathbf{v}}=\frac{\partial\mathbf{v}}{\partial t}+\nabla_{\mathbf{v}}\mathbf{v}. \tag{3.34}$$

In components

$$a^a=\frac{\partial v^a}{\partial t}+\frac{\partial v^a}{\partial x^b}v^b+\gamma^a_{bc}v^bv^c. \tag{3.35}$$

Let $\varphi:\mathcal{B}\to\mathcal{S}$ be a C^1 configuration of $\mathcal{B}$ in $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{S}$ are manifolds. Recall that the deformation gradient is denoted by $\mathbf{F}=T\varphi$. Thus, at each point $\mathbf{X}\in\mathcal{B}$, it is a linear map

$$\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\varphi(\mathbf{X})}\mathcal{S}. \tag{3.36}$$

If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ are

$$F^a{}_A(\mathbf{X})=\frac{\partial\varphi^a}{\partial X^A}(\mathbf{X}). \tag{3.37}$$

The deformation gradient may be viewed as a two-point tensor

$$\mathbf{F}(\mathbf{X}) : T^*_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{X}}\mathcal{B} \to \mathbb{R}; \quad (\alpha, \mathbf{V}) \mapsto \langle \alpha, T_{\mathbf{X}}\varphi \cdot \mathbf{V}\rangle. \tag{3.38}$$

Suppose $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with inner products $\langle\!\langle,\rangle\!\rangle_{\mathbf{X}}$ and $\langle\!\langle,\rangle\!\rangle_{\mathbf{x}}$ based at $\mathbf{X} \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{S}$, respectively. Recall that the transpose of $\mathbf{F}$ is defined by

$$\mathbf{F}^{\mathsf{T}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}, \quad \langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v}\rangle\!\rangle_{\mathbf{x}} = \left\langle\!\left\langle \mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v}\right\rangle\!\right\rangle_{\mathbf{X}} \tag{3.39}$$

for all $\mathbf{V} \in T_{\mathbf{X}}\mathcal{B},\ \mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$. In components

$$(F^{\mathsf{T}}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X}), \tag{3.40}$$

where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively. On the other hand, the ***dual*** of $\mathbf{F}$, a metric independent notion, is defined by

$$\mathbf{F}^*(\mathbf{x}) : T^*_{\mathbf{x}}\mathcal{S} \to T^*_{\mathbf{X}}\mathcal{B}; \quad \langle \mathbf{F}^*(\mathbf{x}) \cdot \alpha, \mathbf{W}\rangle = \langle \alpha, \mathbf{F}(\mathbf{X})\mathbf{W}\rangle \tag{3.41}$$

for all $\alpha \in T^*_{\mathbf{x}}\mathcal{S}, \mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$. Considering bases $\mathbf{e}_a$ and $\mathbf{E}_A$ for $\mathcal{S}$ and $\mathcal{B}$, respectively, one can define the corresponding dual bases $\mathbf{e}^a$ and $\mathbf{E}^A$. The matrix representation of $\mathbf{F}^*$ with respect to the dual bases is the transpose of $F^a{}_A$. $\mathbf{F}$ and $\mathbf{F}^*$ have the following local representations

$$\mathbf{F} = F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A, \qquad \mathbf{F}^* = F^a{}_A dX^A \otimes \frac{\partial}{\partial x^a}. \tag{3.42}$$

The ***right Cauchy–Green deformation tensor*** is defined by

$$\mathbf{C}(X) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}, \qquad \mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X}). \tag{3.43}$$

In components

$$C^A_B = (F^{\mathsf{T}})^A{}_a F^a{}_B. \tag{3.44}$$

It is straightforward to show that

$$\mathbf{C}^\flat = \varphi^*(\mathbf{g}) = \mathbf{F}^*\mathbf{g}\mathbf{F}, \ \text{ i.e. } \quad C_{AB} = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B. \tag{3.45}$$

Let $\varphi_t : \mathcal{B} \to \mathcal{S}$ be a regular motion of $\mathcal{B}$ in $\mathcal{S}$ and $\mathcal{P} \subset \mathcal{B}$ a p-dimensional submanifold. The ***Transport Theorem*** says that for any p-form α on $\mathcal{S}$

$$\frac{d}{dt} \int_{\varphi_t(\mathcal{P})} \alpha = \int_{\varphi_t(\mathcal{P})} \mathbf{L}_{\mathbf{v}}\alpha, \tag{3.46}$$

where $\mathbf{v}$ is the spatial velocity of the motion. In a special case when $\alpha = f dv$ and $\mathcal{P} = \mathcal{U}$ is an open set, one can write

$$\frac{d}{dt} \int_{\varphi_t(\mathcal{P})} f dv = \int_{\varphi_t(\mathcal{P})} \left[\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{v})\right] dv. \tag{3.47}$$

Balance of linear momentum for a body $\mathcal{B}$ is satisfied if for every nice open set $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt} \int_{\varphi_t(\mathcal{U})} \rho \mathbf{v} dv = \int_{\varphi_t(\mathcal{U})} \rho \mathbf{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \mathbf{t} da, \tag{3.48}$$

where $\rho = \rho(\mathbf{x}, t)$ is mass density, $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ is body force vector field and $\mathbf{t} = \mathbf{t}(\mathbf{x}, \hat{\mathbf{n}}, t)$ is the traction vector. Note that according to Cauchy's stress theorem there exists a contra-variant second-order tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ (Cauchy stress tensor) with components σ^{ab} such that $\mathbf{t} = \langle\!\langle \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle\!\rangle$. Note also that $\langle\!\langle , \rangle\!\rangle$ is the inner product induced by the Riemmanian metric $\mathbf{g}$. Equivalently, balance of linear momentum can be written in the undeformed configuration as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathbf{V} dV = \int_{\mathcal{U}} \rho_0 \mathbf{B} dV + \int_{\partial\mathcal{U}} \left\langle\!\!\left\langle \mathbf{P}, \hat{\mathbf{N}} \right\rangle\!\!\right\rangle dA, \tag{3.49}$$

where, $\mathbf{P} = J\varphi^* \sigma$ (the first Piola–Kirchhoff stress tensor) is the Piola transform of Cauchy stress tensor. Note that $\mathbf{P}$ is a two-point tensor with components P^{aA}. Note also that this is the balance of linear momentum in the deformed (physical) space written in terms of some quantities that are defined with respect to the reference configuration.

Let us emphasize that balance of linear momentum has no intrinsic meaning because integrating a vector field is geometrically meaningless, i.e., it is coordinate dependent. Geometrically, forces (interactions) take values in the cotangent bundle of the ambient space manifold (see [5] for a detailed discussion). The ambient space manifold is not linear in general and hence balance of forces cannot be written in an integral form, in general. In classical continuum mechanics, this balance law makes use of the linear (or affine) structure of Euclidean space.

Balance of angular momentum is satisfied for a body $\mathcal{B}$ if for every nice open set $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho \mathbf{x} \times \mathbf{v} dv = \int_{\varphi_t(\mathcal{U})} \rho \mathbf{x} \times \mathbf{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \mathbf{x} \times \langle\!\langle \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle\!\rangle \, da. \tag{3.50}$$

Balance of linear momentum, similar to balance of angular momentum, makes use of the linear structure of Euclidean space and this does *not* transform in a covariant way under a general change of coordinates.

