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Slip at the surface of a sphere translating perpendicular to a plane wall in micropolar fluid

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Abstract. The Stokes axisymmetrical flow caused by a sphere translating in a micropolar fluid perpendicular to a plane wall at an arbitrary position from the wall is presented using a combined analytical-numerical method. A linear slip, Basset type, boundary condition on the surface of the sphere has been used. To solve the Stokes equations for the fluid velocity field and the microrotation vector, a general solution is constructed from fundamental solutions in both cylindrical, and spherical coordinate systems. Boundary conditions are satisfied first at the plane wall by the Fourier transforms and then on the sphere surface by the collocation method. The drag acting on the sphere is evaluated with good convergence. Numerical results for the hydrodynamic drag force and wall effect with respect to the micropolarity, slip parameters and the separation distance parameter between the sphere and the wall are presented both in tabular and graphical forms. Comparisons are made between the classical fluid and micropolar fluid.

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1. Introduction

In recent years there exist several new developments in fluid mechanics that are concerned with structures within fluid, fluids which the classical theory has proved to be inadequate to describe their behavior. The simplest theory considered for structured fluids, the theory for micropolar fluid introduced by Eringen [1]. In micropolar fluids, rigid particles contained in a small volume element can rotate about the center of the volume element described by the microrotation vector. This local rotation of the particles is in addition to the usual rigid body motion of the entire volume element. In micropolar fluid theory, the laws of classical continuum mechanics are augmented with additional equations that account for conservation of microinertia moments and balance of first stress moments that arise due to consideration of the microstructure in a material. Thus, new kinematic variables, e.g., the gyration tensor and microinertia moment tensor, and the concepts of body moments, stress moments, and microstress are combined with classical continuum mechanics. Extensive reviews of the theory and applications can be found in the recent books by Łukaszewicz [2] and Eringen [3]

When one tries to solve the Navier–Stokes equation, it is usually assumed that no slippage arises at the solid-fluid interface. For almost a hundred years scientists and engineers have applied the no-slip boundary condition to fluid flow over a solid surface. While the well accepted no-slip boundary condition has been validated experimentally for a number of macroscopic flows, it remains an assumption not based on physical principles. In fact, nearly two hundred years ago Navier [4] proposed a general boundary condition that permits the possibility of fluid slip at a solid boundary. This boundary condition assumes that the tangential velocity of the fluid relative to the solid at a point on its surface is proportional to the tangential stress acting at that point. Basset [5] derived the expressions for the force and torque exerted by the fluid on a translating and rotating rigid sphere with a slip-flow boundary condition at its surface (e.g., a settling aerosol sphere). Neto et al. [6] summarized a review of experimental studies regarding the phenomenon of slip of Newtonian liquids at solid interfaces, discussed the influence of various factors on the results and proposed new lines of research.

The practical interest in the hydrodynamic interaction between small particles and a wall are important for various applications; e.g., in chemical engineering, for separation processes and filtration, in civil engineering for transport of sediments, in biology for motion of cells in blood vessels. The wall effect on the rate of settling a spherical particle is, in many ways, similar to the effect of a second particle. In order to treat the behavior of a group of particles settling in a container, it is necessary firstly to establish the effect of walls on the particle separately. The interaction of a particle with walls will depend on the particle shape, orientation, and position as well as the geometry of the containing walls. For prescribed translational and angular particle velocities, the macroscopic parameters of primary physical interest are the hydrodynamic forces and torques exerted by the fluid on the particles. Once these parameters are known for a given particle array, one may immediately solve the inverse problem of determining the state of motion of the particle from the known gravitational body forces and torques acting on them. Wall effects on the motion of a solid particle were extensively studied over the past three decades. Classical solutions of the Stokes equations for the flow due to a sphere which translates along or rotates around an axis orthogonal to smooth wall or parallel to the smooth wall, are investigated by many authors [7, 8, 9, 10, 11, 12, 13, 14].

Many biological and industrial processes involve fluid flows in which viscosity is large and/or particle lengths are small. Practical observation shows that, the Reynolds number expresses the relative magnitude of inertia forces to viscous forces, around a small particle moving close to a wall is assumed to be sufficiently small and therefore the Stokesian approximation may be applied. Maude [7] and Brenner [8] analyzed the fluid motion generated by a no-slip sphere moving perpendicular to a solid plane surface or to a free-surface plane. These problems are investigated in the literature by many methods. Happel and Brenner [15] used the reflection method and obtained analytic expression for the drag in a power series of the ratio of the sphere radius to the distance of the sphere center to the wall. Brenner [8], apply the bipolar coordinate method. The advantage of this method enables one to describe the particle surface and the boundary wall very conveniently at any separation distance except at the point of contact. The drag force is determined under the condition of no-slip. Jeffrey [16] treated the Stokes flow between two converging spheres, using lubrication analysis which is appropriate in the case of small spacing between the particle and the boundary. Kohr and Pop [17] discussed the implementation of the singularity method for Stokes flow past or due to the motion of a solid sphere above a plane wall. The boundary collocation method has been used by many authors to solve flow problems in viscous fluids. The collocation series solution technique is developed by Gluckman et al. [18] for unbounded, axisymmetric multispherical Stokes flow, and later it is extended to bounded flows by Leichtberg et al. [19] for coaxial chains of spheres in tubes. A review of application of boundary collocation method in mechanics of continuous media up to late eighties can be found in an article by Kolodziej [20]. Ganatos et al. [21] used a combined analytical-numerical solution to evaluate the drag on a sphere moving perpendicular between two plane parallel boundaries under no-slip condition for viscous flows. Keh and Hsu [22] employed the boundary collocation technique to examine the photophoretic motion of an aerosol sphere perpendicular to an infinite plane wall. The parallel motion of a spherical droplet in a quiescent immiscible fluid at an arbitrary position between two parallel plane walls was studied by Shapira and Haber [23] using the method of reflections and by Keh and Chen [24] using a boundary collocation technique.

The slip boundary condition has been studied by many authors in the literature. Chen and Keh [25] examined the creeping motion of a rigid sphere normal to an infinite plane wall, where the fluid may slip at the solid surfaces. Recently, the slow translational and rotational motions of a slip sphere along the symmetric axis of a circular cylindrical pore was considered by Lu and Lee [26] and parallel to two plane walls at an arbitrary position between them have been investigated by Chen and Keh [27] with the use of the boundary collocation method. Numerical results for the hydrodynamic drag force and torque acting on the particle were obtained for various cases. Chang and Keh [28] studied the slow translational motion of a spherical fluid or solid particle with a slip-flow surface in a viscous fluid perpendicular to two parallel plane walls at an arbitrary position between them.

All results cited above concern viscous fluids. For micropolar fluids, Kucaba-Piętal [29] used the traditional no-slip boundary condition on the surfaces of the sphere and plane wall to study the problem of a sphere moving perpendicular to a smooth plane wall. The slip-flow boundary condition is generally applicable in the micropolar flows. There are only few authors working with the slip boundary condition in microppolar fluids, e.g., Faltas and Saad [30], who considered the axisymmetric motion of a translating sphere with slip in an unbounded incompressible micropolar fluid at rest at infinity.

In micropolar fluids there are six Reynolds' numbers: three for translational viscosities and three for rotational viscosities. Thus for small Reynolds numbers the inertia terms of the field equations can be neglected. Therefore, the equations of motion describing this flow are the Stokes equations for micropolar fluid.

This paper presents the solution of the micropolar fluid flow due to translational motion of a sphere moving perpendicular to a plane wall. A linear slip boundary condition on the surface of the sphere is used. A combined analytical-numerical solution procedure is used. The drag on the translating sphere is evaluated. The effects of the variation of the micropolarity, slip and separation distance parameters on the drag as revealed by numerical studies is shown through figures. Wall effects are then examined.

