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A Sinc-Collocation method for the linear Fredholm integro-differential equations

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Abstract. A Sinc–Collocation method for solving linear integro-differential equations of the Fredholm type is discussed. The integro-differential equations are reduced to a system of algebraic equations and Q-R method is used to establish numerical procedures. The convergence rate of the method is $O\left(e^{-k\sqrt{N}}\right)$. Numerical results are included to confirm the efficiency and accuracy of the method even in the presence of singularities and a comparison with the rationalized Haar wavelet method is made.

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1. Introduction

The boundary value problems in terms of integro-differential equations have many practical applications. The existence and uniqueness of the solutions for these problems were discussed by Agarwal[1, 2]. Hu [13, 14] discussed the extrapolation of the iterated Galerkin approximation to a particular case of Fredholm integro-differential equations. Volk [24] discussed the superconvergence of the iterated Galerkin approximation to Fredholm integro-differential equation. In [18], Neta employed Galerkin's method to obtain a numerical solution to a nonlinear integro-differential equation. Avudainayagam and Vani [4] used Wavelet-Galerkin method to obtain the numerical solution of integro-differential equations. Deeba et al. [6], described and adapted the Adomian's decomposition algorithm to obtain an approximate solution. Wazwaz [25] applied the decomposition method to handle boundary value problems for higher-order integro-differential equations. Hosseini and Shahmorad [12], used the Tau method with arbitrary polynomial bases to solve Fredholm integro-differential equations. Finally, Arikoglu and Ozkol [3], applied the differential transform method to solve boundary value problems for integro-differential equations.

During the last three decades, there have been developed a variety of numerical methods based on the Sinc approximations. These methods are now referred to as Sinc numerical methods [22]. In recent years, Sinc methods have been used in obtaining approximate solutions of a wide class of differential and integral equations [7, 8, 9, 10, 19]. The Sinc methods for ordinary differential equations have many salient features due to the properties of the basis functions and the manner in which the problem is discretized. Of equal practical significance is the fact that the method's implementation requires no modification in the presence of singularities. The approximating discrete system depends only on parameters of the differential equation regardless of whether it is singular or nonsingular.

There are two methods, based on Sinc approximation, to solve this problem: by a Galerkin-type scheme, and by collocation. The Galerkin scheme was developed first [20]; both procedures are described in [21]. Indeed, it is shown in [[21], Chapter 7] that both of these procedures are, in effect, equivalent, in that they converge at the same rate. We only describe Sinc collocation here, since it is easier to apply.

The present work is motivated by the desire to obtain numerical solutions to boundary value problems for second-order Fredholm integro-differential equations via Sinc–Collocation method. We will consider the numerical solution of a class of linear Fredholm integro-differential boundary value problems in the form

$$\begin{gathered}\sum_{i=0}^{n} \mu_i(x)\, u^{(i)}(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt, \qquad x \in J = [a,b] \\ n\, u(a) = \gamma \qquad n\,(n-1)\, u(b) = \beta\end{gathered} \tag{1.1}$$

where $K(x,t)$, $f(x)$, $u(x)$ and $\mu_i(x)$, $i = 0, 1, 2$, are analytic functions and λ is a parameter, and γ and β are real constants. It will always be assumed that (1.1) possesses a unique solution $u \in C^n(J)$.

The organization of the paper is as follows. In Section 2, we review some basic facts about the Sinc approximation that are necessary for the formulation of the discrete linear system. Section 3 is devoted to derivation of the discrete system. Section 4 presents appropriate techniques to treat nonhomogeneous boundary conditions for case $n = 2$. Some numerical examples are presented in Section 5.

2. Preliminaries and fundamentals

The collocation method of the next section depends on the accuracy of approximations obtained by Sinc-interpolation and Sinc-quadrature. A general review of sinc function approximation is given in [15, 21] and the recent papers [8, 9]. Hence, only properties important to the present goals are outlined in this section.