Balance of energy holds for a body $\mathcal{B}$ if, for every nice open set $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho \left(e + \frac{1}{2} \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \right) dv = \int_{\varphi_t(\mathcal{U})} \rho \left(\langle\!\langle \mathbf{b}, \mathbf{v} \rangle\!\rangle + r \right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\langle\!\langle \mathbf{t}, \mathbf{v} \rangle\!\rangle + h \right) da, \tag{3.51}$$

where $e = e(\mathbf{x}, t), r = r(\mathbf{x}, t)$ and $h = h(\mathbf{x}, \hat{\mathbf{n}}, t)$ are internal energy per unit mass, heat supply per unit mass and heat flux, respectively.

4. Geometric linearization of nonlinear elasticity

Marsden and Hughes [7] formulated the theory of linear elasticity by linearizing nonlinear elasticity assuming that reference and ambient space manifolds are Riemannian. Here we review their ideas and obtain some new results. We denote by $\mathcal{C}$ the set of all deformation mappings $\varphi : \mathcal{B} \to \mathcal{S}$. We do not discuss boundary conditions, but assume that deformation mappings satisfy all the displacement (essential) boundary conditions. One can prove that $\mathcal{C}$ is an infinite-dimensional manifold. We will not go into the theory of infinite-dimensional manifolds in detail, however, note that for $\overset{\circ}{\varphi}_t \in \mathcal{C}$, an element of $T_{\overset{\circ}{\varphi}_t}\mathcal{C}$ can be thought of as being tangent to a curve $\varphi_{t,s} \in \mathcal{C}$, parametrized by s, such that $\varphi_{t,0} = \overset{\circ}{\varphi}_t$. This is called a variation of the configuration, and the tangent vector is denoted by $\delta\varphi_t = \mathbf{U} = \frac{d}{ds}\big|_{s=0}\varphi_{t,s}$.

Suppose $\pi : \mathcal{E} \to \mathcal{C}$ is a vector bundle over $\mathcal{C}$ and let $f : \mathcal{C} \to \mathcal{E}$ be a section of this bundle. Let us assume that $\mathcal{E}$ is equipped with a connection $\boldsymbol{\nabla}$. With these assumptions, linearization of $f(\varphi)$ at $\overset{\circ}{\varphi}_t \in \mathcal{C}$ is defined as

$$\mathcal{L}\left(f; \overset{\circ}{\varphi}_t\right) := \boldsymbol{\nabla} f(\overset{\circ}{\varphi}_t) \cdot \mathbf{U}, \quad \mathbf{U} \in T_{\overset{\circ}{\varphi}_t}\mathcal{C}\,. \tag{4.1}$$

In terms of the parallel transport $\boldsymbol{\alpha}_s$ of members of $\mathcal{E}_{\varphi_{t,s}}$ to $\mathcal{E}_{\overset{\circ}{\varphi}_t}$ along a curve $\varphi_{t,s}$ tangent to $\mathbf{U}$ at $\overset{\circ}{\varphi}_t$, this can be written as

$$\boldsymbol{\nabla} f(\overset{\circ}{\varphi}_t) \cdot \mathbf{U} = \frac{d}{ds}\, \boldsymbol{\alpha}_s \cdot f(\varphi_{t,s})\Big|_{s=0}. \tag{4.2}$$

In [7], by using a natural parallel translation on $\mathcal{E}$ obtained from the pointwise parallel translation of two-point tensors over φ, it is shown that deformation gradient has the following linearization about $\overset{\circ}{\varphi}_t$.

$$\mathcal{L}\left(\mathbf{F}; \overset{\circ}{\varphi}\right) = \boldsymbol{\nabla}\mathbf{U}, \tag{4.3}$$

where $\overset{\circ}{\mathbf{F}} = T\,\overset{\circ}{\varphi}_t$. One can think of $\mathbf{F}$ as a vector-valued one-form with the local representation

$$\mathbf{F} = F^a{}_A\ \mathbf{e}_a \otimes dX^A. \tag{4.4}$$

Thus

$$\boldsymbol{\varepsilon} := \mathcal{L}\left(\mathbf{F}; \overset{\circ}{\varphi}\right) = U^a{}_{|A}\ \mathbf{e}_a \otimes dX^A \tag{4.5}$$

can be thought of as a geometric linearized strain, which is a vector-valued one form. See [13] for more discussion on this geometric strain and constitutive equations written in terms of it. Material velocity is linearized at follows.

$$\mathcal{L}\left(\mathbf{V}; \overset{\circ}{\varphi}\right) = \dot{\mathbf{U}}, \tag{4.6}$$

where $\dot{\mathbf{U}}$ is the covariant time derivative of $\mathbf{U}$, i.e.

$$\dot{U}^a = \frac{\partial U^a}{\partial t} + \gamma^a_{bc}\, \overset{\circ}{V}{}^b\ U^c. \tag{4.7}$$

Material acceleration is linearized as follows.

$$\mathcal{L}\left(\mathbf{A}; \overset{\circ}{\varphi}\right) = \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \tag{4.8}$$

where $\boldsymbol{\mathcal{R}}$ is the curvature tensor of $(\mathcal{S}, \mathbf{g})$. In components, the linearized acceleration has the following form

$$\ddot{U}^a + \mathcal{R}^a{}_{bcd} \overset{\circ}{V}{}^b\, U^c \overset{\circ}{V}{}^d\ . \tag{4.9}$$

Proof of this result is lengthy but straightforward. Note that this is a generalization of Jacobi equation. Note also that in [7] it is implicitly assumed that $\boldsymbol{\mathcal{R}} = \mathbf{0}$.

The transpose of the deformation gradient is defined as

$$\langle\!\langle \mathbf{F}\mathbf{W}, \mathbf{z} \rangle\!\rangle_{\mathbf{g}} = \left\langle\!\left\langle \mathbf{W}, \mathbf{F}^{\mathsf{T}}\mathbf{z} \right\rangle\!\right\rangle_{\mathbf{G}} \qquad \forall\ \mathbf{W} \in T_{\mathbf{X}}\mathcal{B},\ \mathbf{z} \in T_{\mathbf{x}}\mathcal{S}\,, \tag{4.10}$$

and its linearization is given by

$$\mathcal{L}\left(\mathbf{F}^{\mathsf{T}}; \overset{\circ}{\varphi}\right) = (\boldsymbol{\nabla}\mathbf{U})^{\mathsf{T}}. \tag{4.11}$$

The right Cauchy–Green strain tensor has the following linearization

$$\mathcal{L}\left(\mathbf{C}; \overset{\circ}{\varphi}_t\right) = \overset{\circ}{\mathbf{F}}{}^{\mathsf{T}}\, \boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^{\mathsf{T}}\, \overset{\circ}{\mathbf{F}}\ . \tag{4.12}$$

Or in component form

$$\mathcal{L}\left(\mathbf{C}; \overset{\circ}{\varphi}_t\right)_{AB} = g_{ab} \overset{\circ}{F}{}^a{}_A\, U^b{}_{|B} + g_{ab} \overset{\circ}{F}{}^b{}_B\, U^a{}_{|A}. \tag{4.13}$$

