Z. angew. Math. Phys. 59 (2008) 1002–1028
0044-2275/08/061002-27
DOI 10.1007/s00033-007-7008-8
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Zeitschrift für angewandte Mathematik und Physik ZAMP

The initial value problem for the cubic nonlinear Klein–Gordon equation

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Abstract. We study the initial value problem for the cubic nonlinear Klein–Gordon equation

$$\begin{cases} u_{tt} + u - u_{xx} = \mu u^3, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0) = u_0, u_t(0) = u_1, x \in \mathbf{R}, \end{cases} \tag{0.1}$$

where $\mu \in \mathbf{R}$ and the initial data are real-valued functions. We obtain a sharp asymptotic behavior of small solutions without the condition of a compact support on the initial data which was assumed in the previous works.

Mathematics Subject Classification (2000). 35Q53, 35Q55.

Keywords. Asymptotics of solutions, Cubic nonlinear Klein–Gordon equation, Initial value problem, The inverse wave modified operator

1. Introduction

We study the initial value problem for the cubic nonlinear Klein–Gordon equation

$$\begin{cases} v_{tt} + v - v_{xx} = \mu v^3, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ v(0) = v_0, v_t(0) = v_1, x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $\mu \in \mathbf{R}$ and the initial data v_0, v_1 are real-valued.

Our purpose is to obtain the large time asymptotic profile of small solutions to the Cauchy problem (1.1) without the restriction of a compact support on the initial data which was assumed in the previous works [3], [15]. Their method is based on the transformation of the equation by virtue of the hyperbolic polar coordinates following [14]. Then the new equation have two nonlinear terms, one of them is the cubic nonlinearity with a critical decay rate and the other one having a better time decay rate however leads to a derivative loss. The application of the hyperbolic polar coordinates implies restricting to the interior of the light cone and so requiring the compactness condition. When $\mu < 0$, the global existence of solutions to (1.1) can be easily proved in the energy space, which is however insufficient for determining the large time asymptotic behavior of solutions. The sharp $\mathbf{L}^{\infty}$-time decay estimates of solutions and non existence of the usual scattering states for equation (1.1) were shown in [8] under the conditions that the initial data are regular and have a compact support.

The initial value problem for the nonlinear Klein–Gordon equation with cubic nonlinearities depending on $v, v_t, v_x, v_{xx}, v_{tx}$ and having a suitable non resonance structure was studied by [13], [18], [19], where small solutions were found in the neighborhood of the free solutions if the initial data are small, regular and decay fast at infinity. Hence the cubic nonlinearities are not necessarily critical, however the resonant nonlinear term v^3 was excluded in these works. In paper [13], the nonlinearities were classified into two types, one of them can be treated by the method of normal forms [24] and the other reveals an additional time decay rate via the operator $x\partial_t + t\partial_x$ [14]. This method was extended to a system of nonlinear Klein–Gordon equations by [26]. Some sufficient conditions on cubic nonlinearities were given in [3], which allow us to prove global existence and to find sharp asymptotics of small solutions to the Cauchy problem (1.1) with small and regular initial data having a compact support. The case of the cubic nonlinearity v^3 was included in [3] as an example, moreover it was proved that the asymptotic profile differs from that of the linear Klein–Gordon equation. Global existence and $\mathbf{L}^{\infty}$-time decay estimates of small solutions to the Klein–Gordon equation with cubic nonlinearity $|v|^2 v$ were obtained in paper [27] if the initial data are complex, smooth and have a compact support. However the large time asymptotics was not found for this case. There are few results concerning the final state problem for the nonlinear Klein–Gordon equation. The modified wave operator which maps the neighborhood of $\mathbf{H}^{3,2}$ into $\mathbf{H}^{1,0}$ was constructed in paper [11] (see also [16] for the case of regular final data with a compact support).

In paper [20] it was proved the completeness of the scattering operator in the energy space for equation (1.1) with cubic nonlinearity μv^3 replaced by a higher order nonlinear term $-|v|^{p-1} v$, $p > 5$ (see also [1], [2], [9] for higher space dimensions). The scattering problem and time decay rates for small solutions of (1.1) with super-critical nonlinearities $|v|^{p-1} v$, or $|v|^p$ were studied in papers [7], [12], [22], [23], [25]. Quadratic nonlinearities in two space dimensions seem to be critical with respect to the large time asymptotic behavior, like cubic nonlinearities in one space dimension. However it was proved in [21] that every quadratic nonlinearity is nonresonance, namely it has super-critical time decay property. Indeed global existence, time decay of small solutions and stability of solutions in the neighborhood of the free solutions were established in [21] by combining the method of normal forms of [24] and the vector field by [14] (see also papers [6], [7], [8] for further developments). In [4], the sharp asymptotic behavior of small solutions to quadratic Klein–Gordon equations in two space dimensions was also obtained via the vector field and hyperbolic polar coordinates implying the compact support condition for the initial data.

We define a new dependent variable $u = \frac{1}{2}\left(v + i\langle i\partial_x\rangle^{-1} v_t\right)$ and initial data

$u_0 = \frac{1}{2}\left(v_0 + i\langle i\partial_x\rangle^{-1} v_1\right)$, where $\langle z\rangle = \sqrt{1+z^2}$. In the case of the real-valued function v the nonlinear Klein–Gordon equation (1.1) can be rewritten as

$$\begin{cases} (\partial_t + i\langle i\partial_x\rangle) u = 4i\mu \langle i\partial_x\rangle^{-1} (\mathrm{Re} u)^3, \ \ (t,x) \in \mathbf{R}\times\mathbf{R}, \\ u(0) = u_0, \ x \in \mathbf{R}, \end{cases} \tag{1.2}$$

where $\mathrm{Re} u = \frac{1}{2}(u + \overline{u})$, $\overline{u}$ is a complex conjugate of u. Then the solution v of (1.1) is represented by $v = 2\mathrm{Re} u$.

We denote the Lebesgue space $\mathbf{L}^p = \{\phi \in \mathcal{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, with the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}} |\phi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. The weighted Sobolev space $\mathbf{H}^{m,s}_p = \{\phi \in \mathbf{L}^p; \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty\}$, for $m, s \in \mathbf{R}$, $1 \le p \le \infty$. For simplicity we write $\mathbf{H}^{m,s} = \mathbf{H}^{m,s}_2$. The index 0 we usually omit if it does not cause a confusion. The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi = \hat{\phi} = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x)\, dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi) d\xi.$$

Our main result is

Theorem 1.1. *Let $u_0 \in \mathbf{H}^{4,1}$ and the norm $\|u_0\|_{\mathbf{H}^{4,1}} = \varepsilon$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (1.2) has a unique global solution*

$$u(t) \in \mathbf{C}\left([0,\infty); \mathbf{H}^{4,1}\right)$$

satisfying the time decay estimate

$$\|u(t)\|_{\mathbf{H}^1_\infty} \le C\varepsilon (1+t)^{-\frac{1}{2}}.$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\left\| u(t) - e^{-i\langle i\partial_x\rangle t} \mathcal{F}^{-1} \widehat{W}_+ e^{\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t} \right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}}$$

and

$$\left\| \mathcal{F} e^{i\langle i\partial_x\rangle t} u(t) - \widehat{W}_+ e^{\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t} \right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}},$$

where $\gamma \in \left(0, \frac{1}{4}\right)$.

From this result it follows that there exists the inverse modified wave operator $(MW_+)^{-1}$ such that

$$(MW_+)^{-1} : u_0 \in \mathbf{H}^{4,1} \to W_+ \in \mathbf{H}^{1,0}.$$

As a consequence of Theorem 1.1 we find the corresponding result concerning the initial value problem (1.1).

Corollary 1.2. *Let* $v_0 \in \mathbf{H}^{4,1}, v_1 \in \mathbf{H}^{3,1}$ *be real valued functions and the norm* $\|v_0\|_{\mathbf{H}^{4,1}} + \|v_1\|_{\mathbf{H}^{3,1}} = \epsilon$. *Then there exists* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$ *the initial value problem (1.1) has a unique global solution*

$$v(t) \in \mathbf{C}\left([0,\infty);\mathbf{H}^{4,1}\right) \cap \mathbf{C}^1\left([0,\infty);\mathbf{H}^{3,1}\right)$$

satisfying the time decay estimate

$$\|v(t)\|_{\mathbf{H}^1_\infty} \le C\varepsilon (1+t)^{-\frac{1}{2}}.$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ *such that*

$$\left\| v(t) - 2Re e^{-i\langle i\partial_x\rangle t} \mathcal{F}^{-1} \widehat{W}_+ e^{\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t} \right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^{\frac{3}{2}} t^{\gamma-\frac{1}{4}},$$

where $\gamma \in \left(0, \frac{1}{4}\right)$.

Remark 1.1. By Corollary 1.2, the second and the third estimates of Lemma 3.1 below, along with the representation (1.5) for the free evolution group we have for $t > 0$

$$\begin{aligned} & e^{-i\langle i\partial_x\rangle t} \mathcal{F}^{-1} \widehat{W}_+ e^{\frac{3i\mu}{2}\langle\xi\rangle^2 \left|\widehat{W}_+\right|^2 \log t} \\ &= \frac{1}{\sqrt{t}} \theta\left(\frac{x}{t}\right) \left\langle i\frac{x}{t}\right\rangle^{-\frac{3}{2}} \widehat{W}_+ \left(\frac{x}{\sqrt{t^2 - x^2}}\right) \\ &\times e^{-i\sqrt{t^2-x^2} - i\frac{\pi}{4} + \frac{3i\mu}{2}\frac{t^2}{(t^2-x^2)}\left|\widehat{W}_+\left(\frac{x}{\sqrt{t^2-x^2}}\right)\right|^2 \log t} \\ &+ O\left(\left\|\widehat{W}_+ e^{\frac{3i\mu}{2}\langle\xi\rangle^2\left|\widehat{W}_+\right|^2 \log t}\right\|_{\mathbf{H}^{1,3}} t^{-\frac{3}{4}}\right), \end{aligned}$$

where the function $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| \ge 1$. Therefore we need a regularity of $\widehat{W}_+$ to obtain a sharp asymptotics of solutions to (1.2) in $\mathbf{L}^\infty$ sense. However we only have $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$. Hence we can not find a sharp asymptotic formula for solutions $u(t)$ in $\mathbf{L}^\infty$ sense.

An important tool for obtaining the time decay estimates of solutions to the nonlinear Klein–Gordon equation is implementation of the operator

$$\mathcal{J} = \langle i\partial_x\rangle e^{-i\langle i\partial_x\rangle t} x e^{i\langle i\partial_x\rangle t} = \mathcal{F}^{-1}\langle\xi\rangle e^{-i\langle\xi\rangle t} i\partial_\xi e^{i\langle\xi\rangle t}\mathcal{F} = \langle i\partial_x\rangle x + it\partial_x,$$

which is analogous to the operator $x + it\partial_x = e^{-\frac{it}{2}\partial_x^2} x e^{\frac{it}{2}\partial_x^2}$ in the case of the nonlinear Schrödinger equations. The operator $\mathcal{J}$ was used previously in paper [12] for constructing the scattering operator for nonlinear Klein–Gordon equations with a super critical nonlinearity. We have $[x, \langle i\partial_x\rangle^\alpha] = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x$, therefore the commutator $[\mathcal{L},\mathcal{J}] = \mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L} = 0$ holds, where $\mathcal{L} = \partial_t + i\langle i\partial_x\rangle$ is a linear part of equation (1.2). Since $\mathcal{J}$ is not a purely differential operator it is apparently difficult to calculate the action of $\mathcal{J}$ on the nonlinearity in equation (1.2). So, instead we use the first order differential operator

$$\mathcal{P} = t\partial_x + x\partial_t$$

which is closely related to $\mathcal{J}$ by the identity $\mathcal{P} = \mathcal{L}x - i\mathcal{J}$ and acts easily on the nonlinearity. Moreover, it almost commutes with $\mathcal{L}$, since $[\mathcal{L}, \mathcal{P}] = -i\langle i\partial_x\rangle^{-1}\partial_x\mathcal{L}$.