In Sinc approximations, the basis is derived from the Whittaker cardinal (Sinc) function

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}, \qquad -\infty < x < \infty$$

and its translates

$$\text{sinc}\left(\frac{x-j\,h}{h}\right), \quad h>0 \quad j=0,\pm1,\pm2,\ldots.$$

If $f(x)$ is defined on the real line, then for $h>0$ the series

$$C(f,h)=\sum_{k=-\infty}^{\infty} f(hk)\,\text{sinc}\left(\frac{x-hk}{h}\right).$$

is called the Whittaker cardinal expansion of f whenever this series converges.

In order to have the Sinc translates defined on a finite interval (a,b) the conformal map

$$\phi(x)=\ln\left(\frac{x-a}{b-x}\right) \tag{2.1}$$

is employed. This map carries the eye-shaped complex domain

$$D_E=\left\{z=x+iy:\left|\arg\left(\frac{z-a}{b-z}\right)\right|<d\le\frac{\pi}{2}\right\},$$

onto the infinite strip

$$D_d=\left\{\zeta=\xi+i\eta:|\eta|<d\le\frac{\pi}{2}\right\}.$$

The basis functions on (a,b) are then given by

$$S(j,h)\circ\phi(x)=\text{sinc}\left(\frac{\phi(x)-j\,h}{h}\right)$$

Notice that these functions exhibit Kronecker delta behavior on the grid points $x_k\in(a,b)$ defined by

$$x_k=\phi_k^{-1}(k\,h)=\frac{a+b\,e^{k\,h}}{1+e^{k\,h}}$$

The interpolation and quadrature formulas for $f(x)$ over $[a,b]$ take the form

$$f(x)\equiv\sum_{k=-N}^{N} f_k\;S(k,h)\circ\phi(x), \tag{2.2}$$

$$\int_a^b f(x)\,d\,x\cong h\sum_{k=-N}^{N}\frac{f_k}{\phi'(x_k)} \tag{2.3}$$

respectively and $f_k=f(x_k)$, and the mesh size is given by

$$h=\sqrt{\frac{2\,\pi\,d}{\alpha\,N}}, \qquad 0<\alpha\le 1$$

where N is suitably chosen and α depends on the asymptotic behavior of $f(x)$. Also, the n-th derivative of the function f at some points x_k can be approximated using finite number of terms as

$$f^{(n)}(x_k) \cong h^{-n} \sum_{k=-N}^{N} \delta^{(n)}_{j\,k}\, f_k$$

where

$$\delta^{(n)}_{j\,k} = \frac{d^n}{d\,\phi^n}\, S(j,h)\circ\phi(x)|_{x=x_k}$$

In particular,

$$\delta^{(0)}_{jk} = [S(j,h)\circ\phi(x)]\,|_{x=x_k} = \begin{cases} 1, & \mathrm{j} = k, \\ 0, & \mathrm{j} \neq k, \end{cases} \tag{2.4}$$

$$\delta^{(1)}_{jk} = \frac{d}{d\phi}\,[S(j,h)\circ\phi(x)]\,|_{x=x_k} = \begin{cases} 0, & \mathrm{j} = k, \\ \frac{(-1)^{k-j}}{k-j}, & \mathrm{j} \neq k, \end{cases} \tag{2.5}$$

and

$$\delta^{(2)}_{jk} = \frac{d^2}{d\phi^2}\,[S(j,h)\circ\phi(x)]\,|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & \mathrm{j} = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & \mathrm{j} \neq k. \end{cases} \tag{2.6}$$

3. The Sinc–Collocation method

We assume that $u(x)$, the solution of (1.1), is approximated by the finite expansion of Sinc basis functions

$$u_m(x) = \sum_{j=-N}^{N} u_j\, S(j,h)\circ\phi(x), \qquad m = 2\,N+1. \tag{3.1}$$

