Z. angew. Math. Phys. 59 (2008) 1029–1038
0044-2275/08/061029-10
DOI 10.1007/s00033-007-6140-9

Zeitschrift für angewandte Mathematik und Physik ZAMP

Exact solutions of Euler equations of ideal gasdynamics via Lie group analysis

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Abstract. In this paper, we explicitly characterize a class of solutions to the first order quasilinear system of partial differential equations (PDEs), governing one dimensional unsteady planar and radially symmetric flows of an adiabatic gas involving shock waves. For this, Lie group analysis is used to identify a finite number of generators that leave the given system of PDEs invariant. Out of these generators, two commuting generators are constructed involving some arbitrary constants. With the help of canonical variables associated with these two generators, the assigned system of PDEs is reduced to an autonomous system, whose simple solutions provide non trivial solutions of the original system. It is interesting to remark that one of the special solutions obtained here, using this approach, is precisely the blast wave solution known in the literature.

Mathematics Subject Classification (2000). 35L50, 35L60, 75L05, 76M60.

Keywords. Gasdynamic Euler equations, quasilinear system of PDEs, Lie group method, shock waves, blast wave.

1. Introduction

Many flow fields involving wave phenomena are governed by quasi linear hyperbolic systems of PDEs [1]. The solutions of the system are either continuously differential functions, or continuous but non-differentiable functions (weak solutions), or discontinuous (shock) solutions. Indeed, for nonlinear systems involving discontinuities such as shocks, we do not normally have the luxury of complete exact solutions, and for analytical work have to rely on some approximate analytical or numerical methods which may be useful to set the scene and provide useful information towards our understanding of the complex physical phenomenon involved. One of the most powerful methods to determine particular solutions to PDEs is based upon the study of their invariance with respect to one parameter Lie group of point transformations (see [2]–[9]). Indeed, with the help of symmetry generators of these equations, one can construct similarity variables which can reduce these equations to ordinary differential equations (ODEs); in some cases, it is possible to solve these ODEs exactly. Besides these similarity solutions, the symmetries admitted by given PDEs enable us to look for appropriate canonical variables which transform the original system to an equivalent one whose simple solutions provide nontrivial solutions of the original system (see Refs. [10], [11], [12] and [13]). Using this procedure, Ames & Donato [14] obtained solutions for the problem of elastic-plastic deformation generated by a torque, and analyzed the evolution of a weak discontinuity in a state characterized by invariant solutions. Donato & Ruggeri[15] used this procedure to study similarity solutions for the system of a monoatomic gas, within the context of the theory of extended thermodynamics, assuming spherical symmetry. In this paper, we use this approach to characterize a class of solutions of the basic equations governing the one dimensional planar and radially symmetric flows of an adiabatic gas involving shock waves. Since, the system involves only two independent variables, we need two commuting Lie vector fields, which are constructed by taking a linear combination of the infinitesimal operators of the Lie point symmetries admitted by the system at hand. It is interesting to note that one of the special exact solutions obtained in this manner is the well known solution to the blast wave problem studied in the theory of explosion in the gaseous media (see Refs. [14], [15], [16], [17] and [18]).

2. Symmetry group analysis

Following [10], [11], [12] and [13], let us assume that the system of N nonlinear partial differential equations

$$F_R\left(x,t,\overline{u},\frac{\partial\overline{u}}{\partial x},\frac{\partial\overline{u}}{\partial t}\right)=0,\qquad R=1,2,\cdots,N, \tag{1}$$

involving two independent variables x, t and the unknown vector $\overline{u}(x,t)$, where $\overline{u}(x,t)=(u_1(x,t),u_2(x,t),\cdots,u_N(x,t))\in R^N$, admits $s-$ parameter Lie group of transformations with infinitesimal operators

