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On some axial Couette flows of non-Newtonian fluids

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Abstract. The velocity fields corresponding to some flows of second grade and Maxwell fluids, induced by a circular cylinder subject to a constantly accelerating translation along its symmetry axis, are presented as Fourier-Bessel series in terms of the eigenfunctions of some suitable boundary value problems. These solutions satisfy both the associate partial differential equations and all imposed initial and boundary conditions. For α or $\lambda \to 0$, they are going to those for a Newtonian fluid. Finally, for comparison, some diagrams corresponding to the solutions for the flow through a circular cylinder are presented for different values of t and of the material constants.

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1. Introduction

Mechanics of non-Newtonian fluids present a special challenge to engineers, physicists and mathematicians. The non-linearity can manifest itself in a variety of ways. Among the many models, which have been used to describe the non-Newtonian behavior exhibited by different liquids, the fluids of differential type [1, 2] and those of rate type [2 4] have received much attention. Two recent and very interesting reviews regarding these models were given by Dunn and Rajagopal [5] and Rajagopal and Srinivasa [6]. Amid these fluids, a major attractiveness was secured by the incompressible second grade and Maxwell fluids whose constitutive equations are [1-5]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \qquad \mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.1}$$

respectively,

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \qquad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu\mathbf{A}_1. \tag{1.2}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ is the extra-stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{L}$ is the velocity gradient, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the

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first two Rivlin–Ericksen tensors, μ is the dynamic viscosity, α_1 and α_2 are the normal stress moduli, λ is the relaxation time and the superposed dot indicates the material time derivative.

In the following we shall study unidirectional motions of the form [7]

$$\mathbf{v} = \mathbf{v}(r,t)\mathbf{e}_z, \tag{1.3}$$

of second grade and Maxwell fluids. In (1.3) $\mathbf{v}$ is the velocity vector and $\mathbf{e}_z$ is the unit vector in the z-direction of the system of cylindrical coordinates r, θ and z. For these flows the constraint of incompressibility is automatically satisfied. On substituting (1.1)–(1.3) into the balance of linear momentum, dropping the body forces and assuming that there is no pressure gradient in the z-direction, we attain to the linear partial differential equations [7]

$$(\mu + \alpha_1 \partial_t)\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\mathbf{v}(r,t) = \rho\partial_t \mathbf{v}(r,t), \tag{1.4}$$

respectively [8],

$$\mu\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\mathbf{v}(r,t) = \rho\partial_t \mathbf{v}(r,t) + \rho\lambda\partial_t^2 \mathbf{v}(r,t). \tag{1.5}$$

The main goal of this work is to establish the analytical solutions corresponding to the unsteady flows (1.3) of the non-Newtonian fluids (1.1) and (1.2), induced by a circular cylinder subject to a constantly accelerating translation along its axis of symmetry. Their governing equations (1.4) and (1.5) can be solved in principle by several methods. The Laplace transform can be applied to eliminate the time variable. However, the inversion procedure for obtaining the solution is not always a trivial matter. Further, the solution so obtained for a second grade fluid does not satisfy the initial condition [9, 10]. This is due to the incompatibility between the prescribed data. Here we used the Fourier sine transform and the solutions that have been obtained satisfy both the governing equations and all imposed initial and boundary conditions. Similar solutions for the flow induced by a constantly accelerating plate in a second grade fluid, have been recently obtained in [9] and [11]. The unsteady flows of a fluid in cylindrical pipes of uniform circular cross-section have applications in medicine, chemical and petroleum industries.

2. Axial Couette flow of a second grade fluid

Let us consider a second grade fluid in the annular region between two infinitely long coaxial cylinders of radii R_0 and $R(> R_0)$. The inner cylinder is held fixed[†] while the outer one is subject, after time zero, to a translation along its axis of symmetry with a constant acceleration A. Due to the shear the fluid is gradually

[†] The case when both cylinders are subject to translations of constant accelerations does not present any more difficulty.

moved, its velocity field being of the form (1.3). The governing equation is (1.4) and the initial and boundary conditions are

$$v(r,0)=0, \quad R_0 \le r \le R \tag{2.1}$$

and

$$v(R_0,t)=0, \quad v(R,t)=At; \quad t>0. \tag{2.2}$$

Making the change of unknown function

$$v(r,t)=\frac{\ln(r/R_0)}{\ln(R/R_0)}At+u(r,t), \tag{2.3}$$

we attain to the next problem with initial and boundary conditions

$$\begin{gathered}(\nu+\alpha\partial_t)\Delta u(r,t)=\partial_t u(r,t)+A\frac{\ln(r/R_0)}{\ln(R/R_0)}; \quad r\in(R_0,R), \quad t>0,\\ u(r,0)=0, \quad r\in[R_0,R]; \quad u(R_0,t)=u(R,t)=0, \quad t>0,\end{gathered} \tag{2.4}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid, $\alpha=\alpha_1/\rho$ and the operator $\Delta=\partial_r^2+\frac{1}{r}\partial_r$.

