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# Travelling wave fronts in nonlocal delayed reaction-diffusion systems and applications

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**Abstract.** This paper is concerned with the travelling wave fronts of nonlocal reaction-diffusion systems with delays. The existence of travelling wave fronts for nonlocal reaction-diffusion systems with delays is established by using Schauder's fixed point theorem and upper-lower solution technique. Then these results are applied to the nonlocal delayed Logistic model and the delayed Belousov-Zhabotinskii reaction-diffusion system. Our results show that the time delay can reduce the minimal wave speed while the nonlocality can increase the minimal wave speed.

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## 1. Introduction

In 1937, Fisher [17] proposed the following reaction-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t)[1-u(x,t)] \tag{1.1}$$

to model the advance of a favorable gene in an infinite one-dimensional habitat in which the process of nature select and random spatial migration were apparent. Since then, the theory of reaction-diffusion equations attracts much attention due to its significant nature in mathematical theory and practical fields, see, e.g., Ashwin et al. [1], Britton [5], Li and Wang [24], Murray [28], Smoller [33], Wang et al. [37, 38], Wu [39], Ye and Li [42].

In fact, (1.1) is an approximate description of the original problem in the work of Fisher [17]. More precisely, the author considered that the gene was distributed on real line $\mathbb{R}$ and assumed that the gene locating at position $x$ affected by the

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gene locating at $y$ with an impact factor $J(x-y)$ and obtained the following model

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]dy + ru(x,t)[1-u(x,t)]. \tag{1.2}$$

The approximation is due to the fact that the nonlocal operator shares many properties of the Laplacian one and in some limiting case reduces to it. Namely, let $J(x-y) = \frac{1}{\varepsilon}P\left(\frac{x-y}{\varepsilon}\right)$, where $P(x)$ is a general mollification function with support $x \in [-1,1]$ and $P$ is an even function. If $u \in C^3(\mathbb{R},\mathbb{R})$, then the Taylor's formula implies that

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{1}{\varepsilon}P\left(\frac{x-y}{\varepsilon}\right)[u(y)-u(x)]dy \\
= \;& \int_{\mathbb{R}} \frac{1}{\varepsilon}P\left(\frac{x-y}{\varepsilon}\right)\left[u'(x)(y-x)+u''(x)(y-x)^2+o((y-x)^2)\right]dy \\
= \;& u''(x)\int_{\mathbb{R}} \frac{1}{\varepsilon}P\left(\frac{x-y}{\varepsilon}\right)(y-x)^2dy + o(\varepsilon^2) \\
\to \;& \alpha u''(x) \text{ as } \varepsilon \to 0,
\end{aligned}$$

where $\alpha > 0$ is given by $\int_{\mathbb{R}} \frac{1}{\varepsilon}P\left(\frac{x-y}{\varepsilon}\right)(y-x)^2dy$. Note that the above derivation implies that the Laplacian operator only averages the neighboring densities, which shows that the Laplacian operator is not sufficiently accurate in some cases, see Hopf [18], Morse and Feshback [27]. At the same time, Murray [28, pp. 244-246] pointed out that (1.1) and the general reaction-diffusion equations

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) \tag{1.3}$$

are strictly only applicable to dilute systems and population models. But in many biological areas, such as the embryological development case, the densities of cells involved are not small and a local or short range diffusive flux proportional to the gradient is not sufficiently accurate. One method in overcoming the shortcoming of the Laplacian operator is to describe these models concerning with the spatial migration by integral equation, such as the following model

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]dy + f(u(x,t)), \tag{1.4}$$

which seems to be the same as the description of the original problem in Fisher [17]. To the best of our knowledge, the model similar to (1.4) arises not only from the biological model, but also from many other practical fields, such as the phase transition model [2], Ising model [14, 15, 30], network model [16], thalamic model [8], activator-inhibitor model [29] and lattice dynamical systems [9, 10], see also the surveys by Chen [7]. For the recent work concerning the nonlocal reaction-diffusion equation, one can refer to Bates et al. [3, 4] and the references cited therein.

We know that the time delay seems to be inevitable in modelling many phenomena, see, e.g., Wu and Zou [40] for the delayed lattice dynamical systems and

Hsu and Yang [19] for the cellular neural networks with multiple time delays. As we have mentioned, these models also can induce equations similar to (1.4). Motivated by these, we consider the following nonlocal reaction-diffusion *systems* with delay

$$\frac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - (J * I)\, u(x,t) + f(u_t(x)), \tag{1.5}$$

where $x \in \mathbb{R}, t \geq 0, u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n$, $I = \text{diag}(1, 1, \cdots, 1)$, $u_t(x)$ is an element in $C([-\tau, 0], \mathbb{R}^n)$ parametrized by $x \in \mathbb{R}$ and is defined by $u(x, t+s)$ for $s \in [-\tau, 0]$ in which $\tau > 0$ denotes the maximal time delay [39], $f$ maps $C([-\tau, 0], \mathbb{R}^n)$ to $\mathbb{R}^n$, $J * u$ means the convolution of $J$ and $u$ with respect to $x \in \mathbb{R}$, which is defined by $J * u = (J_1 * u_1, J_2 * u_2, \cdots, J_n * u_n)^T$, where $J_i$ satisfies $\int_{\mathbb{R}} J_i(x)dx < \infty$ for $i = 1, 2, \cdots, n$, and $J * I = \text{diag}(J_1 * 1, \cdots, J_n * 1) = \text{diag}(\int_{\mathbb{R}} J_1(x)dx, \cdots, \int_{\mathbb{R}} J_n(x)dx)$