2. Field equations

The equations governing the steady flow of an incompressible micropolar fluid under Stokesian assumption in the absence of body force and body couples are given by

$$\nabla \cdot \vec{q} = 0, \tag{2.1}$$

$$\nabla p + (\mu + k)\, \nabla \wedge \nabla \wedge \vec{q} - k\, \nabla \wedge \vec{\nu} = 0, \tag{2.2}$$

$$k\, \nabla \wedge \vec{q} - 2\, k\, \vec{\nu} - \gamma\, \nabla \wedge \nabla \wedge \vec{\nu} + (\alpha + \beta + \gamma)\, \nabla\nabla \cdot \vec{\nu} = 0. \tag{2.3}$$

where $\vec{q}$, $\vec{\nu}$ and p are velocity vector, microrotation vector and pressure, respectively, μ is the viscosity coefficient of the classical viscous fluid, and k, α, β and γ are the new viscosity coefficients for micropolar fluids.

The equations for the stress tensor t_{ij} and the couple stress tensor m_{ij} are defined by the constitutive equations

$$t_{ij} = -p\, \delta_{ij} + \mu\, (q_{i,j} + q_{j,i}) + k\, (q_{j,i} - \epsilon_{ijm}\, \nu_m), \tag{2.4}$$

$$m_{ij} = \alpha\, \nu_{m,m}\, \delta_{ij} + \beta\, \nu_{i,j} + \gamma\, \nu_{j,i}, \tag{2.5}$$

where the comma denotes the partial differentiation, δ_{ij} and ϵ_{ijm} are the Kronecker delta and the alternating tensor, respectively.

3. Statement of the problem

Let us consider a quasi-steady flow of an incompressible micropolar fluid due to a translating sphere, of radius a, at a constant velocity U_z perpendicular to impermeable plane wall, as shown in figure 1. The absence of explicit time dependence classifies the flow of an incompressible fluid as a quasi-steady flow, meaning in general that the instantaneous structure of the flow depends on the instantaneous boundary geometry and boundary conditions, and is independent of the motion at previous times. Thus, if all boundaries are stationary at a particular time, the fluid will also be stationary at that time, independent of the history of fluid and boundary motion.

A linear slip boundary condition, originally introduced by Basset [5] that the tangential velocity of fluid relative to the solid at a point on its surface is proportional to the tangential stress acting at the point, on the surface of the sphere has been used. The equality of the normal components of the velocity at the boundary arises from purely kinematical considerations when there is no mass transfer across the boundary (kinematic condition). The distance of the sphere center to the wall is b. In order to conveniently describe the surfaces of the sphere and plane, we choose the sphere center as the origin and use both the spherical coordinates (r, θ, ϕ) and the cylindrical coordinates (ρ, ϕ, z). The relations between the two coordinate systems are

$$r = (\rho^2 + z^2)^{1/2}, \qquad \theta = \cos^{-1} \frac{z}{(\rho^2 + z^2)^{1/2}}. \tag{3.1}$$

The flow generated is axially symmetric and all the flow functions are independent of ϕ. We can choose the velocity and microrotation vectors as

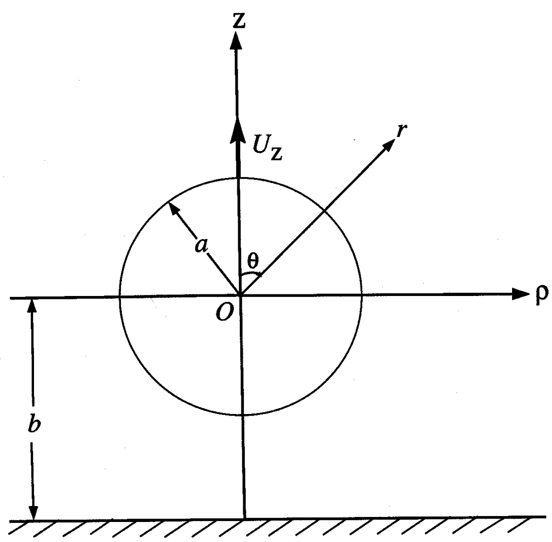

Figure 1. Schematic representation of a sphere translating perpendicularly to a plane wall.

$$\vec{q} = q_\rho(\rho, z)\,\vec{e}_\rho + q_z(\rho, z)\,\vec{e}_z, \tag{3.2}$$

$$\vec{\nu} = \nu_\phi(\rho, z)\,\vec{e}_\phi, \tag{3.3}$$

the stream function $\psi(\rho, z)$ can be used, and the axial and radial velocity components expressed as

$$q_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial z}, \qquad q_z = -\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}. \tag{3.4}$$

Since the speed U_z is supposed to be small, therefore the assumption of the Stokesian flow may be used, the problem is then governed by the following equations

$$0 = -\frac{\partial p}{\partial \rho} - \frac{k}{\rho}\frac{\partial}{\partial z}(\rho\,\nu_\phi) + \frac{\mu + k}{\rho}\frac{\partial}{\partial z}(L_1\psi), \tag{3.5}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{k}{\rho}\frac{\partial}{\partial \rho}(\rho\,\nu_\phi) - \frac{\mu + k}{\rho}\frac{\partial}{\partial \rho}(L_1\psi), \tag{3.6}$$

$$2\,\rho\,\nu_\phi = L_1\psi + \frac{\gamma}{k}\,L_1(\rho\,\nu_\phi), \tag{3.7}$$

where L_1 is the axisymmetric Stokesian operator

$$L_1 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}. \tag{3.8}$$

After elimination of the pressure and the microrotation vector component ν_ϕ from equations (3.5)–(3.7), we get

$$L_1^2\left(L_1 - \ell^2\right)\psi = 0, \tag{3.9}$$

with the microrotation being given by

$$\nu_\phi = \frac{1}{2\,\rho}\left(L_1\psi + \frac{2\,\mu + k}{k\,\ell^2}\,L_1^2\psi\right), \tag{3.10}$$

where $\ell^2 = k\,(2\,\mu + k)/(\gamma\,(\mu + k))$.

To solve equation (3.9), which is equivalent to the Stokes equations of micropolar fluid equations (3.5)–(3.7), the boundary conditions have to be specified.

At a surface of the sphere, we shall assume slip and use the most likely hypothesis [5, 15]. In our case this hypothesis takes the form

$$\beta_1\,(\vec{q} - U_z\,\vec{e}_z) = (\mathbf{I} - \vec{n}\vec{n})\cdot(\vec{n}\cdot\mathbf{t}), \tag{3.11}$$

where $\mathbf{I}$ is the unit dyadic, $\vec{n}$ is the unit normal vector at the particle surface pointing into the fluid and $\mathbf{t}$ is the stress tensor (dyadic), the constant, β_1, termed as the coefficient of sliding friction. This coefficient is a measure of the degree of tangential slip existing between the fluid and solid at its surface. It is assumed to depend only on the nature of the fluid and solid surface. In the limiting case of $\beta_1 = 0$, there is a perfect slip at the surface of the sphere and the solid sphere acts like a spherical gas bubble, while the standard no-slip boundary condition for solids is obtained by letting $\beta_1 \to \infty$.

According to the above remarks the slip boundary condition in cylindrical coordinates and the condition for the microrotation component on the sphere surface, $r = a$, can be expressed as

$$\beta_1\, q_\rho = (1 - n_\rho^2)\, n_\rho\, t_{\rho\rho} - n_\rho\, n_z^2\, t_{zz} + n_z\, \big((1 - n_\rho^2)\, t_{z\rho} - n_\rho^2\, t_{\rho z}\big), \quad (3.12)$$

$$\beta_1\, (q_z - U_z) = (1 - n_z^2)\, n_z\, t_{zz} - n_\rho^2\, n_z\, t_{\rho\rho} + n_\rho\, \big((1 - n_z^2)\, t_{\rho z} - n_z^2\, t_{z\rho}\big), \quad (3.13)$$

$$\nu_\phi = 0, \quad (3.14)$$

where n_ρ and n_z are the local ρ and z components of the unit normal vector $\vec{n}$.

The no-slip boundary conditions, vanishing of normal velocity q_z, and microrotation ν_ϕ along the wall, $z = -b$, are given by

$$q_\rho = 0, \quad (3.15)$$

$$q_z = 0, \quad (3.16)$$

$$\nu_\phi = 0, \quad (3.17)$$

the velocity components as well as the microrotation have to vanish, as $r \to \infty$.