Application of (2.3) to the kernel integral in (1.1) gives

$$\int_a^b K(x,t)\,u(t)\,d\,t \approx h \sum_{j=-N}^{N} \frac{K(x,t_j)}{\phi'(t_j)}\, u_j, \tag{3.2}$$

where u_j denotes an approximate value of $u(x_j)$. If we replace the second term on the right-hand side of (1.1) with the right-hand side of (3.2) we have

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{n} \mu_i(x)\, \frac{d^i}{d\,x^i}\, S(j,h)\circ\phi(x) - h\,\lambda\, \frac{K(x,t_j)}{\phi'(t_j)} \right] u_j = f(x). \tag{3.3}$$

Setting

$$\frac{d^i}{d\phi^i}[S(j,h)\circ\phi(x)] = S^{(i)}_j(x), \qquad 0 \le i \le 2,$$

and noting that

$$\frac{d}{dx}[S(j,h)\circ\phi(x)] = S_j^{(1)}(x)\,\phi'(x),$$

and substituting $x = x_k = \phi(k\,h)$ in (3.3) and applying the collocation to it, we eventually obtain the following theorem

Theorem 3.1. *If the assumed approximate solution of the problem* (1.1) *is* (3.1), *then the discrete Sinc–Collocation system for the determination of the unknown coefficients* $\{u_j, -N < j < N\}$ *is given by*

$$\sum_{j=-N}^{N}\left[\sum_{i=0}^{n} g_i(x_k)\,\frac{\delta_{kj}^{(i)}}{h^i} - h\,\lambda\,\frac{K(x_k,t_j)}{\phi'(t_j)}\right] u_j = f_k, \quad k = -N, -N+1, \ldots, N \tag{3.4}$$

where for $n = 0, 1$ *and* 2 *we have*

$$g_0(x_k) = \mu_0(x_k), \qquad g_2(x_k) = \mu_2(x_k)\ [\phi'(x_k)]^2,$$
$$g_1(x_k) = \mu_1(x_k)\,\phi'(x_k) + \mu_2(x_k)\,\phi''(x_k).$$

To obtain a matrix representation of the equations in (3.4), recall the notation of Toeplitz matrices [11], denoted by $\mathbf{I}^{(i)}$,$0 \le i \le 2$, the $m \times m$ matrices whose jk-th entry is given by (2.4)-(2.6), respectively. We note that

$$\delta_{kj}^{(0)} = \delta_{jk}^{(0)}, \quad \delta_{kj}^{(2)} = \delta_{jk}^{(2)} \quad \text{and} \quad \delta_{kj}^{(1)} = -\delta_{jk}^{(1)}.$$

Let $\mathbf{D}(g(x_j))$ denote the $m \times m$ diagonal matrix with

$$\mathbf{D}(g(x))_{ij} = \begin{cases} g(x_i) & i = j, \\ 0 & i \neq j. \end{cases}$$

Let $\mathbf{u}$ be the m-vector with j-th component given by u_j , and $\mathbf{1}$ is an m-vector each of whose components is 1. In this notation the system in (3.4) takes the matrix form

$$\mathbf{A}\,\mathbf{u} = \Theta, \tag{3.5}$$

where

$$\Theta = \mathbf{D}\,(f)\ \mathbf{1},$$
$$\mathbf{u} = [u_{-N}, u_{-N+1}, \ldots, u_N]^{\tau},$$

and

$$\mathbf{A} = \sum_{i=0}^{n} \frac{1}{h^i}\,\mathbf{I}^{(i)}\,\mathbf{D}\,(g_i) - h\,\lambda\,\frac{K(x_k,t_j)}{\phi'(t_j)}.$$

Now we have a linear system of m equations for the m unknown coefficients, namely, $\{u_j\}_{j=-N}^{N}$. We can obtain the coefficient of the approximate solution by solving this linear system by Q-R method. The solution $\mathbf{u} = (u_{-N}, \ldots, u_N)^{\tau}$ gives the coefficients in the approximate Sinc–Collocation solution $u_m(x)$ of $u(x)$.