In order to solve this problem we shall use, as in [12], the expansion theorem of Steklov. In view of this theorem our solution $u(r,t)$, whose partial derivatives $\partial_r u$ and $\partial_r^2 u$ have to be piecewise continuous, can be written, for each $t>0$, as Fourier-Bessel series absolutely and uniformly convergent in terms of the eigenfunctions

$$B(rr_n)=A_n\left[J_0(rr_n)-\frac{J_0(R_0r_n)}{Y_0(R_0r_n)}Y_0(rr_n)\right], \tag{2.5}$$

of the eigenvalue problem $\Delta\Phi+\gamma^2\Phi=0$, $\Phi(R_0)=\Phi(R)=0$, i.e.,

$$u(r,t)=\sum_{n=1}^{\infty}u_n(t)B(rr_n). \tag{2.6}$$

Here, $J_0(\cdot)$ and $Y_0(\cdot)$ are Bessel functions of order zero of the first and second kind in standard notations, r_n are the positive roots of the transcendental equation

$$J_0(Rr)Y_0(R_0r)-Y_0(Rr)J_0(R_0r)=0$$

and the constants A_n are chosen so that the normalization conditions

$$\int_{R_0}^{R} r[B(rr_n)]^2dr=1, \quad n=1,2,3,\ldots \tag{2.7}$$

are satisfied.

Now, introducing (2.6) into $(2.4)_1$, multiplying then by $rB(rr_p)$, integrating the result with respect to r from R_0 to R and taking into account $(2.4)_{2-4}$, we find that

$$(1+\alpha r_n^2)\dot{u}_n(t)+vr_n^2u_n(t)+AV_n=0, \quad u_n(0)=0; \quad n=1,2,3,\ldots \tag{2.8}$$

where V_n ($n = 1, 2, 3, \dots$) are the modified finite Hankel transforms of $\ln(r/R_0)/\ln(R/R_0)$.

Solving the ordinary differential equation $(2.8)_1$ subject to the initial condition $(2.8)_2$ and using (2.3) and (2.6), we find for the velocity field $v(r,t)$ the expression

$$v(r,t) = \frac{\ln(r/R_0)}{\ln(R/R_0)} At - \frac{A}{\nu} \sum_{n=1}^{\infty} \left[1 - \exp\left(-\frac{\nu r_n^2}{1 + \alpha r_n^2} t \right) \right] \frac{V_n B(r r_n)}{r_n^2}, \tag{2.9}$$

from which the solution for the Navier-Stokes fluid

$$v(r,t) = \frac{\ln(r/R_0)}{\ln(R/R_0)} At - \frac{A}{\nu} \sum_{n=1}^{\infty} \left[1 - \exp(-\nu r_n^2 t) \right] \frac{V_n B(r r_n)}{r_n^2}, \tag{2.10}$$

appears as a limiting case for $\alpha \to 0$.

3. Axial Couette flow of a Maxwell fluid

Suppose now that a Maxwell fluid fills the space between the same infinitely coaxial cylinders of radii R_0 and R. The inner cylinder is again fixed and the outer one is subject, after time zero, to a translation along its axis of symmetry with a constant acceleration A. The governing equation is (1.5), the boundary conditions are the same as in (2.2) while the initial conditions are [8] (the equation (1.5) being of a higher order in t than the equation (1.4))

$$v(r,0) = \partial_t v(r,0) = 0, \quad R_0 \le r \le R. \tag{3.1}$$

Making the same change of unknown function as before we attain to the next partial differential equation

$$\lambda \partial_t^2 u(r,t) + \partial_t u(r,t) + A \frac{\ln(r/R_0)}{\ln(R/R_0)} = \nu \Delta u(r,t); \quad r \in (R_0, R),\ t > 0, \tag{3.2}$$

