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Implications of Shield's inverse deformation theorem for compressible finite elasticity

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Abstract. Some properties of the Shield transformation on elastic strain energy functions are established. It is reflexive, it preserves objectivity and material symmetry for isotropic materials, and it also preserves infinitesimal strain response, ellipticity and Hadamard stability, and the Baker–Ericksen condition. Two new classes of strain energies for compressible isotropic materials are introduced, one of them being the image under the Shield transformation of the class of harmonic strain energies. In view of Shield's Inverse Deformation Theorem, these new classes of strain energies will allow solution in closed form of a variety of problems in finite elastostatics.

Keywords. Finite elasticity, exact solutions, universal deformations.

1. Introduction

Although Shield's Inverse Deformation Theorem was first proved some thirty five years ago its implications, especially with regard to finite deformations of compressible isotropic solids, do not appear to have been fully appreciated. The theorem is rather simple; it states that if a particular deformation is supported without body force for a specific strain energy, then the inverse deformation is supported withour body force for another energy, derived from the first. The theorem applies both to unconstrained materials (compressible materials) and to materials with internal constraints, such as incompressibility. A simpler proof was given later by Carlson and Shield [2].

The second strain energy (W^*) is obtained from the first (W) by a transformation (the *Shield transformation*)

$$W^*(\mathbf{F}) = (\det\mathbf{F})W(\mathbf{F}^{-1}), \tag{1.1}$$

where $\mathbf{F}$ is the deformation gradient tensor. We examine this transformation and establish some of its properties. It is reflexive and it preserves infinitesimal response, ellipticity, strong ellipticity and Hadamard stability, and the Baker–Ericksen condition. It also transpires that W^* is objective if, and only if, W is isotropic (and vice versa, because of reflexivity) and this essentially limits the application of the Inverse Deformation Theorem to isotropic materials.

Our focus in on the applications of the theorem for compressible isotropic elastic materials. Motivated by the theorem, we introduce two new classes of materials. The firs of these is the image under the shield transformation of the class of *harmonic* materials introduced by John [3]. (The image of a harmonic material is not a harmonic material – we call it *coharmonic* material.) One implication of the Inverse Deformation Theorem is that the inverses of deformation solutions for a particular harmonic material are solutions for its dual coharmonic material. A second important implication is that if a deformation is *controllable* (or *universal*) for a class of materials, so that it is supported without body force by *every* strain energy in the class, then its inverse is controllable for the image class. Carroll [4, 5] identified three classes of compressible materials that have many controllable deformations, one of them being the class of harmonic materials. It follows that the inverses of the controllable deformations for harmonic materials are controllable deformations for coharmonic materials. Similarly, the inverses of the controllable deformations for the second class of materials introduced in [4] are controllable for its image class under the Shield transformation. The third class of materials introduced in [4] has the property that the image under the Shield transformation of any material in the class is another material in the class. While this still gives a "two for the price of one" bargain with respect to closed form solutions, it does not give rise to nay new controllable deformations.

The case of plane deformations merits special attention. The Inverse Deformation Theorem was introduced by Adkins [6] for plane deformations and all of its application to date have been for plane problems. John introduced harmonic materials to simplify the plane problem and to allow closed form solutions in terms of harmonic functions. In their treatment of plane strain around elliptical cavities, Varley and Cumberbatch [7] identified four classes that they called harmonic materials of Types 1, 2, 3 and 4. Materials of Type 1 were the harmonic materials and materials of Type 3 were their "Eulerian duals" or Shield transforms, i.e., coharmonic materials. The equations of plane strain also simplified for materials of Type 2 but Ogden [8] pointed out that they are unacceptable on physical grounds because they predict zero stress for *all* (equal) biaxial strain states. Their Shield transforms, materials of Type 4, are similarly flawed.

2. Basic equations

A deformation is a mapping

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) \tag{2.1}$$

that carries a typical material particle from its initial position $\mathbf{X}$ to its final position $\mathbf{x}$. The deformation gradient tensor $\mathbf{F}$ is defined as

$$\mathbf{F} = \mathrm{Grad}\boldsymbol{\chi}. \tag{2.2}$$

It has a positive Jacobian ($J = \det\mathbf{F} > 0$) and it admits polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.3}$$

where $\mathbf{U}$ and $\mathbf{V}$ are positive-definite, symmetric stretch tensors and $\mathbf{R}$ is the rotation tensor.