4. Treatment of boundary condition

In the previous section the development of the Sinc–Collocation technique for homogeneous boundary conditions provided a practical approach since, the sinc functions composed with the various conformal maps, $S(j,h)\circ\phi$, are zero at the endpoints of the interval. For $n=2$, if the boundary conditions are nonhomogeneous, then these conditions need be converted to homogeneous conditions via an interpolation by a known function . Using the transformation

$$y(x) = u(x) - \frac{(b-x)}{2\,(b-a)}\,\gamma - \frac{(x-a)}{2\,(b-a)}\,\beta$$

to the problem (1.1) yields the differential equation

$$\begin{aligned} &\sum_{i=0}^{2} \mu_i(x)\, y^{(i)}(x) = \hat{f}(x) + \lambda \int_a^b K(x,t)\, y(t)\, d\,t, \qquad x \in J = [a,b] \\ &\qquad\qquad y(a) = 0 \qquad y(b) = 0 \end{aligned} \tag{4.1}$$

where

$$\hat{f}(x) = f(x) - \frac{\beta-\gamma}{2\,(b-a)}\,\mu_1(x) - \left(\frac{(\beta-\gamma)\,x - a\,\beta + b\,\gamma}{2\,(b-a)}\right)\mu_0(x) + \lambda \int_a^b K(x,t)\,\left(\frac{(\beta-\gamma)\,t + \gamma\,b - a\,\beta}{2\,(b-a)}\right)d\,t$$

The resulting discrete system for the coefficients in the approximate Sinc solution

$$y_m(x) = \sum_{j=-N}^{N} y_j\, S(j,h)\circ\phi(x) + \frac{(b-x)}{2\,(b-a)}\,\gamma + \frac{(x-a)}{2\,(b-a)}\,\beta \tag{4.2}$$

is exactly the system in (3.4), with f in that system replaced by $\hat{f}$. Notice that if $\gamma=\beta=0$, then the discrete system obtained and the assumed solution (4.2) reduce to (3.4) and (3.1), respectively.

5. Numerical examples

We give several examples to show how this technique can be applied to integro-differential equations. We also compare our method with the rationalized Haar wavelet introduced in [16]. It is shown that the Sinc–Collocation method yields better results.

In the examples, the maximum absolute error at sinc grid points is taken as

$$\|E_s\| = \max_{-N\le i\le N} \left| u_{\text{exact solution}}(x_i) - U_{\text{Sinc–Collocation}}(x_i)\right|,$$

where

$$x_i = \frac{a + b\,e^{i\,h}}{1+e^{i\,h}}.$$

For all examples, we take $d = \pi/2$, $\lambda = 1$, $a = 0$, $b = 1$, $\lambda = 1$, $\mu_2(x) = 1$ and $\alpha = 1$.

Example 1. [16] For the sake of comparison , we consider the same problem discussed by Maleknejad and Mirzaee [16], who used the rationalized Haar wavelet method to obtain his numerical solution. Consider the Fredholm integral equation

$$u(x) = e^x - \frac{e^{x+1} - 1}{x+1} + \int_0^1 e^{x\,t}\, u(t)\, d\,t, \quad 0 \le x \le 1,$$

whose exact solution is

$$u(x) = e^x.$$

Maximum absolute error are tabulated in Table 4.1 for Sinc–Collocation together with the analogous results of Maleknejad and Mirzaee [16].

Table 4.1. Maximum absolute error for example 1

Sinc–Collocation	The rationalized Haar wavelet [16]
0.4756 E-006	0.0413

Example 2. In the case, $\mu_1(x) = 1/x^2$ and $\mu_0(x) = 1/x^3$ equation (1.1) becomes

$$u'' + \frac{1}{x^2}\, u' + \frac{1}{x^3}\, u = f(x) + \int_0^1 K(x,t)\, u(t)\, d\,t, \quad 0 \le x \le 1,$$

If

$$f(x) = \frac{24\,x^3 - x^2 - 3\,x + 2}{x^2} \quad \text{and} \quad K(x,t) = -\pi^3 \left(6\,x + \frac{1}{4}\right) \sin(\pi\, t)$$

and subject to the boundary conditions

$$u(0) = 0 \quad u(1) = 0,$$

then the exact solution is

$$u(x) = x\,(1-x).$$

The maximum absolute error,$\|E_s\|$, is reported in Table 4.2 as N increases from $N = 5$ to $N = 40$.