with the initial and boundary conditions

$$u(r,0) = 0, \quad \partial_t u(r,0) = -A \frac{\ln(r/R_0)}{\ln(R/R_0)}; \quad r \in [R_0, R] \tag{3.3}$$

and

$$u(R_0, t) = u(R, t) = 0, \quad t > 0. \tag{3.4}$$

Introducing (2.6) into (3.2), multiplying then by $rB(rr_p)$, integrating the result after r from R_0 to R and having in mind (3.3) and (3.4) we get

$$\lambda \ddot{u}_n(t) + \dot{u}_n(t) + \nu r_n^2 u_n(t) + A V_n = 0, \quad t > 0, \quad n = 1, 2, 3, \dots \tag{3.5}$$

where

$$u_n(0) = 0 \text{ and } \dot{u}_n(0) = -A V_n;\ n = 1, 2, 3, \dots \tag{3.6}$$

From (3.5), (3.6), (2.6) and (2.3), it easily results the velocity field $v(r,t)$ under form

$$v(r,t) = \frac{\ln(r/R_0)}{\ln(R/R_0)}At - \frac{A}{\nu}\sum_{n=1}^{p}\left[1 - \frac{p_{1n}^2\exp(p_{2n}t) - p_{2n}^2\exp(p_{1n}t)}{p_{2n} - p_{1n}}\lambda\right]\frac{V_nB(rr_n)}{r_n^2} -$$
$$- \frac{A}{\nu}\sum_{n=p+1}^{\infty}\left\{1 - \exp\left(-\frac{t}{2\lambda}\right)\left[\cos\left(\frac{\beta_n t}{2\lambda}\right) + \frac{1-2\nu\lambda r_n^2}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\right]\right\}\cdot$$
$$\cdot\frac{V_nB(rr_n)}{r_n^2}, \tag{3.7}$$

where $p_{1n,2n} = \frac{-1\pm\sqrt{1-4\nu\lambda r_n^2}}{2\lambda}$, $\beta_n = \sqrt{4\nu\lambda r_n^2 - 1}$ and $r_p \le \frac{1}{2\sqrt{\nu\lambda}} < r_{p+1}$.

By letting $\lambda \to 0$ in (3.7) we again attain to the solution (2.10) corresponding to a Navier–Stokes fluid.

4. Axial Couette flow through a circular cylinder

Let us consider a second grade fluid or a Maxwell fluid at rest in an infinite circular cylinder of radius R. After time zero, the cylinder is subject to a translation along its symmetry axis with the constant acceleration A. By the influence of shear the fluid is gradually moved. The governing equations are again (1.4) and (1.5) and the initial conditions are (2.1), respectively, (3.1). The boundary conditions reduce to

$$v(R,t) = At, \quad t > 0 \tag{4.1}$$

and the natural condition

$$|v(0,t)| < \infty, \quad t > 0. \tag{4.2}$$

Taking the limits of Eqs. (2.5) and (2.7) when $R_0 \to 0$ we find the associate eigenfunctions $\sqrt{2}J_0(rr_n)/[RJ_1(Rr_n)]$, (see [13], Eq. (7) pp. 522). The corresponding velocity fields

$$v(r,t) = At - \frac{2A}{\nu R}\sum_{n=1}^{\infty}\left[1 - \exp\left(-\frac{\nu r_n^2}{1+\alpha r_n^2}t\right)\right]\frac{J_0(rr_n)}{r_n^3J_1(Rr_n)}, \tag{4.3}$$

$$v(r,t) = At - \frac{2A}{\nu R}\sum_{n=1}^{\infty}\left[1 - \exp(-\nu r_n^2 t)\right]\frac{J_0(rr_n)}{r_n^3J_1(Rr_n)} \tag{4.4}$$

and

$$v(r,t) = At - \frac{2A}{\nu R}\sum_{n=1}^{p}\left[1 - \frac{p_{1n}^2\exp(p_{2n}t) - p_{2n}^2\exp(p_{1n}t)}{p_{2n}-p_{1n}}\lambda\right]\frac{J_0(rr_n)}{r_n^3 J_1(Rr_n)} -$$
$$-\frac{2A}{\nu R}\sum_{n=p+1}^{\infty}\left\{1 - \exp\left(-\frac{t}{2\lambda}\right)\left[\cos\left(\frac{\beta_n t}{2\lambda}\right) + \frac{1-2\nu\lambda r_n^2}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\right]\right\}\cdot \qquad (4.5)$$
$$\cdot\frac{J_0(rr_n)}{r_n^3 J_1(Rr_n)},$$