Balance of angular momentum in component form reads

$$P^{aA} F^b{}_A = P^{bA} F^a{}_A\,, \tag{4.14}$$

This also implies that

$$\overset{\circ}{P}{}^{aA} \overset{\circ}{F}{}^b{}_A = \overset{\circ}{P}{}^{bA} \overset{\circ}{F}{}^a{}_A. \tag{4.15}$$

Linearization of this relation about $\overset{\circ}{\varphi}$ reads

$$\begin{aligned} &\overset{\circ}{P}{}^{aA} \overset{\circ}{F}{}^b{}_A + \overset{\circ}{P}{}^{aA}\, U^b{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{aA}{}_c{}^B\right) \overset{\circ}{F}{}^b{}_A\, U^c{}_{|B} \\ &= \overset{\circ}{P}{}^{bA} \overset{\circ}{F}{}^a{}_A + \overset{\circ}{P}{}^{bA}\, U^a{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{bA}{}_c{}^B\right) \overset{\circ}{F}{}^a{}_A\, U^c{}_{|B}, \end{aligned} \tag{4.16}$$

Using (4.15) this is simplified to read

$$\overset{\circ}{P}{}^{aA}\, U^b{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{aA}{}_c{}^B\right) \overset{\circ}{F}{}^b{}_A\, U^c{}_{|B} = \overset{\circ}{P}{}^{bA}\, U^a{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{bA}{}_c{}^B\right) \overset{\circ}{F}{}^a{}_A\, U^c{}_{|B}. \tag{4.17}$$

In terms of Cauchy stress this reads

$$\overset{\circ}{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \overset{\circ}{\mathbf{a}} : \boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}\boldsymbol{u} \cdot \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\nabla}\boldsymbol{u} : \overset{\circ}{\mathbf{a}}\ . \tag{4.18}$$

Or in components

$$\overset{\circ}{\sigma}{}^{ac}\, u^b{}_{|c} + \overset{\circ}{a}{}^{ab}{}_c{}^d\, u^c{}_{|d} = \overset{\circ}{\sigma}{}^{bc}\, u^a{}_{|c} + \overset{\circ}{a}{}^{ba}{}_c{}^d\, u^c{}_{|d}, \tag{4.19}$$

where [7]

$$\overset{\circ}{a}{}^{ac}{}_b{}^d = \frac{1}{J}\, F^c{}_A F^d{}_B\, \overset{\circ}{A}{}^{aA}{}_b{}^B \quad \text{and} \quad \mathbf{u} = \mathbf{U} \circ \varphi^{-1}. \tag{4.20}$$

Independent works have been done in the literature of geometric calculus of variations (see [9] and [2] and references therein) on similar problems. There, the idea is to obtain the first and second variations of "energy" of maps between two given Riemannian manifolds. In the following, we make a connection between these efforts and geometric elasticity.

4.1. Linearization of elasticity using variation of maps

Here we follow Nishikawa [9] but with a notation closer to ours. The main motivation for studying variational problems in [9] is to understand geodesics in Riemannian manifolds as minimization problems. Interestingly, these studies are closely related to elasticity. Let us consider two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ and a time-dependent motion $\varphi_t : \mathcal{B} \to \mathcal{S}$. What Nishikawa denotes by $d\varphi$ is $\mathbf{F}$ in our notation, which is an element of $T^*_{\mathbf{X}}\mathcal{B} \otimes T_{\mathbf{x}}\mathcal{S}$, i.e. a vector-valued one-form.[6] One can then define an inner product on $T^*_{\mathbf{X}}\mathcal{B} \otimes T_{\mathbf{x}}\mathcal{S}$ such that in this inner product

$$|\mathbf{F}|^2 = \mathrm{tr}(\mathbf{C}). \tag{4.21}$$

Energy density of the map φ is defined as

$$e(\varphi, \mathbf{X}) = \frac{1}{2}\,\mathrm{tr}[\mathbf{C}(\mathbf{X})]. \tag{4.22}$$

Note that this is a very special energy density, which is not realistic for elastic bodies.[7] Energy of the map φ is then defined as

$$E(\varphi) = \int_{\mathcal{B}} e(\varphi, \mathbf{X})\; dV(\mathbf{X}). \tag{4.24}$$

Consider a reference deformation map $\overset{\circ}{\varphi}_t$ and a C^∞ variation of it $\varphi_{t,s}$ such that $s \in I = (-\epsilon, \epsilon)$ and $\varphi_{t,0} = \overset{\circ}{\varphi}_t$. As in the previous section, let $\mathbf{U}$ be given by

$$\mathbf{U} = \mathbf{U}_t(\mathbf{X}) = \frac{d}{ds}\Big|_{s=0} \varphi_{t,s}(\mathbf{X}). \tag{4.25}$$

First variation of deformation gradient is defined as

$$\boldsymbol{\nabla}_{\frac{\partial}{\partial s}} \mathbf{F}(s)\Big|_{s=0} = \boldsymbol{\nabla}_{\frac{\partial}{\partial s}} \left(\frac{\partial \varphi_{t,s}}{\partial \mathbf{X}}\right)\Bigg|_{s=0} = \boldsymbol{\nabla}\mathbf{U}, \tag{4.26}$$

where the identity [9]

$$\boldsymbol{\nabla}_{\frac{\partial}{\partial s}} \left(\frac{\partial \varphi_{t,s}}{\partial X^A}\right) = \boldsymbol{\nabla}_{\frac{\partial}{\partial X^A}} \left(\frac{\partial \varphi_{t,s}}{\partial s}\right), \tag{4.27}$$

[6] For a discussion on reformulating continuum mechanics using bundle-valued forms see [5].

[7] A modified version which can represent the energy of an elastic body is given by

$$e(\varphi, \mathbf{X}) = \mu E^{AB} E_{AB}, \tag{4.23}$$

where $E_{AB} = \frac{1}{2}(C_{AB} - G_{AB})$ is the Lagrangian (material) strain tensor.

was used. Note that for each $s \in I$ and $\mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$, $\mathbf{F}(s)\mathbf{W} \in T_{\varphi_{t,s}(\mathbf{X})}\mathcal{S}$, i.e. $\mathbf{F}(s)\mathbf{W}$ lies in different tangent spaces for different values of s and this is why covariant derivative with respect to s is used.

Tension field of φ is defined as

$$\boldsymbol{\tau}(\varphi) = tr(\boldsymbol{\nabla}\mathbf{F}), \tag{4.28}$$

or in components

$$\tau^a(\varphi) = F^a{}_{A|B}G^{AB}. \tag{4.29}$$

It can be shown that assuming that $\mathbf{U}$ vanishes on the boundary of $\mathcal{B}$ [9]

$$\frac{d}{ds}E(\varphi_{t,s})\Big|_{s=0} = \int_{\mathcal{B}} \langle \mathbf{F}^{\mathsf{T}}, \boldsymbol{\nabla}\mathbf{U}\rangle dV = -\int_{\mathcal{B}} \langle\!\langle \boldsymbol{\tau}(\varphi), \mathbf{U}\rangle\!\rangle dV, \tag{4.30}$$

where the first integrand on the right-hand side in components reads $\left(F^{\mathsf{T}}\right)^B{}_b U^b{}_{|B}$. A C^∞ map $\varphi_t \in C^\infty(\mathcal{B}, \mathcal{S})$ is called a ***harmonic map*** if its tension field $\boldsymbol{\tau}(\varphi)$ vanishes identically. In other words, φ_t is a harmonic map if for any variation $\varphi_{t,s}$

$$\frac{d}{ds}E(\varphi_{t,s})\Big|_{s=0} = 0. \tag{4.31}$$

In elasticity, this corresponds to an equilibrium configuration in the absence of body and inertial forces.

