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Nonlinear Kelvin-Helmholtz instability of two miscible ferrofluids in porous media

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Abstract. The nonlinear theory of the Kelvin-Helmholtz instability is employed to analyze the instability phenomenon of two ferrofluids through porous media. The effect of both magnetic field and mass and heat transfer is taken into account. The method of multiple scale expansion is employed in order to obtain a dispersion relation for the first-order problem and a Ginzburg–Landau equation, for the higher-order problem, describing the behavior of the system in a nonlinear approach. The stability criterion is expressed in terms of various competing parameters representing the mass and heat transfer, gravity, surface tension, fluid density, magnetic permeability, streaming, fluid thickness and Darcy coefficient. The stability of the system is discussed in both theoretically and computationally, and stability diagrams are drawn.

Keywords. Nonlinear instability, multiple scale methods, ferrofluids, porous media.

1. Introduction

The Rayleigh-Taylor instability (RTI) occurs when a heavy fluid is supported by a lighter one in a gravitational, or equivalently, when a heavy fluid is accelerated by a lighter one. The RTI has been addressed in several studies owing to its importance in stratified system, among which planetary and stellar atmospheres are two examples. The effect of external forces has importance mainly in planetary and stellar systems. Coriolis and centrifugal forces are more common in these systems and play an important role in determining many phenomena including the RTI. The effect of rotation at an angle with gravity was first investigated by Hide [1], who gave a detailed analysis for the case of rotation parallel to gravity. Different properties of fluids have been included in the RTI through theoretical investigations. For instance, Chandrasekhar [2] included the viscosity and Reid [3] added the effects of both viscosity and surface tension. Fluids can also be supposed to be made of a number of layers or to be continuously stratified (for an excellent review see Chandrasekhar [4]). A new and original experiment has been done by Fermigier et al. [5], who have made progress in the development of the RTI, not seen before. Jun and Norman [6] concluded that the magnetic field influences the development of the RTI considerably and that the instability amplifies the field.

Their investigations were intended to study the basic physics of the instability rather than a specific application. The literature of the pure hydrodynamic nonlinear RTI (Hasimoto and Ono [7] and Nayfeh [8]) reveals that the perturbation technique may be chosen so that it renders uniformly valid expansions near the cutoff wave number. The method of multiple scale introduced by Nayfeh [8] has the advantage that it leads to two nonlinear Schrödinger equations describing the finite amplitude wave propagation through the surface. One is valid near the cutoff wave number, while the other equation can be used to study the stability of the system. Malik and Singh [9] studied the motion of inviscid, incompressible, non-conducting ferrofluids with a magnetic field and surface tension under gravity, and demonstrated the formation of bubbles by means of Lagrangian transformations. They showed as well how the magnetic field and the surface tension stabilize the interface to conserve the contours. The formation of bubbles can be inhibited by using a magnetic field with a higher permeability and /or by increasing the strength of the applied magnetic field.

The Kelvin-Helmholtz instability (KHI) occurs when two fluids are in relative motion on either side of a common boundary. The KHI is important in understanding a variety of space and astrophysical phenomena involving plasma flow (e.g. the stability of the solar-wind-magnetosphere interface, interaction between adjacent streams of different velocities in the solar wind, and dynamo generation of cosmic magnetism). The linear theory of the KHI was investigated by Chandrasekhar [4]. He discussed the effect of surface tension, variable density, rotation and applied magnetic field on the behavior of the stability. Lyon [10] added the effect of compressibility and applied electric field, but he neglected the surface tension. The nonlinear development of the KHI has been studied by Drazin [11] for the case where the amplitude of an unstable wave is uniform in space and growing in time. Weissman [12] extended the Drazin [11] work, and treated the case where the amplitude of an unstable wave is dependent on both time and space. Hsieh and Chen [13] first formulated the KHI problem in terms of a variational principle. By choosing a single Fourier mode with time-dependent amplitudes, they derived the evolution equations of the amplitudes. They analyzed and discussed the limiting states and their stability of the evolution equation. Also, they studied a sinusoidal wave state and derived a nonlinear dispersion relation. Elhefnawy [14] studied the nonlinear KHI problem under the influence of an oblique electric field by employing the method of multiple scales. He combined the cases of normal and tangential fields. He found that the nonlinear effects may be stabilized or destabilized depending on both density and dielectric constant.

The magnetic fluids are colloidal suspensions of fine magnetic particles in non-conducting liquids. They behave like a homogeneous continuum and exhibit a variety of surprising phenomena [15]. Magnetic fluids are not found in nature but are artificially synthesized due to widespread interest in diverse applications [16] as well as scientific study. The study of magnetic fluids, when subjected to normal and tangential magnetic fields, has received considerable interest during the present decade, because of their wide range in important industrial applications such as the design of sprays and ink jet printers. Because of the wide range of important industrial applications, there has been a growing interest in recent years in the study of magnetic fluids when subjected to normal and tangential magnetic fields. It was shown by Cowley and Rosensweig [17] that a magnetic field along the normal to a flat interface between a magnetizable and a nonmagnetic fluid has a destabilizing effect; this leads to the appearance of stable regular hexagonal cells at the interface. Zelazo and Melcher [18] considered the propagation of plane waves on the interface between two ferrofluids in the presence of a tangential magnetic field. Their analysis, based on the linear theory, revealed that the magnetic field has a stabilizing effect on the waves. The linear KHI problem in the context of magnetic fluids was investigated by Rosensweig [15]. His analysis revealed that the velocity difference that can be supported by the fluids before the instability sets in, is enhanced if the difference in the permeabilities of the fluids across the interface and the strength of the applied magnetic field are increased. In their investigation of the nonlinear evolution of wave packets on the surface of a magnetic fluid, Malik and Singh [19] showed that the wave train solution of constant amplitude is unstable against modulation if the product of the group velocity rate and the nonlinear interaction coefficient is negative. Furthermore, the magnetic field has a stabilizing influence on the modulation instability for small wave numbers.

Many technological processes involve the parallel flow of fluids of different viscosity and density through porous media. Such parallel flows exist in packed bed reactor in the chemical industry, in petroleum production engineering, in boiling in porous media (countercurrent flow of liquid and vapor), and in many other processes. The instability of a plane interface between two uniform superposed fluids through a porous medium was investigated by Kumar [20], and the KHI for flow in porous media was studied by El-Sayed [21]. They used linear stability analysis to obtain a characteristic equation for the growth rate of the disturbance. A linear theory of the KHI for parallel flow in porous media was introduced by Bau [22] for the Darcian and non-Darcian flows. In both cases, Bau found that the velocities should exceed some critical value for the instability to manifest itself. El-Sayed [23] investigated the RTI problem of a rotating stratified conducting fluid layer through porous medium in the presence of an inhomogeneous magnetic field. This problem corresponds physically (in astrophysics) to the RTI of an equatorial section of a planetary magnetosphere or of stellar atmosphere when rotation and magnetic field are perpendicular to gravity. It is also of great importance in the area of geophysics, civil engineering, soil sciences, ground water hydrology, and petroleum production engineering [24].

The mechanism of heat and mass transfer across an interface is of great importance in numerous industrial and environmental processes. These include the design of many types of contacting equipment, e.g. boilers, condensers, evaporators, gas absorbers, pipelines, chemical reactors, nuclear reactors, and in other problems such as the aeration of rivers and the sea. In most cases of practical importance, the liquid flow is turbulent and the transport across the gas-liquid interface is governed by the liquid side. As a result, characteristics of turbulence in liquid flows near the interface are of significant value in understanding the transport across the gas-liquid interface. Hsieh [25] formulated the general problem of interfacial fluid flow with mass and heat transfer and applied it to discuss the RTI in the problem of film-boiling heat transfer. However, the dispersion relation, in the general case, is too complicated to help us to understand the essential features of the problem. Therefore, he gave a simplified formulation of the problem [26] of interfacial flow with mass and heat transfer by analyzing his previous investigation [25] carefully. Hsieh [26] discussed both the RTI and KHI problems in a plane geometry, taking into account interfacial heat and mass transfer, following his simplified formulation. He obtained the instability criterion in the KHI problem. In studying the nonlinear RTI with mass and heat transfer, Hsieh [27] found that when the heat transfer rate is strong enough, the classically unstable system is stabilized by the nonlinear approach and the effect of mass and heat transfer across the interface. He, also, estimated the size of the bubbles detached from the interface. Recently, Moatimid [28] investigated the stability of two rigidly rotating magnetic fluid columns in the presence of mass and heat transfer. His boundary-value problem leads to a transcendental differential equation. Therefore, the method of multiple scales is adopted to determine the necessary and sufficient conditions for stability. In addition, the stability properties of ferromagnetic fluids in the presence of an oblique field and mass and heat transfer, is investigated by Moatimid [29]. His analysis reveals the case of a uniform magnetic field as well as the periodic one. A most recent work on this topic is introduced by Moatimid [30]. His system is composed of a streaming dielectric fluid sheet of finite thickness embedded between two different streaming finite dielectric fluids. The interfaces permit mass and heat transfer. His analysis reveals that the sheet thickness and mass and heat transfer parameters have a dual influence on the stability picture, especially at small values of the wave number. Elhefnawy [31] studied the nonlinear stability problem of ferromagnetic fluids with mass and heat transfer effect. He showed that the mass and heat transfer coefficient plays an important role in the nonlinear stability of the system.