It is well known that travelling wave solutions play an important role in the parabolic systems due to its significant nature in mathematical theory and practical fields, see Volpert et al. [36] and Wu [39]. The same case holds for the nonlocal reaction-diffusion equation. More precisely, travelling wave solutions can determine the long time behavior of the corresponding initial value problem, see e.g., Chen [7]. At the same time, travelling wave solutions in nonlocal reaction-diffusion equations also were used to describe phenomena in physical process and other fields, see, e.g., Bates et al. [2] for a phase transition model. If the time delay vanishes and $n = 1$, then (1.5) reduces to

$$\frac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - (J * 1)u(x,t) + f(u(x,t)). \tag{1.6}$$

There are many works concerning with travelling wave solutions of (1.6), for example, Carr and Chmaj [6], Chen [7] and Coville et al. [11, 12, 13], Schumacher [32] and the references cited therein. Note that the comparison principle often is invalid for the *delayed system*, so their method for the existence of travelling wave fronts of (1.6) can not directly apply to (1.5). This constitutes the purpose of the current paper.

In order to consider the existence of travelling wave fronts of (1.5), in Section 2, we will transform the existence of travelling wave front of (1.5) to the existence of fixed points of an operator, the technique is similar to that of Wu and Zou [40, 41]. In Section 3, the upper-lower solution technique and the Schauder's fixed point theorem are applied to prove the existence of the fixed point of the operator, see the similar method by Li et al. [22] and Ma [26]. As applications, we give two examples in the last section. We first consider an equation which was established by incorporating time delay into (1.2) and the existence of travelling wave fronts is proved by the results in Section 3. In order to consider the minimal wave speed, we also use the theory of asymptotic spreading developed in [25, 34]. Our results imply that the time delay *decreases* the minimal wave speed which also was reported by Schaaf [31] and Zou [44] for *local* reaction-diffusion equations while nonlocality *increases* the minimal wave speed which was reported by Li et al. [23] for a reaction-diffusion equation with *nonlocal delays*. We also consider a system, which can be regarded as the nonlocal reaction-diffusion system of the Belousov–Zhabotinskii model.

## 2. Preliminaries

In this paper, we will use the usual notations for the standard partial ordering and order interval in $\mathbb{R}^n$. That is, for $u = (u_1, u_2, \cdots, u_n)^T$ and $v = (v_1, v_2, \cdots, v_n)^T$, we denote $u \le v$ if $u_i \le v_i$, $i = 1, 2, \cdots, n$, and $u < v$ if $u \le v$ but $u \ne v$. In particular, we denote $u \ll v$ if $u \le v$ but $u_i \ne v_i, i = 1, 2, \cdots, n$. If $u < v$ holds, then we also denote $(u, v) = \{w \in \mathbb{R}^n, u < w < v\}$, $[u, v] = \{w \in \mathbb{R}^n, u \le w \le v\}$, $[u, v) = \{w \in \mathbb{R}^n, u \le w < v\}$, $(u, v] = \{w \in \mathbb{R}^n, u < w \le v\}$.

A *travelling wave solution* of (1.1) is a solution of the special form $u(x, t) = \Phi(x + ct)$, where the speed parameter $c$ is positive and $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C^1(\mathbb{R}, \mathbb{R}^n)$ is the wave profile. If we denote $\xi = x+ct$ and substitute $u(x, t) = \Phi(\xi)$ into (1.1), then system (1.1) becomes

$$c\Phi'(t) = (J * \Phi)(t) - (J * I)\Phi(t) + f^c(\Phi_t), \tag{2.1}$$

where $f^c(\Phi_t)$ is defined by $f(\Phi(t + cs))$ for $s \in [-\tau, 0]$, and $'$ denotes the differentiation with respect to the travelling wave variable $t$ (replace $\xi$ by $t$ without confusion). Recalling the physical and ecological motivation [2, 17], we require that the travelling wave solution $\Phi$ is nonnegative and satisfies the asymptotic boundary conditions:

$$\lim_{t\to-\infty} \Phi(t) = \Phi_-, \lim_{t\to+\infty} \Phi(t) = \Phi_+. \tag{2.2}$$

If $\Phi$ is nondecreasing, then it is called a *travelling wave front*.

**Remark 2.1.** It is clear that $f^c$ in system (2.1) maps $C([-c\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$.

If our main concern in this section is to establish the existence of travelling wave front of (1.1), then it is equivalent to find a monotone solution of (2.1) which satisfies (2.2). Without loss of generality, let $\Phi_- = (0, 0, \cdots, 0)^T \ll \Phi_+ = (k_1, k_2, \cdots, k_n)^T = K$, then (2.2) can be rewritten as

$$\lim_{t\to-\infty} \Phi(t) = 0, \lim_{t\to+\infty} \Phi(t) = K. \tag{2.3}$$

In this paper, we are interested in monotone solutions of (2.1) and (2.3) when the nonlinearity terms satisfy the following quasimonotonicity (QM).

**(QM)** There exists a matrix $\beta = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n) \gg 0$ such that

$$f^c(\Phi) - f^c(\Psi) + \beta[\Phi(0) - \Psi(0)] \ge (J * I)[\Phi(0) - \Psi(0)],$$

where $\Phi(s)$, $\Psi(s) \in C([-c\tau, 0], \mathbb{R}^n)$ with $0 \le \Psi(s) \le \Phi(s) \le K$ for $s \in [-c\tau, 0]$.