The strain energy W for an elastic solid is a function of the deformation gradient

$$W = W(F). \tag{2.4}$$

It has the invariance (objective) property

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{2.5}$$

for all rotations $\mathbf{Q}$ and this implies a reduced representation

$$W = W(\mathbf{C}); \quad \mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}. \tag{2.6}$$

The strain energy function also exhibits a material symmetry property

$$W(\mathbf{FQ}^T) = W(\mathbf{F}); \qquad W(\mathbf{QCQ}^T) = W(\mathbf{C}) \tag{2.7}$$

for all tensors $\mathbf{Q}$ in the material symmetry group g. We assume that the undeformed state is an undistorted state, so that g is a group of rotations.

The Piola stress $\mathbf{P}$ and the Cauchy stress $\mathbf{T}$ are

$$\mathbf{P} = \partial W/\partial\mathbf{F}; \qquad \mathbf{T} = (\det\mathbf{F})^{-1}\mathbf{PF}^T \tag{2.8}$$

and the equation of equilibrium, with no body force, may be written as

$$\mathrm{Div}\mathbf{P} = \mathbf{o} \quad \text{or} \quad \mathrm{div}\mathbf{T} = \mathbf{o}. \tag{2.9}$$

For isotropic materials, $\mathbf{F}, \mathbf{V}, \mathbf{P}$ and $\mathbf{T}$ admit representations

$$\begin{aligned} \mathbf{F} &= \lambda_i \mathbf{v}^i \otimes \mathbf{u}^i, \qquad & \mathbf{V} &= \lambda_i \mathbf{v}^i \otimes \mathbf{v}^i, \\ \mathbf{P} &= p_i \mathbf{v}^i \otimes \mathbf{u}^i, \qquad & \mathbf{T} &= t_i \mathbf{v}^i \otimes \mathbf{v}^i, \end{aligned} \tag{2.10}$$

where λ_i (the principal values of $\mathbf{U}$ and $\mathbf{V}$), p_i and t_i are the principal stretches, principal forces and principal stresses, respectively.

We will make use of two particular representations of the strain energy for isotropic solids:

$$W = w(\lambda_1, \lambda_2, \lambda_3) \text{ and } W = W(i_1, i_2, i_3). \tag{2.11}$$

Here i_1, i_2 and i_3 are the principal invariants of the stretch tensors or the symmetric combinations of the principal stretches¿

$$i_1 = \mathrm{tr}\mathbf{V} = \lambda_1+\lambda_2+\lambda_3; \quad i_2 = \mathrm{tr}\mathbf{V}^+ = \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2; \quad i_3 = \det\mathbf{V} = \lambda_1\lambda_2\lambda_3, \tag{2.12}$$

where $\mathbf{V}^+$ is the adjoint of $\mathbf{V}$. The function $w(\,)$ in $(2.11)_1$ is completely symmetric in its arguments. The corresponding stress response relations are

$$p_i = \partial w/\partial\lambda_i, \quad t_i = 1/\lambda_j\lambda_k\partial w/\partial\lambda_i \quad (\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i) \tag{2.13}$$

and [4]

$$\mathbf{T} = \partial w/\partial i_3\mathbf{1} + \partial w/\partial i_1\mathbf{V}/i_3 + \partial w/\partial i_2\{(\mathrm{tr}\mathbf{V}^{-1})\mathbf{1} - \mathbf{V}^{-1}\}. \tag{2.14}$$

3. The Shield transformation

In this section we define the Shield transformation of a strain energy $W(\mathbf{F})$ and we establish some of its properties.