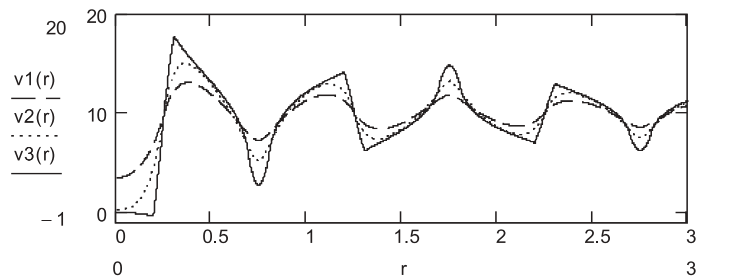
where r_n are positive roots of the transcendental equation $J_0(Rr) = 0$, are also obtained from (2.9), (2.10) and (3.7) for $R_0 \to 0$. Making $\alpha \to 0$ in (4.3) or $\lambda \to 0$ in (4.5) one certainly obtains (4.4).

5. Numerical solutions and conclusions

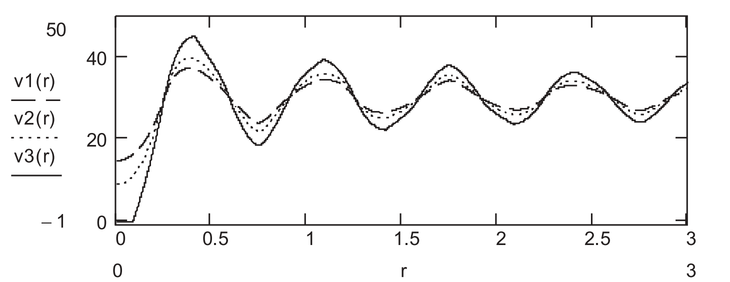
In this paper we established the analytical expressions of the velocity fields corresponding to an axial Couette flow of a second grade fluid and a Maxwell one in cylindrical domains. A circular cylinder that, after time zero, is subject to a linear translation of constant acceleration induces the motion. The solutions corresponding to the flow through a circular cylinder are obtained as a limiting case ($R_0 \to 0$) of those between two cylinders.

Direct computations show that all solutions satisfy both the associate partial differential equations and all imposed initial and boundary conditions, the differentiation of the respective series, term by term, being clearly permissible. Moreover, the similar solutions corresponding to a Navier-Stokes fluid appear as limiting cases of our solutions for α, respectively, $\lambda \to 0$. It is important to note that the solutions for a Maxwell fluid, (3.7) and (4.5), contain sine and cosine terms. This indicates that in contrast with the second grade and Newtonian fluids, whose solutions do not contain such terms, oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.

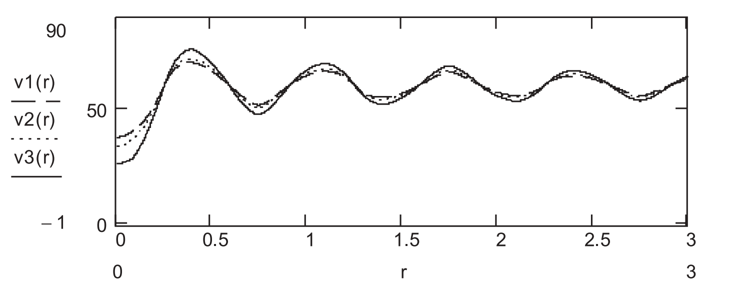
In Figs. 1 and 2, for comparison, the variations of the velocity fields (4.3), (4.4) and (4.5) are plotted for different values of t and of the material constants. The positive roots of the transcendental equation $J_0(Rr) = 0$ have been approximated by $(4n-1)\pi/(4R)$ (see for instance [13], pp.195). For small values of t, one can observe both the differences between the associated diagrams and the more pronounced oscillations corresponding to the Maxwell fluid. The diagrams for a Navier-Stokes fluid, as it was to be expected, are situated between those corresponding to a Maxwell fluid and a second grade one. For large values of t the non-Newtonian effects become weak and the profiles of the three velocity fields are close by.