4. Method of solution

The solution of (3.9) can be expressed in the form

$$\psi = \psi_s + \psi_w, \quad (4.1)$$

where the part ψ_s represents the general solution of the Stokes equation in the spherical coordinates. It can be expressed as an infinite series, and the part ψ_w represents the general solution of the Stokes equation in the cylindrical coordinates regular in the flow field and is given by an integral.

Firstly, we should find the general solutions of the Stokes equation for the micropolar fluid in the cylindrical and spherical coordinates. The stream function ψ can be expressed in the form $\psi = \psi^{(1)} + \psi^{(2)}$, where

$$L_1^2 \psi^{(1)} = 0, \qquad (L_1 - \ell^2)\, \psi^{(2)} = 0. \quad (4.2)$$

The general solution regular in the flow field for the stream function in the spherical coordinates is given by

$$\psi_s = \sum_{n=2}^{\infty} \big(A_n\, r^{-n+1} + B_n\, r^{-n+3} + \sqrt{r}\, C_n\, K_{n-\frac{1}{2}}(r\,\ell)\big)\, \Im_n(\zeta), \quad (4.3)$$

where A_n, B_n and C_n are unknown constants which will be determined using the boundary conditions, K_m is modified Bessel function of order m of the second kind, $\zeta = \cos\theta$ and $\Im_n$ is the Gegenbauer polynomial of the first kind of order n and degree $-1/2$.

The general solution regular in the flow field for the stream function in the cylindrical coordinates, given by the Fourier-Bessel integral

$$\psi_w = \int_0^\infty \Big(A(\tau)\, e^{-\tau z} + \tau\, z\, B(\tau)\, e^{-\tau z} + C(\tau)\, e^{-\xi z}\Big)\, \rho\, J_1(\tau\,\rho)\, d\tau, \quad (4.4)$$

where $A(\tau)$, $B(\tau)$, $C(\tau)$ are unknown functions of the separation variable τ and J_1 is the Bessel function of the first kind of order unity and $\xi = (\tau^2 + \ell^2)^{1/2}$. In the expressions (4.3) and (4.4) and in all subsequent expressions in this paper r is nondimensional with respect to the sphere radius a, i.e., r is r/a and also the parameters ρ, z, ℓ, τ, ξ are ρ/a, z/a, $\ell\, a$, $\tau\, a$, $\xi\, a$, respectively.

Then, the microrotation component can be written as

$$\begin{aligned} a^3\, \nu_\phi = (1-\zeta^2)^{-1/2} \sum_{n=2}^{\infty} \Big((3-2\,n)\, \frac{B_n}{r^n} + \frac{(\mu+k)\,\ell^2}{k\sqrt{r}}\, C_n\, K_{n-\frac{1}{2}}(r\,\ell) \Big)\, \Im_n(\zeta) \\ - \int_0^\infty \Big(\tau^2\, B(\tau)\, e^{-\tau z} - \frac{(\mu+k)\,\ell^2}{k}\, C(\tau)\, e^{-\xi z} \Big)\, J_1(\tau\,\rho)\, d\tau. \end{aligned} \tag{4.5}$$

The corresponding pressure field may be obtained by integration of the Stokes flow equations (3.5) and (3.6), then

$$a^3\, p = -(2\,\mu+k)\, \Big[\sum_{n=2}^{\infty} \frac{(2\,n-3)\, B_n\, P_{n-1}(\zeta)}{n\, r^n} + \int_0^\infty \tau^2\, B(\tau)\, e^{-\tau z}\, J_0(\tau\,\rho)\, d\tau \Big], \tag{4.6}$$

where a constant of integration has been neglected without loss of generality.

The expressions for the axial and radial components of velocity q_ρ and q_z, and the microrotation component ν_ϕ are given by

$$\begin{aligned} a^2\, q_\rho &= \sum_{n=2}^{\infty} [A_n\, A_{1n}(\rho,z) + B_n\, B_{1n}(\rho,z) + C_n\, C_{1n}(\rho,z)] \\ &\quad + \int_0^\infty L(\tau,z)\, \tau\, J_1(\tau\,\rho)\, d\tau, \end{aligned} \tag{4.7}$$

$$\begin{aligned} a^2\, q_z &= \sum_{n=2}^{\infty} [A_n\, A_{2n}(\rho,z) + B_n\, B_{2n}(\rho,z) + C_n\, C_{2n}(\rho,z)] \\ &\quad + \int_0^\infty M(\tau,z)\, \tau\, J_0(\tau\,\rho)\, d\tau, \end{aligned} \tag{4.8}$$

$$\begin{aligned} a^3\, \nu_\phi &= \sum_{n=2}^{\infty} [B_n\, B_{3n}(\rho,z) + C_n\, C_{3n}(\rho,z)] \\ &\quad + \int_0^\infty N(\tau,z)\, \tau\, J_1(\tau\,\rho)\, d\tau, \end{aligned} \tag{4.9}$$

where the functions $A_{in}(\rho,z)$, $B_{in}(\rho,z)$, $C_{in}(\rho,z)$ with $i = 1,2,3$ and $L(\tau,z) - N(\tau,z)$ are listed in Appendix A.

The expressions for the stress components are therefore as follows

$$
\begin{aligned}
a^3\, t_{\rho\rho} &= \sum_{n=2}^{\infty}[A_n\, \alpha_{1n}(\rho,z) + B_n\, \beta_{1n}(\rho,z) + C_n\, \gamma_{1n}(\rho,z)] \\
&\quad + \int_0^{\infty} \big(R(\tau,z)\, \tau\, J_0(\tau\, \rho) + S(\tau,z)\, \tau\, J_1(\tau\, \rho)\big)\, d\tau, \qquad (4.10) \\
a^3\, t_{zz} &= \sum_{n=2}^{\infty}[A_n\, \alpha_{2n}(\rho,z) + B_n\, \beta_{2n}(\rho,z) + C_n\, \gamma_{2n}(\rho,z)] \\
&\quad + \int_0^{\infty} T(\tau,z)\, \tau\, J_0(\tau\, \rho)\, d\tau, \qquad (4.11) \\
a^3\, t_{\rho z} &= \sum_{n=2}^{\infty}[A_n\, \alpha_{3n}(\rho,z) + B_n\, \beta_{3n}(\rho,z) + C_n\, \gamma_{3n}(\rho,z)] \\
&\quad + \int_0^{\infty} Q(\tau,z)\, \tau\, J_1(\tau\, \rho)\, d\tau, \qquad (4.12) \\
a^3\, t_{z\rho} &= \sum_{n=2}^{\infty}[A_n\, \alpha_{4n}(\rho,z) + B_n\, \beta_{4n}(\rho,z) + C_n\, \gamma_{4n}(\rho,z)] \\
&\quad + \int_0^{\infty} W(\tau,z)\, \tau\, J_1(\tau\, \rho)\, d\tau, \qquad (4.13)
\end{aligned}
$$

where the functions $\alpha_{jn}(\rho,z)$, $\beta_{jn}(\rho,z)$, $\gamma_{jn}(\rho,z)$ with $j = 1,2,3,4$ and $R(\tau,z)$ – $W(\tau,z)$ are also listed in Appendix A.

The boundary conditions (3.15)–(3.17) can be easily inverted and integration can be performed using results of Hankel transforms [31]. It is

$$L(\tau,z_p) = -\int_0^{\infty} t \sum_{n=2}^{\infty}[A_n\, A_{1n}(t,z_p) + B_n\, B_{1n}(t,z_p) + C_n\, C_{1n}(t,z_p)]\, J_1(\tau\, t)\, dt, \quad (4.14)$$

$$M(\tau,z_p) = -\int_0^{\infty} t \sum_{n=2}^{\infty}[A_n\, A_{2n}(t,z_p) + B_n\, B_{2n}(t,z_p) + C_n\, C_{2n}(t,z_p)]\, J_0(\tau\, t)\, dt, \quad (4.15)$$

$$N(\tau,z_p) = -\int_0^{\infty} t \sum_{n=2}^{\infty}[B_n\, B_{3n}(t,z_p) + C_n\, C_{3n}(t,z_p)]\, J_1(\tau\, t)\, dt, \quad (4.16)$$

where each of the above integral relations is applied at a plane wall, $z_p = -b$, where b is nondimensional, i.e., b/a.