The aim of this work is to make an extension of our previous study [29] to include the nonlinear effects. Therefore, we shall introduce the nonlinear KHI in magnetic fluids in the presence of a tangential magnetic field and mass and heat transfer across the interface. Because of the instability in a porous medium of a plane interface between two fluids may be of interest in geophysics and biomechanics, the present study is considered through porous media. The analysis includes the linear as well as the nonlinear effects. The plan of this work, which is of the nonlinear KHI type, is outlined as follows: In section 2, we give a description of the problem including the basic equations that govern the motion of our model and also the appropriate boundary conditions. The lines of solutions are reported in section 3. Section 4 is devoted to introduce the nonlinear characteristic equation.

The linear stability analysis is investigated in section 5. The Ginzburg–Landau equation is derived in section 6. Finally, in section 7, we give conclusions for this work based on the obtained results of the stability analysis.

2. Definition of the problem

2.1. Basic equations

The system under consideration is composed of two incompressible magnetic fluids separated by the plane $y = 0$. Each fluid is of infinite horizontal extent. We take the origin o at the mean level of the interface, and the axis oy pointing vertically upwards into the upper fluid. Let the two fluids be confined between two rigid horizontal planes $y = -h_1$ (the lower boundary) and $y = h_2$ (the upper one). The temperatures at $y = -h_1$, $y = h_2$ and $y = 0$ are taken as τ_1, τ_2 and τ_0, respectively. The media are considered as porous, where the resistance term $(-\nu \underline{v})$ is only taken into account, where $\underline{v}$ is the filter velocity of the fluid and ν is the Darcy's coefficient which depends on the ratio of the fluid viscosity to the flow permeability through the voids. The two fluids are influenced by a uniform magnetic field H_0 acting along the positive x-direction, where the axis ox is the mean level of the wave. Also, the fluids are streaming with velocities V_1 and V_2 along the same direction. Acceleration due to gravity acts in the negative y-direction. A schematic diagram of the configuration in the steady state is given in Figure 1.

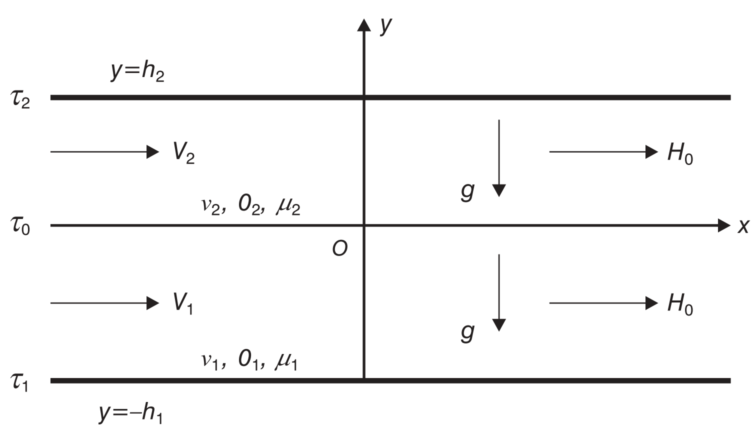

Figure 1. A schematic diagram of the system under consideration.

The upper fluid has density ρ_2 and magnetic permeability μ_2, while the lower one has density ρ_1 and magnetic permeability μ_1. A surface tension exists between the two fluids and is denoted by σ.

Assuming the motion of the system to be irrotational, it can be described by the potential $\phi_1(x, y, t)$ in the lower fluid and $\phi_2(x, y, t)$ in the upper one. Therefore, for incompressible fluids, $\phi_1(x, y, t)$ and $\phi_2(x, y, t)$ satisfy the following Laplace's equations:

$$\nabla^2 \phi_1 = 0, \qquad -h_1 < y < \eta(x, t), \tag{2.1}$$

$$\nabla^2 \phi_2 = 0, \qquad \eta(x, t) < y < h_2, \tag{2.2}$$

where

$$\eta = \gamma e^{i(kx - wt)} + c.c., \tag{2.3}$$

denotes the elevation of the interface at time t, k is the wave number, which is assumed to be real and positive, w is the growth rate, γ is an arbitrary constant, which determines the amplitude of the disturbance of the interface and *c.c.* indicates the complex conjugate of the preceding term. It should be noted that an imaginary part for w indicates a disturbance which either grows with time (instability) or decays with time (stability), depending on whether this imaginary part is positive or negative, respectively.

In case of a magneto-quasistatic system with negligible displacement current, Maxwell's equations in the absence of free currents reduce to Gauss's law $\nabla . \underline{B} = 0$, and Ampère's law (no currents) $\nabla \wedge \underline{H} = \underline{0}$, where $\underline{B} = \mu \underline{H}$ is the magnetic induction vector. From Ampere's law, the magnetic field $\underline{H}$ can be expressed in terms of a magnetic scalar potential $\psi_j(x, y, t)$ in each of the regions occupied by the fluids, i.e.

$$\underline{H}_j = H_0 \underline{e}_x - \nabla \psi_j, \qquad j = 1, 2 \tag{2.4}$$

where $\underline{e}_x$ is the unit vector along the x-direction and the subscripts 1 and 2 refer to quantities in the lower and upper fluids, respectively.

Combining the latter equations (2.4) with Gauss's law, considering μ as a constant, one finds that the magnetic scalar potentials, also, obey the Laplace's equations:

$$\nabla^2 \psi_1 = 0, \qquad -h_1 < y < \eta(x, t), \tag{2.5}$$

$$\nabla^2 \psi_2 = 0, \qquad \eta(x, t) < y < h_2. \tag{2.6}$$

To complete the formulation of the problem, we must define the surface geometry and supplement the magnetic equations with the corresponding boundary conditions. The interface may be represented by the expression

$$S(x, y, t) \equiv y - \eta(x, t) = 0, \tag{2.7}$$

for which the unit outward normal vector is given by

$$\underline{n} = (-\frac{\partial \eta}{\partial x}, 1, 0)(1 + (\frac{\partial \eta}{\partial x})^2)^{-1/2}. \tag{2.8}$$

2.2. Boundary conditions

The boundary conditions represented here are prescribed at the interface $y = \eta(x,t)$. As the interface is deformed, all variables are slightly perturbed from their equilibrium values. Because the interfacial displacement is small, the boundary conditions on the perturbed interfacial variables need to be evaluated at the position of equilibrium position rather than the interface. The development of the weak nonlinear approach, considered here, is based on the idea of linearization of the hydrodynamic equations of motion, under the non-linearizing boundary conditions.

The solutions for both ϕ_j and ψ_j have to satisfy the following relevant boundary conditions for our configuration [15, 26, 31]

(i) On the rigid boundaries $y = h_1$ and $y = -h_2$:

(1) The normal fluid velocities vanish on both the bottom and top boundaries, require

$$\frac{\partial \phi_1}{\partial y} = 0 \qquad \text{on} \qquad y = -h_1, \tag{2.9}$$

$$\frac{\partial \phi_2}{\partial y} = 0 \qquad \text{on} \qquad y = h_2. \tag{2.10}$$

(2) The tangential components of the magnetic field vanish on these boundaries, give

$$\frac{\partial \psi_1}{\partial x} = 0 \qquad \text{on} \qquad y = -h_1, \tag{2.11}$$

$$\frac{\partial \psi_2}{\partial x} = 0 \qquad \text{on} \qquad y = h_2, \tag{2.12}$$

(ii) On the interface $y = \eta(x,t)$:

(1) The tangential components of the magnetic field are equal at the interface. This may be written as $\underline{n} \wedge \parallel \underline{H} \parallel = 0$, or

$$\left\| \frac{\partial \psi}{\partial x} \right\| + \frac{\partial \eta}{\partial x} \left\| \frac{\partial \psi}{\partial y} \right\| = 0, \tag{2.13}$$

where $\|\|$ represents the difference in a quantity as we cross the interface, i.e. $\parallel X \parallel = X_2 - X_1$, where the subscripts refer to upper and lower fluids, respectively.

(2) The normal components of the magnetic induction vector are equal, since we have assumed that there are no free currents at the interface, i.e.

$$\underline{n}.\,\|\underline{B}\| = 0 \qquad \text{or} \qquad \left\| \mu \frac{\partial \psi}{\partial y} \right\| + H_0 \frac{\partial \eta}{\partial x} \|\mu\| = \frac{\partial \eta}{\partial x} \left\| \mu \frac{\partial \psi}{\partial x} \right\|. \tag{2.14}$$

For the hydrodynamic part, let $\underline{v} = V\underline{e}_x + \nabla\phi$ be the velocity vector field of the fluid particles.