The right Cauchy–Green strain tensor for the perturbed motion $\varphi_{t,s}$ is defined as

$$C_{AB}(s) = F^a{}_A(s)F^b{}_B(s)g_{ab}(s). \tag{4.32}$$

Note that $\mathbf{C}(s)$ lies in the same linear space for all $s \in I$, and the first variation of $\mathbf{C}$ can be calculated as,

$$\frac{d}{ds}C_{AB}(s) = \boldsymbol{\nabla}_{\frac{\partial}{\partial s}}F^a{}_A(s)F^b{}_B(s)g_{ab}(s) + F^a{}_A(s)\boldsymbol{\nabla}_{\frac{\partial}{\partial s}}F^b{}_B(s)g_{ab}(s). \tag{4.33}$$

Therefore

$$\frac{d}{ds}\Big|_{s=0}C_{AB}(s) = \overset{\circ}{F}{}^b{}_B\, g_{ab}\, U^a{}_{|A} + \overset{\circ}{F}{}^a{}_A\, g_{ab}\, U^b{}_{|B}, \tag{4.34}$$

which is identical to (4.12).

We know that internal energy density has the following form [7]

$$E = E(\mathbf{X}, \mathbf{C}). \tag{4.35}$$

Thus

$$E(s) = E(\mathbf{X}, \mathbf{C}(s)). \tag{4.36}$$

We also know that

$$E(s) = E(0) + \left[\frac{d}{ds}\Big|_{s=0}E(s)\right]s + o(s), \tag{4.37}$$

where

$$\frac{d}{ds}\Big|_{s=0} E(s) = \frac{\partial \overset{\circ}{E}}{\partial \overset{\circ}{\mathbf{C}}} \cdot \left(\boldsymbol{\nabla}\mathbf{U} \cdot \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} + \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} \cdot \boldsymbol{\nabla}\mathbf{U} \right)$$
$$= \frac{1}{2} \overset{\circ}{\mathbf{S}} \cdot \left(\boldsymbol{\nabla}\mathbf{U} \cdot \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} + \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \quad (4.38)$$

is the linearization of E and where $\overset{\circ}{\mathbf{S}}$ is the second Piola–Kirchhoff stress. Using such ideas one can linearize all the governing equations of nonlinear elasticity about a given reference motion $\overset{\circ}{\varphi}_t$. In this work, we are interested in obtaining the governing equations of linearized elasticity covariantly using energy balance and its symmetry properties.

5. A covariant formulation of linearized elasticity

As in the invariance postulate of the Euclidean case, there are two possibilities for investigating the relation of the covariance postulate and linearization in geometric elasticity: (i) Postulating the invariance of energy balance under spatial diffeomorphisms of the ambient space and then linearizing the energy balance about a given motion (Linearization of Covariance), and (ii) First writing energy balance for a *perturbed* motion and then postulating its invariance under spatial diffeomorphisms of the ambient space (Covariance of Linearized Energy Balance). These two approaches are equivalent to each other for reasons we will mention below, however, we will derive both cases in detail.

5.1. Linearization of covariant energy balance

For the sake of simplicity, we use the material energy balance. Let us first define

$$E(\mathbf{X}, \mathbf{g}) = e(\varphi_t(\mathbf{X}), \mathbf{g}(\varphi_t(\mathbf{X}))). \quad (5.1)$$

We know that under a spatial diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$ [12]

$$E'(\mathbf{X}, \mathbf{g}) = E(\mathbf{X}, \xi_t^* \mathbf{g}), \quad (5.2)$$

where $\xi^* \mathbf{g}$ is the pull-back of $\mathbf{g}$ by ξ_t. Let us assume that $\xi_{t_0} = id$, the identity mapping. At time $t = t_0$

$$\dot{\overline{E'}} = \dot{E} + \frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g}, \quad (5.3)$$

where $\mathbf{W} = \mathbf{w} \circ \varphi_t$[8] and $\mathbf{w} = \frac{\partial}{\partial t}\xi_t$. Material balance of energy for the motions $\mathring{\varphi}_t$ and $\mathring{\varphi}'_t = \xi_t \circ \mathring{\varphi}_t$ respectively, reads (at time $t = t_0$)

$$\int_{\mathcal{U}} \rho_0 \left(\dot{\mathring{E}} + \left\langle\!\left\langle \mathring{\mathbf{V}}, \mathring{\mathbf{A}} \right\rangle\!\right\rangle \right) dV = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\left\langle \mathring{\mathbf{B}}, \mathring{\mathbf{V}} \right\rangle\!\right\rangle + \mathring{R} \right) dV + \int_{\partial\mathcal{U}} \left(\left\langle\!\left\langle \mathring{\mathbf{T}}, \mathring{\mathbf{V}} \right\rangle\!\right\rangle + \mathring{H} \right) dA, \tag{5.4}$$

and[9]

$$\begin{aligned} \int_{\mathcal{U}} \rho_0 \left(\dot{\mathring{E}} + \frac{\partial \mathring{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \left\langle\!\left\langle \mathring{\mathbf{A}}, \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\right\rangle \right) dV &= \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\left\langle \mathring{\mathbf{B}}, \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\right\rangle + \mathring{R} \right) dV \\ &\quad + \int_{\partial\mathcal{U}} \left(\left\langle\!\left\langle \mathring{\mathbf{T}}, \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\right\rangle + \mathring{H} \right) dA. \end{aligned} \tag{5.5}$$

Subtracting (5.4) from (5.5), one obtains

$$\int_{\mathcal{U}} \rho_0 \left(\frac{\partial \mathring{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \left\langle\!\left\langle \mathring{\mathbf{A}}, \mathbf{W} \right\rangle\!\right\rangle \right) dV = \int_{\mathcal{U}} \left\langle\!\left\langle \rho_0 \, \mathring{\mathbf{B}}, \mathbf{W} \right\rangle\!\right\rangle + \int_{\partial\mathcal{U}} \left\langle\!\left\langle \mathring{\mathbf{T}}, \mathbf{W} \right\rangle\!\right\rangle dA. \tag{5.6}$$

Similarly, for the motion $\varphi_{t,s}$

$$\int_{\mathcal{U}} \rho_0 \left(\frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}_s} \mathbf{g} + \langle\!\langle \mathbf{A}, \mathbf{W}_s \rangle\!\rangle \right) dV = \int_{\mathcal{U}} \langle\!\langle \rho_0 \mathbf{B}, \mathbf{W}_s \rangle\!\rangle + \int_{\partial\mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{W}_s \rangle\!\rangle \, dA, \tag{5.7}$$

where $\mathbf{W}_s = \mathbf{w} \circ \varphi_{t,s}$. Therefore, for an arbitrary vector field $\mathbf{w}$ we have

$$\begin{aligned} &\int_{\mathcal{U}} \rho_0 \left[\frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}_s} \mathbf{g} - \frac{\partial \mathring{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \langle\!\langle \mathbf{A}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\left\langle \mathring{\mathbf{A}}, \mathbf{W} \right\rangle\!\right\rangle \right] dV \\ &\quad = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\left\langle \mathring{\mathbf{B}}, \mathbf{W} \right\rangle\!\right\rangle \right) + \int_{\partial\mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\left\langle \mathring{\mathbf{T}}, \mathbf{W} \right\rangle\!\right\rangle \right) dA. \end{aligned} \tag{5.8}$$