Let $C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ be

$$C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) = \{u : u(t) \in C(\mathbb{R}, \mathbb{R}^n), 0 \le u(t) \le K \text{ for all } t \in \mathbb{R}\}.$$

Define the operator $H = (H_1, \cdots, H_n)^T : C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)$ by

$$H(\Phi)(t) = (J * \Phi)(t) + (\beta - J * I)\Phi(t) + f^c(\Phi_t), t \in \mathbb{R}.$$

Thus, (2.1) can be rewritten as

$$c\Phi'(t) = -\beta\Phi(t) + H(\Phi)(t). \tag{2.4}$$

By (2.4), we consider the following integral equation

$$\Phi(t) = \frac{1}{c} e^{-\frac{\beta}{c}t} \int_{-\infty}^{t} e^{\frac{\beta}{c}s} H(\Phi)(s)ds, \tag{2.5}$$

which is well defined if $\Phi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ and $\beta \gg 0$. Moreover, it is easy to prove that the continuous solution $\Phi$ of (2.5) satisfies (2.1). Thus, the existence of monotone solutions of (2.1) and (2.3) is transformed into that of (2.5) and (2.3). For convenience, we use the notation $F = (F_1, \cdots, F_n)^T$ by

$$F(\Phi)(t) = \frac{1}{c} e^{-\frac{\beta}{c}t} \int_{-\infty}^{t} e^{\frac{\beta}{c}s} H(\Phi)(s)ds,$$

then the problem is changed into investigating the fixed point of $F$.

For $\mu \in \left(0, \min_{1 \le i \le n} \left\{\frac{\beta_i}{c}\right\}\right)$, define

$$B_\mu(\mathbb{R}, \mathbb{R}^n) = \left\{ u(t) : u(t) \in C(\mathbb{R}, \mathbb{R}^n) \text{ and } \sup_{t \in \mathbb{R}} |u(t)| e^{-\mu|t|} < \infty \right\},$$

where $|\cdot|$ denotes the super norm in $\mathbb{R}^n$. It is clear that $B_\mu(\mathbb{R}, \mathbb{R}^n)$ is a Banach space equipped with the norm $|\cdot|_\mu$ defined by

$$|u|_\mu = \sup_{t \in \mathbb{R}} |u(t)| e^{-\mu|t|} \text{ for } u \in B_\mu(\mathbb{R}, \mathbb{R}^n).$$

For the reader's convenience, we give the following assumptions on (2.1) and all of them are always imposed throughout this paper.

**(H1)** $f(\widehat{0}) = f(\widehat{K}) = 0$, where $\widehat{u}$ denotes the constant value function in $C([-c\tau, 0], \mathbb{R}^n)$;

**(H2)** For any $u, v \in C([-c\tau, 0], \mathbb{R}^n)$ and $0 \le u, v \le K$, there exists a constant $L > 0$ such that

$$|f^c(u) - f^c(v)| \le L \|u - v\|,$$

in which $\|\cdot\|$ denotes the super norm in $C([-c\tau, 0], \mathbb{R}^n)$;

**(H3)** $(J * u)(t) \ge (J * v)(t)$ if $u, v \in C(\mathbb{R}, \mathbb{R}^n)$ with $u \ge v$;

**(H4)** For $\mu$ given above and $i = 1, 2, \cdots, n$, $\int_{-\infty}^{\infty} J_i(x) e^{\mu|x|} dx < \infty$.

## 3. Main results

Our main purpose in this section is to establish the existence of the monotone solution of (2.1) and (2.3) when the delayed reaction term $f^c(\Phi)$ satisfies the condition (QM). Before formulating the main result, we give the definition of upper and lower solutions of (2.1) as follows (in fact, it depends on the monotone condition (QM)).

**Definition 3.1.** A continuous function $\overline{\Phi}(t) = \left(\overline{\phi}_1(t), \cdots, \overline{\phi}_n(t)\right)^T \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ is called an **upper solution** of (2.1) if it is differentiable on $t \in \mathbb{R} \setminus \mathbb{T}$ and satisfies

$$c\overline{\Phi}'(t) \geq (J * \overline{\Phi})(t) - \overline{\Phi}(t) + f^c(\overline{\Phi}_t), t \in \mathbb{R}\backslash\mathbb{T}, \tag{3.1}$$

where $\mathbb{T} = \{T_1, T_2, \cdots, T_k\}$ with $T_1 < T_2 < \cdots < T_k$. **Lower solution** of (2.1) can be similarly defined by only reversing the inequality in (3.1).

Throughout this section, we assume that (2.1) has an upper solution $\overline{\Phi}(t)$ and a lower solution $\underline{\Phi}(t)$ satisfying the following hypotheses:
**(P1)** $0 < \underline{\Phi}(t) \leq \overline{\Phi}(t) \leq K$;
**(P2)** $\sup_{s \leq t} \underline{\Phi}(s) \leq \overline{\Phi}(t)$ for all $t \in \mathbb{R}$.

Now we formulate our main result as follows.

**Theorem 3.2.** *Assume that (QM) holds. If* (2.1) *has an upper solution* $\overline{\Phi}(t)$ *and a lower solution* $\underline{\Phi}(t)$ *satisfying (P1)-(P2), in addition, we assume that* $f(\widehat{u}) \neq 0$ *if* $u \in \left(0, \inf_{t \in \mathbb{R}} \overline{\Phi}(t)\right] \cup [\sup_{t \in \mathbb{R}} \underline{\Phi}(t), K)$*, then* (2.1) *has a monotone solution* $\Phi(t)$ *satisfying* (2.3)*, which is a travelling wave front of* (1.1)*.*

Before proving the theorem, we give some nice properties of operators $H$ and $F$ defined in Section 2.