$$\zeta_i=X_i(x,t,\overline{u})\frac{\partial}{\partial x}+T_i(x,t,\overline{u})\frac{\partial}{\partial t}+\sum_{j=1}^{N}U_{ij}(x,t,\overline{u})\frac{\partial}{\partial u_j},\ \ i=1,2,\ldots,n \tag{2}$$

such that there exist $r(\leq s)$ infinitesimal generators $\zeta_1,\zeta_2,\cdots\zeta_r$ that form a solvable Lie Algebra. Let us now construct generators $Y_1=\sum_{k=1}^{r}\alpha_k\zeta_k$ and $Y_2=\sum_{k=1}^{r}\beta_k\zeta_k$, where α_k and β_k are constants, to be determined, such that $[Y_1,Y_2]=0$.

Now we introduce the canonical variables τ, ξ and $\bar{\nu}=(\nu_1,\nu_2,\cdots,\nu_N)\in R^N$ related to the infinitesimal generator Y_1, defined by $Y_1\tau=1$, $Y_1\xi=0$, and $Y_1\nu_i=0$, $i=1,2,\cdots,N$. In terms of these canonical variables, the infinitesimal operator Y_1 reduces to $\tilde{Y}_1=\frac{\partial}{\partial\tau}$, i.e., it corresponds to a translation in the variable τ only;

consequently, owing to the invariance, the system (1) must assume the form

$$\tilde{F_R}\left(\xi, \bar{\nu}, \frac{\partial \overline{\nu}}{\partial \xi}, \frac{\partial \overline{\nu}}{\partial \tau}\right) = 0, \qquad R = 1, 2, \cdots, N. \tag{3}$$

In terms of these new variables, the operator Y_2 can be written as

$$\tilde{Y_2} = (Y_2\xi)\frac{\partial}{\partial \xi} + (Y_2\tau)\frac{\partial}{\partial \tau} + \sum_{j=1}^{N}(Y_2\nu_j)\frac{\partial}{\partial \nu_j}, \tag{4}$$

where we require that the condition $Y_2\xi \neq 0$ holds; thus, it is possible to introduce new canonical variables $\eta,\ \tau^*$ and $\bar{w} = (w_1, w_2, \cdots, w_N) \in R^N$ defined by $\tilde{Y_2}\eta = 1$, $\tilde{Y_2}\tau^* = 0$, and $\tilde{Y_2}w_i = 0$, $i = 1, 2, \cdots, N$ which transform (3) to the form

$$\hat{F_R}\left(\bar{w}, \frac{\partial \bar{w}}{\partial \eta}, \frac{\partial \overline{w}}{\partial \tau^*}\right) = 0, \qquad R = 1, 2, \cdots, N. \tag{5}$$

The resulting system (5) is an autonomous system associated with (1) to illustrate the method, outlined as above, we consider the system of Euler equations of ideal gasdynamics in the next section.

3. Euler equations of ideal gas dynamics

The equations governing the one dimensional unsteady planar and radially symmetric flows of an adiabatic gas with adiabatic index γ in the absence of viscosity, heat conduction and body forces can be written in the form [1]:

$$\begin{aligned} &\rho_t + u\rho_x + \rho u_x + \frac{m\rho u}{x} = 0,\\ &u_t + uu_x + \rho^{-1}p_x = 0,\\ &p_t + up_x + \gamma p u_x + \frac{m\gamma p u}{x} = 0, \end{aligned} \tag{6}$$

where t is the time, x the spatial coordinate being either axial in flows with planar ($m = 0$) geometry or radial in cylindrically ($m = 1$) and spherically ($m = 2$) symmetric flows. The state variable u denotes the gas velocity, p the pressure, and ρ the density. By a straight forward analysis, it is found that the Lie groups of point transformations that leave the system (6) invariant constitute a 4 - dimensional Lie algebra generated by the following infinitesimal operators:

$$\begin{aligned} &\zeta_1 = \rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}, &&\zeta_2 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2p\frac{\partial}{\partial p},\\ &\zeta_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2p\frac{\partial}{\partial p}, &&\zeta_4 = \frac{\partial}{\partial t}. \end{aligned}$$