Note that [12]

$$\int_{\partial\mathcal{U}} \left\langle\!\left\langle \mathring{\mathbf{T}}, \mathbf{W} \right\rangle\!\right\rangle dA = \int_{\mathcal{U}} \left(\left\langle\!\left\langle \mathrm{Div}\, \mathring{\mathbf{P}}, \mathbf{W} \right\rangle\!\right\rangle + \mathring{\boldsymbol{\tau}} : \boldsymbol{\omega} + \mathring{\boldsymbol{\tau}} : \mathbf{k} \right) dV, \tag{5.9}$$

$$\int_{\partial\mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{W}_s \rangle\!\rangle \, dA = \int_{\mathcal{U}} \left(\langle\!\langle \mathrm{Div}\, \mathbf{P}, \mathbf{W}_s \rangle\!\rangle + \boldsymbol{\tau} : \boldsymbol{\omega}_s + \boldsymbol{\tau} : \mathbf{k}_s \right) dV, \tag{5.10}$$

where, $\mathring{\boldsymbol{\tau}} = \mathring{\mathbf{P}}\mathring{\mathbf{F}}$ and $\boldsymbol{\tau} = \mathbf{PF}$ are Kirchhoff stresses and $\boldsymbol{\omega}$ and $\mathbf{k}$ have the coordinate representations $k_{ab} = \frac{1}{2}\left(W_{a|b} + W_{b|a}\right)$ and $\omega_{ab} = \frac{1}{2}\left(W_{a|b} - W_{b|a}\right)$ with similar

[8] Not to be confused with the $\mathbf{W}$ in §2.

[9] Note that we assume that $\mathring{\mathbf{A}}' - \mathring{\mathbf{B}}' = \xi_{t*}(\mathring{\mathbf{A}} - \mathring{\mathbf{B}})$ and hence at $t = t_0$, $\mathring{\mathbf{A}}' - \mathring{\mathbf{B}}' = \mathring{\mathbf{A}} - \mathring{\mathbf{B}}$.

representations for $\boldsymbol{\omega}_s$ and $\mathbf{k}_s$. Note also that

$$\mathcal{L}\left(\langle\!\langle \mathbf{A}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \mathbf{W} \right\rangle\!\!\right\rangle ; \overset{\circ}{\varphi}_t\right) = \left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle, \tag{5.11}$$

$$\mathcal{L}\left(\langle\!\langle \mathbf{B}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \mathbf{W} \right\rangle\!\!\right\rangle ; \overset{\circ}{\varphi}_t\right) = \langle\!\langle \mathcal{L}(\mathbf{B}), \mathbf{W} \rangle\!\rangle + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle, \tag{5.12}$$

$$\mathcal{L}\left(\boldsymbol{\tau} : (\boldsymbol{\omega}_s + \mathbf{k}_s) - \overset{\circ}{\boldsymbol{\tau}} : (\boldsymbol{\omega} + \mathbf{k}); \overset{\circ}{\varphi}_t\right) = \left[\overset{\circ}{\mathbf{P}}\ \boldsymbol{\nabla}\mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U}\right) \overset{\circ}{\mathbf{F}}\right] : (\boldsymbol{\omega} + \mathbf{k}) + \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\nabla}_{\mathbf{U}} \boldsymbol{\nabla}\mathbf{W}, \tag{5.13}$$

$$\mathcal{L}\left(\langle\!\langle \operatorname{Div} \mathbf{P}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \operatorname{Div} \overset{\circ}{\mathbf{P}}, \mathbf{W} \right\rangle\!\!\right\rangle ; \overset{\circ}{\varphi}_t\right) = \left\langle\!\!\left\langle \operatorname{Div}\left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U}\right), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \operatorname{Div} \overset{\circ}{\mathbf{P}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle, \tag{5.14}$$

where

$$\mathcal{L}(\mathbf{B}) = \boldsymbol{\nabla}_{\mathbf{U}}\ \overset{\circ}{\mathbf{B}} + \frac{\partial}{\partial s}\Big|_{s=0} \mathbf{B}(s, \varphi_{t,s}). \tag{5.15}$$

And

$$\mathcal{L}(\mathbf{W}) = \boldsymbol{\nabla}_{\frac{\partial}{\partial s}} \mathbf{W}_s\Big|_{s=0} = \boldsymbol{\nabla}\mathbf{W} \cdot \mathbf{U} = \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W}. \tag{5.16}$$

For the derivative of the internal energy we have the following linearization.

$$\mathcal{L}\left(\frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}_s} \mathbf{g} - \frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g}; \overset{\circ}{\varphi}_t\right) = \boldsymbol{\nabla}_{\mathbf{U}} \left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right) : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} : \boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{W}} \mathbf{g}). \tag{5.17}$$

Thus, (5.8) is now simplified to read

$$\begin{aligned} \int_{\mathcal{U}} \rho_0 \Bigg[\boldsymbol{\nabla}_{\mathbf{U}} \left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right) &: \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} : \boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{W}} \mathbf{g}) + \left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \mathbf{W} \right\rangle\!\!\right\rangle \\ &+ \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle \Bigg] dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathcal{L}(\mathbf{B}), \mathbf{W} \rangle\!\rangle + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle\right) \\ &+ \int_{\mathcal{U}} \Bigg\{ \left\langle\!\!\left\langle \operatorname{Div}\left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U}\right), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \operatorname{Div} \overset{\circ}{\mathbf{P}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle \\ &+ \left[\overset{\circ}{\mathbf{P}}\ \boldsymbol{\nabla}\mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U}\right) \overset{\circ}{\mathbf{F}}\right] : (\boldsymbol{\omega} + \mathbf{k}) + \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\nabla}_{\mathbf{U}} (\boldsymbol{\omega} + \mathbf{k}) \Bigg\} dV. \end{aligned} \tag{5.18}$$

Using the governing equations of the reference motion, this is simplified to read

$$\int_{\mathcal{U}} \rho_0 \left[\nabla_{\mathbf{U}} \left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} \right) : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \nabla_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle \right] dV$$

$$= \int_{\mathcal{U}} \rho_0 \left\langle\!\left\langle \mathcal{L}(\mathbf{B}), \mathbf{W} \right\rangle\!\right\rangle + \int_{\mathcal{U}} \Bigg\{ \left\langle\!\!\left\langle \mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \nabla \mathbf{U} \right), \mathbf{W} \right\rangle\!\!\right\rangle$$

$$+ \left[\overset{\circ}{\mathbf{P}} \, \nabla \mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \nabla \mathbf{U} \right) \overset{\circ}{\mathbf{F}} \right] : (\boldsymbol{\omega} + \mathbf{k}) \Bigg\} dV. \tag{5.19}$$

Arbitrariness of $\mathbf{W}$ and $\mathcal{U}$ implies that

$$\mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \nabla \mathbf{U} \right) + \rho_0 \mathcal{L}(\mathbf{B}) = \rho_0 \left(\ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}) \right), \tag{5.20}$$

$$\nabla_{\mathbf{U}} \overset{\circ}{\boldsymbol{\tau}} = 2\rho_0 \nabla_{\mathbf{U}} \left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} \right) = \overset{\circ}{\mathbf{P}} \, \nabla \mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \nabla \mathbf{U} \right) \overset{\circ}{\mathbf{F}}, \tag{5.21}$$

$$\left[\overset{\circ}{\mathbf{P}} \, \nabla \mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \nabla \mathbf{U} \right) \overset{\circ}{\mathbf{F}} \right]^{\mathsf{T}} = \overset{\circ}{\mathbf{P}} \, \nabla \mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \nabla \mathbf{U} \right) \overset{\circ}{\mathbf{F}}, \tag{5.22}$$

which are the governing equations of linearized elasticity. Therefore, we have proven the following proposition.