**Lemma 3.3.** *Assume that (QM) holds. Then*
**(i)** $H(\Psi)(t) \leq H(\Phi)(t)$, $F(\Psi)(t) \leq F(\Phi)(t)$ *for* $t \in \mathbb{R}$;
**(ii)** $0 \leq H(\Phi)(t) \leq \beta K$, $0 \leq F(\Phi)(t) \leq K$ *for* $t \in \mathbb{R}$;
**(iii)** $H(\Phi)(t)$ *and* $F(\Phi)(t)$ *is nondecreasing if* $\Phi(t)$ *is nondecreasing on* $(-\infty, \infty)$, *where* $\Psi(t), \Phi(t)$ *satisfy* $0 \leq \Psi(t) \leq \Phi(t) \leq K$ *for* $t \in \mathbb{R}$.

*Proof.* By the conditions (QM) and (H3), then

$$\begin{aligned} & H(\Phi)(t) - H(\Psi)(t) \\ = \; & [J * (\Phi - \Psi)](t) + (\beta - J * I)(\Phi - \Psi)(t) + f^c(\Phi_t) - f^c(\Psi_t) \\ \geq \; & 0, \; t \in \mathbb{R}, \end{aligned}$$

which also implies that $F(\Phi)(t) \geq F(\Psi)(t)$ holds for all $t \in \mathbb{R}$. (ii) and (iii) are direct consequences of (i). The proof is complete. □

Now we define the closed set $\Gamma$ consisting of wave profiles by

$$\Gamma = \left\{ \Phi(t) \in C(\mathbb{R}, \mathbb{R}^n) \;\middle|\; \begin{array}{l} \text{(a) } \Phi(t) \text{ is nondecreasing} \\ \text{(b) } \underline{\Phi}(t) \leq \Phi(t) \leq \overline{\Phi}(t) \end{array} \right\}.$$

From the definition of $\Gamma$ and the condition (P2), the following result is obvious.

**Lemma 3.4.** $\Gamma$ *is a bounded, closed, nonempty and convex subset of* $C(\mathbb{R}, \mathbb{R}^n)$ *with respect to the decay norm* $|\cdot|_\mu$.

**Lemma 3.5.** *$F : C_{[0,K]}(\mathbb{R},\mathbb{R}^n) \to C(\mathbb{R},\mathbb{R}^n)$ is continuous with respect to the norm $|\cdot|_\mu$.*

*Proof.* For any $\Phi = (\phi_1,\cdots,\phi_n)^T, \Psi = (\psi_1,\cdots,\psi_n)^T \in C_{[0,K]}(\mathbb{R},\mathbb{R}^n)$,

$$\begin{aligned}
& |F_i(\Phi)(t) - F_i(\Psi)(t)|\, e^{-\mu|t|} \\
\leq\ & \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)}\, |H_i(\Phi)(s) - H_i(\Psi)(s)|\, ds \\
\leq\ & \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)} \int_{-\infty}^{\infty} J_i(s-x)\, |\phi_i(x) - \psi_i(x)|\, dxds \\
& + \frac{e^{-\mu|t|}}{c}\left|\beta_i - \int_{-\infty}^{\infty} J_i(x)dx\right| \int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)}\, |\phi_i(s) - \psi_i(s)|\, ds \\
& + \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)}\, |f_i^c(\Phi_s) - f_i^c(\Psi_s)|\, ds \\
=:\ & I + II + III,
\end{aligned}$$

in which the definitions of $I, II, III$ are clear. Furthermore,

$$\begin{aligned}
I &= \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)} \int_{-\infty}^{\infty} J_i(s-x)\, |\phi_i(x) - \psi_i(x)|\, dxds \\
&= \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)} \int_{-\infty}^{\infty} J_i(s-x)\, |\phi_i(x) - \psi_i(x)|\, e^{-\mu|x|}e^{\mu|x|}dxds \\
&\leq |\Phi - \Psi|_\mu \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)} \int_{-\infty}^{\infty} J_i(s-x)e^{\mu|x|}dxds \\
&\leq |\Phi - \Psi|_\mu \int_{-\infty}^{\infty} J_i(x)e^{\mu|x|}dx \int_{-\infty}^{t} \frac{e^{-\mu|t|}e^{-\frac{\beta_i}{c}(t-s)}}{c} e^{\mu|s|}ds \\
&\leq |\Phi - \Psi|_\mu \left[\frac{1}{\beta_i - \mu c} + \frac{1}{\beta_i + \mu c}\right] \int_{-\infty}^{\infty} J_i(x)e^{\mu|x|}dx
\end{aligned}$$

since $\mu < \frac{\beta_i}{c}$ holds. Similarly, we can prove

$$\begin{aligned}
II &= \frac{e^{-\mu|t|}}{c}\left|\beta_i - \int_{-\infty}^{\infty} J_i(x)dx\right| \int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)}\, |\phi_i(s) - \psi_i(s)|\, ds \\
&\leq |\Phi - \Psi|_\mu \left|\beta_i - \int_{-\infty}^{\infty} J_i(x)dx\right| \left[\frac{1}{\beta_i - \mu c} + \frac{1}{\beta_i + \mu c}\right],
\end{aligned}$$

and

$$\begin{aligned}
III &= \frac{e^{-\mu|t|}}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)}\, |f_i^c(\Phi_s) - f_i^c(\Psi_s)|\, ds \\
&\leq L\, |\Phi - \Psi|_\mu\, e^{\mu c\tau} \left[\frac{1}{\beta_i - \mu c} + \frac{1}{\beta_i + \mu c}\right]
\end{aligned}$$

by conditions (H2) and $\mu < \frac{\beta_i}{c}$. Define $M_i$ by

$$M_i = \left[\frac{1}{\beta_i - \mu c} + \frac{1}{\beta_i + \mu c}\right]\left[\int_{-\infty}^{\infty} J_i(x)e^{\mu|x|}dx + \left|\beta_i - \int_{-\infty}^{\infty} J_i(x)dx\right| + Le^{\mu c\tau}\right].$$