For the convenience of the reader we briefly explain our strategy. Since the nonlinear Klein–Gordon equation (1.1) is a relativistic version of the nonlinear Schrödinger equation

$$\begin{cases} iv_t + \frac{1}{2}v_{xx} = \mu|v|^2 v, \ \ (t,x) \in \mathbf{R}\times\mathbf{R}, \\ v(0) = v_0, x \in \mathbf{R}. \end{cases} \tag{1.3}$$

we can expect that the methods developed for (1.3) could be also useful for the nonlinear Klein–Gordon equations. The inverse modified wave operator for (1.3) was constructed in [10], where the main idea was to translate equation (1.3) into another one multiplying both sides of (1.3) by the operator $\mathcal{F}e^{\frac{i}{2}t\partial_x^2}$. Also we used the factorization of the free Schrödinger evolution group

$$\begin{aligned} e^{-\frac{i}{2}t\partial_x^2}\phi &= \frac{1}{\sqrt{2\pi it}}\int e^{\frac{ix^2}{2t} - \frac{ixy}{t} + \frac{iy^2}{2t}}\phi(y)\,dy \\ &= e^{\frac{ix^2}{2t}}\frac{1}{\sqrt{it}}\mathcal{F}\left(e^{\frac{iy^2}{2t}}\phi\right)\left(\frac{x}{t}\right) = \mathcal{MDFM}\phi, \end{aligned}$$

where $\mathcal{M} = e^{\frac{ix^2}{2t}}, \mathcal{D}\phi(y) = \frac{1}{\sqrt{it}}\phi\left(\frac{x}{t}\right)$. Then we obtained from (1.3)

$$\begin{aligned} i\left(\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v\right)_t &= \mu\mathcal{F}\overline{\mathcal{M}}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{\mathcal{M}}|v|^2 v \\ &= \mu t^{-1}\mathcal{F}\overline{\mathcal{M}}\mathcal{F}^{-1}\left|\mathcal{F}\overline{\mathcal{M}}e^{\frac{i}{2}t\partial_x^2}v\right|^2\mathcal{F}\overline{\mathcal{M}}e^{\frac{i}{2}t\partial_x^2}v \\ &= \mu t^{-1}\left|\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v\right|^2\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v + R, \end{aligned}$$

where the nonlinearity is decomposed into the resonant term $\mu t^{-1}\left|\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v\right|^2\mathcal{F}$ $e^{\frac{i}{2}t\partial_x^2}v$ and the remainder R. The resonant term can be canceled by the change of the dependent variable $\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v$ with a new variable

$$\left(\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v\right)e^{i\int_1^t \mu\tau^{-1}\left|\mathcal{F}e^{\frac{i}{2}t\partial_x^2}v\right|^2 d\tau}.$$

In this way the $\mathbf{L}^\infty$ - estimate of $\mathcal{F}e^{\frac{i}{2}(\partial_x^2)t}v$ follows. We also follow this idea in the present. We multiply both sides of (1.2) by $\langle i\partial_x\rangle\mathcal{F}e^{it\langle i\partial_x\rangle}$ and put $\varphi = \langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u$ to get

$$\begin{aligned} \varphi_t &= 4i\mu\mathcal{F}e^{i\langle i\partial_x\rangle t}(\mathrm{Re}u)^3 \\ &= \frac{3i\mu}{2t}|\varphi|^2\varphi + \frac{i\mu}{2}\mathcal{F}e^{i\langle i\partial_x\rangle t}\left(u^3 + \overline{u}^3 + 3|u|^2\overline{u}\right) \\ &\quad + O\left(t^{-\frac{5}{4}}\|u\|^3_{\mathbf{H}^{4,1}}\right). \end{aligned} \tag{1.4}$$

Thus we see that the nonlinearity in the right-hand side of (1.4) is decomposed into the resonant term $\frac{3i\mu}{2t}|\varphi|^2\varphi$, nonresonance cubic nonlinearities and the remainder

$O\left(t^{-\frac{5}{4}}\|u\|_{\mathbf{H}^{4,1}}^3\right)$. This fact is exploited in Lemma 3.3 and Lemma 3.1 below. Then in Lemma 3.4 we will show that the nonresonance terms have a better time decay property through the integration by parts. Furthermore the first resonant term of the right-hand side of (1.4) can be removed when we multiply both sides by $e^{-\int_1^t i\frac{3\mu}{2}|\varphi|^2\tau^{-1}d\tau}$. Then the $\mathbf{L}^\infty$-estimate of $\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u$ follows and the energy method yields a-priori estimate of $\|u\|_{\mathbf{H}^{4,1}}$.

We now decompose the free evolution group $e^{-i\langle i\partial_x\rangle t} = \mathcal{F}^{-1}e^{-it\langle\xi\rangle}\mathcal{F}$ of (1.2) into the main and remainder terms similarly to the factorization of the free Schrödinger evolution group. We denote the dilation operator by

$$\mathcal{D}_\omega\phi = |\omega|^{-\frac{1}{2}}\phi\left(x\omega^{-1}\right),\ \ (\mathcal{D}_\omega)^{-1} = \mathcal{D}_{\frac{1}{\omega}}.$$

Define the multiplication factor

$$M(t) = e^{-it\langle ix\rangle\theta(x)},$$

where $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| \geq 1$. We introduce the operator

$$\mathcal{B}\phi = e^{-i\frac{\pi}{4}}\langle ix\rangle^{-\frac{3}{2}}\theta(x)\phi\left(x\langle ix\rangle^{-1}\right).$$

Thus $\mathcal{D}_tM(t)\mathcal{B}\phi$ is a well-known leading term of the large time asymptotics of solutions of the linear Klein–Gordon equation $(\partial_t + i\langle i\partial_x\rangle)u = 0$ with initial data ϕ. The inverse operator $\mathcal{B}^{-1}$ acts on the functions $\phi(x)$ defined on $(-1,1)$ as follows

$$\mathcal{B}^{-1}\phi = e^{i\frac{\pi}{4}}\langle\xi\rangle^{-\frac{3}{2}}\phi\left(\xi\langle\xi\rangle^{-1}\right)$$

for all $\xi \in \mathbf{R}$. We now introduce the operators

$$\mathcal{V}(t) = \mathcal{B}^{-1}\overline{M}(t)\mathcal{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$$

and

$$\mathcal{W}(t) = (1-\theta)\mathcal{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$$

so that we have the representation for the free Klein–Gordon evolution group

$$\begin{aligned} e^{-i\langle i\partial_x\rangle t}\mathcal{F}^{-1} &\equiv \mathcal{F}^{-1}e^{-it\langle\xi\rangle} = \mathcal{D}_tM(t)\mathcal{B}\mathcal{V}(t) + \mathcal{D}_t\mathcal{W}(t) \\ &= \mathcal{D}_tM(t)\mathcal{B} + \mathcal{D}_tM(t)\mathcal{B}(\mathcal{V}(t)-1) + \mathcal{D}_t\mathcal{W}(t). \end{aligned} \tag{1.5}$$

The first term of the right-hand side of (1.5) describes the leading term of the large time asymptotics inside of the light cone. The second term is a remainder inside of the light cone and the last term represents the large time asymptotics outside of the light cone. We also have

$$\begin{aligned} \mathcal{F}e^{i\langle i\partial_x\rangle t} &\equiv e^{it\langle\xi\rangle}\mathcal{F} = \mathcal{V}^{-1}(t)\mathcal{B}^{-1}\overline{M}(t)\mathcal{D}_t^{-1} + \mathcal{W}^{-1}(t)\mathcal{D}_t^{-1} \\ &= \mathcal{B}^{-1}\overline{M}(t)\mathcal{D}_t^{-1} + \left(\mathcal{V}^{-1}(t)-1\right)\mathcal{B}^{-1}\overline{M}(t)\mathcal{D}_t^{-1} + \mathcal{W}^{-1}(t)\mathcal{D}_t^{-1}, \end{aligned}$$

where the right-inverse operators

$$\mathcal{V}^{-1}(t) = e^{it\langle\xi\rangle}\mathcal{F}\mathcal{D}_tM(t)\mathcal{B}$$

and

$$\mathcal{W}^{-1}(t) = e^{it\langle\xi\rangle}\mathcal{F}\mathcal{D}_t(1-\theta).$$

We prove our main result in the next section. In Section 3 we prove several lemmas involved in the proof of the main result.

2. Proof of Theorem 1.1

We introduce the function space

$$\mathbf{X}_T = \left\{\phi \in \mathbf{C}\left([0,T];\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_T} < \infty\right\},$$

where

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[0,T]}\Big(\langle t\rangle^{-\gamma}\|\phi(t)\|_{\mathbf{H}^4} + \langle t\rangle^{-\gamma}\|\mathcal{J}\phi(t)\|_{\mathbf{H}^2} + \langle t\rangle^{-3\gamma}\|\mathcal{J}\phi(t)\|_{\mathbf{H}^3} + \langle t\rangle^{\frac{1}{2}}\|\phi(t)\|_{\mathbf{H}^1_\infty}\Big),$$

and $\gamma > 0$ is small.

The local existence in the function space $\mathbf{X}_T$ can be proved by a standard contraction mapping principle. We state it without a proof.

Theorem 2.1. *Let $u_0 \in \mathbf{H}^{4,1}$ and the norm $\|u_0\|_{\mathbf{H}^{4,1}} = \varepsilon$. Then there exist $\varepsilon_0 > 0$ and $T > 1$ such that for all $0 < \varepsilon < \varepsilon_0$ the initial value problem (1.2) has a unique local solution $u \in \mathbf{C}\left([0,T];\mathbf{H}^{4,1}\right)$ with the estimate $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$.*

Let us prove that the existence time T can be extended to infinity which then yields the result of Theorem 1.1. By contradiction, we assume that there exists a minimal time $T > 0$ such that the a-priori estimate $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$ does not hold, namely, we have $\|u\|_{\mathbf{X}_T} \le \sqrt{\varepsilon}$. We apply the energy method to (1.2) to obtain

$$\frac{d}{dt}\|u\|_{\mathbf{H}^4} \le C\|u\|^2_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^3} \le C\varepsilon\langle t\rangle^{-1}\|u\|_{\mathbf{H}^4}.$$

Hence by Theorem 2.1

$$\|u\|_{\mathbf{H}^4} \le C\varepsilon\langle t\rangle^{\gamma}. \tag{2.1}$$

Next we use the commutator relations

$$\mathcal{L}\mathcal{P} = \left(\mathcal{P} - i\langle i\partial_x\rangle^{-1}\partial_x\right)\mathcal{L},$$

$$[x, \langle i\partial_x\rangle^{\alpha}] = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x, [\mathcal{P}, \langle i\partial_x\rangle^{\alpha}] = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x\partial_t,$$
$$\partial_t u = 4i\mu\langle i\partial_x\rangle^{-1}(\mathrm{Re}u)^3 - i\langle i\partial_x\rangle u$$

to get

$$\mathcal{L}\mathcal{P}u = 12i\mu\langle i\partial_x\rangle^{-1}(\mathrm{Re}u)^2\,\mathrm{Re}\mathcal{P}u + 4\mu\langle i\partial_x\rangle^{-2}\partial_x(\mathrm{Re}u)^3 - 12i\mu\langle i\partial_x\rangle^{-3}\partial_x\left((\mathrm{Re}u)^2\,\mathrm{Re}\left(4i\mu\langle i\partial_x\rangle^{-1}(\mathrm{Re}u)^3 - i\langle i\partial_x\rangle u\right)\right).$$

Then by the energy method (i.e., multiplying both sides of the above equation by $\langle i\partial_x\rangle^4 \overline{\mathcal{P}u}$, taking the real part and integrating over the space) we obtain

$$\begin{aligned}\frac{d}{dt}\|\mathcal{P}u\|_{\mathbf{H}^2} &\le C\|u\|^2_{\mathbf{H}^1_\infty}\|\mathcal{P}u\|_{\mathbf{H}^1} + C\|u\|^2_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^1} + C\|u\|^4_{\mathbf{L}^\infty}\|u\|_{\mathbf{L}^2}\\ &\le C\varepsilon\langle t\rangle^{-1}\|\mathcal{P}u\|_{\mathbf{H}^2} + C\varepsilon^2\langle t\rangle^{\gamma-1}.\end{aligned}$$

Therefore by Theorem 2.1 it follows

$$\|\mathcal{P}u\|_{\mathbf{H}^2} \le C\varepsilon\langle t\rangle^{\gamma}. \tag{2.2}$$

The energy estimate and the identity $\mathcal{L}x = x\mathcal{L} - i\langle i\partial_x\rangle^{-1}\partial_x$ imply

$$\begin{aligned}\frac{d}{dt}\|xu\|_{\mathbf{H}^2} &\le C\|u\|^2_{\mathbf{H}^1_\infty}\|xu\|_{\mathbf{H}^1} + \|u\|_{\mathbf{H}^2}\\ &\le C\varepsilon\langle t\rangle^{-1}\|xu\|_{\mathbf{H}^2} + C\varepsilon^2\langle t\rangle^{\gamma}\end{aligned}$$

which yields

$$\|xu\|_{\mathbf{H}^2} \le C\varepsilon\langle t\rangle^{\gamma+1}. \tag{2.3}$$

Then by the identity

$$\mathcal{J} = i\mathcal{P} - ix\mathcal{L} - \langle i\partial_x\rangle^{-1}\partial_x \tag{2.4}$$

and (2.1) we obtain

$$\|\mathcal{J}u\|_{\mathbf{H}^2} \le C\varepsilon\langle t\rangle^{\gamma}. \tag{2.5}$$