Proposition 5.1. *Linearization of covariant energy balance is equivalent to linearization of all the field equations of elasticity.*

5.2. Covariance of linearized energy balance

Next, let us first linearize energy balance about a reference motion and then postulate its invariance under arbitrary spatial diffeomorphisms. Subtracting the balance of energy for the motion $\overset{\circ}{\varphi}_t$ from that of $\varphi_{t,s}$ yields

$$\int_{\mathcal{U}} \rho_0 \left(\dot{E} - \dot{\overset{\circ}{E}} + \left\langle\!\left\langle \mathbf{A}, \mathbf{V} \right\rangle\!\right\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{V}} \right\rangle\!\!\right\rangle \right) dV = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\left\langle \mathbf{B}, \mathbf{V} \right\rangle\!\right\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{V}} \right\rangle\!\!\right\rangle \right) dV$$

$$+ \int_{\mathcal{U}} \rho_0 \left(R - \overset{\circ}{R} \right) dV + \int_{\partial\mathcal{U}} \left(\left\langle\!\left\langle \mathrm{Div}\, \mathbf{P}, \mathbf{V} \right\rangle\!\right\rangle - \left\langle\!\!\left\langle \mathrm{Div}\, \overset{\circ}{\mathbf{P}}, \overset{\circ}{\mathbf{V}} \right\rangle\!\!\right\rangle \right) dV$$

$$+ \int_{\mathcal{U}} \left(\boldsymbol{\tau} : \nabla \mathbf{V} - \overset{\circ}{\boldsymbol{\tau}} : \nabla \overset{\circ}{\mathbf{V}} \right) dV + \int_{\partial\mathcal{U}} (H - \overset{\circ}{H}) dA = 0. \tag{5.23}$$

Now let us linearize the integrands. Body force power has the following linearization

$$\begin{aligned}\mathcal{L}\left(\langle\!\langle \mathbf{B},\mathbf{V}\rangle\!\rangle - \left\langle\!\!\left\langle \mathring{\mathbf{B}},\mathring{\mathbf{V}}\right\rangle\!\!\right\rangle;\mathring{\varphi}_t\right) &= \frac{d}{ds}\Big|_{s=0}\langle\!\langle \boldsymbol{\alpha}_s\cdot\mathbf{B},\boldsymbol{\alpha}_s\cdot\mathbf{V}\rangle\!\rangle \\ &= \left\langle\!\!\left\langle \frac{d}{ds}\Big|_{s=0}\boldsymbol{\alpha}_s\cdot\mathbf{B},\mathbf{V}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathring{\mathbf{B}},\frac{d}{ds}\Big|_{s=0}\boldsymbol{\alpha}_s\cdot\mathbf{V}\right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \mathcal{L}(\mathbf{B}),\mathring{\mathbf{V}}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathring{\mathbf{B}},\dot{\mathbf{U}}\right\rangle\!\!\right\rangle. \end{aligned} \tag{5.24}$$

Similarly, inertial force power has the following linearization

$$\begin{aligned}\mathcal{L}\left(\langle\!\langle \mathbf{A},\mathbf{V}\rangle\!\rangle - \left\langle\!\!\left\langle \mathring{\mathbf{A}},\mathring{\mathbf{V}}\right\rangle\!\!\right\rangle;\mathring{\varphi}_t\right) &= \frac{d}{ds}\Big|_{s=0}\langle\!\langle \boldsymbol{\alpha}_s\cdot\mathbf{A},\boldsymbol{\alpha}_s\cdot\mathbf{V}\rangle\!\rangle \\ &= \left\langle\!\!\left\langle \frac{d}{ds}\Big|_{s=0}\boldsymbol{\alpha}_s\cdot\mathbf{A},\mathbf{V}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathbf{A},\frac{d}{ds}\Big|_{s=0}\boldsymbol{\alpha}_s\cdot\mathbf{V}\right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}},\mathbf{U},\mathring{\mathbf{V}}),\mathring{\mathbf{V}}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathring{\mathbf{A}},\dot{\mathbf{U}}\right\rangle\!\!\right\rangle, \end{aligned} \tag{5.25}$$

where $\boldsymbol{\mathcal{R}}$ is the curvature tensor of the ambient space manifold. Traction power is linearized as follows

$$\mathcal{L}\left(\boldsymbol{\tau}:\boldsymbol{\nabla}\mathbf{V}-\mathring{\boldsymbol{\tau}}:\boldsymbol{\nabla}\mathring{\mathbf{V}};\mathring{\varphi}_t\right) = \left[\mathring{\mathbf{P}}\ \boldsymbol{\nabla}\mathbf{U}+\left(\mathring{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)\mathring{\mathbf{F}}\right]:\boldsymbol{\nabla}\mathring{\mathbf{V}}+\mathring{\boldsymbol{\tau}}:\boldsymbol{\nabla}\dot{\mathbf{U}}, \tag{5.26}$$

$$\mathcal{L}\left(\langle\!\langle \mathrm{Div}\,\mathbf{P},\mathbf{V}\rangle\!\rangle - \left\langle\!\!\left\langle \mathrm{Div}\,\mathring{\mathbf{P}},\mathring{\mathbf{V}}\right\rangle\!\!\right\rangle;\mathring{\varphi}_t\right) = \left\langle\!\!\left\langle \mathrm{Div}\left(\mathring{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right),\mathring{\mathbf{V}}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathrm{Div}\,\mathring{\mathbf{P}},\dot{\mathbf{U}}\right\rangle\!\!\right\rangle. \tag{5.27}$$

Internal energy part of energy balance is linearized as follows. Note that

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}}\rho_0 E dV = \int_{\mathcal{U}}\rho_0\frac{\partial E}{\partial\mathbf{C}}:\varphi^*(\mathbf{L}_{\mathbf{v}}\mathbf{g})dV &= \int_{\mathcal{U}}\rho_0\varphi^*\left(\frac{\partial E}{\partial\mathbf{g}}\right):\varphi^*(\mathbf{L}_{\mathbf{v}}\mathbf{g})dV \\ &= \int_{\mathcal{U}}\rho_0\frac{\partial E}{\partial\mathbf{g}}:\mathbf{L}_{\mathbf{v}}\mathbf{g}\ dV. \end{aligned} \tag{5.28}$$

Thus[10]

$$\mathcal{L}\left(\frac{d}{dt}\int_{\mathcal{U}}\rho_0 E dV;\mathring{\varphi}_t\right) = \int_{\mathcal{U}}\left[\boldsymbol{\nabla}_{\mathbf{U}}\left(\rho_0\frac{\partial E}{\partial\mathbf{g}}\right):\mathfrak{L}_{\mathbf{v}}\mathbf{g}+\rho_0\frac{\partial E}{\partial\mathbf{g}}:\boldsymbol{\nabla}_{\mathbf{U}}(\mathfrak{L}_{\mathbf{v}}\mathbf{g})\right]dV. \tag{5.29}$$

[10] Note that because $\mathbf{g}$ is time independent $\mathbf{L}_{\mathbf{v}}\mathbf{g}=\mathfrak{L}_{\mathbf{v}}\mathbf{g}$.