Then, we have the following estimate

$$|F_i(\Phi)(t) - F_i(\Psi)(t)|\, e^{-\mu|t|} \le M_i\, |\Phi - \Psi|_\mu \text{ for any } t \in \mathbb{R},$$

which implies that

$$\sup_{t\in\mathbb{R}} |F_i(\Phi)(t) - F_i(\Psi)(t)|\, e^{-\mu|t|} \le M_i\, |\Phi - \Psi|_\mu\,.$$

Let $M = \sum_{1\le i\le n}\{M_i\}$, then

$$|F(\Phi)(t) - F(\Psi)(t)|_\mu \le M\, |\Phi - \Psi|_\mu\,.$$

This implies that $F$ is continuous with respect to the norm $|\cdot|_\mu$. The proof is complete. □

**Lemma 3.6.** *Assume that (QM), (P1) and (P2) hold. Then* $F : \Gamma \to \Gamma$.

*Proof.* From Lemma 3.3, $F(\Phi)(t)$ is monotone with respect to $t \in \mathbb{R}$ if $\Phi(t) \in \Gamma$ holds. So, it suffices to prove that

$$\underline{\Phi}(t) \le F(\Phi)(t) \le \overline{\Phi}(t) \text{ for any } \Phi \in \Gamma. \tag{3.2}$$

Furthermore, Lemma 3.3 implies that (3.2) holds if

$$F(\underline{\Phi})(t) \le \underline{\Phi}(t) \le \overline{\Phi}(t) \le F(\overline{\Phi})(t). \tag{3.3}$$

In fact, the definition of upper solution implies that

$$H(\overline{\Phi})(t) \le c\overline{\Phi}'(t) + \beta\overline{\Phi}(t),\ t \in \mathbb{R}\backslash\mathbb{T}.$$

Let $T_0 = -\infty, T_{k+1} = +\infty$, then

$$\begin{aligned} F(\overline{\Phi})(t) &= \frac{1}{c}\int_{-\infty}^{t} e^{-\frac{\beta}{c}(t-s)} H(\overline{\Phi})(s)ds \\ &\le \frac{1}{c}\left\{\left(\sum_{j=1}^{i-1}\int_{T_{j-1}}^{T_j} + \int_{T_{i-1}}^{t}\right) e^{-\frac{\beta}{c}(t-s)}\left[c\overline{\Phi}'(s) + \beta\overline{\Phi}(s)\right] ds\right\} \\ &= \overline{\Phi}(t) \end{aligned}$$

for $T_{i-1} < t < T_i$ with $i = 1, 2, \cdots, k+1$.

Similarly, we can prove that $F(\underline{\Phi})(t) \ge \underline{\Phi}(t)$ for all $t \in \mathbb{R}\backslash\mathbb{T}$. By the continuity of $F(\overline{\Phi})(t), \overline{\Phi}(t)$, $F(\underline{\Phi})(t)$ and $\underline{\Phi}(t)$ for $t \in \mathbb{R}$, we know that (3.3) holds for all $t \in \mathbb{R}$. The proof is complete. □

**Lemma 3.7.** *Assume that (QM), (P1) and (P2) hold. Then* $F : \Gamma \to \Gamma$ *is compact with respect to the decay norm* $|\cdot|_\mu$.

*Proof.* For any $\Phi(t) \in \Gamma$ and $n \in \mathbb{N}$, define $F^n(\Phi)(t)$ by

$$F^n(\Phi)(t) = \begin{cases} F(\Phi)(-n) \text{ if } t < -n, \\ F(\Phi)(t) \text{ if } t \in [-n, n], \\ F(\Phi)(n) \text{ if } t > n. \end{cases}$$

Then $F^n(\Phi)(t)$ is compact once $F(\Phi)(t)|_{[-n,n]}$ is compact. And the equicontinuity and uniform boundedness of $F^n(\Phi)(t)$ in $B_\mu(\mathbb{R}, \mathbb{R}^n)$ are clear, so $F^n(\Phi)(t)$ is a precompact subset with respect to the decay norm since the Ascoli–Arzela lemma implies that $F(\Phi)(t)|_{[-n,n]}$ is compact in the sense of decay norm. Furthermore

$$|F^n(\Phi)(t) - F(\Phi)(t)|\, e^{-\mu|t|} \leq |K|\, e^{-\mu n} \to 0 \text{ as } n \to \infty.$$

Thus $F^n(\Phi)(t)$ convergence to $F(\Phi)(t)$ in the sense of decay norm, so the compactness of $F^n(\Phi)(t)$ implies that $F(\Phi)(t)$ is precompact. The proof is complete. □

*The Proof of Theorem 3.2.* By Lemmas 3.4-3.7 and the Schauder's fixed point theorem, there exists $\Phi^* \in \Gamma$ such that

$$\Phi^*(t) = F\left(\Phi^*\right)(t) = \frac{1}{c}\int_{-\infty}^{t} e^{-\frac{\beta}{c}(t-s)} H(\Phi^*)(s)ds, \tag{3.4}$$

which implies that $\Phi^*(t)$ is a fixed point of $F$ in $\Gamma$, so it is a monotone solution of (2.1). In the rest, we will show that $\lim_{t\to-\infty}\Phi^*(t) = 0$ and $\lim_{t\to\infty}\Phi^*(t) = K$.