By Lemma 3.2, (2.1), (2.3) and the relation (2.4) we find that

$$\begin{aligned}\|u\|_{\mathbf{H}^2_\infty} &\le C\langle t\rangle^{-\frac{1}{2}}\|u\|^{\frac{1}{2}}_{\mathbf{H}^4}\left(\|u\|^{\frac{1}{2}}_{\mathbf{H}^4} + \|\mathcal{J}u\|^{\frac{1}{2}}_{\mathbf{H}^3}\right)\\ &\le C\varepsilon\langle t\rangle^{2\gamma-\frac{1}{2}} + C\varepsilon^{\frac{1}{2}}\langle t\rangle^{\frac{\gamma}{2}-\frac{1}{2}}\|\mathcal{P}u\|^{\frac{1}{2}}_{\mathbf{H}^3}.\end{aligned}$$

Then by the energy method, Theorem 2.1 and (2.2) we obtain

$$\begin{aligned}&\frac{d}{dt}\|\mathcal{P}u\|_{\mathbf{H}^3} \le C\|u\|^2_{\mathbf{H}^1_\infty}\|\mathcal{P}u\|_{\mathbf{H}^2} + C\|u\|_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^2_\infty}\|\mathcal{P}u\|_{\mathbf{H}^1}\\ &\quad + C\|u\|^2_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^2} + C\|u\|^4_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^1}\\ &\le C\varepsilon\langle t\rangle^{-1}\|\mathcal{P}u\|_{\mathbf{H}^3} + C\varepsilon^2\langle t\rangle^{2\gamma-1}\|\mathcal{P}u\|_{\mathbf{H}^1}\\ &\quad + C\varepsilon^{\frac{3}{2}}\langle t\rangle^{\frac{\gamma}{2}-1}\|\mathcal{P}u\|^{\frac{1}{2}}_{\mathbf{H}^3}\|\mathcal{P}u\|_{\mathbf{H}^1}\\ &\le C\varepsilon\langle t\rangle^{-1}\|\mathcal{P}u\|_{\mathbf{H}^3} + C\varepsilon^2\langle t\rangle^{3\gamma-1} + C\varepsilon^{\frac{5}{2}}\langle t\rangle^{\frac{3}{2}\gamma-1}\|\mathcal{P}u\|^{\frac{1}{2}}_{\mathbf{H}^3}\\ &\le C\varepsilon\langle t\rangle^{-1}\|\mathcal{P}u\|_{\mathbf{H}^3} + C\varepsilon^2\langle t\rangle^{3\gamma-1}.\end{aligned}$$

Therefore

$$\|\mathcal{P}u\|_{\mathbf{H}^3} \le C\varepsilon\langle t\rangle^{3\gamma}. \tag{2.6}$$

By identity (2.4) and (2.6) we see that

$$\|\mathcal{J}u\|_{\mathbf{H}^3} \le C\varepsilon\langle t\rangle^{3\gamma}. \tag{2.7}$$

We use Lemma 3.3 to obtain for the new variable $\varphi(t) = \langle \xi \rangle \mathcal{F} e^{i \langle i \partial_x \rangle t} u(t)$

$$\partial_t \varphi = i t^{-1} e^{it\langle \xi \rangle} \sum_{j=1}^{4} \lambda_j \mathcal{D}_{\omega_j} e^{-it\omega_j \langle \xi \rangle} \overline{\varphi}^{3-\alpha_j} \varphi^{\alpha_j} + O\left(t^{-\frac{5}{4}} \|\varphi\|_{\mathbf{H}^{1,3}}^3 \right) \tag{2.8}$$

in $\mathbf{L}^r$, $2 \le r \le \infty$. Then we apply Lemma 3.4 to show that the first term of the right hand side of (2.8) is decomposed into the resonant and nonresonance terms, and then we obtain

$$\sup_{t \ge 1} \|\varphi(t)\|_{\mathbf{L}^\infty} \le C\varepsilon. \tag{2.9}$$

By the decomposition of the free evolutions group we have the identity

$$\begin{aligned} \langle i\partial_x \rangle u(t) &= \left(\mathcal{D}_t M(t) \mathcal{B} \mathcal{V}(t) + \mathcal{D}_t \mathcal{W}(t) \right) \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) \\ &= \mathcal{D}_t M(t) \mathcal{B} \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) \\ &+ \left(\mathcal{D}_t M(t) \mathcal{B} \langle \xi \rangle^{-\frac{3}{2}} \langle \xi \rangle^{\frac{3}{2}} (\mathcal{V}(t) - 1) + \mathcal{D}_t \mathcal{W}(t) \right) \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t). \end{aligned} \tag{2.10}$$

By the second and the third estimates of Lemma 3.1 we find that the last term of the right-hand side of (2.10) is a remainder term. Indeed we have the estimate

$$\begin{aligned} \|u(t)\|_{\mathbf{H}^1_\infty} &\le C \left\| \mathcal{D}_t M(t) \mathcal{B} \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) \right\|_{\mathbf{L}^\infty} \\ &\quad + C\varepsilon \langle t \rangle^{-\frac{5}{4}} \left\| \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) \right\|_{\mathbf{H}^{1,3}} \\ &\le C \langle t \rangle^{-\frac{1}{2}} \sup_{t \ge 1} \|\varphi(t)\|_{\mathbf{L}^\infty} + C \langle t \rangle^{-\frac{5}{4}} \|\mathcal{J} u(t)\|_{\mathbf{H}^3}, \end{aligned}$$

which along with the estimates (2.7) and (2.9) yields

$$\sup_{t \ge 1} \langle t \rangle^{\frac{1}{2}} \|u(t)\|_{\mathbf{H}^1_\infty} \le C\varepsilon. \tag{2.11}$$

From (2.1), (2.5), (2.7) and (2.11) it follows that

$$\|u\|_{\mathbf{X}_T} \le C\varepsilon < \sqrt{\varepsilon}$$

which implies the desired contradiction. Thus there exists a unique global solution $u \in \mathbf{C}\left([0, \infty); \mathbf{H}^{4,1} \right)$ of (1.2) with the time decay estimate.

We now prove the asymptotics of solutions. By equation (2.8) as in the proof of Lemma 3.4 we have

$$\|\psi(t) - \psi(s)\|_{\mathbf{L}^\infty} + \|\psi(t) - \psi(s)\|_{\mathbf{L}^2} \le C\varepsilon^{\frac{3}{2}} \int_s^t \tau^{\gamma - \frac{5}{4}} d\tau \le C\varepsilon^{\frac{3}{2}} s^{\gamma - \frac{1}{4}}$$

with $\gamma \in \left(0, \frac{1}{4} \right)$, where

$$\begin{aligned} \psi(t) &= \varphi e^{-i\lambda_1 |\varphi|^2 \log t} \\ &= \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) e^{\frac{3i\mu}{2} \langle \xi \rangle^2 \left| \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) \right|^2 \log t}. \end{aligned}$$

Thus we see that there exists a unique final state $\psi_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\|\psi(t) - \psi_+\|_{\mathbf{H}^{0,1}_\infty} + \|\psi(t) - \psi_+\|_{\mathbf{H}^{0,1}} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}}.$$

We consider the asymptotics of the phase function

$$\Phi(t) = \frac{3i\mu}{2}\int_1^t \left(|\varphi(\tau)|^2 - |\varphi(t)|^2\right)\frac{d\tau}{\tau} = \frac{3i\mu}{2}\int_1^t \left(|\psi(\tau)|^2 - |\psi(t)|^2\right)\frac{d\tau}{\tau}.$$

By a direct calculation we have

$$\Phi(t) - \Phi(s) = \frac{3i\mu}{2}\left(\int_s^t \left(|\psi(\tau)|^2 - |\psi(t)|^2\right)\frac{d\tau}{\tau} + \left(|\psi(t)|^2 - |\psi(s)|^2\right)\log s\right),$$

where $1 < s < \tau < t$. Hence

$$\begin{aligned}
&\|\Phi(t) - \Phi(s)\|_{\mathbf{L}^\infty} \\
&\le C\int_s^t \|\psi(\tau) - \psi(t)\|_{\mathbf{L}^\infty}\left(\|\psi(\tau)\|_{\mathbf{L}^\infty} + \|\psi(t)\|_{\mathbf{L}^\infty}\right)\frac{d\tau}{\tau} \\
&\quad + C\|\psi(s) - \psi(t)\|_{\mathbf{L}^\infty}\left(\|\psi(s)\|_{\mathbf{L}^\infty} + \|\psi(t)\|_{\mathbf{L}^\infty}\right)\log s \\
&\le C\varepsilon^{\frac{5}{2}}\int_s^t \tau^{\gamma-\frac{5}{4}}d\tau + C\varepsilon^{\frac{5}{2}}s^{\gamma-\frac{1}{4}}\log s
\end{aligned}$$

from which it follows that there exists a unique real valued function Φ_+ such that

$$\|\Phi(t) - i\Phi_+\|_{\mathbf{L}^\infty} \le C\varepsilon^{\frac{3}{2}}t^{\gamma-\frac{1}{4}}.$$

Similarly,

$$\|\Phi(t) - i\Phi_+\|_{\mathbf{L}^2} \le C\varepsilon^{\frac{3}{2}}t^{\gamma-\frac{1}{4}}.$$

Therefore we have the asymptotics of the phase function

$$\begin{aligned}
\frac{3i\mu}{2}\int_1^t |\psi(\tau)|^2\frac{d\tau}{\tau} &= \frac{3i\mu}{2}\langle\xi\rangle^2\int_1^t \left|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(\tau)\right|^2\frac{d\tau}{\tau} \\
&= i\Phi_+ + \frac{3i\mu}{2}|\psi_+|^2\log t \\
&\quad + (\Phi(t) - i\Phi_+) + \frac{3i\mu}{2}\left(\langle\xi\rangle^2\left|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)\right|^2 - |\psi_+|^2\right)\log t.
\end{aligned}$$

We also have

$$\begin{aligned}
&\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \psi_+e^{\left(i\Phi_+ + \frac{3i\mu}{2}|\psi_+|^2\log t\right)} \\
&= \left(\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \psi_+e^{\frac{3i\mu}{2}\int_1^t|\psi(\tau)|^2\frac{d\tau}{\tau}}\right) \\
&\quad + \psi_+\left(e^{\frac{3i\mu}{2}\int_1^t|\psi(\tau)|^2\frac{d\tau}{\tau}} - e^{\left(i\Phi_+ + \frac{3i\mu}{2}|\psi_+|^2\log t\right)}\right) \\
&= \left(\psi(t)e^{\frac{3i\mu}{2}\langle\xi\rangle^2\int_1^t|\psi(t)|^2\frac{d\tau}{\tau}} - \psi_+e^{\frac{3i\mu}{2}\int_1^t|\psi(\tau)|^2\frac{d\tau}{\tau}}\right) \\
&\quad + \psi_+\left(e^{\frac{3i\mu}{2}\int_1^t|\psi(\tau)|^2\frac{d\tau}{\tau}} - e^{\left(i\Phi_+ + \frac{3i\mu}{2}|\psi_+|^2\log t\right)}\right).
\end{aligned}$$

Collecting these estimates we find

$$\begin{aligned}&\left\| \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) - \psi_+ e^{\left(i\Phi_+ + \frac{3i\mu}{2}|\psi_+|^2 \log t\right)} \right\|_{\mathbf{L}^2} \\ &\le C \|\psi(t) - \psi_+\|_{\mathbf{L}^2} + \|\psi_+\|_{\mathbf{L}^\infty} \|\Phi(t) - i\Phi_+\|_{\mathbf{L}^2} \\ &\quad + \|\psi_+\|_{\mathbf{L}^\infty} \|\psi(t) - \psi_+\|_{\mathbf{L}^\infty} \left(\|\psi_+\|_{\mathbf{L}^2} + \|\psi(t)\|_{\mathbf{L}^2}\right) \log t \\ &\le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}}\end{aligned}$$

and similarly,

$$\left\| \langle \xi \rangle \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) - \psi_+ e^{\left(i\Phi_+ + \frac{3i\mu}{2}|\psi_+|^2 \log t\right)} \right\|_{\mathbf{L}^\infty} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}}.$$

Therefore we have

$$\left\| \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) - \widehat{W}_+ e^{\frac{3i\mu}{2}\langle \xi \rangle^2 \left|\widehat{W}_+\right|^2 \log t} \right\|_{\mathbf{H}^{0,1}} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}} \tag{2.12}$$

and

$$\left\| \mathcal{F} e^{i\langle i\partial_x \rangle t} u(t) - \widehat{W}_+ e^{\frac{3i\mu}{2}\langle \xi \rangle^2 \left|\widehat{W}_+\right|^2 \log t} \right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}},$$

where $\widehat{W}_+ = \langle \xi \rangle^{-1} \psi_+ \exp(i\Phi_+)$. Estimate (2.12) means that

$$\left\| u(t) - e^{-i\langle i\partial_x \rangle t} \mathcal{F}^{-1} \widehat{W}_+ e^{\frac{3i\mu}{2}\langle \xi \rangle^2 \left|\widehat{W}_+\right|^2 \log t} \right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^{\frac{3}{2}} t^{\gamma - \frac{1}{4}}.$$

Theorem 1.1 is now proved.