(3) The conservation of mass across the interface, yields

$$\left\| \rho\left(\frac{\partial S}{\partial t} + \underline{v}.\nabla S\right) \right\| = 0, \tag{2.15}$$

or

$$\left\| \rho\frac{\partial \phi}{\partial y} \right\| - \frac{\partial \eta}{\partial x}\left\| \rho V \right\| - \frac{\partial \eta}{\partial t}\left\| \rho \right\| = \frac{\partial \eta}{\partial x}\left\| \rho\frac{\partial \phi}{\partial x} \right\|. \tag{2.16}$$

(4) The interfacial condition for energy is

$$L^*\rho_1\left(\frac{\partial S}{\partial t} + \underline{v}.\underline{\nabla} S\right) = F(\eta), \tag{2.17}$$

where L^* is the latent heat of the transformation from the fluid of density ρ_1 to the fluid of density ρ_2 and $F(\eta)$ is a function of the instantaneous profile of the interface and is determined by the heat transfer relation at equilibrium.

Physically, the left-hand side of equation (2.17) represents the latent released during the phase transformation, while $F(\eta)$ on the right-hand side of equation (2.17) represents the net heat flux across the interface, so that the energy will be conserved.

Let us consider a specific equilibrium state. Since the two fluids are confined between two parallel planes $y = -h_1$ and $y = h_2$, the heat fluxes, then in the positive y-direction in the two regions 1 and 2 are $K_1(\tau_1 - \tau_0)/h_1$ and $K_2(\tau_0 - \tau_2)/h_2$, where K_1 and K_2 represent the lower and upper thermal conductivities, respectively. As in [26], we denote

$$F(y) = \frac{K_2(\tau_0 - \tau_2)}{h_2 - y} - \frac{K_1(\tau_1 - \tau_0)}{h_1 + y}, \tag{2.18}$$

and we expand it about $y = 0$, by Maclaurin series, such as

$$F(y) = F(0) + \eta F'(0) + \frac{1}{2!}\eta^2 F''(0) + \frac{1}{3!}\eta^3 F'''(0) + ... \tag{2.19}$$

It is clear that $F(0)$ represents the net heat flux from the interface into the fluid regions. Since it is an equilibrium state, we have $F(0) = 0$, so that

$$\frac{K_2(\tau_0 - \tau_2)}{h_2} = \frac{K_1(\tau_1 - \tau_0)}{h_1} = G \quad \text{(say)}, \tag{2.20}$$

indicating that in the equilibrium state, the heat fluxes are equal across the interface between the two fluids.

Substituting (2.7) and (2.18)–(2.20) into (2.17), we have

$$\rho_1\left(\frac{\partial\phi_1}{\partial y}-\frac{\partial\eta}{\partial t}-\frac{\partial\eta}{\partial x}\frac{\partial\phi_1}{\partial x}\right)=\alpha(\eta+\alpha_2\eta^2+\alpha_3\eta^3), \tag{2.21}$$

where

$$\alpha=\frac{G}{L^*}\left(\frac{1}{h_1}+\frac{1}{h_2}\right)\qquad \alpha_2=\frac{1}{h_2}-\frac{1}{h_1}\qquad\text{and}\qquad \alpha_3=\frac{1}{h_2^2}-\frac{1}{h_1h_2}+\frac{1}{h_1^2}.$$

If we state that the lower fluid is hotter than the upper one ($\tau_1>\tau_0>\tau_2$), then L^* and G are both positive. In the opposite, if the upper fluid is hotter than the lower one ($\tau_2>\tau_0>\tau_1$), then L^* and G are both negative. In both cases α is positive.

(5) Finally, at the boundary between the two fluids, the fluids and the magnetic stresses must be balanced. The components of these stresses consist of the hydrodynamic pressure, surface tension, mass and heat transfer effects and magnetic stresses. These stresses lead to the following conservation of the momentum balance:

$$\left\|\rho(\underline{v}.\nabla S)\left(\frac{\partial S}{\partial t}+\underline{v}.\nabla S\right)\right\|+\left(\left\|\hat{P}\right\|-\frac{1}{2}\left\|\mu(H_n^2-H_t^2)\right\|+\sigma\nabla.\underline{n}\right)|\nabla S|^2=0, \tag{2.22}$$

where $\hat{P}$ is the pressure, H_n and H_t represent the normal and tangential components of the magnetic field, respectively.

By eliminating the pressure by Bernoulli's equation and using equations (2.4), (2.7), (2.16) and (2.21), condition (2.22) may be rewritten as

$$\begin{aligned}&\alpha(\eta+\alpha_2\eta^2+\alpha_3\eta^3)\left(1+\left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-1}\left\{\left\|\frac{\partial\phi}{\partial y}\right\|-\frac{\partial\eta}{\partial x}\left(\left\|\frac{\partial\phi}{\partial x}\right\|+\|V\|\right)\right\}\\&-\|\rho\|\,g\eta-\left\|\rho\frac{\partial\phi}{\partial t}\right\|-\left\|\rho V\frac{\partial\phi}{\partial x}\right\|-\|\nu\phi\|-\sigma\frac{\partial^2\eta}{\partial x^2}\left(1+\left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2}\\&-H_0\left\|\mu\frac{\partial\psi}{\partial x}\right\|-\frac{1}{2}\left\|\mu\left(\frac{\partial\psi}{\partial y}\right)^2\right\|+\frac{1}{2}\left\|\mu\left(\frac{\partial\psi}{\partial x}\right)^2\right\|-2H_0\frac{\partial\eta}{\partial x}\left\|\mu\frac{\partial\psi}{\partial y}\right\|\\&-H_0^2\left(\frac{\partial\eta}{\partial x}\right)^2\|\mu\|+2\frac{\partial\eta}{\partial x}\left\|\mu\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\right\|+2H_0\left(\frac{\partial\eta}{\partial x}\right)^2\left\|\mu\frac{\partial\psi}{\partial x}\right\|=0.\end{aligned} \tag{2.23}$$

3. Lines of solution

The analysis of linear stability as presented by Chandrasekhar [4] depends on neglecting the nonlinear terms from the equations of motion as well as from the boundary conditions. Therefore, a dispersion relation should be arisen without nonlinear terms. The idea for the weak nonlinear approach is slightly some departure from the linearity technique. At this stage, the nonlinear problem will contain the linear description with some additional terms that make a correction of the main solution. The weakly nonlinear description given here depends on neglecting the nonlinear terms from the equations of motion and applying the appropriate nonlinear boundary conditions. Therefore, the dispersion relation should be extended to include the nonlinear terms.

To solve the linearized equations of motion for the fluid phases under consideration, two dimensional finite disturbances are introduced to the boundary-value problem. As a customary in hydrodynamic stability theory [4], all quantities have exponential time dependence and a periodic spatial dependence. Also, in view of a standard Fourier decomposition, we may similarly assume that the bulk solutions are in the form

$$\phi_j(x,y,t) = \hat{\phi}_j(y) e^{i(kx-wt)}, \qquad j = 1,2, \tag{3.1}$$

$$\psi_j(x,y,t) = \hat{\psi}_j(y) e^{i(kx-wt)}, \qquad j = 1,2, \tag{3.2}$$

where $\hat{\phi}_j(y)$ and $\hat{\psi}_j(y)$ are arbitrary functions of y.

Substituting (3.1) and (3.2) into equations (2.1), (2.2), (2.5) and (2.6). The solutions which are consistent with the foregoing nonlinear boundary conditions stated in subsection 2.2, may be represented in the form:

$$\phi_1 = \frac{\cosh\, k(y+h_1)}{k\,\sinh\, kh_1(1+k\eta\,\coth\, kh_1)}\left[\frac{\partial\eta}{\partial t} + ikV_1\eta + \frac{\alpha}{\rho_1}(\eta+\alpha_2\eta^2+\alpha_3\eta^3)\right], \tag{3.3}$$

$$\phi_2 = -\frac{\cosh\, k(y-h_2)}{k\,\sinh\, kh_2(1-k\eta\,\coth\, kh_2)}\left[\frac{\partial\eta}{\partial t} + ikV_2\eta + \frac{\alpha}{\rho_2}(\eta+\alpha_2\eta^2+\alpha_3\eta^3)\right], \tag{3.4}$$

$$\psi_1 = \frac{iH_0}{\Delta}(\mu_2-\mu_1)(1-k\eta\,\coth\, kh_2)\sinh\, kh_2\,\sinh\, k(y+h_1)\,\eta, \tag{3.5}$$

$$\psi_2 = -\frac{iH_0}{\Delta}(\mu_2-\mu_1)(1+k\eta\,\coth\, kh_1)\sinh\, kh_1\,\sinh\, k(y-h_2)\,\eta, \tag{3.6}$$

where

$$\begin{aligned}\Delta =& \mu_1\cosh\, kh_1\,\sinh\, kh_2(1+k\eta\,\tanh\, kh_1)(1-k\eta\,\coth\, kh_2)\\ &+\mu_2\cosh\, kh_2\,\sinh\, kh_1(1-k\eta\,\tanh\, kh_2)(1+k\eta\,\coth\, kh_1).\end{aligned}$$

The above distributions of the velocity and magnetic potentials contain nonlinear terms in the elevation parameter η. As the nonlinear terms are ignored, the linear profile arises and equivalent to those obtained by Rosensweig [15], Chandrasekhar [4] and Elhefnawy [31].