Therefore, material balance of energy for the perturbed motion reads

$$\begin{aligned}
&\int_{\mathcal{U}} \left[\boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial E}{\partial \mathbf{g}} \right) : \mathfrak{L}_{\mathbf{v}} \mathbf{g} + \rho_0 \frac{\partial E}{\partial \mathbf{g}} : \boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{v}} \mathbf{g}) \right] dV \\
&\quad + \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \mathcal{L}(\mathbf{B}), \mathring{\mathbf{V}} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathring{\mathbf{A}} - \mathring{\mathbf{B}}, \dot{\mathbf{U}} \right\rangle\!\!\right\rangle \right) dV \\
&= \int_{\mathcal{U}} \rho_0 \ \mathbf{d}R \cdot \mathbf{U} \ dV + \int_{\partial \mathcal{U}} \mathbf{d}H \cdot \mathbf{U} dA \\
&\quad + \int_{\mathcal{U}} \left\{ \left[\mathring{\mathbf{P}} \ \boldsymbol{\nabla} \mathbf{U} + \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla} \mathbf{U} \right) \mathring{\mathbf{F}} \right] : \boldsymbol{\nabla} \ \mathring{\mathbf{V}} + \mathring{\boldsymbol{\tau}} : \boldsymbol{\nabla} \dot{\mathbf{U}} \right\} dV \\
&\quad + \int_{\mathcal{U}} \left(\left\langle\!\!\left\langle \operatorname{Div} \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla} \mathbf{U} \right), \mathring{\mathbf{V}} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \operatorname{Div} \mathring{\mathbf{P}}, \dot{\mathbf{U}} \right\rangle\!\!\right\rangle \right) dV. \qquad (5.30)
\end{aligned}$$

Under a spatial diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$, we have

$$\mathbf{U}' = \xi_{t*} \mathbf{U}. \qquad (5.31)$$

Let us now find the transformed linearized velocity. Note that for the variations $\varphi_{t,s}$, velocity is defined as

$$\mathbf{V}_s = \frac{\partial}{\partial t} \varphi_{t,s}. \qquad (5.32)$$

Thus

$$\mathcal{L}(\mathbf{V}) = \boldsymbol{\nabla}_{\frac{\partial}{\partial s}} \mathbf{V}_s \big|_{s=0} = \dot{\mathbf{U}}. \qquad (5.33)$$

For the motion $\varphi'_{t,s} = \xi_t \circ \varphi_{t,s}$, velocity is defined as

$$\mathbf{V}'_s = \frac{\partial}{\partial t} \varphi'_{t,s} = \xi_{t*} \mathbf{V}_s + \mathbf{w}_t \circ \varphi_{t,s}. \qquad (5.34)$$

Note that

$$\mathcal{L}(\mathbf{V}') = \boldsymbol{\nabla}_{\xi_{t*} \frac{\partial}{\partial s}} \mathbf{V}'_s \big|_{s=0} = \xi_{t*} \left(\boldsymbol{\nabla}_{\frac{\partial}{\partial s}} \mathbf{V}_s \right)_{s=0} + \xi_{t*} \left[\boldsymbol{\nabla}_{\frac{\partial}{\partial s}} (\xi_t^* \mathbf{w} \circ \varphi_{t,s}) \right]_{s=0}. \qquad (5.35)$$

Thus

$$\mathcal{L}(\mathbf{V}') = \xi_{t*} \dot{\mathbf{U}} + \xi_{t*} \left[\boldsymbol{\nabla}_{\mathbf{U}} (\xi_t^* \mathbf{w} \circ \varphi_t) \right]. \qquad (5.36)$$

Hence at $t = t_0$

$$\dot{\mathbf{U}}' = \dot{\mathbf{U}} + \mathbf{Z}, \qquad (5.37)$$

where $\mathbf{Z} = \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} = \boldsymbol{\nabla} \mathbf{W} \cdot \mathbf{U}$. We assume that [7]

$$\begin{aligned}
\mathring{\mathbf{A}}' - \mathring{\mathbf{B}}' &= \xi_{t*} (\mathring{\mathbf{A}} - \mathring{\mathbf{B}}), \qquad (5.38) \\
\mathbf{A}'_s - \mathbf{B}'_s &= \xi_{t*} (\mathbf{A}_s - \mathbf{B}_s) \qquad \forall\ s \in\ I. \qquad (5.39)
\end{aligned}$$

(5.38) implies that at $t = t_0$, $\mathring{\mathbf{A}}' - \mathring{\mathbf{B}}' = \mathring{\mathbf{A}} - \mathring{\mathbf{B}}$. Because (5.39) holds for every $s \in I$, it holds, in particular, for its linearization, i.e. at $t = t_0$

$$\ddot{\mathbf{U}}' + \boldsymbol{\mathcal{R}}' (\mathring{\mathbf{V}}', \mathbf{U}', \mathring{\mathbf{V}}') - \mathcal{L}(\mathbf{B}') = \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \mathcal{L}(\mathbf{B}). \qquad (5.40)$$

Now under this spatial reframing, the perturbed energy balance (5.30) at time $t = t_0$ reads

$$\begin{aligned}
&\int_{\mathcal{U}} \left[\boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial E}{\partial \mathbf{g}} \right) : (\mathfrak{L}_{\mathbf{v}}\mathbf{g} + \mathfrak{L}_{\mathbf{w}}\mathbf{g}) + \rho_0 \frac{\partial E}{\partial \mathbf{g}} : [\boldsymbol{\nabla}_{\mathbf{U}}(\mathfrak{L}_{\mathbf{v}}\mathbf{g}) + \boldsymbol{\nabla}_{\mathbf{U}}(\mathfrak{L}_{\mathbf{w}}\mathbf{g})] \right] dV \\
&\qquad + \int_{\mathcal{U}} \rho_0 \Bigg(\left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \mathcal{L}(\mathbf{B}), \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\!\right\rangle \\
&\qquad + \left\langle\!\!\left\langle \mathring{\mathbf{A}} - \mathring{\mathbf{B}}, \dot{\mathbf{U}} + \mathbf{Z} \right\rangle\!\!\right\rangle \Bigg) dV \\
&= \int_{\mathcal{U}} \left\{ \left[\mathring{\mathbf{P}}\ \boldsymbol{\nabla}\mathbf{U} + \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \mathring{\mathbf{F}} \right] : (\boldsymbol{\nabla}\mathring{\mathbf{V}} + \boldsymbol{\nabla}\mathbf{W}) + \mathring{\boldsymbol{\tau}} : (\boldsymbol{\nabla}\dot{\mathbf{U}} + \boldsymbol{\nabla}\mathbf{Z}) \right\} dV \\
&\quad + \int_{\mathcal{U}} \left(\left\langle\!\!\left\langle \operatorname{Div}\left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \operatorname{Div} \mathring{\mathbf{P}}, \dot{\mathbf{U}} + \mathbf{Z} \right\rangle\!\!\right\rangle \right) dV.
\end{aligned} \tag{5.41}$$