3. Lemmas

In the next lemma we obtain the large time asymptotics for the free Klein–Gordon evolution group.

Lemma 3.1. *The estimates*

$$\|\mathcal{V}(t)\phi\|_{\mathbf{H}^{1,1-\gamma}} \le C \|\phi\|_{\mathbf{H}^{1,4}}$$

$$\left\| \langle \xi \rangle^{\frac{3}{2}} (\mathcal{V}(t) - 1)\phi \right\|_{\mathbf{L}^\infty} \le C \|\phi\|_{\mathbf{H}^{1,3}} t^{-\frac{1}{4}}$$

$$\|\mathcal{W}(t)\phi\|_{\mathbf{L}^r} \le C \|\phi\|_{\mathbf{H}^{1,3}} t^{-\frac{1}{2}}$$

and

$$\left\| \left(\mathcal{V}^{-1}(t) - 1\right)\phi \right\|_{\mathbf{L}^\infty} \le C t^{-\frac{1}{4}} \|\phi\|_{\mathbf{H}^{1,\frac{3}{4}+\gamma}}$$

hold for $2 \le r \le \infty$, $0 < \gamma < 1$ *provided the right hand sides are finite.*

Remark 3.1. The first and the last estimates of Lemma 3.1 will be used to estimate the remainder in equation (2.8). The second and the third estimates of Lemma 3.1 imply that the last term of the right-hand side of (2.10) is a remainder in our functional space.

Proof. Changing the variable of integration $x = \xi \langle \xi \rangle^{-1}$ we see that

$$\left\| \mathcal{B}^{-1}\phi \right\|^2_{\mathbf{L}^2(\mathbf{R})} = \int_{\mathbf{R}} \left| \phi \left(\xi \langle \xi \rangle^{-1} \right) \right|^2 \langle \xi \rangle^{-3} d\xi = \int_{-1}^{1} |\phi(x)|^2 dx = \|\phi\|^2_{\mathbf{L}^2(-1,1)} .$$

By a direct computation

$$\partial_\xi \mathcal{B}^{-1}\phi(x) = e^{i\frac{\pi}{4}} \partial_\xi \langle \xi \rangle^{-\frac{3}{2}} \phi \left(\xi \langle \xi \rangle^{-1} \right)$$
$$= -\frac{3}{2} e^{i\frac{\pi}{4}} \xi \langle \xi \rangle^{-\frac{5}{2}} \phi \left(\xi \langle \xi \rangle^{-1} \right) + e^{i\frac{\pi}{4}} \langle \xi \rangle^{-\frac{3}{2}} \langle \xi \rangle^{-3} \phi' \left(\xi \langle \xi \rangle^{-1} \right)$$

and

$$\mathcal{B}^{-1} \langle ix \rangle^{\frac{3}{2}} \partial_x \langle ix \rangle^{\frac{3}{2}} \phi(x) = \frac{3}{2} \mathcal{B}^{-1} \left(-\langle ix \rangle x \phi(x) + \langle ix \rangle^3 \phi'(x) \right)$$
$$= -\frac{3}{2} e^{i\frac{\pi}{4}} \langle \xi \rangle^{-\frac{3}{2}} \left\langle i\xi \langle \xi \rangle^{-1} \right\rangle \xi \langle \xi \rangle^{-1} \phi \left(\xi \langle \xi \rangle^{-1} \right)$$
$$+ \frac{3}{2} e^{i\frac{\pi}{4}} \langle \xi \rangle^{-\frac{3}{2}} \left\langle i\xi \langle \xi \rangle^{-1} \right\rangle^3 \phi' \left(\xi \langle \xi \rangle^{-1} \right)$$
$$= -\frac{3}{2} e^{i\frac{\pi}{4}} \xi \langle \xi \rangle^{-\frac{5}{2}} \phi \left(\xi \langle \xi \rangle^{-1} \right) + e^{i\frac{\pi}{4}} \langle \xi \rangle^{-\frac{3}{2}} \langle \xi \rangle^{-3} \phi' \left(\xi \langle \xi \rangle^{-1} \right).$$

Therefore

$$\partial_\xi \mathcal{B}^{-1} = \mathcal{B}^{-1} \langle ix \rangle^{\frac{3}{2}} \partial_x \langle ix \rangle^{\frac{3}{2}}$$

and similarly,

$$\langle \xi \rangle^{1-\gamma} \mathcal{B}^{-1} = \mathcal{B}^{-1} \langle ix \rangle^{-1+\gamma} .$$

We also have

$$\partial_x \mathcal{B} = \mathcal{B} \langle \xi \rangle^{\frac{3}{2}} \partial_\xi \langle \xi \rangle^{\frac{3}{2}} .$$

Thus the identity

$$\langle \xi \rangle^{1-\gamma} \partial_\xi \mathcal{V}(t) = \langle \xi \rangle^{1-\gamma} \partial_\xi \mathcal{B}^{-1} \mathcal{B} \mathcal{V}(t) \mathcal{B}^{-1} \mathcal{B}$$
$$= \mathcal{B}^{-1} \langle ix \rangle^{\frac{1}{2}+\gamma} \partial_x \langle ix \rangle^{\frac{3}{2}} \mathcal{B} \mathcal{V}(t) \mathcal{B}^{-1} \mathcal{B}$$

holds. In order to prove the first estimate of the lemma we consider the operator $\mathcal{B}\mathcal{V}(t)\mathcal{B}^{-1}$. Changing the variable of integration $y = \xi \langle \xi \rangle^{-1}$ we obtain

$$\mathcal{B}\mathcal{V}(t)\mathcal{B}^{-1}\phi = \overline{M}(t) \mathcal{D}_t^{-1} \mathcal{F}^{-1} e^{-it\langle \xi \rangle} \mathcal{B}^{-1}\phi$$
$$= C |t|^{\frac{1}{2}} \int_{\mathbf{R}} e^{it(\langle ix \rangle + x\xi - \langle \xi \rangle)} \phi \left(\xi \langle \xi \rangle^{-1} \right) \langle \xi \rangle^{-\frac{3}{2}} d\xi$$
$$= C |t|^{\frac{1}{2}} \int_{-1}^{1} e^{-it\left(\frac{1-xy}{\langle iy \rangle} - \langle ix \rangle \right)} \phi(y) \langle iy \rangle^{-\frac{3}{2}} dy$$

since $dy = \langle \xi \rangle^{-3} d\xi = \langle \xi \rangle^{-3/2} \langle iy \rangle^{3/2} d\xi$. Then changing the variable of integration

$$(1-xy) \langle iy \rangle^{-1} - \langle ix \rangle = \eta^2, \; 2\eta d\eta = (y-x) \langle iy \rangle^{-3} dy$$

we find

$$\mathcal{BV}(t)\,\mathcal{B}^{-1}\phi = C\,|t|^{\frac{1}{2}}\int_{-1}^{1} e^{-it\left(\frac{1-xy}{\langle iy\rangle}-\langle ix\rangle\right)}\phi(y)\,\langle iy\rangle^{-\frac{3}{2}}\,dy$$
$$= C\,|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{-it\eta^{2}}\psi(y)\,\frac{\eta d\eta}{y-x}$$

for all $x\in(-1,1)$, $t>0$, where $\psi(y)=\phi(y)\langle iy\rangle^{\frac{3}{2}}$. Now we differentiate the last identity with respect to x

$$\partial_x\mathcal{BV}(t)\,\mathcal{B}^{-1}\phi = C\,|t|^{\frac{1}{2}}\int_{\mathbf{R}} e^{-it\eta^{2}}\left(\psi'(y)\,y_x+\psi(y)\,\frac{y_x-1}{x-y}\right)\frac{\eta d\eta}{y-x}$$
$$= C\,|t|^{\frac{1}{2}}\int_{-1}^{1} e^{-it\left(\frac{1-xy}{\langle iy\rangle}-\langle ix\rangle\right)}\left(\psi'(y)\,g_1(x,y)+\psi(y)\,g_2(x,y)\right)dy,$$

where

$$g_1(x,y)\equiv y_x\langle iy\rangle^{-3}=\left(x\langle ix\rangle^{-1}-y\langle iy\rangle^{-1}\right)(x-y)^{-1}$$

and

$$g_2(x,y)\equiv\frac{y_x-1}{x-y}\langle iy\rangle^{-3}=\frac{g_1(x,y)-g_1(y,y)}{x-y}.$$

Then we have

$$\left\|\langle ix\rangle^{2+\gamma}\partial_x\mathcal{BV}(t)\,\mathcal{B}^{-1}\phi\right\|_{\mathbf{L}^2}^{2}$$
$$\le C\,|t|\int_{-1}^{1}\int_{-1}^{1} e^{it\left(\langle iz\rangle^{-1}-\langle iy\rangle^{-1}\right)}\psi'(y)\,\overline{\psi'(z)}G_1(t,y,z)\,dydz$$
$$+C\,|t|\int_{-1}^{1}\int_{-1}^{1} e^{it\left(\langle iz\rangle^{-1}-\langle iy\rangle^{-1}\right)}\psi(y)\,\overline{\psi(z)}G_2(t,y,z)\,dydz$$

with kernels

$$G_j(t,y,z)=\int_{-1}^{1} e^{itx\left(\frac{z}{\langle iz\rangle}-\frac{y}{\langle iy\rangle}\right)}g_j(x,y)\,g_j(x,z)\,\langle ix\rangle^{4+2\gamma}\,dx.$$

By virtue of the mean value theorem we have

$$f(x)-f(y)=\int_0^1 f^{(1)}(x\theta+y(1-\theta))\,d\theta\,(x-y).$$

Therefore

$$\frac{d^k}{dx^k}\left(\frac{f(x)-f(y)}{x-y}\right)=\int_0^1 f^{(k+1)}(x\theta+y(1-\theta))\,\theta^k d\theta.$$

Taking $f(x) = x\langle ix\rangle^{-1}$, we find $\left|f^{(k+1)}(x)\right| \le C\langle ix\rangle^{-3-2k}$ and

$$\left|\frac{d^k}{dx^k}g_1(x,y)\right| \le C\int_0^1 \frac{\theta^k d\theta}{\langle i(x\theta + y(1-\theta))\rangle^{3+2k}}$$
$$\le C\int_0^1 \frac{\theta^k d\theta}{\left(\theta\langle ix\rangle^2 + (1-\theta)\langle iy\rangle^2\right)^{\frac{3}{2}+k}}$$
$$\le C\langle ix\rangle^{-1-\gamma-2k}\langle iy\rangle^{\gamma-2}\int_0^1 \frac{d\theta}{(1-\theta)^{1-\frac{\gamma}{2}}\theta^{\frac{1+\gamma}{2}}} \le C\langle ix\rangle^{-1-\gamma-2k}\langle iy\rangle^{\gamma-2}$$

for all $x, y \in (-1,1)$, where $\gamma \in (0,1)$. In the same manner

$$\left|\frac{d^k}{dx^k}g_2(x,y)\right| \le C\langle iy\rangle^{\frac{\gamma}{2}-2}\int_0^1 \frac{\theta^k d\theta}{\langle i(x\theta + y(1-\theta))\rangle^{3+\frac{\gamma}{2}+2k}}$$
$$\le C\langle iy\rangle^{\frac{\gamma}{2}-2}\int_0^1 \frac{\theta^k d\theta}{\left(\theta\langle ix\rangle^2 + (1-\theta)\langle iy\rangle^2\right)^{\frac{3}{2}+\frac{\gamma}{4}+k}}$$
$$\le \langle ix\rangle^{-1-\gamma-2k}\langle iy\rangle^{\gamma-4}\int_0^1 \frac{d\theta}{(1-\theta)^{1-\frac{\gamma}{4}}\theta^{\frac{1+\gamma}{2}}} \le C\langle ix\rangle^{-1-\gamma-2k}\langle iy\rangle^{\gamma-4}$$

for all $x, y \in (-1,1)$, where $\gamma \in (0,1)$. Applying the identity

$$e^{itx\left(z\langle iz\rangle^{-1} - y\langle iy\rangle^{-1}\right)}$$
$$= \left(1 + t^2\left(y\langle iy\rangle^{-1} - z\langle iz\rangle^{-1}\right)^2\right)^{-1}\left(1-\partial_x^2\right)e^{itx\left(z\langle iz\rangle^{-1} - y\langle iy\rangle^{-1}\right)}$$

we integrate two times by parts

$$|G_1(t,y,z)| \le C\left(1 + t^2\left(y\langle iy\rangle^{-1} - z\langle iz\rangle^{-1}\right)^2\right)^{-1}$$
$$\times\int_{-1}^1 \left|\left(1-\partial_x^2\right)\left(g_1(x,y)\,g_1(x,z)\langle ix\rangle^{4+2\gamma}\right)\right| dx$$
$$\le \frac{C\langle iy\rangle^{\frac{\gamma}{2}-2}\langle iz\rangle^{\frac{\gamma}{2}-2}}{1 + t^2\left(y\langle iy\rangle^{-1} - z\langle iz\rangle^{-1}\right)^2}\int_{-1}^1 \langle ix\rangle^{\gamma-2}dx \le \frac{C\langle iy\rangle^{\frac{\gamma}{2}-2}\langle iz\rangle^{\frac{\gamma}{2}-2}}{1 + t^2(y-z)^2}$$

and similarly

$$|G_2(t,y,z)| \le \frac{C\langle iy\rangle^{\frac{\gamma}{2}-4}\langle iz\rangle^{\frac{\gamma}{2}-4}}{1 + t^2(y-z)^2}.$$