Table 4.2. $\|E_s\|$ for Example 2

N	$\|E_s\|$
5	4.4382 E-003
10	2.1190 E-004
20	6.1169 E-006
30	1.6203 E-007
40	1.8806 E-008

Example 3. In the case, $\mu_1(x) = 0$ and $\mu_0(x) = 1$ equation (1.1) becomes

$$u'' + u = f(x) + \int_0^1 K(x,t)\, u(t)\, dt, \quad 0 < x < 1,$$

If

$$f(x) = -\frac{1+3x}{4x^2} - \frac{1}{2} + 4x^2 + \sqrt{x}\,(1-x) \quad \text{and} \quad K(x,t) = 3\sqrt{t} - 15x^2$$

and subject to the boundary conditions

$$u(0) = 0 \quad u(1) = 0,$$

then the exact solution is

$$u(x) = \sqrt{x}\,(1-x).$$

The maximum absolute error, $\|E_s\|$, is reported in Table 4.3 as N increases from $N = 5$ to $N = 100$.

Table 4.3. $\|E_s\|$ for Example 3

N	$\|E_s\|$
5	1.6506 E-002
10	4.4348 E-003
20	6.4311 E-004
40	3.8304 E-005
50	4.1437 E-006
100	1.8104 E-007

Example 4. In the case, $\mu_1(x) = \frac{2x}{1-x^2}$ and $\mu_0(x) = \frac{2}{1-x^2}$ equation (1.1) becomes

$$u'' + \frac{2x}{1-x^2}\, u' + \frac{2}{1-x^2}\, u = f(x) + \int_0^1 K(x,t)\, u(t)\, dt, \quad 0 < x < 1,$$

If

$$f(x) = \frac{x^4 - 4x^2 + 4x - 3}{1-x^2} \quad \text{and} \quad K(x,t) = \frac{x^2+1}{3-e}\, e^t$$

and subject to the boundary conditions

$$u(0) = 0 \quad u(1) = 0,$$

then the exact solution is

$$u(x) = x\,(1-x).$$

The maximum absolute error, $\|E_s\|$, is reported in Table 4.4 as N increases from $N = 5$ to $N = 45$.

Table 4.4. $\|E_s\|$ for Example 4

N	$\|E_s\|$
5	4.1278 E-003
10	1.3366 E-004
20	2.0737 E-006
30	1.2983 E-007
40	1.4446 E-008
45	1.1377 E-008

All computations were carried out using **Matlab** on a personal computer.

6. Conclusion

In this paper,we use Sinc–Collocation method for solving Fredholm integro-differential equations. The formulation reduces to previous results for treating second order differential equation [5] when $\lambda = 0$ and First kind Fredholm integral equation [19] when $\mu_i = 0, \forall\, i$. The numerical results demonstrate the reliability and efficiency of using the method to solve such problems. The results of examples $2, 3$ and 4 clearly indicate that our methods are accurate even when singularities occur at the boundaries. The Sinc–Collocation technique introduced in this paper is also suitable for Fredholm integro-differential equations with other kinds of initial conditions. It seems that this method is also applicable for Volterra integro-differential equations, which is left as a future work.

In standard Sinc methods, the errors as known to be $\mathrm{O}\left(e^{-k\sqrt{N}}\right)$ with some $k > 0$, where N is the number of nodes or bases used in the method. However for a certain class of problems, using the proper transformation, the error can be improved to $\mathrm{O}\left(e^{-k'\,N/\log(N)}\right)$ with some $k' > 0$ [17, 23]. The integro-differential equations belonging to this class of problems will be considered in future publications.

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