4. Nonlinear characteristic equation

In what follows, we shall derive the nonlinear equation which governs the surface elevation. To relax the mathematical manipulation, without loss of generality, we shall restrict the analysis to the case of two equal layers, i.e. $h_1 = h_2 = h$. Combining equations (3.3), (3.4), (3.5), (3.6) and (2.23), after lengthy but straightforward calculations, one finds the following nonlinear equation:

$$D(w,k)\eta = \lambda(w,k)\eta^2 + \beta(w,k)\eta^3, \tag{4.1}$$

where

$$\begin{aligned} D(w,k) =& (\rho_1+\rho_2)w^2 + \left[-2k(\rho_1 V_1 + \rho_2 V_2) + i(2\alpha+\nu_1+\nu_2)\right] w \\ &+ \Big[-k\Sigma \tanh kh - \mu_0 k^2 H_0^2 \tanh^2 kh - \alpha(\frac{\nu_1}{\rho_1} + \frac{\nu_2}{\rho_2}) \\ &+ k^2(\rho_1 V_1^2 + \rho_2 V_2^2) - ik((\alpha+\nu_1)V_1 + (\alpha+\nu_2)V_2)\Big], \end{aligned}$$

$$\begin{aligned} \lambda(w,k) =& k \coth kh \Big[(\rho_1-\rho_2)w^2 + \left[-2k(\rho_1 V_1 - \rho_2 V_2) + i(\nu_1-\nu_2)\right] w \\ &+ \Big[\alpha(\frac{\nu_2}{\rho_2} - \frac{\nu_1}{\rho_1}) + k^2(\rho_1 V_1^2 - \rho_2 V_2^2) + \alpha^2(\frac{1}{\rho_2} - \frac{1}{\rho_1})\tanh^2 kh \\ &+ k^2 H_2 H_0^2 \tanh kh - ik((\alpha+\nu_1)V_1 - (\alpha+\nu_2)V_2)\Big]\Big], \end{aligned}$$

$$\begin{aligned} \beta(w,k) =& k^2 \coth^2 kh \Big[-(\rho_1+\rho_2)w^2 \\ &+ \Big[2k(\rho_1 V_1 + \rho_2 V_2) - i(2\alpha+\nu_1+\nu_2 + 2\alpha\frac{\alpha_3}{k^2}\tanh^2 kh\Big] w \\ &+ \Big[\alpha(\frac{\nu_1}{\rho_1} + \frac{\nu_2}{\rho_2})(1 + \frac{\alpha_3}{k^2}\tanh^2 kh) - k^2(\rho_1 V_1^2 + \rho_2 V_2^2) + \frac{3}{2}\sigma k^3 \tanh^3 kh \\ &+ k^2 H_0^2 H_3 \tanh^2 kh \\ &+ i\left(\alpha\frac{\alpha_3}{k}(V_1+V_2)\tanh^2 kh + k((\alpha+\nu_1)V_1 + (\alpha+\nu_2)V_2\right)\Big]\Big]. \end{aligned}$$

Σ, μ_0, H_2 and H_3 are given by

$$\Sigma = (\rho_1-\rho_2)g + \sigma k^2, \qquad \mu_0 = \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)}$$

$$\begin{aligned} H_2 =& \frac{(\mu_1-\mu_2)}{2(\mu_1+\mu_2)^2}\Big[(4 - \coth kh + (-5 + 3\operatorname{sech}^2 kh)\tanh kh)\mu_1^2 \\ &+ 2(4 + \coth kh - 3(1+\operatorname{sech}^2 kh)\tanh kh)\mu_1\mu_2 \\ &+ (4 - \coth kh + (-5 + 3\operatorname{sech}^2 kh)\tanh kh)\mu_2^2\Big], \end{aligned}$$

and

$$
\begin{aligned}
H_3 =& \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^3}\Big[(5-6\text{sech}^2\ kh+2\text{sech}^4\ kh-4\text{tanh}\ kh))\mu_1^2 \\
&+2(5+2csch^2\ kh-2(2+\cosh\ 2kh)\text{sech}^4\ kh-4\text{tanh}\ kh)\mu_1\mu_2 \\
&+(5-6\text{sech}^2\ kh+2\text{sech}^4 kh-4\text{tanh}\ kh)\mu_2^2\Big].
\end{aligned}
$$

As a limit case, for two semi-infinite layers, $h_1, h_2 \to \infty$, one finds that the contribution of the magnetic fields intensities of the first and third-orders are the same. This is in agreement with the result recently obtained by El-Dib [32].

Equation (4.1) is more general than those obtained by Bau [22], Hsieh [26] and Elhefnawy [31]. In addition to the nonlinear contribution, it includes the magnetic field influence.

According to the Grimshaw theory [33], the nonlinear terms of the characteristic equation (4.1) consist of two parts. The first part contains the interaction of the second harmonic term. The second one represents the cubic interactions of the primary harmonic term. Neglecting the higher orders of η, the linearized form of equation (4.1) yields

$$D(w,k)\eta = 0. \tag{4.2}$$

Equation (4.2) represents the linear dispersion relation which is corresponding to the linear differential equation

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)\eta = 0, \tag{4.3}$$

where L is a linear operator involving the temporal and spatial partial derivatives. Firstly, we study equation (4.3), then return to equation (4.1) to incorporate the nonlinear effects. Consider a uniform harmonic wave train solutions of equation (4.3) in the light of (2.3). The existence of the harmonic wave trains in a dispersive medium and the correspondence between the wave number and frequency leads to several physical consequences. Combining equations (4.3) and (2.3), gives

$$L(-iw, ik)\eta = 0, \tag{4.4}$$

from which, it follows that

$$D(w,k)\gamma = 0. \tag{4.5}$$

As $\gamma \neq 0$, we obtain

$$
\begin{aligned}
D(w,k) =& (\rho_1+\rho_2)w^2 + \left[-2k(\rho_1 V_1+\rho_2 V_2) + i(2\alpha+\nu_1+\nu_2)\right] w \\
&+ \Big[-k\Sigma\ \tanh\ kh - k^2\mu_0 H_0^2\ \tanh^2 kh - \alpha\left(\frac{\nu_1}{\rho_1}+\frac{\nu_2}{\rho_2}\right) \\
&+ k^2(\rho_1 V_1^2+\rho_2 V_2^2) - ik((\alpha+\nu_1)V_1+(\alpha+\nu_2)V_2)\Big] = 0
\end{aligned} \tag{4.6}
$$

Equation (4.6) represents the linear dispersion relation that is satisfied by the values of w and k. To develop the nonlinear effects, for the amplitude of the progressive waves, we need to go to the full nonlinear equation (4.1) taking into account the linear dispersion relation (4.6). Before going to the nonlinear analysis, let us investigate the linear stability analysis.

5. Linear stability analysis

The goal of this item is to analyze the stability throughout the linear approach. Therefore, we consider the linear dispersion relation (4.6). This equation may be rewritten in the form

$$a_0 w^2 + (a_1 + ib_1)w + (a_2 + ib_2) = 0, \tag{5.1}$$

where

$$a_0 = \rho_1 + \rho_2, \qquad a_1 = -2k(\rho_1 V_1 + \rho_2 V_2), \qquad b_1 = 2\alpha + \nu_1 + \nu_2$$
$$a_2 = -k\Sigma \tanh kh - k^2 \mu_0 H_0^2 \tanh^2 kh + \alpha\left(\frac{\nu_1}{\rho_1} + \frac{\nu_2}{\rho_2}\right) + k^2(\rho_1 V_1^2 + \rho_2 V_2^2)$$
$$b_2 = -k((\alpha + \nu_1)V_1 + (\alpha + \nu_2)V_2).$$

Before dealing with the dispersion relation (5.1), we first consider some special cases:

(i) Consider the case where the media are non-porous, in addition, in absence of the rate of interfacial mass and heat transfer. In this case, the system is stable if

$$a_1^2 - 4a_0 a_2 \geq 0, \tag{5.2}$$

or

$$H_0^2 \geq H_{C1}, \tag{5.3}$$

where

$$H_{C1} = -\frac{\Sigma \coth kh}{\mu_0 k} + \frac{\rho_1 \rho_2 (V_1 - V_2)^2 \coth^2 kh}{\mu_0(\rho_1 + \rho_2)}.$$