Then

$$\begin{aligned}
&\left\|\langle ix\rangle^{2+\gamma}\partial_x\mathcal{B}\mathcal{V}(t)\mathcal{B}^{-1}\phi\right\|_{\mathbf{L}^2}^2 \le C|t|\int_{-1}^{1}\int_{-1}^{1}\left(\left|\langle iy\rangle^{\frac{\gamma}{2}-2}\psi'(y)\right|\left|\langle iz\rangle^{\frac{\gamma}{2}-2}\psi'(z)\right|\right.\\
&\left.+\left|\langle iy\rangle^{\frac{\gamma}{2}-4}\psi(y)\right|\left|\langle iz\rangle^{\frac{\gamma}{2}-4}\psi(z)\right|\right)\frac{dydz}{1+t^2(y-z)^2}\\
&= C\int_{\mathbf{R}}\frac{|t|\,dy}{1+t^2y^2}\int_{|z|<1,|y+z|<1}\left(\left|\langle i(y+z)\rangle^{\frac{\gamma}{2}-2}\psi'(y+z)\right|\left|\langle iz\rangle^{\frac{\gamma}{2}-2}\psi'(z)\right|\right.\\
&\left.+\left|\langle i(y+z)\rangle^{\frac{\gamma}{2}-4}\psi(y+z)\right|\left|\langle iz\rangle^{\frac{\gamma}{2}-4}\psi(z)\right|\right)dz\\
&\le C\left\|\langle ix\rangle^{\frac{\gamma}{2}-2}\partial_x\left(\langle ix\rangle^{\frac{3}{2}}\phi(x)\right)\right\|_{\mathbf{L}^2}^2 + C\left\|\langle ix\rangle^{\frac{\gamma-5}{2}}\phi(x)\right\|_{\mathbf{L}^2}^2 \le C\left\|\mathcal{B}^{-1}\phi\right\|_{\mathbf{H}^{1,4}}^2.
\end{aligned}$$

Since

$$\langle\xi\rangle^{1-\gamma}\partial_\xi\mathcal{V}(t) = \mathcal{B}^{-1}\langle ix\rangle^{\frac{1}{2}+\gamma}\partial_x\langle ix\rangle^{\frac{3}{2}}\mathcal{B}\mathcal{V}(t)\mathcal{B}^{-1}\mathcal{B}$$

we have the first estimate of the lemma. To prove the second estimate of the lemma (that is the estimate of the remainder in the asymptotics of the free evolution group inside of the light cone) we represent

$$\begin{aligned}
\langle\xi\rangle^{\frac{3}{2}}(\mathcal{V}(t)-1)\langle\xi\rangle^{-2}\phi = \frac{\sqrt{it}}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it\left(\langle\xi\rangle^{-1}+\frac{\xi}{\langle\xi\rangle}\eta-\langle\eta\rangle\right)}\phi(\eta)\langle\eta\rangle^{-2}d\eta\\
-\langle\xi\rangle^{-\frac{1}{2}}\phi(\xi).
\end{aligned}$$

In view of the asymptotics (see [5])

$$\frac{\sqrt{it}}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it\left(\langle\xi\rangle^{-1}+\frac{\xi}{\langle\xi\rangle}\eta-\langle\eta\rangle\right)}\langle\eta\rangle^{-2}d\eta = \langle\xi\rangle^{-\frac{1}{2}}+O\left(t^{-\frac{1}{2}}\right)$$

changing the variable of integration $\eta = y\langle iy\rangle^{-1}$ and putting $\chi = \xi\langle\xi\rangle^{-1}$ we have

$$\begin{aligned}
&\langle\xi\rangle^{\frac{3}{2}}(\mathcal{V}(t)-1)\langle\xi\rangle^{-2}\phi + O\left(t^{-\frac{1}{2}}\right)\\
&= \frac{\sqrt{it}}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it\left(\langle\xi\rangle^{-1}+\frac{\xi}{\langle\xi\rangle}\eta-\langle\eta\rangle\right)}(\phi(\eta)-\phi(\xi))\langle\eta\rangle^{-2}d\eta\\
&= \frac{\sqrt{it}}{\sqrt{2\pi}}\int_{-1}^{1} e^{it\left(\langle i\chi\rangle+\frac{\chi y-1}{\langle iy\rangle}\right)}\left(\phi\left(y\langle iy\rangle^{-1}\right)-\phi\left(\chi\langle i\chi\rangle^{-1}\right)\right)\langle iy\rangle^{-1}dy.
\end{aligned}$$

Then via the identity

$$e^{it\frac{\chi y-1}{\langle iy\rangle}} = A\langle iy\rangle^{3}\partial_y\left((y-\chi)e^{it\frac{\chi y-1}{\langle iy\rangle}}\right)$$

with $A = \left(\langle iy \rangle^3 - it\,(y-\chi)^2 \right)^{-1}$ we integrate by parts with respect to y

$$\begin{aligned}
&\langle \xi \rangle^{\frac{3}{2}} \left(\mathcal{V}(t) - 1 \right) \langle \xi \rangle^{-2} \phi + O\left(t^{-\frac{1}{2}} \right) \\
&= -\frac{\sqrt{it}}{\sqrt{2\pi}} \int_{-1}^{1} e^{it\frac{\chi y - 1}{\langle iy \rangle}} \left(A^2 \left(3y\,(y-\chi) \langle iy \rangle + 2it\,(y-\chi)^2 \right) \right. \\
&\quad \left. -2y\,(y-\chi)\,A \right) \left(\phi\left(y \langle iy \rangle^{-1} \right) - \phi\left(\chi \langle i\chi \rangle^{-1} \right) \right) \langle iy \rangle^2 \, dy \\
&\quad -\frac{\sqrt{it}}{\sqrt{2\pi}} \int_{-1}^{1} e^{it\frac{\chi y - 1}{\langle iy \rangle}} (y-\chi)\,A\phi'\left(y \langle iy \rangle^{-1} \right) \langle iy \rangle^{-1} \, dy.
\end{aligned}$$

We have

$$\begin{aligned}
&\left\| \langle iy \rangle^{-3} \phi'\left(y \langle iy \rangle^{-1} \right) \right\|_{\mathbf{L}^2_y(-1,1)} = \left(\int_{-1}^{1} \langle iy \rangle^{-6} \left| \phi'\left(y \langle iy \rangle^{-1} \right) \right|^2 dy \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbf{R}} \langle \xi \rangle^3 \left| \phi'(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} = \|\phi'\|_{\mathbf{H}^{0,\frac{3}{2}}}.
\end{aligned}$$

Furthermore by the inequality

$$\left| \phi\left(y \langle iy \rangle^{-1} \right) - \phi\left(\chi \langle i\chi \rangle^{-1} \right) \right| \le C \|\phi'\|_{\mathbf{H}^{0,\frac{3}{2}}} |y-\chi|^{\frac{1}{2}}$$

for all $\chi, y \in (-1,1)$ we have

$$\begin{aligned}
&\left| \langle \xi \rangle^{\frac{3}{2}} \left(\mathcal{V}(t) - 1 \right) \langle \xi \rangle^{-2} \phi \right| + O\left(t^{-\frac{1}{2}} \right) \\
&\le C t^{\frac{1}{2}} \|\phi\|_{\mathbf{L}^\infty} \int_{-1}^{1} \frac{|y-\chi|}{\langle iy \rangle^3 + t\,(y-\chi)^2} dy \\
&\quad + C t^{\frac{1}{2}} \|\phi'\|_{\mathbf{H}^{0,\frac{3}{2}}} \int_{-1}^{1} \frac{\langle iy \rangle^2 |y-\chi|^{\frac{1}{2}}}{\langle iy \rangle^3 + t\,(y-\chi)^2} dy \\
&\quad + C t^{\frac{1}{2}} \|\phi'\|_{\mathbf{H}^{0,\frac{3}{2}}} \left(\int_{-1}^{1} \left(\frac{\langle iy \rangle^2 |y-\chi|}{\langle iy \rangle^3 + t\,(y-\chi)^2} \right)^2 dy \right)^{\frac{1}{2}}.
\end{aligned}$$

We find

$$\begin{aligned}
&\int_{-1}^{1} \frac{|y-\chi| + \langle iy \rangle^2 |y-\chi|^{\frac{1}{2}}}{\langle iy \rangle^3 + t\,(y-\chi)^2} dy + \left(\int_{-1}^{1} \frac{\langle iy \rangle^4 (y-\chi)^2}{\left(\langle iy \rangle^3 + t\,(y-\chi)^2 \right)^2} dy \right)^{\frac{1}{2}} \\
&\le C t^{-\frac{3}{4}}.
\end{aligned}$$

This yields the second estimate of the lemma.

To prove the third estimate (i.e., estimates of the free evolution group outside of the light cone) we write for all $|x| > 1$ changing $\xi = y \langle iy \rangle^{-1}$

$$\mathcal{W}(t)\phi = \frac{\sqrt{t}}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{it(x\xi - \langle \xi \rangle)} \phi(\xi) \, d\xi$$
$$= \frac{\sqrt{t}}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{it\frac{xy-1}{\langle iy \rangle}} \phi\left(y \langle iy \rangle^{-1}\right) \langle iy \rangle^{-3} \, dy.$$

Employing the identity

$$e^{it\frac{xy-1}{\langle iy \rangle}} = \frac{\langle iy \rangle^3}{it(x-y)} \partial_\eta \left(e^{it\frac{xy-1}{\langle iy \rangle}}\right)$$

we integrate by parts with respect to y

$$\mathcal{W}(t)\phi$$
$$= \frac{1}{i\sqrt{2t\pi}} \int_{-1}^{1} e^{it\frac{xy-1}{\langle iy \rangle}} \left(\frac{1}{(y-x)^2} \phi\left(y \langle iy \rangle^{-1}\right) + \frac{\langle iy \rangle^{-3}}{y-x} \phi'\left(y \langle iy \rangle^{-1}\right) \right) dy.$$

Since

$$(y-x)^{-2} \le ||y|-1|^{-\frac{1}{2}} \langle x \rangle^{-1} \langle iy \rangle^{-3}$$

and

$$|y-x|^{-1} \le ||y|-1|^{-\frac{1}{4}} \langle x \rangle^{-1} \langle iy \rangle^{-\frac{3}{2}},$$

then

$$|\mathcal{W}(t)\phi| \le Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^{0,3}_\infty} \langle x \rangle^{-1} \int_{-1}^{1} ||y|-1|^{-\frac{1}{2}} \, dy$$
$$+ Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^{1,3}} \langle x \rangle^{-1} \left(\int_{-1}^{1} ||y|-1|^{-\frac{1}{2}} \, dy \right)^{\frac{1}{2}}.$$

Therefore

$$\|\mathcal{W}(t)\phi\|_{\mathbf{L}^r} \le Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^{1,3}},$$

where $2 \le r \le \infty$. This yields the third estimate of the lemma.