The expressions (4.14)–(4.16) can be now rewritten as

$$L(\tau, z_p) = \sum_{n=2}^{\infty} [A_n\, e_{1n}(\tau, z_p) + B_n\, f_{1n}(\tau, z_p) + C_n\, h_{1n}(\tau, z_p)], \tag{4.17}$$

$$M(\tau, z_p) = \sum_{n=2}^{\infty} [A_n\, e_{2n}(\tau, z_p) + B_n\, f_{2n}(\tau, z_p) + C_n\, h_{2n}(\tau, z_p)], \tag{4.18}$$

$$N(\tau, z_p) = \sum_{n=2}^{\infty} [B_n\, f_{3n}(\tau, z_p) + C_n\, h_{3n}(\tau, z_p)], \tag{4.19}$$

where the expressions for $e_{in}(\tau, z_p)$, $f_{in}(\tau, z_p)$, $h_{in}(\tau, z_p)$ are given in Appendix B.

Now, we have to determine the functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ which appear in the above equations. The boundary conditions on the wall are used, therefore

$$\begin{aligned} A(\tau) = \frac{1}{\Delta} \big[\ell^2\, (\mu + k) \big(\tau\, b\, L(\tau, z_p) - \tau\, b\, M(\tau, z_p) - M(\tau, z_p) \big) \\ + k\, \tau\, \big(b\, (\xi - \tau)\, N(\tau, z_p) - \tau\, L(\tau, z_p) + \xi\, M(\tau, z_p) - N(\tau, z_p) \big) \big]\, e^{-\tau b}, \end{aligned} \tag{4.20}$$

$$B(\tau) = \frac{1}{\Delta} \big[\ell^2\, (\mu + k) \big(L(\tau, z_p) - M(\tau, z_p) \big) + k\, (\xi - \tau)\, N(\tau, z_p) \big]\, e^{-\tau b}, \tag{4.21}$$

$$C(\tau) = \frac{k\, \tau}{\Delta} \big[\tau\, L(\tau, z_p) - \tau\, M(\tau, z_p) + N(\tau, z_p) \big]\, e^{-\xi b}, \tag{4.22}$$

where $\Delta = \ell^2\, (\mu + k) + k\, \tau\, (\tau - \xi)$.

To determine the unknown constants A_n, B_n and C_n, we apply the boundary conditions (3.12)–(3.14) at the sphere surface to these velocity and microrotation components to give

$$\begin{aligned} 0 = \sum_{n=2}^{\infty} \big[A_n \left(A^*_{1n}(\rho, z) + a_{1n}(\rho, z) \right) + B_n \left(B^*_{1n}(\rho, z) + b_{1n}(\rho, z) \right) \\ + C_n \left(C^*_{1n}(\rho, z) + c_{1n}(\rho, z) \right) \big], \end{aligned} \tag{4.23}$$

$$\begin{aligned} U_z\, a^2 = \sum_{n=2}^{\infty} \big[A_n \left(A^*_{2n}(\rho, z) + a_{2n}(\rho, z) \right) + B_n \left(B^*_{2n}(\rho, z) + b_{2n}(\rho, z) \right) \\ + C_n \left(C^*_{2n}(\rho, z) + c_{2n}(\rho, z) \right) \big], \end{aligned} \tag{4.24}$$

$$\begin{aligned} 0 = \sum_{n=2}^{\infty} \big[A_n\, a_{3n}(\rho, z) + B_n \left(B_{3n}(\rho, z) + b_{3n}(\rho, z) \right) \\ + C_n \left(C_{3n}(\rho, z) + c_{3n}(\rho, z) \right) \big], \end{aligned} \tag{4.25}$$

where the expressions for $a_{in}(\rho, z)$, $b_{in}(\rho, z)$ and $c_{in}(\rho, z)$ are listed in Appendix B, these expressions containing several integrals, which are very complicated to evaluate them analytically, so that they may be calculated numerically.

To satisfy the boundary conditions (4.23)–(4.25) exactly along the surface of the sphere, the solution of the entire infinite array of unknown constants A_n, B_n and

C_n is required. However, the collocation method [21, 22, 29] satisfies the boundary conditions at a finite number of discrete points on the half-circular generating arc of the sphere (from $\theta = 0$ to π) and truncates the infinite series in (4.23)–(4.25) into finite ones. If the spherical boundary is approximated by satisfying the conditions of (4.23)–(4.25) at M discrete points (values of θ) on its generating arc, then infinite series are truncated after Mth terms, resulting in a system of $3M$ simultaneous linear algebraic equations in the truncated form of equations (4.23)–(4.25). This matrix equation can be numerically solved to yield the $3M$ unknown constants A_n, B_n and C_n required in the truncated equations for the flow field. The accuracy of the boundary-collocation/truncation method can be improved to any degree by taking a sufficiently large value of M. Naturally, the truncation error vanishes as $M \to \infty$, and the overall accuracy of the solution depends only on the numerical integration required in evaluating the matrix elements.

The hydrodynamic drag force exerted by the fluid on the particle can be determined by using the simple formula derived by authors [32, 33]. Therefore,

$$F_z = 4\,\pi\,(2\,\mu + k)\,\lim_{r\to\infty} \frac{\psi}{r\,a\,\sin^2\theta}, \tag{4.26}$$

$$= \frac{2\,\pi\,(2\,\mu + k)}{a}\,B_2. \tag{4.27}$$

For comparison purposes we note that for the case where the wall is absent ($b \to \infty$), so that the fluid is infinite. The formula for the drag on a translating sphere with slip in a micropolar fluid at rest at infinity is obtained by Faltas and Saad [30] as

$$F_z^{\infty} = -\frac{6\,\pi\,a\,U_z\,(2\,\mu + k)\,(\mu + k)\,(1 + \ell)\,(1 + \lambda_1)}{(\mu + k)\,(2\,\lambda_1 + 3)\,(1 + \ell) - k\,(1 + \lambda_1)}, \tag{4.28}$$

where $\lambda_1 = \beta_1\,a/(2\,\mu + k)$.

Having regard to (4.28) we write (4.27) in the form

$$F_z = -\frac{6\,\pi\,a\,U_z\,(2\,\mu + k)\,(\mu + k)\,(1 + \ell)\,(1 + \lambda_1)}{(\mu + k)\,(2\,\lambda_1 + 3)\,(1 + \ell) - k\,(1 + \lambda_1)}\,K, \tag{4.29}$$

where the wall correction factor K is defined by

$$K = \frac{\text{Drag in the presence of the wall}}{\text{Drag in an infinite medium}}.$$

When specifying the points along the semi-circular generating arc of the sphere where the boundary conditions are to be exactly satisfied, the first point that should be chosen is $\theta = \pi/2$, since this point defines the projected area of the sphere normal to the direction of motion. In addition, the points $\theta = 0$ and $\theta = \pi$ are also important because they control the gap between the sphere and the plane. However, an examination of the systems of linear algebraic equations for the unknown constants A_n, B_n and C_n shows that the coefficient matrix becomes singular if these points are used. To overcome the difficulty of singularity and to preserve the geometric symmetry of the spherical boundary about the equatorial plane $\theta = \pi/2$, points at $\theta = \varepsilon$, $\pi/2 \mp \varepsilon$, $\pi - \varepsilon$ are taken to be four basic collocation points. Additional points along the boundary are selected as mirror-image pairs about the plane $\theta = \pi/2$ to divide the θ coordinate into equal parts. The optimum value of ε in this work is found to be $0.01°$, with which the numerical results of the hydrodynamic drag force acting on the sphere can converge satisfactorily. All of these results were obtained by choosing the number of collocation points ($M = 20$) to show their convergence.