(ii) Consider the case where the media are porous, but in absence of the rate of interfacial mass and heat transfer. Applying Routh-Hurwitz [34] stability criterion to equation (5.1), we obtain the stability conditions (in other words, to have the imaginary part of w less than zero) as

$$b_1 > 0 \qquad \text{and} \qquad a_1 b_2 b_2 - a_0 b_2^2 - a_2 b_1^2, \geq 0. \tag{5.4}$$

Since ν_1 and ν_2 are always positive, the first condition in (5.4) is trivially satisfied, while the second one is satisfied if

$$H_0^2 \geq H_{C2}, \tag{5.5}$$

where

$$H_{C2} = -\frac{\Sigma \coth kh}{\mu_0 k} + \frac{(\rho_1 \nu_2^2 + \rho_2 \nu_1^2)(V_1 - V_2)^2 \coth^2 kh}{\mu_0 (\nu_1 + \nu_2)^2}.$$

(iii) Consider the case where the media are non-porous, but in presence of the rate of interfacial mass and heat transfer. Applying the same condition as in case (ii), it follows that the stability criterion may be written as

$$H_0^2 \geq H_{C3}, \tag{5.6}$$

where

$$H_{C3} = -\frac{\Sigma \coth kh}{\mu_0 k} + \frac{\rho_1 \rho_2 (V_1 - V_2)^2 \coth^2 kh}{\mu_0 (\rho_1 + \rho_2)} \left(1 + \frac{(\rho_1 - \rho_2)^2}{4 \rho_1 \rho_2}\right).$$

It is worthwhile to note that although the parameter α does not appear in H_{C3}, but H_{C3} differs from H_{C1} by the additional last term. This term appears due to the presence of mass and heat transfer. Thus condition (5.6) is still valid for infinitesimal α and when $\alpha = 0$, the last term is absent. Also, when $V_1 = V_2$ (the RTI problem) or $\rho_1 = \rho_2$, then the last term disappears. These results are already reported by Hsieh [26]. On the other hand, we see that the last term in H_{C3} is always positive. It means that the mass and heat transfer has a destabilizing influence. Therefore, however α is small or large, the mass and heat transfer has a linearly destabilizing influence on the KHI problem.

Now, we consider the general case of two superposed magnetic fluids in the presence of interfacial mass and heat transfer through porous media. Since α, ν_1 and ν_2 are always positive, the stability criterion becomes

$$H_0^2 \geq H_{C4}, \tag{5.7}$$

where

$$H_{C4} = -\frac{\Sigma \coth kh}{\mu_0 k} + \frac{(\rho_1 (\alpha + \nu_2)^2 + \rho_2 (\alpha + \nu_1)^2)(V_1 - V_2)^2 \coth^2 kh}{\mu_0 (2\alpha + \nu_1 + \nu_2)^2} - \frac{\alpha (\rho_1 \nu_2 + \rho_2 \nu_1) \coth^2 kh}{\mu_0 \rho_1 \rho_2 k^2}.$$

It should be noted that the porous media show, frequently, the presence of the parameter α on the stability criterion.

From the foregoing results, in all cases, it is apparent that the uniform streaming has a destabilizing influence. This role is enhanced as the relative motion between the two fluid layers is increased. This result is in agreements with all studies in the linear stability theory, for example (see Chandrasekhar [4]). In contrast, the magnetic field has a stabilizing influence on the wave motion. This theoretical result was first obtained and confirmed experimentally by Zelazo and Melcher [18] (see also Rosensweig [15]). It is also clear that the mass and heat transfer parameter has no implication on the stability criterion in the case of the RTI through the non-porous media as pointed out by Hsieh [26].

Before dealing with some numerical estimation, it is convenient to write the above stability conditions in an appropriate dimensionless form. This can be done in a number of ways depending primarily on the choice of the characteristic length. Consider the following dimensionless forms: The characteristic length = h, the characteristic time = $\sqrt{\frac{h}{g}}$ and the characteristic mass = $\frac{\sigma}{g}$. The other dimensionless quantities are given by:

$$\begin{aligned} &k = \frac{k^*}{h},\ \rho_j = \frac{\sigma\rho_j^*}{gh^2},\ \nu_j = \frac{\sigma\nu_j^*}{h^2\sqrt{gh}},\ V_j = V_j^*\sqrt{gh}, \\ &\alpha = \frac{\sigma\alpha^*}{h^2\sqrt{gh}},\ H_0^2 = \frac{\sigma}{h}H_0^{*2},\ H_{Cr} = \frac{\sigma}{h}H_{Cr}^*, \qquad j = 1,2,\ r = 1,2,3,4 \end{aligned} \tag{5.8}$$

where the superposed asterisks refer to the dimensionless quantities. From now on, it will be omitted for simplicity. To this end, the critical magnetic field intensities may be written as:

$$H_{C1} = -\frac{\hat{\Sigma}\ \coth\,k}{\mu_0 k} + \frac{\rho_1\rho_2(V_1 - V_2)^2\coth^2 k}{\mu_0(\rho_1+\rho_2)}, \tag{5.9}$$

$$H_{C2} = -\frac{\hat{\Sigma}\ \coth\,k}{\mu_0 k} + \frac{(\rho_1\nu_2^2 + \rho_2\nu_1^2)(V_1 - V_2)^2\coth^2 k}{\mu_0(\nu_1+\nu_2)^2}, \tag{5.10}$$

$$H_{C3} = -\frac{\hat{\Sigma}\ \coth\,k}{\mu_0 k} + \frac{\rho_1\rho_2(V_1 - V_2)^2\coth^2 k}{\mu_0(\rho_1+\rho_2)}, \tag{5.11}$$

$$\begin{aligned} H_{C4} = &-\frac{\hat{\Sigma}\ \coth\,k}{\mu_0 k} + \frac{(\rho_1(\alpha+\nu_2)^2 + \rho_2(\alpha+\nu_1)^2)(V_1 - V_2)^2\coth^2 k}{\mu_0(2\alpha+\nu_1+\nu_2)^2} \\ &- \frac{\alpha(\rho_1\nu_2 + \rho_2\nu_1)\coth^2 k}{\mu_0 k^2\rho_1\rho_2}, \end{aligned} \tag{5.12}$$

where

$$\hat{\Sigma} = \rho_1 - \rho_2 + k^2.$$

In what follows, we shall make a numerical estimation for the stability picture for the surface waves propagating in the miscible magnetic fluids throughout porous media. In order to screen this examination, numerical calculation for the transition curves (5.9), (5.10), (5.11) and (5.12) are made for the variation of the magnetic field intensity H_0^2 versus the wave number k. As we conclude, these transition curves separate the stable from instable regions. The region above the

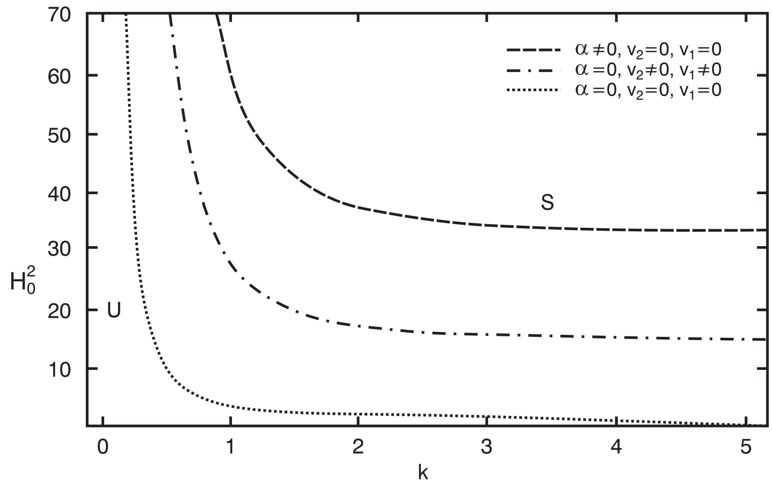

Figure 2. Represents the stability diagram on the $(H_0^2 - k)$-plane according to equations (5.9), (5.10), and (5.11) for a system having the particulars $\rho_1 = 0.0123$, $\rho_2 = 0.9987$, $\mu_1 = 1$, $\mu_2 = 5$, $V_1 = 20$, $V_2 = 2$, $\nu_1 = 1$, $\nu_2 = 2$, and $\alpha = 1$.