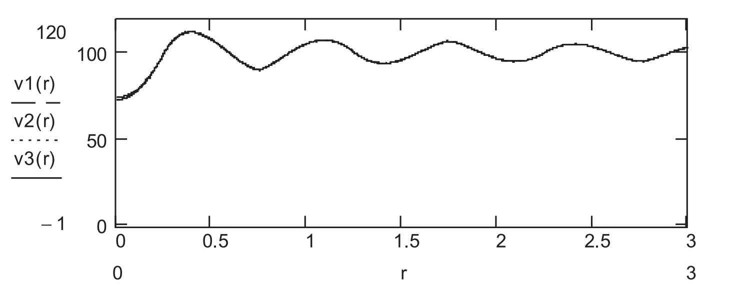
a) $\bar{t} = 5s$.



b) $\bar{t} = 15s$.

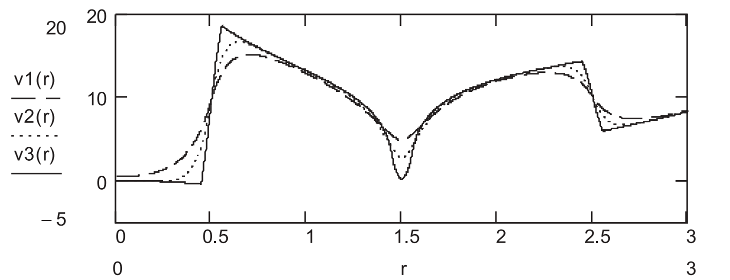


c) $\bar{t} = 30s$.

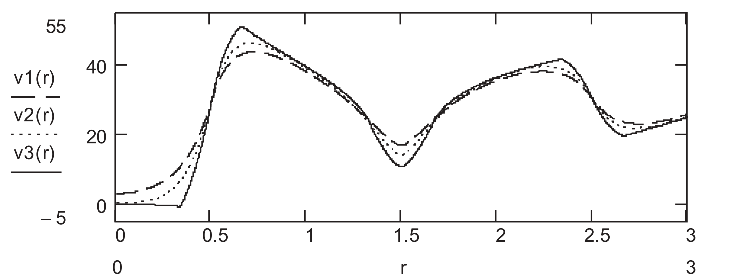


d) $\bar{t} = 50s$.

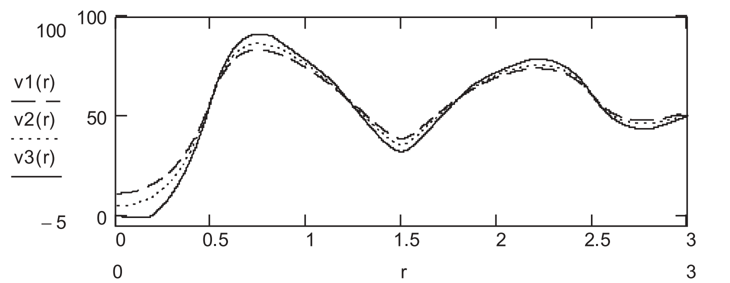
Figure 1. Velocity profiles $v(r,t)$ corresponding to the relations (4.3) - curves v1, (4.4) - curves v2 and (4.5) - curves v3, for $A = 2$, $\nu = 0.0011746$ (glycerin), $R = 0.25$, $\lambda = 10$ and $\alpha = 0.009$.



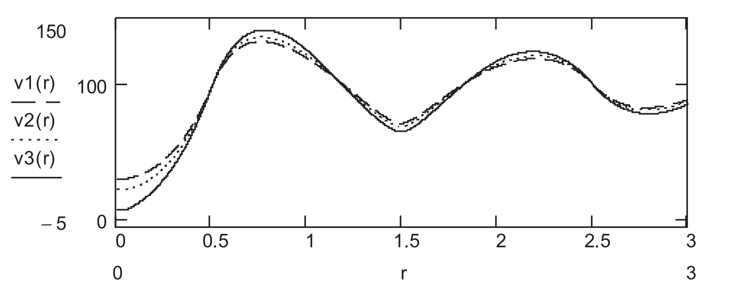
a) $\bar{t} = 5s$.



b) $\bar{t} = 15s$.



c) $\bar{t} = 30s$.



d) $\bar{t} = 50s$.

Figure 2. Velocity profiles $v(r, t)$ corresponding to the relations (4.3) - curves v1, (4.4) - curves v2 and (4.5) - curves v3, for $A = 2$, $\nu = 0.0011746$ (glycerin), $R = 0.5$, $\lambda = 10$ and $\alpha = 0.009$.

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