In order to construct generators Y_1, Y_2 such that $[Y_1, Y_2] = 0$, let

$$\begin{aligned}
Y_1 &= \alpha_1\zeta_1 + \alpha_2\zeta_2 + \alpha_3\zeta_3 + \alpha_4\zeta_4, \\
&= (\alpha_2 t + \alpha_4)\frac{\partial}{\partial t} + \alpha_3 x\frac{\partial}{\partial x} + \alpha_1\rho\frac{\partial}{\partial \rho} + (\alpha_3 - \alpha_2)\, u\frac{\partial}{\partial u} \\
&\quad + (\alpha_1 - 2\alpha_2 + 2\alpha_3)\, p\frac{\partial}{\partial p}, \\
Y_2 &= \beta_1\zeta_1 + \beta_2\zeta_2 + \beta_3\zeta_3 + \beta_4\zeta_4, \\
&= (\beta_2 t + \beta_4)\frac{\partial}{\partial t} + \beta_3 x\frac{\partial}{\partial x} + \beta_1\rho\frac{\partial}{\partial \rho} + (\beta_3 - \beta_2)\, u\frac{\partial}{\partial u} \\
&\quad + (\beta_1 - 2\beta_2 + 2\beta_3)\, p\frac{\partial}{\partial p},
\end{aligned}$$

where $\alpha_2\beta_4 - \alpha_4\beta_2 = 0$ and $\alpha_1, \alpha_3, \beta_1, \beta_3$ are arbitrary constants. Since the system is invariant under the group generated by the generator Y_1, we introduce canonical variables $\bar{\tau}$, $\bar{\xi}$, $\bar{R}$, $\bar{U}$ and $\bar{P}$ such that $Y_1\bar{\tau} = 1$, $Y_1\bar{\xi} = 0$, $Y_1\bar{R} = 0$, $Y_1\bar{U} = 0$ and $Y_1\bar{P} = 0$. This implies that when $\alpha_2, \alpha_3 \neq 0$, we have

$$\begin{aligned}
&\bar{\tau} = (1/\alpha_2)\log(\alpha_2 t + \alpha_4), \quad \bar{\xi} = (\alpha_2 t + \alpha_4)\, x^{-\alpha_2/\alpha_3}, \\
&\bar{R} = \rho\, x^{-\alpha_1/\alpha_3}, \quad \bar{U} = u\, x^{(\alpha_2-\alpha_3)/\alpha_3}, \quad \bar{P} = p\, x^{(2\alpha_2-\alpha_1-2\alpha_3)/\alpha_3}.
\end{aligned} \tag{7}$$

In terms of these new variables, Y_2 becomes

$$\begin{aligned}
\bar{Y}_2 = \frac{\beta_2}{\alpha_2}\frac{\partial}{\partial\bar{\tau}} + \frac{\beta_2\alpha_3 - \beta_3\alpha_2}{\alpha_3}\bar{\xi}\frac{\partial}{\partial\bar{\xi}} + \frac{\beta_1\alpha_3 - \beta_3\alpha_1}{\alpha_3}\bar{R}\frac{\partial}{\partial\bar{R}} + \frac{\alpha_2\beta_3 - \beta_2\alpha_3}{\alpha_3}\bar{U}\frac{\partial}{\partial\bar{U}} \\
+ \frac{(\alpha_3\beta_1 - \alpha_1\beta_3) + 2(\alpha_2\beta_3 - \beta_2\alpha_3)}{\alpha_3}\bar{P}\frac{\partial}{\partial\bar{P}}.
\end{aligned}$$