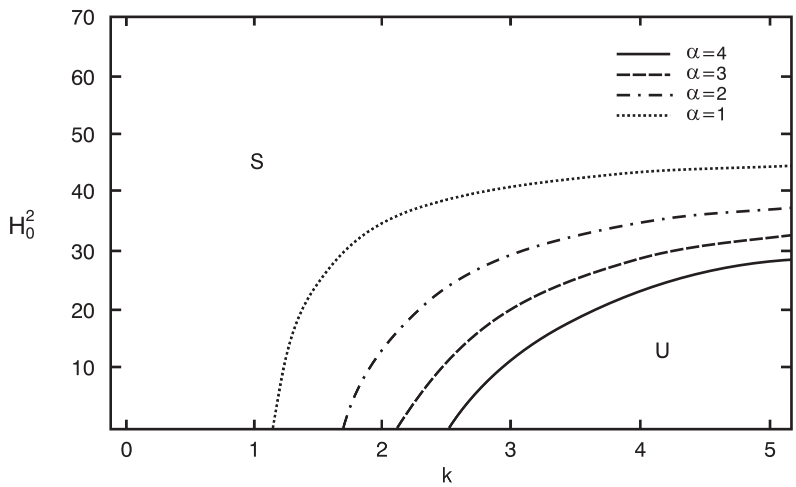

Figure 3. As in Figure 2, to indicate equation (5.12), but when the densities are interchanged and for various values of α.

transition curve is stable, while the lower area is unstable. The stable region is referred by the letter S, while the letter U stands for the unstable one.

The special cases are discussed numerically through Figure 2. This figure includes the cases (i), (ii) and (iii). The inspection of this figure shows the destabilizing influence of both mass and heat transfer (α) and the porosity of the medium, which is represented by the Darcy's coefficients ν_1 and ν_2. It is also observed that

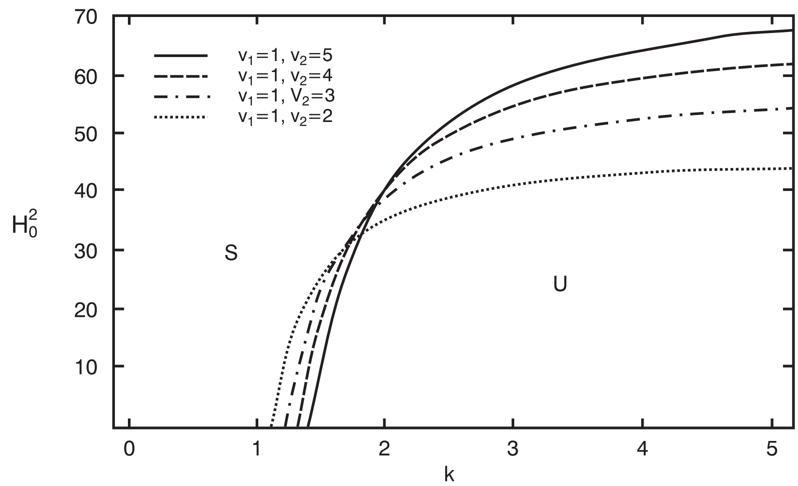

Figure 4. As in Figure 3, but when $\alpha = 1$ and for various values of ν_2.

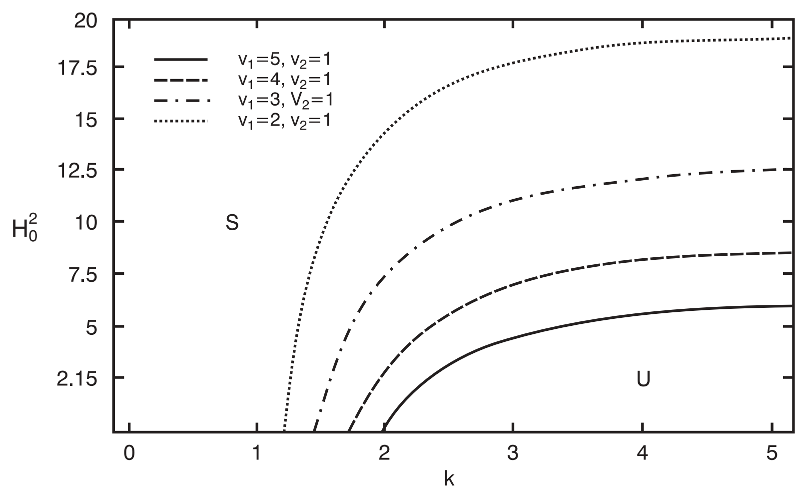

Figure 5. As in Figure 4, but when $\nu_2 = 1$ and for various values of ν_1.

the instability is enhanced when the media are considered as non-porous.

The general case is pictured in Figure 3. The inspection of this figure shows that the parameter α has a stabilizing influence. Actually, this role depends on the structure of the media. It follows that the parameter α has a dual role in the stability criterion. In the case of non-porous media, α has a destabilizing influence and vice versa in the porous media. The effect of the Darcy's coefficients ν_1 and ν_2 are displayed in Figures 4 and (5). From Figure 4, where $\nu_2 > \nu_1$, a dual role in the stability criterion is found. For large values of k, a destabilizing influence is observed and vice versa for small k. As $\nu_1 > \nu_2$ and for all values of the wave number k, a stabilizing influence is seen as shown in Figure 5.

6. The nonlinear Ginzburg–Landau equation

In order to investigate the nonlinear stability criteria of the system under consideration, we make a modulation to the problem so that the linear dispersion relation $D(w, k)$ represents the slowly modulated wave train. At this stage, we may use the expansion procedure formed by the method of multiple scales [8]. The underlying idea of this method is to make an expansion, representing the solution of the problem, as a function of two or more independent variables. Consider a small parameter δ which measures the ratio of a typical wave length or time scale of the modulation. The independent variables x and t may be expanded to introduce alternative independent variables as:

$$T_n = \delta^n t, \qquad X_n = \delta^n x, \qquad n = 0, 1, 2 \tag{6.1}$$

Therefore, defining T_0, X_0 as variables appropriate to fast variation, T_1, X_1, T_2 and X_2 are slow ones. The differential operators can now be expressed as the derivative expansion

$$\frac{\partial}{\partial t} = -w\frac{\partial}{\partial \theta_0} + \delta\frac{\partial}{\partial T_1} + \delta^2\frac{\partial}{\partial T_2} + ..., \tag{6.2}$$

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial \theta_0} + \delta\frac{\partial}{\partial X_1} + \delta^2\frac{\partial}{\partial X_2} + ..., \tag{6.3}$$

where $\theta_0 = kX_0 - wT_0$.

The operator L then becomes

$$L\left[(-iw, ik) + \delta\left(\frac{\partial}{\partial T_1}, \frac{\partial}{\partial X_1}\right) + \delta^2\left(\frac{\partial}{\partial T_2}, \frac{\partial}{\partial X_2}\right) + ...\right] \tag{6.4}$$

The expression for the operator L may be expanded in powers of δ. This may be achieved by making use of Taylor's theory about $(-iw, ik)$ up to $O(\delta^2)$. Therefore, we may obtain

$$L \to L_0 + \delta L_1 + \delta^2 L_2 + ..., \tag{6.5}$$

combining (6.5) and (4.3), yields

$$(L_0 + \delta L_1 + \delta^2 L_2 + ...)\eta = 0. \tag{6.6}$$

It follows that

$$(D_0 + \delta D_1 + \delta^2 D_2 + ...)\gamma = 0, \tag{6.7}$$

where $D_0 \equiv 0$ describes the linear dispersion relation,

$$D_1 \equiv i\left\{\frac{\partial D}{\partial w}\frac{\partial}{\partial T_1} - \frac{\partial D}{\partial k}\frac{\partial}{\partial X_1}\right\}$$
$$D_2 \equiv i\left\{\frac{\partial D}{\partial w}\frac{\partial}{\partial T_2} - \frac{\partial D}{\partial k}\frac{\partial}{\partial X_2}\right\} - \frac{1}{2}\frac{\partial^2 D}{\partial w^2}\frac{\partial^2}{\partial T_1^2} + \frac{\partial^2 D}{\partial w \partial k}\frac{\partial^2}{\partial X_1 \partial T_1} - \frac{1}{2}\frac{\partial^2 D}{\partial k^2}\frac{\partial^2}{\partial X_1^2}$$

At this end, the nonlinearity of η, in terms of δ, may be written as:

$$\eta = \sum_{n=1}^{3} \delta^2 \eta_n(\theta_0, X_1, X_2; T_1, T_2) + 0(\delta^4). \tag{6.8}$$

Combining equations (4.1), (6.5) and (6.8), then equating like powers of δ on both sides, one finds the following three orders in δ as:

$$L_0\eta_1 = 0, \tag{6.9}$$

$$L_0\eta_2 = -L_1\eta_1 + \lambda\ \eta_2^2, \tag{6.10}$$

$$L_0\eta_3 = -L_1\eta_2 - L_2\eta_1 + 2\lambda\ \eta_1\eta_2 + \beta\ \eta_1^3. \tag{6.11}$$

In the lowest order approximation, we may assume the following quasi-monochromatic wave solution

$$\eta_1 = \gamma(X_1, X_2; T_1, T_2)e^{i\theta_0} + c.c. \tag{6.12}$$

Equation (6.10) then becomes

$$L_0\eta_2 = -i\left[\frac{\partial D}{\partial w}\frac{\partial \gamma}{\partial T_1} - \frac{\partial D}{\partial k}\frac{\partial \gamma}{\partial X_1}\right]e^{i\theta_0} + \lambda(\gamma^2 e^{2i\theta_0} + 2\gamma\bar{\gamma}) + c.c., \tag{6.13}$$

where $\bar{\gamma}$ denotes the complex conjugate of γ.