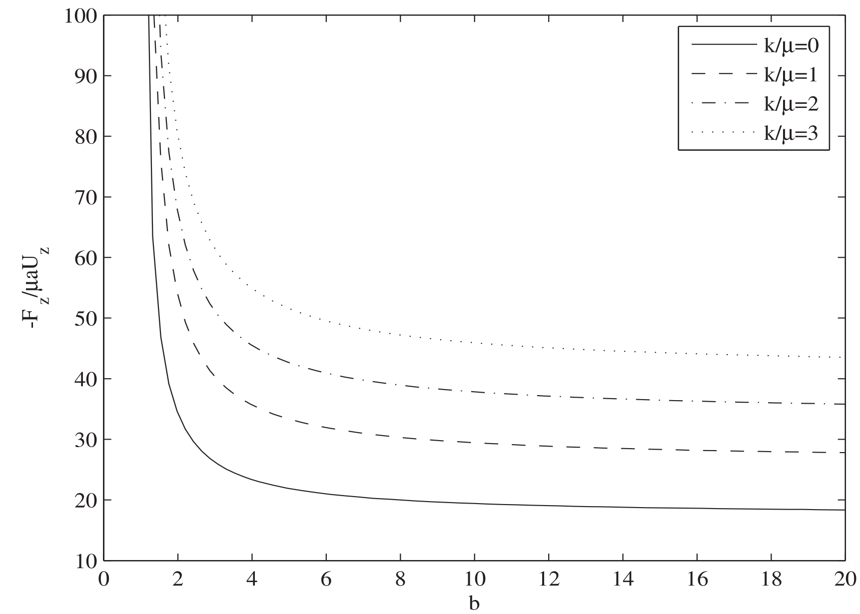

Figure 2. Variation of the drag coefficient with b for different values of k/μ for $\beta_1\, a/\mu = 10$.

Figures 2 and 3 indicate the variation of the nondimensional drag coefficient $-F_z/\mu\, a\, U_z$ with the separation distance parameter b between the particle and the walls, for different values of the micropolarity coefficient k/μ and the sliding friction parameter $\beta_1\, a/\mu$ when the parameter $\gamma/\mu\, a^2 = 0.3$. Obviously, the value of drag coefficient become infinite when the separation distance between the sphere and a plane wall vanishes ($b = 1$), where the tangential stresses, normal velocity and microrotation vanish, and tends to a constant value as b is greater than unity and there is an increase in the values of the drag coefficient when it is compared with the viscous fluid. As the micropolarity coefficient k/μ increases the drag coefficient increases and it can be seen that the drag coefficient is to be finite in both the perfect slip and nonslip limits in the entire range of the separation

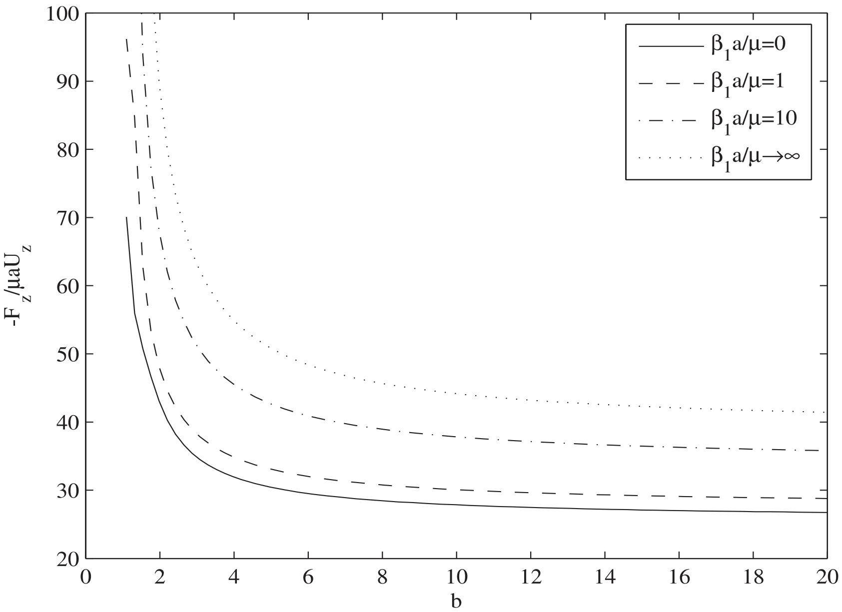

Figure 3. Variation of the drag coefficient with b for different values of $\beta_1\, a/\mu$ for $k/\mu = 2$.

distance parameter. In Table 1, numerical values of the wall correction factor K are presented for different values of b, k/μ with various values of $\beta_1\, a/\mu$. As expected, $K = 1$ as $b \to \infty$ and will become infinite in the limit $b = 1$ for any given values of $\beta_1\, a/\mu$ and k/μ.

5. Conclusion

In this paper, we have presented a combined analytical-numerical solution procedure for the Stokes flow caused by a sphere translating with slip axisymmetrically in a micropolar fluid perpendicular to a plane wall at an arbitrary position from the wall. The result for the drag force acting on the sphere by the fluid indicates that the solution procedure converges rapidly and accurate solutions can be obtained for various cases of the micropolarity, separation distance and slip parameters. Analysis shows that the drag force and wall correction factor is finite for all values of the slip parameter, in addition, they are increasing monotonically with an increase in the slip parameter but increasing and decreasing as the micropolarity parameter increasing, respectively. On the other hand, they become infinite when the sphere touches a plane wall and the wall correction factor tends to unity as the sphere translating far away from the wall for any given values of the sliding friction and micropolarity parameters. However, the wall correction factor of a micropolar fluid is smaller than a classical fluid.

Table 1. Wall correction factor for a sphere translating with slip perpendicular to a plane wall for various values of the micropolarity, separation distance and slip parameters.

$\beta_1 a/\mu$	b	K			
		$k/\mu = 0$	$k/\mu = 1$	$k/\mu = 3$	$k/\mu = 6$
0	1.2	3.4742	2.1246	2.0982	2.0694
	1.5	2.1325	1.8278	1.8080	1.7897
	5	1.1968	1.1816	1.1786	1.1765
	10	1.0911	1.0839	1.0827	1.0820
1	1.2	3.7290	2.1391	2.1047	2.0727
	1.5	2.2942	1.8517	1.8214	1.7977
	5	1.2020	1.1950	1.1869	1.1819
	10	1.0920	1.0909	1.0872	1.0848
10	1.2	4.9010	3.2195	3.1578	3.1069
	1.5	2.8126	2.2195	2.1578	2.1069
	5	1.2582	1.2350	1.2208	1.2090
	10	1.1155	1.1112	1.1046	1.0990
∞	1.2	6.3434	5.4587	5.3297	4.9967
	1.5	3.2054	2.9769	2.8203	2.6709
	5	1.2851	1.2612	1.2550	1.2509
	10	1.1262	1.1242	1.1218	1.1202

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Appendix A

The functions appearing in equations (4.7)–(4.13) are defined as

$$A_{1n}(\rho,z) = -\rho^{-1}\,(n+1)\,(\rho^2+z^2)^{-n/2}\,\Im_{n+1}(\omega), \tag{A.1}$$

$$B_{1n}(\rho,z) = -\rho^{-1}\,(\rho^2+z^2)^{-(n-2)/2}\,\big((n+1)\,\Im_{n+1}(\omega) - 2\,\omega\,\Im_n(\omega)\big), \tag{A.2}$$

$$\begin{aligned} C_{1n}(\rho,z) = {} & -\rho^{-1}\,(\rho^2+z^2)^{-1/4}\,\big((n+1)\,K_{n-\frac{1}{2}}(\chi)\,\Im_{n+1}(\omega) \\ & + \omega\,\chi\,K_{n-\frac{3}{2}}(\chi)\,\Im_n(\omega)\big), \end{aligned} \tag{A.3}$$

$$A_{2n}(\rho,z) = -(\rho^2+z^2)^{-(n+1)/2}\,P_n(\omega), \tag{A.4}$$

$$B_{2n}(\rho,z) = -(\rho^2+z^2)^{-(n-1)/2}\,\big(2\,\Im_n(\omega) + P_n(\omega)\big), \tag{A.5}$$