Now, we introduce canonical variables τ, ξ, R, U and P such that $\bar{Y}_2\tau = 0$, $\bar{Y}_2\xi = 1$, $\bar{Y}_2 R = 0$, $\bar{Y}_2 U = 0$ and $\bar{Y}_2 P = 0$; thus, the corresponding characteristic conditions yield

$$\begin{aligned}
&\xi = (\alpha_3/A)\log\left(\bar{\xi}\right), \quad \tau = \bar{\tau} - (\beta_2/\alpha_2)\xi, \\
&R = \bar{R}\,\bar{\xi}^{\,(\alpha_1\beta_3-\alpha_3\beta_1)/A}, \quad U = \bar{U}\,\bar{\xi}, \quad P = \bar{P}\,\bar{\xi}^{\,2+((\alpha_1\beta_3-\alpha_3\beta_1)/A)},
\end{aligned} \tag{8}$$

where $A = \alpha_3\beta_2 - \alpha_2\beta_3 \neq 0$. In view of (7) and (8), we are led to the following transformations

$$\begin{aligned}
\tau = (-\beta_3/A)\log\{(\alpha_2 t + \alpha_4)\, x^{-\beta_2/\beta_3}\}, \quad \xi = (\alpha_3/A)\log\{(\alpha_2 t + \alpha_4)\, x^{-\alpha_2/\alpha_3}\}, \\
\rho = R(\xi,\tau)x^L(\alpha_2 t + \alpha_4)^K, \quad u = U(\xi,\tau)x(\alpha_2 t + \alpha_4)^{-1}, \\
p = P(\xi,\tau)x^{L+2}(\alpha_2 t + \alpha_4)^{K-2},
\end{aligned} \tag{9}$$

where $L = (\alpha_1\beta_2 - \alpha_2\beta_1)/A$, $K = (\alpha_3\beta_1 - \alpha_1\beta_3)/A$ with $\beta_3 \neq 0$, and R, U, P are arbitrary functions of τ and ξ. Using (9) in (6) we get

$$\begin{aligned}
&(\alpha_2\alpha_3 - U\alpha_2)\frac{\partial R}{\partial \xi} + (\beta_2 U - \beta_3\alpha_2)\frac{\partial R}{\partial \tau} + \beta_2 R\frac{\partial U}{\partial \tau} - \alpha_2 R\frac{\partial U}{\partial \xi}\\
&\qquad + (K\alpha_2 + LU + U + mU)\,AR = 0,\\
&(\alpha_2\alpha_3 - U\alpha_2)\frac{\partial U}{\partial \xi} + (\beta_2 U - \beta_3\alpha_2)\frac{\partial U}{\partial \tau} + \frac{\beta_2}{R}\frac{\partial P}{\partial \tau} - \frac{\alpha_2}{R}\frac{\partial P}{\partial \xi}\\
&\qquad + \left(U^2 - \alpha_2 U\right)A + (L+2)\,A\frac{P}{R} = 0,\\
&(\alpha_2\alpha_3 - U\alpha_2)\frac{\partial P}{\partial \xi} + (\beta_2 U - \beta_3\alpha_2)\frac{\partial P}{\partial \tau} + \beta_2\gamma P\frac{\partial U}{\partial \tau} - \alpha_2\gamma P\frac{\partial U}{\partial \xi}\\
&\qquad + (K\alpha_2 - 2\alpha_2 + LU + 2U + \gamma U + m\gamma U)\,AP = 0,
\end{aligned} \tag{10}$$

where A is same as in (8). The above equations can be solved completely when $U \equiv$ constant. We, therefore, consider the following cases:

Case-I

Let $U \equiv$ constant $\neq \alpha_3$. Then the equations $(10)_{1,3}$ have the closed form solutions as

$$\begin{aligned}
R(\xi,\tau) &= R_1(\eta)\exp\{-((K\alpha_2 + (L+1+m)U)/(\alpha_2(\alpha_3 - U)))A\xi\}, \qquad (11)\\
P(\xi,\tau) &= P_1(\eta)\exp\{-((K\alpha_2 - 2\alpha_2 + (L+2+\gamma+m\gamma)U)/(\alpha_2(\alpha_3 - U)))A\xi\},
\end{aligned}$$

where

$$\eta = \tau - \frac{\beta_2 U - \alpha_2\beta_3}{\alpha_2(\alpha_3 - U)}\xi = \frac{1}{\alpha_3 - U}\log\left(x\,(\alpha_2 t + \alpha_4)^{-U/\alpha_2}\right),$$

and $R_1(\eta)$ is an arbitrary function of η. Using (11) into $(10)_2$, we get the compatibility conditions for U and $P_1(\eta)$ as

$$U = 2\alpha_2/(\gamma + 1 + m(\gamma - 1)) \qquad \text{or} \qquad U = \alpha_2, \tag{12}$$

and

$$P_1'(\eta) + (\alpha_1 + 2\alpha_3 + mU - U)\,P_1(\eta) + (\alpha_3 - U)\,U\,(U - \alpha_2)\,R_1(\eta) = 0.$$

Thus, in view of (9), (11) and (12), the solution of the system (6) can be expressed as follows.

Case-Ia:

When $U = \alpha_2 \neq \alpha_3$, the solution of the system (6) takes the form

$$\begin{aligned}
\rho &= R_1(\eta)\;x^{(\alpha_1+(m+1)\alpha_2)/(\alpha_3-\alpha_2)}\;(\alpha_2 t + \alpha_4)^{-(\alpha_1+(m+1)\alpha_3)/(\alpha_3-\alpha_2)}, \qquad (13)\\
u &= \alpha_2 x/(\alpha_2 t + \alpha_4), \quad p = C\,(\alpha_2 t + \alpha_4)^{-(m+1)\gamma},
\end{aligned}$$

where C is an arbitrary constant, $R_1(\eta)$ is an arbitrary function of η, and

$$\eta = (1/\alpha_3 - \alpha_2)\log\left(x/\left(\alpha_2 t + \alpha_4\right)\right).$$

Case-Ib:

When $U = 2\alpha_2/\Gamma \neq \alpha_3$, where $\Gamma = \gamma + 1 + m(\gamma - 1) \neq 0$, the solution of the system (6) takes the form

$$\rho = R_1(\eta)x^{A_1}(\alpha_2 t + \alpha_4)^{-A_2}, \quad u = (2\alpha_2/\Gamma)x/(\alpha_2 t + \alpha_4) \tag{14}$$
$$p = \left\{C - A_3 \int R_1(\eta)e^{A_4\eta}d\eta\right\}(\alpha_2 t + \alpha_4)^{-2(m+1)\gamma/\Gamma},$$

where C is an arbitrary constant, $R_1(\eta)$ is an arbitrary function of η, and

$$\eta = \frac{\Gamma}{\Gamma\alpha_3 - 2\alpha_2}\log\left(x(\alpha_2 t + \alpha_4)^{-2/\Gamma}\right),$$
$$A_1 = \frac{\Gamma\alpha_1 + 2(m+1)\alpha_2}{\Gamma\alpha_3 - 2\alpha_2}, \quad A_2 = \frac{2(\alpha_1 + (m+1)\alpha_3)}{\Gamma\alpha_3 - 2\alpha_2},$$
$$A_3 = \frac{2(\alpha_2)^2(m+1)(1-\gamma)(\Gamma\alpha_3 - 2\alpha_2)}{\Gamma^3}, \quad A_4 = \frac{\Gamma(\alpha_1 + 2\alpha_3) + 2(m-1)\alpha_2}{\Gamma}.$$