We now prove the last estimate. Changing $y = \eta \langle \eta \rangle^{-1}$ we have

$$\mathcal{V}^{-1}(t) \langle \xi \rangle^{-\frac{3}{4}-\gamma} \phi = \frac{\sqrt{t}}{\sqrt{2i\pi}} \int_{-1}^{1} e^{it(\langle \xi \rangle - \xi y - \langle iy \rangle)} \phi\left(y \langle iy \rangle^{-1}\right) \langle iy \rangle^{-\frac{3}{4}+\gamma} \, dy$$
$$= \frac{\sqrt{t}}{\sqrt{2i\pi}} \int_{\mathbf{R}} e^{it\left(\langle \xi \rangle - \frac{\xi\eta+1}{\langle \eta \rangle}\right)} \phi(\eta) \langle \eta \rangle^{-\frac{9}{4}-\gamma} \, d\eta.$$

In view of the asymptotics (see [5])

$$\frac{\sqrt{t}}{\sqrt{2i\pi}} \int_{\mathbf{R}} e^{it\left(\langle \xi \rangle - \frac{\xi\eta+1}{\langle \eta \rangle}\right)} \langle \eta \rangle^{-\frac{9}{4}-\gamma} \, d\eta = \langle \xi \rangle^{-\frac{3}{4}-\gamma} + O\left(t^{-\frac{1}{2}}\right)$$

we get

$$\begin{aligned}&\left(\mathcal{V}^{-1}(t)-1\right)\langle\xi\rangle^{-\frac{3}{4}-\gamma}\phi+O\left(t^{-\frac{1}{2}}\right)\\&=\frac{\sqrt{t}}{\sqrt{2i\pi}}\int_{\mathbf{R}}e^{it\left(\langle\xi\rangle-\frac{\xi\eta+1}{\langle\eta\rangle}\right)}\left(\phi(\eta)-\phi(\xi)\right)\langle\eta\rangle^{-\frac{9}{4}-\gamma}d\eta.\end{aligned}$$

Then via the identity

$$e^{-it\frac{\xi\eta+1}{\langle\eta\rangle}}=B\langle\eta\rangle^{3}\partial_{\eta}\left((\eta-\xi)e^{-it\frac{\xi\eta+1}{\langle\eta\rangle}}\right)$$

with $B=\left(\langle\eta\rangle^{3}+it(\eta-\xi)^{2}\right)^{-1}$ we integrate by parts with respect to η

$$\begin{aligned}&\left(\mathcal{V}^{-1}(t)-1\right)\langle\xi\rangle^{-\frac{3}{4}-\gamma}\phi+O\left(t^{-\frac{1}{2}}\right)\\&=-\frac{\sqrt{t}}{\sqrt{2i\pi}}\int_{\mathbf{R}}e^{it\left(\langle\xi\rangle-\frac{\xi\eta+1}{\langle\eta\rangle}\right)}\left(B^{2}\left(3\eta(\eta-\xi)\langle\eta\rangle^{\frac{7}{4}-\gamma}+2it(\eta-\xi)^{2}\langle\eta\rangle^{\frac{3}{4}-\gamma}\right)\right.\\&\quad\left.+\frac{3-4\gamma}{8}\eta\langle\eta\rangle^{-\frac{5}{4}-\gamma}(\eta-\xi)B\right)\left(\phi(\eta)-\phi(\xi)\right)d\eta\\&\quad-\frac{\sqrt{t}}{\sqrt{2i\pi}}\int_{\mathbf{R}}e^{it\left(\langle\xi\rangle-\frac{\xi\eta+1}{\langle\eta\rangle}\right)}(\eta-\xi)B\phi'(\eta)\langle\eta\rangle^{\frac{3}{4}-\gamma}d\eta.\end{aligned}$$

Furthermore by the inequality

$$\left|\left(\phi(\eta)-\phi(\xi)\right)\right|\le C\|\phi\|_{\mathbf{H}^{1}}|\eta-\xi|^{\frac{1}{2}}$$

we find

$$\begin{aligned}&\left|\left(\mathcal{V}^{-1}(t)-1\right)\langle\xi\rangle^{-\frac{3}{4}-\gamma}\phi\right|+O\left(t^{-\frac{1}{2}}\right)\\&\le Ct^{\frac{1}{2}}\|\phi\|_{\mathbf{L}^{\infty}}\int_{\mathbf{R}}\frac{|\eta-\xi|\langle\eta\rangle^{-\frac{1}{4}-\gamma}d\eta}{\langle\eta\rangle^{3}+t(\eta-\xi)^{2}}+Ct^{\frac{1}{2}}\|\phi\|_{\mathbf{H}^{1}}\int_{\mathbf{R}}\frac{\langle\eta\rangle^{\frac{3}{4}-\gamma}|\eta-\xi|^{\frac{1}{2}}d\eta}{\langle\eta\rangle^{3}+t(\eta-\xi)^{2}}\\&\quad+Ct^{\frac{1}{2}}\|\phi\|_{\mathbf{H}^{1}}\left(\int_{\mathbf{R}}\frac{\langle\eta\rangle^{\frac{3}{2}-2\gamma}(\eta-\xi)^{2}d\eta}{\left(\langle\eta\rangle^{3}+t(\eta-\xi)^{2}\right)^{2}}\right)^{\frac{1}{2}}.\end{aligned}$$

We have

$$\begin{aligned}&\int_{\mathbf{R}}\frac{|\eta-\xi|\langle\eta\rangle^{-\frac{1}{4}-\gamma}+|\eta-\xi|^{\frac{1}{2}}\langle\eta\rangle^{\frac{3}{4}-\gamma}}{\langle\eta\rangle^{3}+t(\eta-\xi)^{2}}d\eta+\left(\int_{\mathbf{R}}\frac{\langle\eta\rangle^{\frac{3}{2}-2\gamma}(\eta-\xi)^{2}d\eta}{\left(\langle\eta\rangle^{3}+t(\eta-\xi)^{2}\right)^{2}}\right)^{\frac{1}{2}}\\&\le Ct^{-\frac{3}{4}}.\end{aligned}$$

This yields the forth estimate of the lemma. Lemma 3.1 is proved. □

We next prove the time decay estimate in terms of the operator $\mathcal{J}$.

Lemma 3.2. *The estimate is valid*

$$\|\varphi\|_{\mathbf{L}^\infty} \le C \langle t\rangle^{-\frac{1}{2}} \|\varphi\|^{\frac{1}{2}}_{\mathbf{H}^{\frac{3}{2}}} \left(\|\varphi\|^{\frac{1}{2}}_{\mathbf{H}^{\frac{3}{2}}} + \|\mathcal{J}\varphi\|^{\frac{1}{2}}_{\mathbf{H}^{\frac{1}{2}}} \right)$$

for all $t \ge 0$, provided that the right-hand side is finite.

Proof. Since $\|\varphi\|_{\mathbf{L}^1} \le C \|x\varphi\|^{\frac{1}{2}}_{\mathbf{L}^2} \|\varphi\|^{\frac{1}{2}}_{\mathbf{L}^2}$, by applying the $\mathbf{L}^\infty - \mathbf{L}^1$ time decay estimate of the free evolution group $e^{-i\langle i\partial_x\rangle t}$ (see Lemma 1 in [17]) we get

$$\begin{aligned} \|\varphi\|_{\mathbf{L}^\infty} &= \left\| e^{-i\langle i\partial_x\rangle t} e^{i\langle i\partial_x\rangle t}\varphi \right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}} \left\| \langle i\partial_x\rangle^{\frac{3}{2}} e^{i\langle i\partial_x\rangle t}\varphi \right\|_{\mathbf{L}^1} \\ &\le Ct^{-\frac{1}{2}} \left\| x \langle i\partial_x\rangle^{\frac{3}{2}} e^{i\langle i\partial_x\rangle t}\varphi \right\|^{\frac{1}{2}}_{\mathbf{L}^2} \left\| \langle i\partial_x\rangle^{\frac{3}{2}} e^{i\langle i\partial_x\rangle t}\varphi \right\|^{\frac{1}{2}}_{\mathbf{L}^2} \\ &\le Ct^{-\frac{1}{2}} \|\mathcal{J}\varphi\|^{\frac{1}{2}}_{\mathbf{H}^{\frac{1}{2}}} \|\varphi\|^{\frac{1}{2}}_{\mathbf{H}^{\frac{3}{2}}} \end{aligned}$$

for all $t > 0$. Then by the Sobolev inequality we have $\|\varphi\|_{\mathbf{L}^\infty} \le C \|\varphi\|_{\mathbf{H}^1}$. Thus the desired estimate follows. Lemma 3.2 is proved. □

In the next lemma we obtain the asymptotics for the nonlinear term. Denote $\lambda_1 = \frac{3\mu}{2}$, $\lambda_2 = \frac{\mu}{2i}$, $\lambda_3 = \frac{3i\mu}{2}$, $\lambda_4 = -\frac{\mu}{2}$, $\omega_j = 2\alpha_j - 3$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 1$, $\alpha_4 = 0$.

Lemma 3.3. *Let $\phi \in \mathbf{H}^{4,1}$. Then the asymptotic formula for large time t holds*

$$\begin{aligned} &\mathcal{F}e^{i\langle i\partial_x\rangle t} \langle i\partial_x\rangle \mathcal{N}\left(e^{-i\langle i\partial_x\rangle t}\phi\right) \\ &\quad = it^{-1}e^{it\langle\xi\rangle} \sum_{j=1}^{4} \lambda_j \mathcal{D}_{\omega_j} e^{-it\omega_j\langle\xi\rangle} \langle\xi\rangle^3 \overline{\widehat{\phi}}^{3-\alpha_j}(\xi)\, \widehat{\phi}^{\alpha_j}(\xi) + O\left(t^{-\frac{5}{4}} \|\phi\|^3_{\mathbf{H}^{4,1}}\right) \end{aligned}$$

uniformly with respect to $\xi \in \mathbf{R}$.

Proof. First we prove the following representation

$$\mathcal{F}e^{i\langle i\partial_x\rangle t}\left(\overline{u}^{3-\alpha}(t)\, u^\alpha(t)\right) = -e^{-\frac{i\pi\alpha}{2}} t^{-1} e^{it\langle\xi\rangle} \mathcal{D}_\omega e^{-it\omega\langle\xi\rangle} \langle\xi\rangle^3 \overline{\varphi}^{3-\alpha}\varphi^\alpha + R, \quad (3.1)$$

where $\varphi = \mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)$, $\omega = 2\alpha - 3$, $\alpha \in [0,3]$, $\alpha \neq \frac{3}{2}$, and

$$\|R\|_{\mathbf{L}^\infty} \le Ct^{-\frac{5}{4}} \|\varphi\|^3_{\mathbf{H}^{1,4}}.$$

We introduce the operators

$$\mathcal{Q}(t) = \mathcal{B}\mathcal{V}(t) + \mathcal{W}(t) = \overline{M}\mathcal{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$$

so that we have the representation for the free evolution group

$$e^{-i\langle i\partial_x\rangle t}\mathcal{F}^{-1} = \mathcal{D}_t M \mathcal{Q}(t).$$

We also have

$$\mathcal{F}e^{i\langle i\partial_x\rangle t} = \mathcal{Q}^{-1}(t)\, \overline{M}\mathcal{D}_t^{-1},$$

where the inverse operator

$$\mathcal{Q}^{-1}(t) = \mathcal{V}^{-1}(t)\,\mathcal{B}^{-1} + \mathcal{W}^{-1}(t) = e^{it\langle\xi\rangle}\mathcal{F}\mathcal{D}_t M.$$

Since $\mathcal{D}_{\omega t} = \mathcal{D}_\omega \mathcal{D}_t$, $\mathcal{D}_\omega^{-1} = \mathcal{D}_{\frac{1}{\omega}}$ and $\mathcal{F}\mathcal{D}_{\frac{1}{\omega}} = \mathcal{D}_\omega \mathcal{F}$ we find

$$\begin{aligned}\mathcal{Q}^{-1}(t)\,M^{\omega-1} &= e^{it\langle\xi\rangle}\mathcal{F}\mathcal{D}_t M^\omega = e^{it\langle\xi\rangle}\mathcal{D}_\omega\mathcal{F}\mathcal{D}_{\omega t}M^\omega \\ &= e^{it\langle\xi\rangle}\mathcal{D}_\omega e^{-it\omega\langle\xi\rangle}\mathcal{Q}^{-1}(\omega t)\end{aligned}$$

with $\omega \neq 0$. Therefore putting

$$u(t) = U(t)\,\mathcal{F}^{-1}\varphi = \mathcal{D}_t M\mathcal{Q}(t)\,\varphi$$

and taking $\omega = 2\alpha - 3$ we get

$$\begin{aligned}&\mathcal{F}\mathcal{U}(-t)\left(\overline{u}^{3-\alpha}(t)\,u^\alpha(t)\right)\\ &= \mathcal{Q}^{-1}(t)\,\overline{M}\mathcal{D}_t^{-1}\left(\overline{\mathcal{D}_t M\mathcal{Q}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{D}_t M\mathcal{Q}(t)\,\varphi\right)^\alpha\\ &= t^{-1}\mathcal{Q}^{-1}(t)\,M^{2\alpha-4}\left(\overline{\mathcal{Q}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{Q}(t)\,\varphi\right)^\alpha\\ &= t^{-1}e^{it\langle\xi\rangle}\mathcal{D}_\omega e^{-it\omega\langle\xi\rangle}\mathcal{Q}^{-1}(\omega t)\left(\overline{\mathcal{Q}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{Q}(t)\,\varphi\right)^\alpha.\end{aligned}$$