Definition. The Shield transformation takes W to its image W^*, with

$$W^*(\mathbf{F}) = (\det\mathbf{F})W(\mathbf{F}^{-1}). \tag{3.1}$$

This transformation has the following properties:
a) It is reflexive.
b) It preserves objectivity and material symmetry *for isotropic materials.*
c) It preserves strong ellipticity and Hadamard stability.
d) It preserves the Baker–Ericksen condition.
e) It preserves infinitesimal strain response.

a) Reflexivity
It should be obvious that the transformation is reflexive, i.e., that the image of W^* is W. Indeed, (3.1) is easily inverted to give

$$W(\mathbf{F}) = (\det\mathbf{F})W^*(\mathbf{F}^{-1}). \tag{3.2}$$

b) Objectivity
For any rotation $\mathbf{Q}$, we have

$$W^*(\mathbf{QF}) = \det(\mathbf{QF})W((\mathbf{QF})^{-1}) = (\det\mathbf{F})W(\mathbf{F}^{-1}\mathbf{Q}^T) = (\det\mathbf{F})W(\mathbf{F}^{-1}) = W^*(\mathbf{F}). \tag{3.3}$$

where the third step invokes the property $(2.7)_1$. Since this invariance property must hold for all rotations, the symmetry group g must be orthogonal group, i.e., $W(\mathbf{F})$ must be an isotropic strain energy. Shield [1] was aware of this connection between isotropy of W and objectivity of W^*. The proof of his important theorem on inverse deformations, discussed in the next section, does not invoke isotropy. He did discuss its applicability for anisotropic materials, saying "Inverse deformation results for anisotropic materials can be obtained if the class of deformations is restricted" and going on to discuss plane deformations of transversely isotropic materials. For such deformations, of course, the material is effectively isotropic. The reflexivity property shows that there is a similar relationship between objectivity of W and isotropy of W^*. Thus, the transformation preserves objectivity and material symmetry for isotropic materials.

(c) Strong ellipticity and Hadamard stability
We now introduce a suffix notation with the summation convention and we write

$$F_{iA} = x_{i,A}, \qquad F_{Ai} = X_{A,i}. \tag{3.4}$$

The strain energy W is elliptic at $\mathbf{F}$ if, and only if,

$$\det\{\partial^2 W/\partial x_{i,A}\partial x_{j,B} M_A M_B\} \neq 0 \qquad (\text{all } \mathbf{M} \neq \mathbf{o}). \tag{3.5}$$

It is Hadamard stable at $\mathbf{F}$ if

$$\partial^2 W/\partial x_{i,A}\partial x_{j,B} m_i m_j M_A M_B \geq 0 \qquad (\text{all } \mathbf{m} \neq \mathbf{o} \text{ and } \mathbf{M} \neq \mathbf{o}), \tag{3.6}$$

i.e., if the acoustic tensor at $\mathbf{F}$ is positive semi-definite, and it is strongly elliptic at $\mathbf{F}$ if the acoustic tensor is positive definite, so that strict inequality prevails in (3.5). Clearly, strong ellipticity implies ellipticity.

Repeated differentiation of $W^*(\mathbf{F})$, given in (3.1), with repeated use of formula

$$\partial X_{A,i}/\partial x_{j,B} = -X_{B,i} X_{A,j}, \tag{3.7}$$

which follows from the indentity

$$X_{A,i} x_{i,C} = \delta_{AC}, \tag{3.8}$$

leads to

$$\begin{aligned} \partial^2 W^*/\partial x_{i,A}\partial x_{j,B} &= JW(X_{A,i}X_{B,j} - X_{A,j}X_{B,i}) \\ - J\partial W/\partial X_{C,k}&(X_{A,i}X_{B,k}X_{C,j} + X_{A,k}X_{B,j}X_{C,i} - X_{A,j}X_{B,k}X_{C,i} - X_{A,k}X_{B,i}X_{C,j}) \\ &+ J\partial^2 W/\partial X_{C,k}\partial X_{D,l} X_{A,k}X_{B,l}X_{C,i}X_{D,j}. \end{aligned} \tag{3.9}$$

It follows that

$$\partial^2 W^*/\partial x_{i,A}\partial x_{j,B} M_A M_B = J\partial^2 W/\partial X_{C,k} X_{D,l} n_k n_l X_{C,i} X_{D,j} \tag{3.10}$$

and

$$\partial^2 W^*/\partial x_{i,A}\partial x_{j,B} m_i m_j M_A M_B = \partial^2 W/\partial X_{A,i} X_{B,j} n_i n_j N_A N_B, \tag{3.11}$$

with

$$n_i = X_{A,i} M_A; \quad N_A = X_{A,i} m_i. \tag{3.12}$$

Since $\mathbf{F}^{-1}$ is nonsingular, it follows that W^* is elliptic, strongly elliptic or Hadamard stable at $\mathbf{F}$ if, and only if, W is elliptic, strongly elliptic or Hadamard stable at $\mathbf{F}^{-1}$.