Case-II

Let $U \equiv \alpha_3$. Then $(10)_{1,3}$ imply that

$$R(\xi, \tau) = R_1(\xi)\exp\{-(K\alpha_2 + (L+1+m)U)\tau\},$$
$$P(\xi, \tau) = P_1(\xi)\exp\{-(K\alpha_2 - 2\alpha_2 + (L+2+\gamma+m\gamma)U)\tau\}, \tag{15}$$

where τ and ξ are same as defined in (9), and $R_1(\xi)$ and $P_1(\xi)$ are arbitrary functions of ξ. Moreover, on using (15) into $(10)_2$, we get the compatibility conditions for U and $P_1(\xi)$ as

$$U = \alpha_3 = 2\alpha_2/\Gamma, \tag{16}$$
$$P_1'(\xi) + \left(\beta_1 + 2\beta_3 + \frac{2(m-1)\beta_2}{\Gamma}\right)P_1(\xi)$$
$$+\frac{2\alpha_2(\gamma-1)(m+1)(\alpha_3\beta_2 - \alpha_2\beta_3)}{\Gamma^2}R_1(\xi) = 0.$$

Thus, in view of the equations (9), (15) and (16), the solution of the system (6) can be written as

$$\rho = R_1(\xi)x^{B_1}(\alpha_2 t + \alpha_4)^{-B_2}, \quad u = (2\alpha_2/\Gamma)\,x/(\alpha_2 t + \alpha_4), \tag{17}$$
$$p = \left\{C - B_3\int R_1(\xi)e^{B_4\xi}d\xi\right\}(\alpha_2 t + \alpha_4)^{-2(m+1)\gamma/\Gamma},$$

where C is an arbitrary constant, $R_1(\xi)$ is an arbitrary function of ξ, and

$$\xi = (\Gamma/(\Gamma\beta_3 - 2\beta_2))\log\left(x(\alpha_2 t + \alpha_4)^{-2/\Gamma}\right)$$

with B_i $(i = 1, 2, 3, 4)$ defined as follows

$$B_1 = \frac{\Gamma\beta_1 + 2(m+1)\beta_2}{\Gamma\beta_3 - 2\beta_2}, \quad B_2 = \frac{2(\beta_1 + (m+1)\beta_3)}{\Gamma\beta_3 - 2\beta_2},$$
$$B_3 = \frac{2(\alpha_2)^2(m+1)(1-\gamma)(\Gamma\beta_3 - 2\beta_2)}{\Gamma^3}, \quad B_4 = \frac{\Gamma(\beta_1 + 2\beta_3) + 2(m-1)\beta_2}{\Gamma}.$$

It may be remarked that the solution (13) (respectively, (14)) involves arbitrary parameters, α_1, α_2 and α_3 with $\alpha_2 \neq \alpha_3$ (respectively, $\alpha_3 \neq 2\alpha_2/\Gamma$), whereas the solution (17) depends on the parameters, α_2, β_1, β_2 and β_3. In fact, solution (17) is exactly the same as (14) i.e., the constants B_1, B_2, B_3, B_4 and ξ are exactly the same as A_1, A_2, A_3, A_4 and η when α_1, α_2 and α_3 are replaced by β_1, β_2 and β_3.

4. Solution with shocks

As is well known that a shock wave may be initiated in the flow region, and once it is formed, it will propagate by separating the portions of the continuous region. At shock, the correct generalized solution satisfies the Rankine–Hugoniot (RH) jump conditions. Let $x = X(t)$ be the shock location in the $x - t$ plane propagating in to the medium where $\rho = \rho_0(x)$, $u \equiv 0$ and $p = p_0 =$ constant. If the shock speed $V = dX/dt$ is very large compared with the sound speed $a_0 = \sqrt{\gamma p_0/\rho_0(x)}$, and the medium behind the shock is given by the solution (13) or (14), then at the shock front the following relations hold [1]:

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0(X(t)), \;\; u = \frac{2}{\gamma+1}V, \;\; p = \frac{2}{\gamma+1}\rho_0(X(t))V^2. \tag{18}$$