$$C_{2n}(\rho,z) = -(\rho^2+z^2)^{-3/4}\,\big(K_{n-\frac{1}{2}}(\chi)\,P_n(\omega) - \chi\,K_{n-\frac{3}{2}}(\chi)\,\Im_n(\omega)\big), \tag{A.6}$$

$$B_{3n}(\rho,z) = \rho^{-1}\,(3-2\,n)\,(\rho^2+z^2)^{-(n-1)/2}\,\Im_n(\omega), \tag{A.7}$$

$$C_{3n}(\rho,z) = \rho^{-1}\,k^{-1}\,\chi^2\,(\mu+k)\,(\rho^2+z^2)^{-3/4}\,K_{n-\frac{1}{2}}(\chi)\,\Im_n(\omega), \tag{A.8}$$

$$L(\tau,z) = -A(\tau)\,e^{-\tau z} + B(\tau)\,(1-\tau\,z)\,e^{-\tau z} - \xi\,\tau^{-1}\,C(\tau)\,e^{-\xi z}, \tag{A.9}$$

$$M(\tau,z) = -A(\tau)\,e^{-\tau z} - B(\tau)\,\tau\,z\,e^{-\tau z} - C(\tau)\,e^{-\xi z}, \tag{A.10}$$

$$N(\tau,z) = -\tau\,B(\tau)\,e^{-\tau z} + \ell^2\,k^{-1}\,\tau^{-1}\,(\mu+k)\,C(\tau)\,e^{-\xi z}, \tag{A.11}$$

$$\begin{aligned} \alpha_{1n}(\rho,z) = {} & \rho^{-2}\,(2\,\mu+k)\,(\rho^2+z^2)^{-(n+1)/2}\big[(n\,\rho^2+\rho^2+z^2)\,P_{n-1}(\omega) \\ & - (2\,n\,\rho^2 + 2\,\rho^2 + z^2)\,\omega\,P_n(\omega)\big], \end{aligned} \tag{A.12}$$

$$\begin{aligned} \beta_{1n}(\rho,z) = {} & \rho^{-2}\,(2\,\mu+k)\,(n-1)\,(\rho^2+z^2)^{-n/2} \\ & \cdot\big[\big(z^2 - (n-2)\,(2\,n\,\rho^2+z^2)\big)\,\omega\,P_n(\omega) \\ & + \big((n-3)\,[(n+3)\,\rho^2+z^2] + (7+3\,n^{-1})\,\rho^2\big)\,P_{n-1}(\omega)\big], \end{aligned} \tag{A.13}$$

$$\begin{aligned} \gamma_{1n}(\rho,z) = {} & \rho^{-2}\,(2\,\mu+k)\,(n-1)^{-1}\,(\rho^2+z^2)^{-5/4}\big[\big([\omega^2\,\rho^2\,\chi^2 \\ & + (n-1)\,(n\,\rho^2+z^2)]\,K_{n-\frac{1}{2}}(\chi) \\ & + [z^2 + (2-n)\,\rho^2 + (2\,n-3)\,(\rho^2+z^2)^{-1}\,\rho^4]\,\chi\,K_{n-\frac{3}{2}}(\chi)\big)\,P_{n-1}(\omega) \\ & - \big([\chi^2\,\rho^2 + (n-1)\,(2\,n\,\rho^2 + 2\,\rho^2 + z^2)]\,K_{n-\frac{1}{2}}(\chi) \\ & + (z^2+2\,\rho^2)\,\chi\,K_{n-\frac{3}{2}}(\chi)\big)\,\omega\,P_{n-1}(\omega)\big], \end{aligned} \tag{A.14}$$

$$\alpha_{2n}(\rho,z) = (2\,\mu+k)\,(\rho^2+z^2)^{-(n+2)/2}\big[(2\,n+1)\,\omega\,P_n(\omega) - n\,P_{n-1}(\omega)\big], \tag{A.15}$$

$$\begin{aligned} \beta_{2n}(\rho,z) = {} & (2\,\mu+k)\,(\rho^2+z^2)^{-n/2}\big[(2\,n-3)\,\omega\,P_n(\omega) \\ & - (1-n^{-1})\,(n-3)\,P_{n-1}(\omega)\big], \end{aligned} \tag{A.16}$$

$$\begin{aligned} \gamma_{2n}(\rho,z) = {} & (2\,\mu+k)\,(n-1)^{-1}\,(\rho^2+z^2)^{-5/4}\big[\big([\chi^2 + (2\,n+1)\,(n-1)]\,K_{n-\frac{1}{2}}(\chi) \\ & + \chi\,K_{n-\frac{3}{2}}(\chi)\big)\,\omega\,P_n(\omega) - \big([\omega^2\,\chi^2 + n\,(n-1)]\,K_{n-\frac{1}{2}}(\chi) \\ & + [2 - n + \rho^2\,(2\,n-3)\,(z^2+2\,\rho^2)^{-1}]\,\chi\,K_{n-\frac{3}{2}}(\chi)\big)\,P_{n-1}(\omega)\big], \end{aligned} \tag{A.17}$$

$$\begin{aligned} \alpha_{3n}(\rho,z) = {} & \rho^{-1}\,(2\,\mu+k)\,(\rho^2+z^2)^{-(n+1)/2}\,\big[n\,\omega\,P_{n-1}(\omega) \\ & + (n\,\rho^2+\rho^2-n\,z^2)\,(\rho^2+z^2)^{-1}\,P_n(\omega)\big] = \alpha_{4n}(\rho,z), \end{aligned} \tag{A.18}$$

$$\begin{aligned} \beta_{3n}(\rho,z) = {} & \rho^{-1}\,(2\,\mu+k)\,(n-1)^{-1}\,(\rho^2+z^2)^{-(1+n)/2} \\ & \cdot\big[(n^2-3\,n+3)\,(\rho^2+z^2)\,\omega\,P_{n-1}(\omega) \\ & + [z^2\,(n-3) + n\,(n-2)\,(\rho^2-z^2)]\,P_n(\omega)\big] = \beta_{4n}(\rho,z), \end{aligned} \tag{A.19}$$

$$\begin{aligned}\gamma_{3n}(\rho,z) = {} & \rho^{-1}\,(2\,\mu+k)\,(n-1)^{-1}\,(\rho^2+z^2)^{-7/4}\\ & \cdot\Big[\big([z^2\,\chi^2+n\,(n-1)\,(\rho^2+z^2)]\,K_{n-\frac{1}{2}}(\chi)\\ & +\rho^2\,(2\,n-3)\,\chi\,K_{n-\frac{3}{2}}(\chi)\big)\,\omega\,P_{n-1}(\omega)\Big]+\big(\rho^2\,\chi\,K_{n-\frac{3}{2}}(\chi)\\ & -[z^2\,\chi^2+(1-n)\,(n\,\rho^2+\rho^2-n\,z^2)]\,K_{n-\frac{1}{2}}(\chi)\big)\,P_n(\omega)\Big], \quad (A.20)\\ \gamma_{4n}(\rho,z) = {} & \rho^{-1}\,(2\,\mu+k)\,(n-1)^{-1}\,(\rho^2+z^2)^{-7/4}\\ & \cdot\Big[\big([n\,(n-1)\,(\rho^2+z^2)-\rho^2\,\chi^2]\,K_{n-\frac{1}{2}}(\chi)\\ & +\rho^2\,(2\,n-3)\,\chi\,K_{n-\frac{3}{2}}(\chi)\big)\,\omega\,P_{n-1}(\omega)\Big]+\big(\rho^2\,\chi\,K_{n-\frac{3}{2}}(\chi)\\ & +[\rho^2\,\chi^2-(1-n)\,(n\,\rho^2+\rho^2-n\,z^2)]\,K_{n-\frac{1}{2}}(\chi)\big)\,P_n(\omega)\Big], \quad (A.21)\end{aligned}$$

$$R(\tau,z) = -(2\mu+k)\big(\tau A(\tau)e^{-\tau z}-\tau B(\tau)(2-\tau z)e^{-\tau z}+\xi C(\tau)e^{-\xi z}\big), \quad (A.22)$$

$$S(\tau,z) = \rho^{-1}(2\mu+k)\big(A(\tau)e^{-\tau z}-B(\tau)(1-\tau z)e^{-\tau z}+\xi\tau^{-1}C(\tau)e^{-\xi z}\big), \quad (A.23)$$

$$T(\tau,z) = (2\mu+k)\big(\tau A(\tau)e^{-\tau z}+B(\tau)\tau^2 z e^{-\tau z}+\xi C(\tau)e^{-\xi z}\big), \quad (A.24)$$

$$Q(\tau,z) = (2\mu+k)\big(\tau A(\tau)e^{-\tau z}-\tau B(\tau)(1-\tau z)e^{-\tau z}+\xi^2\tau^{-1}C(\tau)e^{-\xi z}\big), \quad (A.25)$$

$$W(\tau,z) = (2\mu+k)\big(\tau A(\tau)e^{-\tau z}-\tau B(\tau)(1-\tau z)e^{-\tau z}+\tau C(\tau)e^{-\xi z}\big), \quad (A.26)$$

where $\omega = z\,(\rho^2+z^2)^{-1/2}$, $\chi=\ell\,(\rho^2+z^2)^{1/2}$, and P_n is the Legendre polynomial of order n.