Equation (6.13) contains secular terms, corresponding to the factor $e^{i\theta_0}$. The elimination of this term, leads to the following solvability condition

$$\frac{\partial D}{\partial w}\frac{\partial \gamma}{\partial T_1} - \frac{\partial D}{\partial k}\frac{\partial \gamma}{\partial X_1} = 0, \tag{6.14}$$

which gives a complex relation. With the aid of this condition, a uniformly valid expansion of η_2 arises in the form

$$\eta_2 = \frac{\lambda}{\Omega}\gamma^2 e^{2i\theta_0} + c.c. \tag{6.15}$$

The non zero denominator may be derived from the linear dispersion relation by replacing both w and k by $2w$ and $2k$, respectively. The vanishing of the denominator Ω refers to the second harmonic resonance. In general, the harmonic resonance may occur if (w, k) and (nw, nk), when n is a positive integer, satisfying the same dispersion relation [8].

Combing equations (6.11), (6.12) and (6.15), the uniformly third order solution yields the following solvability condition

$$i\left(\frac{\partial D}{\partial w}\right)\frac{\partial \gamma}{\partial T_2} - i\left(\frac{\partial D}{\partial k}\right)\frac{\partial \gamma}{\partial X_2} - \frac{1}{2}\left(\frac{\partial^2 D}{\partial w^2}\right)\frac{\partial^2 \gamma}{\partial T_1^2} - \frac{1}{2}\left(\frac{\partial^2 D}{\partial k^2}\right)\frac{\partial^2 \gamma}{\partial X_1^2} + \frac{\partial^2 D}{\partial w \partial k}\frac{\partial^2 \gamma}{\partial T_1 \partial X_1} = \left(2\frac{\lambda^2}{\Omega} + 3\beta\right)\gamma^2\bar{\gamma}. \tag{6.16}$$

It is well known that the nonlinear Schrödinger equation is a generic equation describing unidirectional wave modulation. It has been used to describe the spatial and temporal evolution of the envelope of a sinusoidal wave with phase $(kX_0 - wT_0)$, drawing potential energy from some background field. In fact, the nonlinear Schrödinger equation generally describes the competition between nonlinearity and dispersion. To derive the equation of amplitude evolution, we proceed to solve the second and third-order problems. Following the procedure developed by Nayfeh [8], the non secularity conditions for the existence of the uniformly valid solutions in the second-order, as given by equation (6.14), may be written as

$$\frac{\partial \gamma}{\partial T_1} + V_g\frac{\partial \gamma}{\partial X_1} = 0, \qquad \text{provided that} \qquad \frac{\partial D}{\partial w} \neq 0, \tag{6.17}$$

where $V_g = -\left(\frac{\partial D}{\partial k}\right)\left(\frac{\partial D}{\partial w}\right)^{-1}$ is the group velocity of the wave train.

Equation (6.17) implies that the wave moves with group velocity in the second order approximation. This means that the amplitude γ depends on the slow variables X_1, T_1 through the combination $(X_1 - V_g T_1)$.

To develop the amplitude modulation for the progressive waves, we need to go to the third-order problem as given by equation (6.6). It is interesting to observe that equation (6.16) has a singularity at $\Omega = 0$. This corresponds to the case of second harmonic resonance. We should remark that the analysis given in this paper is not valid in the neighborhood of such resonance.

The solvability conditions (6.14) and (6.16) may be simplified and combined together to produce a single equation. By using equation (6.17), the derivatives in T_1 may be eliminated from equation (6.16). Therefore, let us write

$$\frac{\partial^2 \gamma}{\partial X_1 \partial T_1} = -V_g\frac{\partial^2 \gamma}{\partial X_1^2}, \qquad \frac{\partial^2 \gamma}{\partial T_1^2} = V_g^2\frac{\partial^2 \gamma}{\partial X_1^2}. \tag{6.18}$$

Substituting (6.18) into (6.16), dividing through $\left(\frac{\partial D}{\partial w}\right)$, and using (6.1), one gets

$$i\left(\frac{\partial \gamma}{\partial t} + V_g\frac{\partial \gamma}{\partial x}\right) + P\frac{\partial^2 \gamma}{\partial x^2} = \delta^2 Q\gamma^2\bar{\gamma}, \tag{6.19}$$

where P (the group velocity rate) and Q (the nonlinear interaction coefficient) are given by

$$P = \frac{1}{2}\frac{d^2V_g}{dk^2} = -\frac{1}{2}\left(V_g^2\frac{\partial^2 D}{\partial w^2} + 2V_g\frac{\partial^2 D}{\partial w\partial k} + \frac{\partial^2 D}{\partial k^2}\right)\left(\frac{\partial D}{\partial w}\right)^{-1},$$
$$Q = \left(2\frac{\lambda^2}{\Omega} + 3\beta\right)\left(\frac{\partial D}{\partial w}\right)^{-1}.$$

Introducing the Grander–Morikawa transformation [14].

$$\xi = \delta(x - V_g t) \qquad \text{and} \qquad \tau = \delta^2 t, \tag{6.20}$$

equation (6.19) is then reduced to

$$i\frac{\partial\gamma}{\partial\tau} + P\frac{\partial^2\gamma}{\partial\xi^2} = Q\gamma^2\bar{\gamma}. \tag{6.21}$$

Equation (6.21) is the well-known Ginzburg–Landau equation. The coefficients P and Q are complex, so that

$$P = P_r + iP_i \qquad \text{and} \qquad Q = Q_r + iQ_i, \tag{6.22}$$

P is simply the derivatives of the characteristic function $D(w,k)$, while Q represents the nonlinear interaction term. These terms are lengthy and not included here. They are available from the author on request.

An equation similar to (6.21) was derived, for waves in cylinder wakes, by Fujimura et al. [35]. Kinds and solitons in the generalized Ginzburg–Landau equation are discussed by Malomed and Nepomnyashchy [36]. Landman [37] studied a particular class of solution of equation (6.21), which he called quasi-steady solutions, and found that their spatial variation may be periodic, quasi-periodic, or apparently chaotic. Rotenberry and Saffman [38] used Landman's formalism to study the quasi-steady solutions of the Ginzburg–Landau equation for compliant walls. The stability of the Ginzburg–Landau equation (6.21) is discussed by Lange and Newell [39]. If the solution of this equation is linearly perturbed, the perturbations are stable under the conditions

$$Q_i < 0 \qquad \text{and} \qquad P_rQ_r + P_iQ_i > 0. \tag{6.23}$$

Otherwise, the system is unstable. The absence of the imaginary parts P_i and Q_i in the above criteria reduces to those obtained by Nayfeh [8] and others. Finally, the transition curves separate the stable from the unstable regions are corresponding to

$$Q_i = 0 \qquad \text{and} \qquad P_iQ_i + P_rQ_r = 0 \tag{6.24}$$

These marginal curves may be borne out by numerical estimation. Before dealing with these calculations, it is convenient to write the stability condition (6.23) in an appropriate dimensionless form: consider the characteristic length, time and mass are $h, \frac{1}{w}, \frac{\sigma}{w^2}$, respectively. The other dimensionless quantities are given by:

$$k = \frac{\hat{k}}{h},\ \rho_j = \frac{\hat{\rho}_j \sigma}{w^2 h^3},\ \nu_j = \frac{\hat{\nu}_j \sigma}{w h^3},\ V_j = \hat{V}_j h w, \\ \alpha = \frac{\hat{\alpha}\sigma}{w h^3},\ \alpha_3 = \frac{\hat{\alpha}_3}{h^2} \quad \text{and} \quad H_0^2 = \frac{\hat{H}_0^2 \sigma}{h},\ (j = 1, 2), \tag{6.25}$$

where the superposed hats refer to the dimensionless quantities, it will be omitted for simplicity. At this stage, the stability can therefore be discussed by dividing the $(H_0^2 - k)$ plane into stable and unstable regions. After lengthy but straightforward calculations, the transition curve $Q_i = 0$, may be arranged in a third-degree polynomial on H_0^2 as:

$$A_1 (H_0^2)^3 + A_2 (H_0^2)^2 + A_3 (H_0^2) + A_4 = 0, \tag{6.26}$$

while the transition curve $P_r Q_r + P_i Q_i$, can be arranged in a fifth-degree polynomial on H_0^2 as:

$$C_1 (H_0^2)^5 + C_2 (H_0^2)^4 + C_3 (H_0^2)^3 + C_4 (H_0^2)^2 + C_5 (H_0^2) + C_6 = 0, \tag{6.27}$$

where $A's$ and $C's$ are functions of $\rho_1, \rho_2, \mu_1, \mu_2, \nu_1, \nu_2, V_1, V_2, g, \alpha, \alpha_3$ and k.