First, by the monotonicity and boundedness of $\Phi^*(t)$, we know $\lim_{t\to\pm\infty}\Phi^*(t)$ exists and denote it by $\lim_{t\to\pm\infty}\Phi^*(t) = \Phi^*_\pm$, it is clear that $0 \leq \Phi^*_- \leq \inf_{t\in\mathbb{R}} \overline{\Phi}(t)$ and $\sup_{t\in\mathbb{R}} \underline{\Phi}(t) \leq \Phi^*_+ \leq K$ hold. On the other hand, if we take limit in (2.1) as $t \to \pm\infty$, then $f(\widehat{\Phi^*_\pm}) = 0$ holds. Hence $\Phi^*_- = 0$ and $\Phi^*_+ = K$ are true since $f(\widehat{u}) \neq 0$ for any $u \in \left(0, \inf_{t\in\mathbb{R}} \overline{\Phi}(t)\right] \cup [\sup_{t\in\mathbb{R}} \underline{\Phi}(t), K)$. The proof is complete. □

## 4. Applications

In this section, we shall consider the travelling wave fronts of two examples.

**Example 4.1.** Let us consider the following delayed differential equation

$$\frac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - u(x,t) + ru(x, t-\tau)\left[1 - u(x,t)\right], \ t \in \mathbb{R}, \tag{4.1}$$

where $r$, $\tau$ are positive constants. $J$ satisfies the conditions (H3)-(H4) and $\int_{-\infty}^{\infty} J(x)dx = 1$, in addition, $J \in C(\mathbb{R}, \mathbb{R})$ is an even function, we also assume that $(J * e^{\lambda\cdot})(x) < +\infty$ for any $\lambda, x \in \mathbb{R}$. In fact, (4.1) can be regarded as the second version of the classical Logistic equation with delay which was widely studied in the literature and we can refer to Wu and Zou [41], Zou [44] and the references cited therein.

Let $c > 0$. Substituting $u(x,t) = \phi(x+ct) = \phi(\xi)$ into (4.1)(the variable $\xi$ still is denoted by $t$ without confusion), then

$$c\phi'(t) = (J * \phi)(t) - \phi(t) + r\phi(t - c\tau)\left[1 - \phi(t)\right], \ t \in \mathbb{R}. \tag{4.2}$$

Just like the theory established in Section 3, we are interested in the monotone solution of (4.2) satisfying

$$\lim_{t\to-\infty} \phi(t) = 0, \ \lim_{t\to\infty} \phi(t) = 1. \tag{4.3}$$

For any $c > 0$ and $\phi \in C([-c\tau, 0], \mathbb{R})$ with $0 \le \phi \le 1$, define $f^c(\phi)$ as

$$f^c(\phi) = r\phi(-c\tau)\left[1 - \phi(0)\right].$$

**Lemma 4.2.** *The function $f^c(\phi)$ satisfies* (QM).

The lemma can be easily proved, so we omit it here.

For $\lambda \in \mathbb{R}^+$, define

$$\Delta(\lambda, c) \triangleq J * e^{\lambda \cdot} - 1 - c\lambda + re^{-\lambda c\tau}. \tag{4.4}$$

**Lemma 4.3.** *There exists $c^* > 0$ such that* (4.4) *has two distinct positive roots for $c > c^*$ and* (4.4) *has no real roots for $c < c^*$. More precisely, there exist $\lambda_2(c) > \lambda_1(c) > 0$ such that*

$$\Delta(\lambda, c) = \begin{cases} > 0 & \text{for } 0 < \lambda < \lambda_1(c), \\ = 0 & \text{for } \lambda = \lambda_1(c), \lambda_2(c), \\ < 0 & \text{for } \lambda_1(c) < \lambda < \lambda_2(c), \\ > 0 & \text{for } \lambda > \lambda_2(c). \end{cases}$$

Define the continuous functions

$$\overline{\phi}(t) = \min\{1, e^{\lambda_1(c)t}\}, \underline{\phi}(t) = \max\left\{0, e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\right\},$$

where $q > 1$ is large enough and

$$\eta \in \left(1, \min\left\{2, \frac{\lambda_2}{\lambda_1}\right\}\right). \tag{4.5}$$

It is easy to see that $\overline{\phi}(t)$ and $\underline{\phi}(t)$ satisfy the conditions (P1)-(P2) given in Section 3. Now we prove $\overline{\phi}(t)$ and $\underline{\phi}(t)$ are upper and lower solutions of (4.2).

**Lemma 4.4.** *$\overline{\phi}(t)$ is the upper solution of* (4.2) *for $t \in \mathbb{R}$.*

*Proof.* To this end, we only need to prove

$$c\overline{\phi}'(t) \ge (J * \overline{\phi})(t) - \overline{\phi}(t) + r\overline{\phi}(t - c\tau)\left[1 - \overline{\phi}(t)\right] \text{ for } t \ne 0. \tag{4.6}$$

If $t > 0$, then $\overline{\phi}(t) = 1$ and (4.6) is clear.