Appendix B

The functions appearing in equations (4.17)–(4.19) are defined as

$$e_{1n}(\tau, z_p) = \frac{1}{n!}\left(\frac{\tau\,|z_p|}{z_p}\right)^{n-1} e^{-\tau |z_p|}, \tag{B.1}$$

$$f_{1n}(\tau, z_p) = \frac{1}{n!}\left(\frac{\tau\,|z_p|}{z_p}\right)^{n-3} \big((2\,n-3)\,\tau\,|z_p| - n\,(n-2)\big)\, e^{-\tau |z_p|}, \tag{B.2}$$

$$h_{1n}(\tau, z_p) = -\int_0^\infty t\, C_{1n}(t, z_p)\, J_1(\tau\, t)\, dt, \tag{B.3}$$

$$e_{2n}(\tau, z_p) = \frac{\tau^{n-1}}{n!}\left(\frac{|z_p|}{z_p}\right)^{n} e^{-\tau |z_p|}, \tag{B.4}$$

$$f_{2n}(\tau, z_p) = \frac{\tau^{n-3}}{n!}\left(\frac{|z_p|}{z_p}\right)^{n} \big((2\,n-3)\,\tau\,|z_p| - (n-1)\,(n-3)\big)\, e^{-\tau |z_p|}, \tag{B.5}$$

$$h_{2n}(\tau, z_p) = -\int_0^\infty t\, C_{2n}(t, z_p)\, J_0(\tau\, t)\, dt, \tag{B.6}$$

$$f_{3n}(\tau, z_p) = \frac{3-2\,n}{n!}\left(\frac{\tau\,|z_p|}{z_p}\right)^{n-2} e^{-\tau |z_p|}, \tag{B.7}$$

$$h_{3n}(\tau, z_p) = -\int_0^\infty t\, C_{3n}(t, z_p)\, J_1(\tau\, t)\, dt, \tag{B.8}$$

where the integrations in equations (B.3), (B.6) and (B.8) can be performed numerically after the substitution of equations (A.3), (A.6) and (A.8). Also the functions appearing in equations (4.23)–(4.25) are defined as

$$\begin{aligned} A^*_{1n}(\rho, z) = {} & \rho^{-1}\,(\rho^2+z^2)^{-n/2}\big[\big(1+(1+n)\,\lambda^{-1}\big)\,\omega\, P_n(\omega) \\ & - \big(1+(1+n)\,\omega^2\,\lambda^{-1}\big)\, P_{n-1}(\omega)\big], \end{aligned} \tag{B.9}$$

$$\begin{aligned} B^*_{1n}(\rho, z) = {} & \rho^{-1}\,(n-1)^{-1}\,(\rho^2+z^2)^{-(n-2)/2}\big[\big(n-3+n\,(n-2)\,\lambda^{-1}\big)\,\omega\, P_n(\omega) \\ & - \big(n-1-2\,z^2\,(\rho^2+z^2)^{-1}+n\,(n-2)\,\omega^2\,\lambda^{-1}\big)\, P_{n-1}(\omega)\big], \end{aligned} \tag{B.10}$$

$$\begin{aligned} C^*_{1n}(\rho, z) = {} & \rho^{-1}\,(\rho^2+z^2)^{-1/4}\big[\big([1+(1+n)\,\lambda^{-1}]\, K_{n-\frac{1}{2}}(\chi) \\ & + (n-1)^{-1}\,(1+\lambda^{-1})\,\chi\, K_{n-\frac{3}{2}}(\chi)\big)\,\omega\, P_n(\omega) \\ & - \big([1+(1+n)\,\omega^2\,\lambda^{-1}]\, K_{n-\frac{1}{2}}(\chi) \\ & + (n-1)^{-1}\,(1+\lambda^{-1})\,\omega^2\,\chi\, K_{n-\frac{3}{2}}(\chi)\big)\, P_{n-1}(\omega)\big], \end{aligned} \tag{B.11}$$

$$\begin{aligned} A^*_{2n}(\rho, z) = {} & (\rho^2+z^2)^{-(n+1)/2}\big[(1+n)\,\lambda^{-1}\,\omega\, P_{n-1}(\omega) \\ & - \big(1+(1+n)\,\lambda^{-1}\big)\, P_n(\omega)\big], \end{aligned} \tag{B.12}$$

$$\begin{aligned} B^*_{2n}(\rho, z) = {} & (n-1)^{-1}\,(\rho^2+z^2)^{-(n-1)/2}\big[\big(3-n-n\,(n-2)\,\lambda^{-1}\big)\, P_n(\omega) \\ & - \big(2-n\,(n-2)\,\lambda^{-1}\big)\,\omega\, P_{n-1}(\omega)\big], \end{aligned} \tag{B.13}$$

$$C^*_{2n}(\rho, z) = (\rho^2+z^2)^{-3/4}\big[\big((1+n)\,\lambda^{-1}\, K_{n-\frac{1}{2}}(\chi)$$

$$+ (n-1)^{-1} (1+\lambda^{-1})\, \chi\, K_{n-\frac{3}{2}}(\chi)\big)\, \omega\, P_{n-1}(\omega)$$
$$-\big([1+(1+n)\,\lambda^{-1}]\, K_{n-\frac{1}{2}}(\chi)$$
$$+ (n-1)^{-1} (1+\lambda^{-1})\, \chi\, K_{n-\frac{3}{2}}(\chi)\big)\, P_n(\omega)\big], \tag{B.14}$$

$$\begin{bmatrix} a_{1n}(\rho,z) \\ b_{1n}(\rho,z) \\ c_{1n}(\rho,z) \end{bmatrix} = \int_0^\infty \Big\{ H_1(\tau,z) \begin{bmatrix} e_{1n}(\tau,z_p) \\ f_{1n}(\tau,z_p) \\ h_{1n}(\tau,z_p) \end{bmatrix} + H_2(\tau,z) \begin{bmatrix} e_{2n}(\tau,z_p) \\ f_{2n}(\tau,z_p) \\ h_{2n}(\tau,z_p) \end{bmatrix}$$
$$+ H_3(\tau,z) \begin{bmatrix} 0 \\ f_{3n}(\tau,z_p) \\ h_{3n}(\tau,z_p) \end{bmatrix} \Big\}\, \tau\, J_0(\tau\,\rho)\, d\tau + \int_0^\infty \Big\{ H_4(\tau,z) \begin{bmatrix} e_{1n}(\tau,z_p) \\ f_{1n}(\tau,z_p) \\ h_{1n}(\tau,z_p) \end{bmatrix}$$
$$+ H_5(\tau,z) \begin{bmatrix} e_{2n}(\tau,z_p) \\ f_{2n}(\tau,z_p) \\ h_{2n}(\tau,z_p) \end{bmatrix} + H_6(\tau,z) \begin{bmatrix} 0 \\ f_{3n}(\tau,z_p) \\ h_{3n}(\tau,z_p) \end{bmatrix} \Big\}\, \tau\, J_1(\tau\,\rho)\, d\tau, \tag{B.15}$$