I. Let the medium behind the shock be represented by the solution (13). Then equations (18) imply

$$R_1(\eta_s)X(t)^{\left[\frac{\alpha_1+(m+1)\alpha_2}{\alpha_3-\alpha_2}\right]}(\alpha_2 t+\alpha_4)^{-\left[\frac{\alpha_1+(m+1)\alpha_3}{(\alpha_3-\alpha_2)}\right]} = \frac{\gamma+1}{\gamma-1}\rho_0(X(t)),$$
$$\frac{\alpha_2 X(t)}{\alpha_2 t+\alpha_4} = \frac{2}{\gamma+1}V, \qquad C\,(\alpha_2 t+\alpha_4)^{-(m+1)\gamma} = \frac{2}{\gamma+1}\rho_0(X(t))V^2, \tag{19}$$

where C is an arbitrary constant and $\eta_s = (\alpha_3-\alpha_2)^{-1}\log\left(X(t)\,(\alpha_2 t+\alpha_4)^{-1}\right)$. From $(19)_2$, the shock speed $V(t)$ can be written as $V = ((\gamma+1)\alpha_2/2)\;X(t)/(\alpha_2 t+\alpha_4)$, implying thereby that

$$X(t) = X_0\,(T/T_0)^{(\gamma+1)/2}\,, \tag{20}$$

where $T = \alpha_2 t + \alpha_4$ and $T_0 = \alpha_2 t_0 + \alpha_4$, with X_0 and t_0 being related with the position and time of the shock. Thus, on using (20) in equations $(19)_{1,3}$, we find that $R_1(\eta_s)$ and $\rho(X(t))$ must have the following forms:

$$R_1(\eta_s) = \frac{2C\hat{R}_{10}}{(\gamma-1)\alpha_2^2}\left(\frac{T}{T_0}\right)^{\frac{(1-\gamma)[\alpha_1-(m+3)\alpha_2+2(m+2)\alpha_3]}{2(\alpha_3-\alpha_2)}},$$
$$\rho_0(X(t)) = \rho_c\,(X(t)/X_0)^{\frac{2}{\gamma+1}(1-2\gamma-m\gamma)}\,, \tag{21}$$

where $\rho_c = \dfrac{2CT_0^{2-m\gamma-\gamma}}{(\gamma+1)(\alpha_2)^2(X_0)^2}$ and $\hat{R}_{10} = X_0^{-B_1}T_0^{B_2}$ with

$$B_1 = \frac{\alpha_1 + (m-1)\alpha_2 + 2\alpha_3}{\alpha_3 - \alpha_2},$$
$$B_2 = \frac{\alpha_1 - (2-m\gamma-\gamma)\alpha_2 + (3-m\gamma-\gamma+m)\alpha_3}{\alpha_3 - \alpha_2}.$$

In view of (20), η_s can be written as

$$\eta_s = \frac{1}{\alpha_3-\alpha_2}\log\left(\frac{X_0}{T_0}\left(\frac{T}{T_0}\right)^{\frac{\gamma-1}{2}}\right),$$

and hence $R_1(\eta_s) = R_{10}\exp\left[-\alpha_1 + (m+3)\alpha_2 - 2(m+2)\alpha_3\right]\eta_s$, where $R_{10} = \dfrac{2C}{(\gamma-1)(\alpha_2)^2}X_0^{2(m+1)}T_0^{-(m+1)(\gamma+1)}$. Thus, for a shock, $X(t) = X_0(T/T_0)^{(\gamma+1)/2}$, propagating into a nonuniform region $\rho(x) = \rho_c(x/X_0)^{\frac{2}{\gamma+1}(1-2\gamma-m\gamma)}$, $u = 0$, $p = p_0$, the downstream flow given by (13) takes the form

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_c\left(\frac{x}{X_0}\right)^{-2(m+2)}\left(\frac{T}{T_0}\right)^{m+3}, \quad u = \frac{\alpha_2 x}{T}, \tag{22}$$
$$p = \frac{\gamma+1}{2}\rho_c\left(\frac{\alpha_2 X_0}{T_0}\right)^2\left(\frac{T}{T_0}\right)^{-(m+1)\gamma}.$$