If $t < 0$, then $\overline{\phi}(t) = e^{\lambda_1(c)t}$ and

$$(J * \overline{\phi})(t) - \overline{\phi}(t) - c\overline{\phi}'(t) + r\overline{\phi}(t - c\tau)\left[1 - \overline{\phi}(t)\right]$$

$$\begin{aligned}
&\le \left(J * \left(e^{\lambda_1(c)\cdot}\right)\right)(t) - e^{\lambda_1(c)t} - c(e^{\lambda_1(c)t})' + re^{\lambda_1(c)(t-c\tau)}\left[1 - e^{\lambda_1(c)t}\right] \\
&\le \left(J * \left(e^{\lambda_1(c)\cdot}\right)\right)(t) - e^{\lambda_1(c)t} - c(e^{\lambda_1(c)t})' + re^{\lambda_1(c)(t-c\tau)} \\
&= 0.
\end{aligned}$$

The proof is complete. □

**Lemma 4.5.** *If $q > 1$ is large enough, then $\underline{\phi}(t)$ is the lower solution of* (4.2) *for $t \in \mathbb{R}$.*

*Proof.* Assume that there exists $t_1 \in \mathbb{R}$ such that $0 = e^{\lambda_1(c)t_1} - qe^{\eta\lambda_1(c)t_1}$. Then $t_1 < 0$ is small enough since $q > 1$ is large enough. In order to prove the lemma, it suffices to show

$$c\underline{\phi}'(t) \le (J*\underline{\phi})(t) - \underline{\phi}(t) + r\underline{\phi}(t - c\tau)(1 - \underline{\phi}(t)) \text{ for } t \ne t_1. \tag{4.7}$$

If $t > t_1$, then $\underline{\phi}(t) = 0$ and (4.7) is clear.

If $t < t_1$, then we only need to prove

$$\begin{aligned}
& c\left(e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\right)' \\
\le\ & \left(J*\left(e^{\lambda_1(c)\cdot} - qe^{\eta\lambda_1(c)\cdot}\right)\right)(t) - \left[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\right] \\
& + r\left(e^{\lambda_1(c)(t-c\tau)} - qe^{\eta\lambda_1(c)(t-c\tau)}\right)\left[1 - \left(e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\right)\right],
\end{aligned}$$

by Lemma 4.3, this is equivalent to

$$\Delta(\eta\lambda_1, c)qe^{\eta\lambda_1(c)t} \le -r\left(e^{\lambda_1(c)(t-c\tau)} - qe^{\eta\lambda_1(c)(t-c\tau)}\right)\left(e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\right). \tag{4.8}$$

Note that $t_1 < 0$ is small enough, then (4.8) is clear by (4.5), which implies that (4.7) holds for $t < t_1$. The proof is complete. □

By what we have done, the following result is obvious.

**Theorem 4.6.** *Assume that $c > c^*$ holds. Then* (4.1) *has a travelling wave front $\phi(t)$ connecting* 0 *with* 1. *Furthermore,* $\lim_{t\to-\infty} \phi(t)e^{-\lambda_1(c)t}$ *exists and is positive.*

**Theorem 4.7.** *For $c = c^*$,* (4.1) *has a travelling wave front $\phi(x + ct)$ connecting* 0 *with* 1 *in the sense that* $\lim_{\xi\to-\infty} \phi(\xi) = 0, \lim_{\xi\to\infty} \phi(\xi) = 1$.

*Proof.* By the definition of the travelling wave front and $c > c^*$, $|\phi'(t)|$ is uniformly bounded and the travelling wave fronts are equicontinuous for any $c > c^*$.

Then the standard argument about equicontinuity and uniform boundedness of travelling wave fronts (see Theorem 3.4 in Thieme and Zhao [34]) implies that there exists a monotone function $\phi(t)$ such that

$$\lim_{t\to-\infty} \phi(t) = 0, \lim_{t\to\infty} \phi(t) = 1.$$

The proof is complete. □

Moreover, by the result of asymptotic spreading in Liang and Zhao [25], Thieme and Zhao [34], it is easy to prove that the following result holds.

**Theorem 4.8.** *For* $0 < c < c^*$, (4.2) *and* (4.3) *has no monotone solution.*

It is clear that the above Theorems 4.6-4.8 imply that $c^*$ is the *minimal wave speed* in the sense that (4.1) has *no* nontrivial travelling wave fronts if $c < c^*$ while (4.1) has a monotone travelling wave front for all $c \geq c^*$. We now consider how the delay and nonlocality affect the minimal wave speed.

Form the definition of $\Delta(\lambda, c)$ and Lemma 4.3, it is easy to see that the time delay reduces the minimal wave speed $c^*$, see also [31, 44], which is the following conclusion.

**Proposition 4.9.** *The time delay reduces the minimal wave speed* $c^* > 0$. *Namely,* $c^* > 0$ *can be sufficiently small if* $\tau$ *is large enough.*

We further consider the effect of nonlocal diffusion for the existence of travelling wave fronts. Let $J_1(x)$ and $J_2(x)$ satisfy the assumptions of $J(x)$ in (4.1). Moreover, denote $c_1^*$ and $c_2^*$ as $c^*$ in Lemma 4.3 for $J_1(x)$ and $J_2(x)$ respectively. We call the nonlocality of $J_1(x)$ is *stronger* than $J_2(x)$ if

$$0 \leq \int_0^x J_1(s)ds \leq \int_0^x J_2(s)ds \leq \frac{1}{2} \text{ for all } x > 0.$$

It is easy to see that $c_1^* \geq c_2^*$ if the nonlocality of $J_1(x)$ is stronger than $J_2(x)$ since $J_1(x)$ and $J_2(x)$ are even functions. Moreover, $c^*$ can be arbitrary large if the nonlocality of $J(x)$ is sufficiently strong. We formulate this as following proposition.

**Proposition 4.10.** *The nonlocality increases the minimal wave speed.*

**Remark 4.11.** The above two propositions have significant sense in biological science since $c^*$ is related to the asymptotic speed of spreading [25, 34], and this will be further studied in our recent papers.