$$\begin{bmatrix} a_{2n}(\rho,z) \\ b_{2n}(\rho,z) \\ c_{2n}(\rho,z) \end{bmatrix} = \int_0^\infty \Big\{ H_7(\tau,z) \begin{bmatrix} e_{1n}(\tau,z_p) \\ f_{1n}(\tau,z_p) \\ h_{1n}(\tau,z_p) \end{bmatrix} + H_8(\tau,z) \begin{bmatrix} e_{2n}(\tau,z_p) \\ f_{2n}(\tau,z_p) \\ h_{2n}(\tau,z_p) \end{bmatrix}$$
$$+ H_9(\tau,z) \begin{bmatrix} 0 \\ f_{3n}(\tau,z_p) \\ h_{3n}(\tau,z_p) \end{bmatrix} \Big\}\, \tau\, J_0(\tau\,\rho)\, d\tau + \int_0^\infty \Big\{ H_{10}(\tau,z) \begin{bmatrix} e_{1n}(\tau,z_p) \\ f_{1n}(\tau,z_p) \\ h_{1n}(\tau,z_p) \end{bmatrix}$$
$$+ H_{11}(\tau,z) \begin{bmatrix} e_{2n}(\tau,z_p) \\ f_{2n}(\tau,z_p) \\ h_{2n}(\tau,z_p) \end{bmatrix} + H_{12}(\tau,z) \begin{bmatrix} 0 \\ f_{3n}(\tau,z_p) \\ h_{3n}(\tau,z_p) \end{bmatrix} \Big\}\, \tau\, J_1(\tau\,\rho)\, d\tau, \tag{B.16}$$

$$\begin{bmatrix} a_{3n}(\rho,z) \\ b_{3n}(\rho,z) \\ c_{3n}(\rho,z) \end{bmatrix} = \int_0^\infty \Big\{ H_{13}(\tau,z) \begin{bmatrix} e_{1n}(\tau,z_p) \\ f_{1n}(\tau,z_p) \\ h_{1n}(\tau,z_p) \end{bmatrix} + H_{14}(\tau,z) \begin{bmatrix} e_{2n}(\tau,z_p) \\ f_{2n}(\tau,z_p) \\ h_{2n}(\tau,z_p) \end{bmatrix}$$
$$+ H_{15}(\tau,z) \begin{bmatrix} 0 \\ f_{3n}(\tau,z_p) \\ h_{3n}(\tau,z_p) \end{bmatrix} \Big\}\, \tau\, J_1(\tau\,\rho)\, d\tau, \tag{B.17}$$

where

$$H_1(\tau,z) = \frac{2\,\tau\,\rho\,z^2}{\lambda\,\Delta\,(\rho^2+z^2)}\,[k\,\tau\,\xi\, e^{-\delta} + \vartheta_1\, e^{-\sigma}], \tag{B.18}$$

$$H_2(\tau,z) = \frac{-2\,\tau\,\rho\,z^2}{\lambda\,\Delta\,(\rho^2+z^2)}\,[k\,\tau\,\xi\, e^{-\delta} + \vartheta_2\, e^{-\sigma}], \tag{B.19}$$

$$H_3(\tau,z) = \frac{2\,k\,\tau\,\rho\,z^2}{\lambda\,\Delta\,(\rho^2+z^2)}\,[\xi\, e^{-\delta} + \vartheta_3\, e^{-\sigma}], \tag{B.20}$$

$$H_4(\tau,z) = \frac{1}{\lambda\,\Delta\,(\rho^2+z^2)}\,[k\,\tau\,\vartheta_4\, e^{-\delta} + \vartheta_1\,\vartheta_5\, e^{-\sigma}], \tag{B.21}$$

$$H_5(\tau, z) = \frac{-1}{\lambda\,\Delta\,(\rho^2+z^2)}\,[k\,\tau\,\vartheta_4\,e^{-\delta} + \vartheta_2\,\vartheta_5\,e^{-\sigma}], \tag{B.22}$$

$$H_6(\tau, z) = \frac{k}{\lambda\,\Delta\,(\rho^2+z^2)}\,[\vartheta_4\,e^{-\delta} + \vartheta_3\,\vartheta_5\,e^{-\sigma}], \tag{B.23}$$

$$H_7(\tau, z) = \frac{-1}{\lambda\,\Delta\,(\rho^2+z^2)}\,[\tau\,\vartheta_6\,e^{-\delta} + \{\vartheta_1\,\vartheta_7 + \lambda\,\ell^2\,(\mu+k)\,(\rho^2+z^2)\}\,e^{-\sigma}], \tag{B.24}$$

$$H_8(\tau, z) = \frac{1}{\lambda\,\Delta\,(\rho^2+z^2)}\,[\tau\,\vartheta_6\,e^{-\delta} + \{\vartheta_2\,\vartheta_7 + \lambda\,\ell^2\,(\mu+k)\,(\rho^2+z^2)\}\,e^{-\sigma}], \tag{B.25}$$

$$H_9(\tau, z) = \frac{-1}{\lambda\,\Delta\,(\rho^2+z^2)}\,[\vartheta_6\,e^{-\delta} - \{\vartheta_6 - k\,\tau\,(\delta-\sigma)\,\vartheta_7\}\,e^{-\sigma}], \tag{B.26}$$

$$H_{10}(\tau, z) = \frac{-\rho}{\lambda\,\Delta\,(\rho^2+z^2)}\,[k\,\tau\,\vartheta_8\,e^{-\delta} + \vartheta_1\,\vartheta_9\,e^{-\sigma}], \tag{B.27}$$

$$H_{11}(\tau, z) = \frac{\rho}{\lambda\,\Delta\,(\rho^2+z^2)}\,[k\,\tau\,\vartheta_8\,e^{-\delta} + \vartheta_2\,\vartheta_9\,e^{-\sigma}], \tag{B.28}$$

$$H_{12}(\tau, z) = \frac{k\,\rho}{\lambda\,\Delta\,(\rho^2+z^2)}\,[\vartheta_8\,e^{-\delta} - \vartheta_3\,\vartheta_9\,e^{-\sigma}], \tag{B.29}$$

$$H_{13}(\tau, z) = \frac{\ell^2\,\tau\,(\mu+k)}{\Delta}\,[e^{-\delta} - e^{-\sigma}], \tag{B.30}$$

$$H_{14}(\tau, z) = \frac{-\ell^2\,\tau\,(\mu+k)}{\Delta}\,[e^{-\delta} - e^{-\sigma}], \tag{B.31}$$

$$H_{15}(\tau, z) = \frac{1}{\Delta}\,[\ell^2\,(\mu+k)\,e^{-\delta} + k\,\tau\,(\tau-\xi)\,e^{-\sigma}], \tag{B.32}$$

with

$$\begin{aligned}
&\lambda = \beta_1\,a\,(\rho^2+z^2)^{1/2}/(2\,\mu+k), \quad \sigma = \tau\,(z+b), \quad \delta = \xi\,(z+b),\\
&\vartheta_1 = \ell^2\,(\mu+k)\,(\sigma-1) - k\,\tau^2,\ \vartheta_2 = \ell^2\,\sigma\,(\mu+k) - k\,\tau\,\xi,\ \vartheta_3 = \tau\,(\delta-\sigma) - \xi,\\
&\vartheta_4 = z\,(\xi^2\,\rho^2 - \tau^2\,z^2) - \xi\,[\lambda\,\rho^2 + (1+\lambda)\,z^2],\\
&\vartheta_5 = (\tau\,z - \lambda)\,(\rho^2+z^2) - z^2\,(1+2\,\tau\,z),\\
&\vartheta_6 = k\,\tau\,[\lambda\,(\rho^2+z^2) + 2\,\xi\,z\,\rho^2], \quad \vartheta_7 = \lambda\,(\rho^2+z^2) + 2\,\tau\,z\,\rho^2,\\
&\vartheta_8 = \xi^2\,\rho^2 - \tau^2\,z^2 - \xi\,z, \quad \vartheta_9 = \tau\,(\rho^2-z^2) - z.
\end{aligned}$$

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