II. Similarly, when the medium behind a shock is represented by (14), conditions (18) imply that the speed of such a shock is given by

$$X(t) = X_0\left(\frac{\alpha_2 t + \alpha_4}{\alpha_2 t_0 + \alpha_4}\right)^{\delta}, \tag{23}$$

where $\delta = (\gamma+1)/\Gamma$, and the following conditions for $R_1(\eta_s)$ and $\rho_0(x)$ must hold:

$$A_3\int R_1(\eta_s)e^{A_4\eta_s}d\eta_s + A_5R_1(\eta_s)\left(\alpha_2 t + \alpha_4\right)^{A_6} - C = 0,$$
$$\rho_0(X(t)) = \frac{\gamma-1}{\gamma+1}X_0^{A_1}\left(\alpha_2 t_0 + \alpha_4\right)^{\delta A_1 - A_2}\left(\frac{\alpha_2 t + \alpha_4}{\alpha_2 t_0 + \alpha_4}\right)^{\delta A_1 - A_2},$$

where A_3 is same as in (14), $\eta_s = \frac{\Gamma}{\Gamma\alpha_3 - 2\alpha_2}\ln\left(X_0\left(\alpha_2 t + \alpha_4\right)^{\frac{\gamma-1}{\Gamma}}\right)$ and

$$A_5 = \frac{2(\gamma-1)(\alpha_2)^2X_0^{2+A_1}}{\Gamma^2\left(\alpha_2 t_0 + \alpha_4\right)^{(2+A_1)(\delta A_1 - A_2)}}, \quad A_6 = \delta A_1 - A_2 + \frac{2(m+\gamma)}{\Gamma}.$$

The solution of the above integral equation exits when $C = 0$, and it can be expressed as

$$R_1(\eta) \propto \exp\left\{-(\alpha_1 + (1-m)\alpha_3 + (4m/\Gamma)\alpha_2)\eta\right\}. \tag{24}$$

Thus, in view of (24), equations (14), which describe the flow downstream from the shock $x = X(t)$, yield

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_c\left(\frac{x}{X_0}\right)^{(m-1)}\left(\frac{\alpha_2 t+\alpha_4}{\alpha_2 t_0+\alpha_4}\right)^{-4m/\Gamma}, \quad u = \frac{2\alpha_2}{\Gamma}\frac{x}{(\alpha_2 t+\alpha_4)},$$
$$p = \left(\frac{2(\gamma+1)\rho_c\alpha_2 X_0}{\Gamma^2(\alpha_2 t_0+\alpha_4)}\right)^2\left(\frac{x}{X_0}\right)^{(m+1)}\left(\frac{\alpha_2 t+\alpha_4}{\alpha_2 t_0+\alpha_4}\right)^{-2(m+1)(\gamma+1)/\Gamma}, \tag{25}$$
$$\rho_0(X(t)) = \rho_c\left(\frac{\alpha_2 t+\alpha_4}{\alpha_2 t_0+\alpha_4}\right)^{\frac{m\gamma-3m-\gamma-1}{\Gamma}}, \quad X(t) = X_0\left(\frac{\alpha_2 t+\alpha_4}{\alpha_2 t_0+\alpha_4}\right)^{(\gamma+1)/\Gamma},$$

where Γ is same as in (14). It is interesting to note that the above solution (25) is exactly the same as the one obtained in the literature, using different approaches, for describing a blast wave (strong shock) propagating into a medium the density of which varies according to a power law of the distance measured from the source of explosion (see, Refs. [16], [17] and Chapter 2 in [18]).

Acknowledgement

Research support from ISRO-IIT Bombay, Space Technology Cell (Ref. No. 05-IS001) is gratefully acknowledged. The authors are thankful to the learned referee for making certain points more explicit. Authors also express their thanks to Mr. Manoj Pandey in helping them prepare the final manuscript.

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(Received: December 20, 2006; revised: September 4, 2007)

Published Online First: December 22, 2007