Since $\mathcal{Q}(t) = \mathcal{B}\mathcal{V}(t)$ for $|x| \leq 1$ and $\mathcal{Q}(t) = \mathcal{W}(t)$ for $|x| \geq 1$ we then have

$$\begin{aligned}&\left(\overline{\mathcal{Q}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{Q}(t)\,\varphi\right)^\alpha\\ &= \left(\overline{\mathcal{B}\mathcal{V}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{B}\mathcal{V}(t)\,\varphi\right)^\alpha + \left(\overline{\mathcal{W}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{W}(t)\,\varphi\right)^\alpha\end{aligned}$$

for all $x \in \mathbf{R}$. In the same manner we obtain

$$\begin{aligned}&\mathcal{Q}^{-1}(\omega t)\left(\overline{\mathcal{Q}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{Q}(t)\,\varphi\right)^\alpha\\ &= \mathcal{V}^{-1}(\omega t)\,\mathcal{B}^{-1}\left(\overline{\mathcal{B}\mathcal{V}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{B}\mathcal{V}(t)\,\varphi\right)^\alpha\\ &\quad+\mathcal{W}^{-1}(\omega t)\left(\overline{\mathcal{W}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{W}(t)\,\varphi\right)^\alpha.\end{aligned}$$

Applying the identity

$$\mathcal{B}^{-1}\left(\overline{\mathcal{B}\varphi}\right)^{3-\alpha}\left(\mathcal{B}\varphi\right)^\alpha = -e^{-i\frac{\pi}{2}\alpha}\langle\xi\rangle^3\left(\overline{\varphi(\xi)}\right)^{3-\alpha}\varphi^\alpha(\xi)$$

we get

$$\begin{aligned}&\mathcal{F}\mathcal{U}(-t)\left(\overline{u}^{3-\alpha}(t)\,u^\alpha(t)\right)\\ &= -e^{-i\frac{\pi}{2}\alpha}t^{-1}e^{it\langle\xi\rangle}\mathcal{D}_\omega e^{-it\omega\langle\xi\rangle}\mathcal{V}^{-1}(\omega t)\langle\xi\rangle^3\left(\overline{\mathcal{V}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{V}(t)\,\varphi\right)^\alpha\\ &\quad+t^{-1}e^{it\langle\xi\rangle}\mathcal{D}_\omega e^{-it\omega\langle\xi\rangle}\mathcal{W}^{-1}(\omega t)\left(\overline{\mathcal{W}(t)\,\varphi}\right)^{3-\alpha}\left(\mathcal{W}(t)\,\varphi\right)^\alpha\\ &= -e^{-i\frac{\pi}{2}\alpha}t^{-1}e^{it\langle\xi\rangle}\mathcal{D}_\omega e^{-it\omega\langle\xi\rangle}\langle\xi\rangle^3\,\overline{\varphi}^{3-\alpha}\varphi^\alpha + R,\end{aligned}$$

where the remainder

$$
\begin{aligned}
R &= -e^{-i\frac{\pi}{2}\alpha}t^{-1}e^{it\langle\xi\rangle}\mathcal{D}_{\omega}e^{-it\omega\langle\xi\rangle} \\
&\times\left(\mathcal{V}^{-1}(\omega t)\langle\xi\rangle^{3}\left(\overline{\mathcal{V}(t)\varphi}\right)^{3-\alpha}(\mathcal{V}(t)\varphi)^{\alpha}-\langle\xi\rangle^{3}\overline{\varphi}^{3-\alpha}\varphi^{\alpha}\right) \\
&+t^{-1}e^{it\langle\xi\rangle}\mathcal{D}_{\omega}e^{-it\omega\langle\xi\rangle}\mathcal{W}^{-1}(\omega t)\left(\overline{\mathcal{W}(t)\varphi}\right)^{3-\alpha}(\mathcal{W}(t)\varphi)^{\alpha}.
\end{aligned}
$$

By Lemma 3.1 we obtain

$$
\begin{aligned}
&\left\|\left(\mathcal{V}^{-1}(\omega t)-1\right)\langle\xi\rangle^{3}\left(\overline{\mathcal{V}(t)\varphi}\right)^{3-\alpha}(\mathcal{V}(t)\varphi)^{\alpha}\right\|_{\mathbf{L}^{\infty}} \\
&\le Ct^{-\frac{1}{4}}\left\|\langle\xi\rangle^{\frac{15}{4}+\gamma}\left(\overline{\mathcal{V}(t)\varphi}\right)^{3-\alpha}(\mathcal{V}(t)\varphi)^{\alpha}\right\|_{\mathbf{H}^{1}} \\
&\le Ct^{-\frac{1}{4}}\left\|\langle\xi\rangle^{\frac{11}{8}+\gamma}\mathcal{V}(t)\varphi\right\|_{\mathbf{L}^{\infty}}^{2}\|\mathcal{V}(t)\varphi\|_{\mathbf{H}^{1,1-\gamma}}\le Ct^{-\frac{1}{4}}\|\varphi\|_{\mathbf{H}^{1,4}}^{3}
\end{aligned}
$$

if $0<\gamma<\frac{1}{8}$. Then

$$
\begin{aligned}
&\left\|\langle\xi\rangle^{3}\left(\left(\overline{\mathcal{V}(t)\varphi}\right)^{3-\alpha}(\mathcal{V}(t)\varphi)^{\alpha}-\overline{\varphi}^{3-\alpha}\varphi^{\alpha}\right)\right\|_{\mathbf{L}^{\infty}} \\
&\le C\left(\left\|\langle\xi\rangle^{\frac{3}{2}}(\mathcal{V}(t)-1)\varphi\right\|_{\mathbf{L}^{\infty}}^{2}+\left\|\langle\xi\rangle^{\frac{3}{2}}\varphi\right\|_{\mathbf{L}^{\infty}}^{2}\right)\|(\mathcal{V}(t)-1)\varphi\|_{\mathbf{L}^{\infty}} \\
&\le Ct^{-\frac{1}{4}}\|\varphi\|_{\mathbf{H}^{1,4}}^{3}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\mathcal{W}^{-1}(\omega t)\left(\overline{\mathcal{W}(t)\varphi}\right)^{3-\alpha}(\mathcal{W}(t)\varphi)^{\alpha}\right\|_{\mathbf{L}^{\infty}} \\
&\le C\sqrt{t}\|\mathcal{W}(t)\varphi\|_{\mathbf{L}^{3}}^{3}\le Ct^{-1}\|\varphi\|_{\mathbf{H}^{1,4}}^{3}.
\end{aligned}
$$

Thus we find the estimate for the remainder $R(t)$

$$
\begin{aligned}
\|R(t)\|_{\mathbf{L}^{\infty}} &\le Ct^{-1}\left\|\left(\mathcal{V}^{-1}(\omega t)-1\right)\langle\xi\rangle^{3}\left(\overline{\mathcal{V}(t)\varphi}\right)^{3-\alpha}(\mathcal{V}(t)\varphi)^{\alpha}\right\|_{\mathbf{L}^{\infty}} \\
&+Ct^{-1}\left\|\langle\xi\rangle^{3}\left(\left(\overline{\mathcal{V}(t)\varphi}\right)^{3-\alpha}(\mathcal{V}(t)\varphi)^{\alpha}-\overline{\varphi}^{3-\alpha}\varphi^{\alpha}\right)\right\|_{\mathbf{L}^{\infty}} \\
&+Ct^{-1}\left\|\mathcal{W}^{-1}(\omega t)\left(\overline{\mathcal{W}(t)\varphi}\right)^{3-\alpha}(\mathcal{W}(t)\varphi)^{\alpha}\right\|_{\mathbf{L}^{\infty}}\le Ct^{-\frac{5}{4}}\|\varphi\|_{\mathbf{H}^{1,4}}^{3}.
\end{aligned}
$$

Therefore representation (3.1) is true. We now take $u(t)=e^{-i\langle i\partial_x\rangle t}\phi$ in (3.1) to find

$$\mathcal{F}e^{i\langle i\partial_x\rangle t}\langle i\partial_x\rangle \mathcal{N}(u(t)) = -i\sum_{j=1}^{4}\lambda_j e^{\frac{i\pi\alpha_j}{2}}\mathcal{F}e^{i\langle i\partial_x\rangle t}\left(\overline{u}^{3-\alpha_j}(t)\,u^{\alpha_j}(t)\right)$$

$$= it^{-1}e^{it\langle\xi\rangle}\sum_{j=1}^{4}\lambda_j\mathcal{D}_{\omega_j}e^{-it\omega_j\langle\xi\rangle}\langle\xi\rangle^3\,\overline{\widehat{\phi}}^{3-\alpha_j}(\xi)\,\widehat{\phi}^{\alpha_j}(\xi) + O\left(t^{-\frac{5}{4}}\|\phi\|^3_{\mathbf{H}^{4,1}}\right).$$

Lemma 3.3 is proved. □

In order to prove that the second term $\frac{1}{2}i\mu\mathcal{F}e^{i\langle i\partial_x\rangle t}\left(u^3+\overline{u}^3+3|u|^2\overline{u}\right)$ of the right-hand side of (1.4) is a remainder we now consider the following ordinary differential equations for $t\geq 1$ depending on a parameter $\xi\in\mathbf{R}$

$$\begin{cases} \frac{d\phi}{dt} = i\lambda_1 t^{-1}|\phi|^2\phi + \frac{i}{t}e^{it\langle\xi\rangle}\sum_{j=2}^{4}\lambda_j\mathcal{D}_{\omega_j}e^{-it\omega_j\langle\xi\rangle}\phi^{\alpha_j}\overline{\phi}^{3-\alpha_j} + h(t), \\ \phi(1,\xi) = \phi^0, \end{cases} \tag{3.2}$$

where $\lambda_1\in\mathbf{R}$, λ_2, λ_3, $\lambda_4\in\mathbf{C}$, $\omega_j = 2\alpha_j - 3$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 1$, $\alpha_4 = 0$, and $h(t) = O\left(\varepsilon^3 t^{-\frac{5}{4}}\right)$ in $\mathbf{L}^\infty$. Note that the first term in the right-hand side of (3.2) is divergent when integrating over an unbounded time interval.

Lemma 3.4. *Let the initial data $\phi^0\in\mathbf{L}^\infty$ with a norm $\left\|\phi^0\right\|_{\mathbf{L}^\infty} = \varepsilon$. Then there exists $\varepsilon_0 > 0$ such that for all $0<\varepsilon<\varepsilon_0$ the initial value problem (3.2) has a unique solution $\phi\in\mathbf{C}([1,\infty);\mathbf{L}^\infty)$ satisfying the a-priori estimate*

$$\sup_{t\geq 1}\|\phi(t)\|_{\mathbf{L}^\infty} \leq C\left\|\phi^0\right\|_{\mathbf{L}^\infty}.$$

Proof. We consider the linearized version of (3.2)

$$\begin{cases} \frac{d\phi_n}{dt} = i\lambda_1 t^{-1}|\phi_{n-1}|^2\phi_n + \frac{i}{t}e^{it\langle\xi\rangle}\sum_{j=2}^{4}\lambda_j\mathcal{D}_{\omega_j}e^{-it\omega_j\langle\xi\rangle}\phi_{n-1}^{\alpha_j}\overline{\phi}_{n-1}^{3-\alpha_j} + h(t), \\ \phi_n(1,\xi) = \phi^0(\xi), \end{cases} \tag{3.3}$$

where $n\in\{0\}\cup\mathbf{N}$ and

$$\phi_0(t,\xi) = \phi^0(\xi).$$

We apply the contraction mapping principle to (3.3) in the set

$$\mathbf{Y}_\varepsilon = \left\{g\in\mathbf{C}([1,\infty);\mathbf{L}^\infty) : \|g\|_{\mathbf{Y}} = \sup_{t\geq 1}\|g(t)\|_{\mathbf{L}^\infty} + \sup_{t\geq 1}t\|\partial_t g(t)\|_{\mathbf{L}^\infty} \leq 4\varepsilon\right\},$$

where $\varepsilon = \|\phi_0\|_{\mathbf{L}^\infty}$. Multiplying both sides of (3.3) by $e^{-i\lambda_1|\phi_{n-1}|^2\log t}$ we obtain

$$\frac{d}{dt}\phi_n e^{-i\lambda_1|\phi_{n-1}|^2\log t}$$
$$= \frac{i}{t}e^{-i\lambda_1|\phi_{n-1}|^2\log t}e^{it\langle\xi\rangle}\sum_{j=2}^{4}\lambda_j\mathcal{D}_{\omega_j}e^{-it\omega_j\langle\xi\rangle}\phi_{n-1}^{\alpha_j}\overline{\phi}_{n-1}^{3-\alpha_j} + e^{-i\lambda_1|\phi_{n-1}|^2\log t}h(t)$$