**Example 4.12.** Let us consider the travelling wave fronts of the following delayed system

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - u(x,t) + u(x,t)\left[1 - u(x,t) - rv(x,t-\tau)\right], \\ \frac{\partial v(x,t)}{\partial t} = (J * v)(x,t) - v(x,t) - bu(x,t)v(x,t), \end{cases} \tag{4.9}$$

where $r \in (0,1)$ and $b > 0$ are constants. $J$ is similar to that of Example 4.1. In fact, (4.9) can be regarded as the second version of the well-known Belousov–Zhabotinskii reaction model with delay. The travelling wave front of the (delayed) Belousov–Zhabotinskii reaction model was widely studied in the literature, we can refer to Ma [26], Wu and Zou [41] and Ye and Wang [43], and for more of the model, we also can see Kanel [20], Kapel [21], Troy [35], etc.

Similar to [20, 21, 26, 35, 41, 43], we are interested in (4.9) with the following asymptotic boundary conditions:

$$\begin{cases} \lim_{x\to-\infty} u(x,t) = 0, \lim_{x\to-\infty} v(x,t) = 1, \\ \lim_{x\to\infty} u(x,t) = 1, \lim_{x\to\infty} v(x,t) = 0. \end{cases} \tag{4.10}$$

By (4.9) and (4.10), we get the wave system

$$\begin{cases} c\phi_1'(t) = (J*\phi_1)(t) - \phi_1(t) + \phi_1(t)\left[1 - \phi_1(t) - r\phi_2(t-c\tau)\right], \\ c\phi_2'(t) = (J*\phi_2)(t) - \phi_2(t) - b\phi_1(t)\phi_2(t), \end{cases} \tag{4.11}$$

and the boundary conditions:

$$\begin{cases} \lim_{t\to-\infty} \phi_1(t) = 0, \lim_{t\to-\infty} \phi_2(t) = 1, \\ \lim_{t\to\infty} \phi_1(t) = 1, \lim_{t\to\infty} \phi_2(t) = 0. \end{cases} \tag{4.12}$$

By making simple change of variables $\phi_1^* = \phi_1, \phi_2^* = 1 - \phi_2$, and omitting the asterisks for the sake of notational simplicity, (4.11) and (4.12) become

$$\begin{cases} c\phi_1'(t) = (J*\phi_1)(t) - \phi_1(t) + \phi_1(t)\left[1 - r - \phi_1(t) + r\phi_2(t-c\tau)\right], \\ c\phi_2'(t) = (J*\phi_2)(t) - \phi_2(t) + b\phi_1(t)\left[1 - \phi_2(t)\right], \end{cases} \tag{4.13}$$

and

$$\begin{cases} \lim_{t\to-\infty} \phi_1(t) = 0, \lim_{t\to-\infty} \phi_2(t) = 0, \\ \lim_{t\to\infty} \phi_1(t) = 1, \lim_{t\to\infty} \phi_2(t) = 1. \end{cases} \tag{4.14}$$

Namely, $K = (1,1)^T$. For $(\phi_1, \phi_2)^T \in C([-c\tau, 0], \mathbb{R}^2)$, define $f^c = (f_1, f_2)^T$ by

$$\begin{cases} f_1(\phi_1, \phi_2) = \phi_1(0)[1 - r - \phi_1(0) + r\phi_2(-c\tau)], \\ f_2(\phi_1, \phi_2) = b\phi_1(0)[1 - \phi_2(0)]. \end{cases}$$

**Lemma 4.13.** *The function $f^c$ satisfies (QM).*

The lemma is obvious, so we omit the proof.

Define

$$\Delta(\lambda, c) = J * e^{\lambda \cdot} - 1 - c\lambda + (1 - r). \tag{4.15}$$

**Lemma 4.14.** *There exists $c^* > 0$ such that* (4.15) *has two distinct positive roots $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) < \lambda_2(c)$ for any $c > c^*$ and* (4.15) *has no real root if $c < c^*$.*

Define the continuous functions

$$\begin{aligned} \overline{\phi}_1(t) &= \min\{e^{\lambda_1 t}, 1\}, \ \overline{\phi}_2(t) = \min\left\{e^{\lambda_1 (t + c\tau)}, 1\right\}, \\ \underline{\phi}_1(t) &= \max\left\{(1-r)(e^{\lambda_1 t} - q e^{\eta \lambda_1 t}), 0\right\}, \ \underline{\phi}_2(t) = 0, \end{aligned}$$

where $\eta \in \left(1, \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\right\}\right)$ and $q > 1$ is a constant.

**Lemma 4.15.** *Assume that $c > c^*$ and $be^{-\lambda_1 c\tau} \le 1 - r$. Then $(\overline{\phi}_1, \overline{\phi}_2)^T$ is an upper solution and $(\underline{\phi}_1, \underline{\phi}_2)^T$ is a lower solution of* (4.13) *if $q > 1$ is large enough.*

The proof is similar to these of Lemmas 4.4 and 4.5 and is omitted here.

It is clear that $(\overline{\phi}_1, \overline{\phi}_2)^T$ and $(\underline{\phi}_1, \underline{\phi}_2)^T$ satisfy all the conditions of Theorem 3.2. Thus, the following result holds.

**Theorem 4.16.** *Assume that $\lambda_1(c)$, $c^*$ are defined in Lemma 4.14 and $be^{-\lambda_1(c)c\tau} \leq 1 - r$ holds for given $c$ satisfying $c > c^*$. Then (4.9) and (4.10) has a travelling wave front with wave speed $c$.*

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