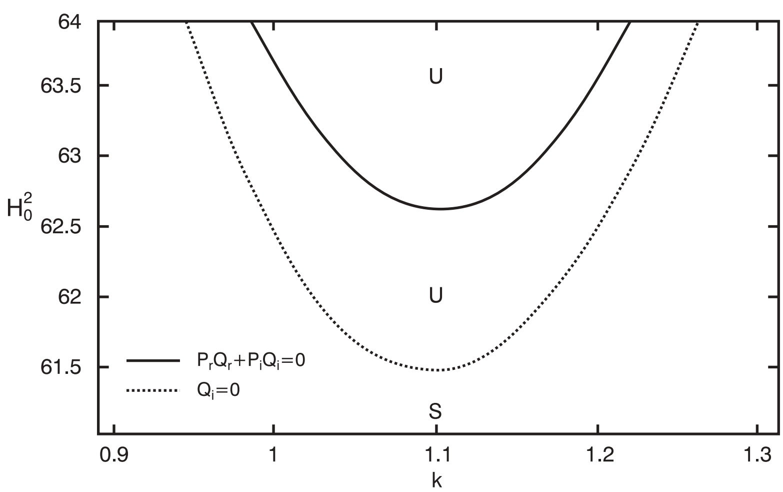

Figure 6. Represents the stability diagram on the $(H_0^2 - k)$-plane according to equations (5.12), and (6.24) for a system having the particulars $\rho_1 = 0.5432$, $\rho_2 = 0.0123$, $\mu_1 = 1$, $\mu_2 = 5$, $V_1 = 0.2$, $V_2 = 0.1$, $\nu_1 = 0.52$, $\nu_2 = 0.70$, $\alpha_3 = 0.1$, $g = 100$, and $\alpha = 1$.

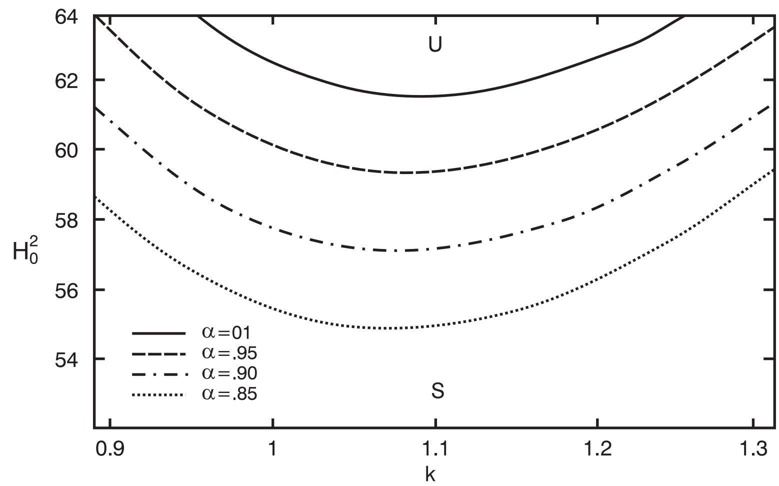

Figure 7. As in Figure 6, but for various values of α.

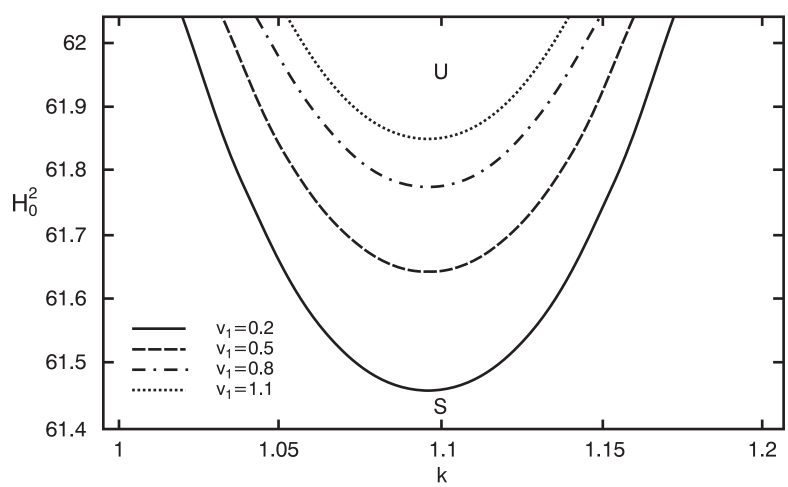

Figure 8. As in Figure 6, but for various values of V_1.

In what follows, we consider the linear as well as the nonlinear stability criteria, through the Figures 6, 7, 8 and 9. The transition curves (6.24) are plotted in Figure 6. As shown in this figure, the solid curve represents one root from equation (6.27), where the other roots are either negative or complex. At the same time, the dotted curve represents one root from equation (6.26) for the same reason as stated above. The linear curve, equation (5.7), is not found in this figure, since it lies in the negative part. Therefore, in the light of the linear theory, the whole plane becomes stable. So, the transition curves (6.24) generally describe the competition between linearity and dispersion. The inspection of this figure shows

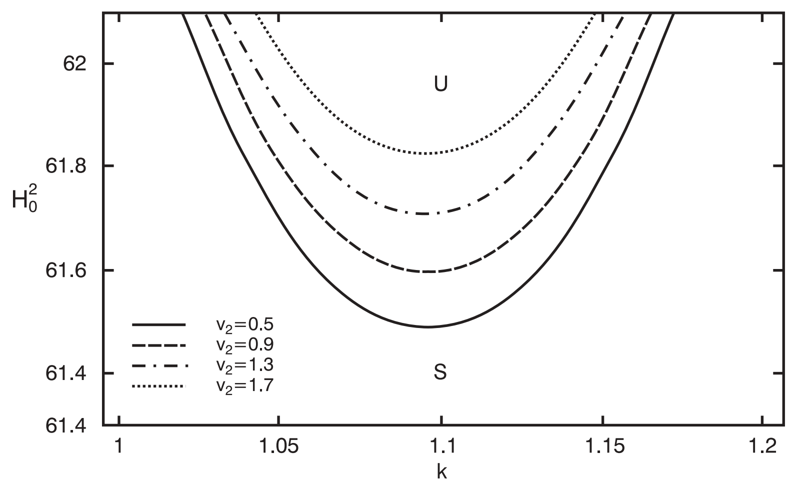

Figure 9. As in Figure 6, but for various values of V_2.

that the nonlinear effects partition the stable region into stable and unstable parts. It is also found that the stability is governed by the transition curve $Q_i = 0$. The influence of the parameter α is computed through Figure 7. In this figure, the transition curve $Q_i = 0$ is only graphed. It is observed that the parameter α has a stabilizing influence. This result is in agreement with the case of linear scope as reported previously. During the pervious analysis in the linear approach, we see that the streaming has a destabilizing influence regardless of the values of V_1 or V_2. Figures 8 and 9 are devoted to indicate the influence of streaming. In contrast to the linear theory, it is found that the streaming has a stabilizing influence.

7. Conclusion

The instability of two layers of incompressible magnetic fluids in the presence of a tangential magnetic field and mass heat transfer is studied with allowance for small but finite, disturbances and for spatial as well as temporal development. No free currents at the surface of separation are assumed. Also, in the stationary state, the fluids are uniformly streaming parallel to each other. Because of the great importance of the practical applications of the porous media, this work examines a few representative porous media configurations. By using the method of multiple scale, we obtain a dispersion relation in the linear approximation and a generalized formulation of the amplitude equation in the nonlinear approximation. In this problem, the effects of mass and heat transfer are revealed through one single parameter α, and the porous influence through Darcy's coefficients ν_1 and ν_2.

From the linearized problem, we have shown that both the tangential magnetic field and the surface tension are stabilizing, while the streaming velocity is strictly destabilizing. As shown by many researches [25, 31], in the non-porous media, we have shown that the parameter α has a destabilizing influence. Several special cases are reported. In contrast with the case of non-porous media, the parameter α has a stabilizing influence through the porous medium. We have, also, shown that Darcy's coefficients play a dual role in the stability picture.

From the nonlinear stability analysis, we have obtained a Ginzburg–Landau equation. The transition curves $Q_i = 0$ and $P_i Q_i + P_r Q_r = 0$ are rearranged, to be plotted in the $(H_0^2 - k)$-plane, in third and fifth-degrees in H_0^2, respectively. The numerical calculations, for some range of the wave number k, showed that the stability criteria are governed by the transition curve $Q_i = 0$. In contrast to the linear theory, we have seen that the streaming velocities have stabilizing influence. It is also found that the mass and heat transfer has the same effect as in the linear theory through the porous media.

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