We assume that $\phi_{n-1} \in \mathbf{Y}_{\varepsilon}$. Denote

$$\nu_j(\xi) \equiv \langle \xi \rangle - \omega_j \left\langle \xi \omega_j^{-1} \right\rangle = \langle \xi \rangle - \sqrt{\omega_j^2 + \xi^2} \operatorname{sign} \omega_j,$$

then we get $e^{it\langle \xi \rangle} \mathcal{D}_{\omega_j} e^{-it\omega_j \langle \xi \rangle} = e^{it\nu_j(\xi)} \mathcal{D}_{\omega_j}$ and using the identity

$$e^{it\nu_j(\xi)} = B_j \partial_t \left(t e^{it\nu_j(\xi)} \right)$$

with $B_j(t,\xi) = (1 + it\nu_j(\xi))^{-1}$ we represent

$$\begin{aligned} &\frac{d}{dt} \left(\phi_n e^{-i\lambda_1 |\phi_{n-1}|^2 \log t} - ite^{-i\lambda_1 |\phi_{n-1}|^2 \log t} \sum_{j=2}^{4} \lambda_j B_j e^{it\nu_j(\xi)} \mathcal{D}_{\omega_j} \phi_{n-1}^{\alpha_j} \overline{\phi}_{n-1}^{3-\alpha_j} \right) \\ &= -i \sum_{j=2}^{4} t\lambda_j e^{it\nu_j(\xi)} \partial_t \left(t^{-1} e^{-i\lambda_1 |\phi_{n-1}|^2 \log t} B_j \mathcal{D}_{\omega_j} \phi_{n-1}^{\alpha_j} \overline{\phi}_{n-1}^{3-\alpha_j} \right) \\ &\quad + e^{-i\lambda_1 |\phi_{n-1}|^2 \log t} h(t). \end{aligned} \tag{3.4}$$

We have

$$\begin{aligned} |B_j(t,\xi)| &\leq \frac{1}{1 + t|\nu_j(\xi)|} \leq Ct^{-1} |\nu_j(\xi)|^{-1} \frac{t|\nu_j(\xi)|}{1 + t|\nu_j(\xi)|} \\ &\leq Ct^{-1} |\nu_j(\xi)|^{-1} \leq Ct^{-1} \end{aligned}$$

and

$$|\partial_t B_j(t,\xi)| \leq C \frac{|\nu_j(\xi)|}{1 + t^2 |\nu_j(\xi)|^2} \leq Ct^{-2} |\nu_j(\xi)|^{-1} \leq Ct^{-2}.$$

Then integrating (3.4) with respect to time we find

$$\|\phi_n(t)\|_{\mathbf{L}^\infty} \leq \varepsilon + C\varepsilon^3$$

and by (3.3)

$$\begin{aligned} t \|\partial_t \phi_n(t)\|_{\mathbf{L}^\infty} &\leq C \left(\|\phi_{n-1}(t)\|_{\mathbf{L}^\infty}^2 \|\phi_n(t)\|_{\mathbf{L}^\infty} + \|\phi_{n-1}(t)\|_{\mathbf{L}^\infty}^3 \right) + C\varepsilon^3 t^{-\frac{1}{4}} \\ &\leq C\varepsilon^3. \end{aligned}$$

Therefore

$$\sup_{t\geq 1} \|\phi_n\|_{\mathbf{L}^\infty} + \sup_{t\geq 1} t \|\partial_t \phi_n\|_{\mathbf{L}^\infty} \leq \varepsilon + C\varepsilon^3 \leq 2\varepsilon$$

since $\varepsilon > 0$ is sufficiently small. Therefore $\phi_n \in \mathbf{Y}_{\varepsilon}$ for any n. In the same manner by (3.4)

$$\begin{aligned}
&\left|\phi_{n+1}(t)\, e^{-i\lambda_1|\phi_n|^2\log t} - \phi_n(t)\, e^{-i\lambda_1|\phi_{n-1}|^2\log t}\right| \\
\le &\left| ite^{-i\lambda_1|\phi_n|^2\log t}\sum_{j=2}^{4}\lambda_j B_j e^{it\nu_j(\xi)}\mathcal{D}_{\omega_j}\phi_n^{\alpha_j}\overline{\phi}_n^{3-\alpha_j}\right. \\
&\left. -ite^{-i\lambda_1|\phi_{n-1}|^2\log t}\sum_{j=2}^{4}\lambda_j B_j e^{it\nu_j(\xi)}\mathcal{D}_{\omega_j}\phi_{n-1}^{\alpha_j}\overline{\phi}_{n-1}^{3-\alpha_j}\right| \\
&+C\int_1^t\left|\sum_{j=2}^{4} t\lambda_j e^{it\nu_j(\xi)}\partial_t\left(t^{-1}e^{-i\lambda_1|\phi_n|^2\log t}B_j\mathcal{D}_{\omega_j}\phi_n^{\alpha_j}\overline{\phi}_n^{3-\alpha_j}\right.\right. \\
&\left.\left. -t^{-1}e^{-i\lambda_1|\phi_{n-1}|^2\log t}B_j\mathcal{D}_{\omega_j}\phi_{n-1}^{\alpha_j}\overline{\phi}_{n-1}^{3-\alpha_j}\right)\right| \\
&+\left|\left(e^{-i\lambda_1|\phi_n|^2\log t} - e^{-i\lambda_1|\phi_{n-1}|^2\log t}\right)h(t)\right| dt \\
\le &C\varepsilon^2\left(\sup_{t\ge 1}\|\phi_n-\phi_{n-1}\|_{\mathbf{L}^\infty} + \sup_{t\ge1} t\,\|\partial_t(\phi_n-\phi_{n-1})\|_{\mathbf{L}^\infty}\right)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&|\phi_{n+1}(t,\xi) - \phi_n(t,\xi)| = \left|(\phi_{n+1}(t)-\phi_n(t))\, e^{-i\lambda_1|\phi_n|^2\log t}\right| \\
\le &\left|\phi_{n+1}(t)\, e^{-i\lambda_1|\phi_n|^2\log t} - \phi_n(t)\, e^{-i\lambda_1|\phi_{n-1}|^2\log t}\right| \\
&+\left|\phi_n(t)\left(e^{-i\lambda_1|\phi_n|^2\log t} - e^{-i\lambda_1|\phi_{n-1}|^2\log t}\right)\right| \\
\le &C\varepsilon^2\left(\sup_{t\ge 1}\|\phi_n-\phi_{n-1}\|_{\mathbf{L}^\infty} + \sup_{t\ge1} t\,\|\partial_t(\phi_n-\phi_{n-1})\|_{\mathbf{L}^\infty}\right).
\end{aligned} \tag{3.5}$$

We also have

$$\begin{aligned}
&|\partial_t\phi_{n+1}(t,\xi) - \partial_t\phi_n(t,\xi)| \\
\le &Ct^{-1}\varepsilon^2\left(\sup_{t\ge 1}\|\phi_n-\phi_{n-1}\|_{\mathbf{L}^\infty} + \sup_{t\ge1} t\,\|\partial_t(\phi_n-\phi_{n-1})\|_{\mathbf{L}^\infty}\right).
\end{aligned} \tag{3.6}$$

Thus by (3.5) and (3.6) we get

$$\|\phi_{n+1}-\phi_n\|_{\mathbf{Y}} \le \frac{1}{2}\|\phi_n-\phi_{n-1}\|_{\mathbf{Y}}$$

which means that $\{\phi_n\}$ is a Cauchy sequence in $\mathbf{Y}_\varepsilon$. Lemma 3.4 is proved. □

4. Remark

In this section we consider briefly the Klein–Gordon equation with dissipative cubic nonlinearity

$$\begin{cases} v_{tt} + v - v_{xx} = -v_t^3, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ v(0) = v_0, v_t(0) = v_1, x \in \mathbf{R}. \end{cases} \tag{4.1}$$

The usual energy method yields the space-time estimate of solutions

$$\int_0^t \|v_t(\tau)\|_{\mathbf{L}^4}^4 \, d\tau \le C\left(\|v_0\|_{\mathbf{H}^1}^2 + \|v_1\|_{\mathbf{L}^2}^2\right)$$

which implies dissipativity. Sunagawa studied in [28] the asymptotic behavior of small solutions of (4.1) under the condition that the initial data are regular, small and have a compact support. He showed that solutions have a more rapid time decay of order $(t\log(1+t))^{-\frac{1}{2}}$. By applying our method to the problem (4.1) we can remove the compactness of the initial data. If we put $u = \frac{1}{2}\left(v + i\langle i\partial_x\rangle^{-1} v_t\right)$, then for the real-valued function v the nonlinear Klein–Gordon equation (4.1) can be written as

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u), \ \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{4.2}$$

where $\mathcal{L} = \partial_t + i\langle i\partial_x\rangle$, and $\mathcal{N}(u) = -\frac{1}{2}i\langle i\partial_x\rangle^{-1}(-i\langle i\partial_x\rangle(u-\overline{u}))^3$. In the same way as in the proof of Theorem 1.1 we find that

$$\begin{aligned} &\mathcal{N}(u) = \frac{1}{2}\langle i\partial_x\rangle^{-1}\left(\left(\langle i\partial_x\rangle u - \overline{\langle i\partial_x\rangle u}\right)^3\right) \\ &= \frac{1}{2}\langle i\partial_x\rangle^{-1}\left((\langle i\partial_x\rangle u)^3 - \left(\overline{\langle i\partial_x\rangle u}\right)^3 - 3|\langle i\partial_x\rangle u|^2\langle i\partial_x\rangle u + 3|\langle i\partial_x\rangle u|^2\overline{\langle i\partial_x\rangle u}\right) \\ &= -\frac{3}{2}\langle i\partial_x\rangle^{-1}|\langle i\partial_x\rangle u|^2\langle i\partial_x\rangle u + R, \end{aligned}$$

where R is a remainder. By Lemma 3.3 we have

$$\begin{aligned} &\left(\langle\xi\rangle^m \mathcal{F}e^{i\langle i\partial_x\rangle t}u\right)_t \\ &= -\frac{3}{2}t^{-1}\langle\xi\rangle^{5-2m}\left|\langle\xi\rangle^m \mathcal{F}e^{i\langle i\partial_x\rangle t}u\right|^2\langle\xi\rangle^m \mathcal{F}e^{i\langle i\partial_x\rangle t}u \\ &\quad -\frac{1}{2}\langle\xi\rangle^m \mathcal{F}e^{i\langle i\partial_x\rangle t}\left((\langle i\partial_x\rangle u)^3 - \left(\overline{\langle i\partial_x\rangle u}\right)^3 - 3|\langle i\partial_x\rangle u|^2\langle i\partial_x\rangle u\right) \\ &\quad +O\left(t^{-\frac{5}{4}}\|u\|_{\mathbf{H}^{3+m,1}}^3\right), \end{aligned}$$

where $m \ge 3$. We let $v(t) = \langle\xi\rangle^m \mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)$. Then v satisfies the ordinary differential equation

$$v_t = -\frac{3}{2}t^{-1}\langle\xi\rangle^{5-2m}|v|^2 v + R.$$

We define

$$w(t) = \frac{v(1)}{\sqrt{1 + 3\langle\xi\rangle^{5-2m}|v(1)|^2 \log t}}$$

which is a solution of the Cauchy problem

$$\begin{cases} \frac{dw}{dt} = -\frac{3}{2}t^{-1}\langle\xi\rangle^{5-2m}|w|^2 w, \\ w(1) = v(1). \end{cases}$$

We can find a solution v in the neighborhood of w. Then the estimate

$$\left\| \langle\xi\rangle^m \mathcal{F} e^{i\langle i\partial_x\rangle t} u(t) \right\|_{\mathbf{L}^\infty} \le C \frac{|v(1)|}{\sqrt{1 + 3\langle\xi\rangle^{5-2m}|v(1)|^2 \log t}} \le C(\log(1+t))^{-\frac{1}{2}}$$

follows, which leads to the time decay rate $(t\log(1+t))^{-\frac{1}{2}}$ for solutions.

Acknowledgments

The authors would like to thank the referee for useful comments.

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(Received: January 23, 2007; revised: July 13, 2007)

Published Online First: November 24, 2007