Subtracting (5.30) from (5.41) yields

$$\begin{aligned}
&\int_{\mathcal{U}} \left[\boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial \mathring{E}}{\partial \mathbf{g}} \right) : \mathfrak{L}_{\mathbf{w}}\mathbf{g} + \rho_0 \frac{\partial \mathring{E}}{\partial \mathbf{g}} : \boldsymbol{\nabla}_{\mathbf{U}}(\mathfrak{L}_{\mathbf{w}}\mathbf{g}) \right] dV \\
&\qquad + \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \mathcal{L}(\mathbf{B}), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathring{\mathbf{A}} - \mathring{\mathbf{B}}, \mathbf{Z} \right\rangle\!\!\right\rangle \right) dV \\
&\quad = \int_{\mathcal{U}} \left\{ \left[\mathring{\mathbf{P}}\ \boldsymbol{\nabla}\mathbf{U} + \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \mathring{\mathbf{F}} \right] : \boldsymbol{\nabla}\mathbf{W} + \mathring{\boldsymbol{\tau}} : \boldsymbol{\nabla}\mathbf{Z} \right\} dV \\
&\qquad + \int_{\mathcal{U}} \left(\left\langle\!\!\left\langle \operatorname{Div}\left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \operatorname{Div} \mathring{\mathbf{P}}, \mathbf{Z} \right\rangle\!\!\right\rangle \right) dV.
\end{aligned} \tag{5.42}$$

Using the governing equations of the motion $\mathring{\varphi}_t$, i.e.

$$\operatorname{Div} \mathring{\mathbf{P}} + \rho_0 \mathring{\mathbf{B}} = \rho_0 \mathring{\mathbf{A}}, \tag{5.43}$$

$$\mathring{\boldsymbol{\tau}} = 2\rho_0 \frac{\partial \mathring{E}}{\partial \mathbf{g}}, \tag{5.44}$$

$$\mathring{\boldsymbol{\tau}} = \mathring{\boldsymbol{\tau}}^{\mathsf{T}}, \tag{5.45}$$

(5.42) is simplified to read

$$\begin{aligned}
&\int_{\mathcal{U}} \boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial \mathring{E}}{\partial \mathbf{g}} \right) : \mathfrak{L}_{\mathbf{w}}\mathbf{g}\ dV + \int_{\mathcal{U}} \rho_0 \left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \mathcal{L}(\mathbf{B}), \mathbf{W} \right\rangle\!\!\right\rangle dV \\
&= \int_{\mathcal{U}} \left[\mathring{\mathbf{P}}\ \boldsymbol{\nabla}\mathbf{U} + \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \mathring{\mathbf{F}} \right] : \boldsymbol{\nabla}\mathbf{W} dV + \int_{\mathcal{U}} \left\langle\!\!\left\langle \operatorname{Div}\left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \mathbf{W} \right\rangle\!\!\right\rangle dV.
\end{aligned} \tag{5.46}$$

Now arbitrariness of $\mathbf{W}$ and $\mathcal{U}$ implies that

$$\operatorname{Div}\left(\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)+\rho_0\mathcal{L}(\mathbf{B})=\rho_0\left(\ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}},\mathbf{U},\overset{\circ}{\mathbf{V}})\right), \tag{5.47}$$

$$\boldsymbol{\nabla}_{\mathbf{U}}\overset{\circ}{\boldsymbol{\tau}}=2\rho_0\boldsymbol{\nabla}_{\mathbf{U}}\left(\frac{\partial\overset{\circ}{E}}{\partial\mathbf{g}}\right)=\overset{\circ}{\mathbf{P}}\,\boldsymbol{\nabla}\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)\overset{\circ}{\mathbf{F}}, \tag{5.48}$$

$$\left[\overset{\circ}{\mathbf{P}}\,\boldsymbol{\nabla}\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\right]^{\mathsf{T}}=\overset{\circ}{\mathbf{P}}\,\boldsymbol{\nabla}\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\,. \tag{5.49}$$

Thus, we have proved the following proposition.

Proposition 5.2. *Covariance of the linearized energy balance is equivalent to linearization of all the field equations of elasticity.*

In other words, one can covariantly obtain all the governing equations of linearized elasticity by postulating covariance of the linearized energy balance.

Remark. As we mentioned previously, the linearization of covariant energy balance and covariance of linearized energy balance give the same linearized governing equations. This is due to the fact that the operations of linearization and transformation under a diffeomorphism commute with each other. For the case of velocity, for example, this can be represented as the commutative property of the following diagram:

$$\begin{array}{ccc}
\mathbf{V} & \xrightarrow{\xi_t} & \mathbf{V}'=\xi_{t*}\mathbf{V}+\mathbf{W}_t \\
{\scriptstyle\mathcal{L}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{L}} \\
\dot{\mathbf{U}} & \xrightarrow[\xi_t]{} & \dot{\mathbf{U}}'=\xi_{t*}\dot{\mathbf{U}}+\xi_{t*}\left[\boldsymbol{\nabla}_{\mathbf{U}}(\xi_t^*\mathbf{w})\right]
\end{array}$$

6. Conclusions

The main motivation for the present work is to understand the connection between governing equations of linearized elasticity and energy balance and its invariance (or covariance). We first looked at the case where the ambient space is Euclidean. Having a reference motion $\overset{\circ}{\varphi}_t$, we quadratized the energy balance about $\overset{\circ}{\varphi}_t$. This leads to two identities: linearized energy balance and quadratized energy balance. We showed that postulating invariance of the linearized energy balance under isometries of the ambient space will give all the governing equations of linearized elasticity. Classical linear elasticity corresponds to choosing a stress-free reference motion. For such reference motions all the terms in the linearized energy balance are identically zero and the quadratized energy balance is identical to what is called "energy balance" or power theorem in classical linear elasticity.

We then studied the case where the ambient space is a Riemannian manifold

$(\mathcal{S}, \mathbf{g})$. We first reviewed some previous ideas in geometric linearization of nonlinear elasticity and presented some new results. We also showed the close connection between these ideas and those of geometric calculus of variations. We considered two notions of covariance: (i) linearization of covariant energy balance and (ii) covariance of the linearized energy balance. We showed that postulating either (i) or (ii) will give all the governing equations of linearized elasticity. Of course, (ii) is more interesting. In other words, if one postulates invariance of the linearized energy balance about a reference motion $\overset{\circ}{\varphi}_t$ under spatial diffeomorphisms of $\mathcal{S}$ (the same diffeomorphism acts on $\overset{\circ}{\varphi}_t$ and its variations $\varphi_{t,s}$), one obtains all the governing equations of linearized elasticity. In this sense, linearized elasticity can be covariantly derived.

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(Received: September 11, 2007)

Published Online First: March 26, 2008