d) The Baker–Ericksen inequality is

$$(t_i - t_j)(\lambda_i - \lambda_j) > 0 \qquad (\lambda_i \neq \lambda_j), \tag{3.13}$$

i.e., the orderings of the principal stresses and principal stretches are the same. With the principal stress response equation $(2.13)_2$, this becomes

$$(\lambda_i \partial w/\partial \lambda_i - \lambda_j \partial w/\partial \lambda_j)(\lambda_i - \lambda_j) > 0 \qquad (\lambda_i \neq \lambda_j, \text{ no summation}). \tag{3.14}$$

The image of the strain energy $(2.11)_1$ under the Shield transformation is

$$W^* = w^*(\lambda_1, \lambda_2, \lambda_3) = \lambda_1\lambda_2\lambda_3 w(1/\lambda_1, 1/\lambda_2, 1/\lambda_3) \tag{3.15}$$

The principal stresses are

$$\begin{aligned} t_i &= 1/\lambda_j\lambda_k \partial w^*/\partial\lambda_i \quad (\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i) \\ &= w(1/\lambda_1, 1/\lambda_2, 1/\lambda_3) - 1/\lambda_i w_i(1/\lambda_1, 1/\lambda_2, 1/\lambda_3) \quad (\text{no summation}), \end{aligned} \tag{3.16}$$

where $w_i(\,)$ denotes the partial derivative of $w(\,)$ with respect to its ith argument. The Barker–Ericksen inequality becomes

$$(1/\lambda_j w_j - 1/\lambda_i w_i)(\lambda_i - \lambda_j) > 0 \quad (\lambda_i \neq \lambda_j, \text{ no summation}). \tag{3.17}$$

Bearing in mind that the arguments of $w_j(\,)$ in this inequality are the inverse principal stretches, the substitution $\mu_i = 1/\lambda_i$ gives

$$\{\mu_j w_j(\mu_i, \mu_2, \mu_3) - \mu_i w_i(\mu_i, \mu_2, \mu_3)\}(1/\mu_i - 1/\mu_j) > 0 \qquad (\mu_i \neq \mu_j, \text{ no summation}), \tag{3.18}$$

which is equivalent to (3.14). Thus, the Shield transformation preserves the Baker–Ericksen condition in the sense that W^* meets the Baker–Ericksen condition at a particular defomation if, and only if, W meets the condition at the inverse deformation.

e) Infinitesimal deformations

Elastic response in infinitesimal deformation in determined by the fourth-order elasticity tensor, which has component form

$$(\partial^2 W/\partial \mathbf{F} \partial \mathbf{F})_{iAjB} = \partial^2 W/\partial x_{i,A} \partial x_{j,B} \quad \text{at } \mathbf{F} = \mathbf{1},\ x_{i,A} = \delta_{i,A}. \tag{3.19}$$

It follows immediately from (3.9), setting $\mathbf{F} = \mathbf{1}$, and making use of the fact that the energy and the stress vanish in the undeformed state, that the response of W^* in infinitesimal deformation is the same as that of W.

This may also be shown directly from $(2.13)_1$ and (3.16). Indeed, for both W^* and W the Lamé constants μ and β are given by

$$2\mu = w_{11}(1,1,1) - w_{12}(1,1,1); \quad \beta = w_{12}(1,1,1), \tag{3.20}$$

where the subscript again denote partial differentiation and the symmetry of the strain energy function $w(\,)$ implies that $w_{12} = w_{13}$ at $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

4. Shield's Inverse Deformation Theorem

The motivation for our discussion of the Shield transformation in the previous section is the following

Inverse Deformation Theorem. *Let W be a strain energy and W^* its image under the Shield transformation. If a deformation*

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) \tag{4.1}$$

is supported by W without body force, then the inverse deformation, given implicitly by

$$\mathbf{X} = \boldsymbol{\chi}(\mathbf{x}), \tag{4.2}$$

is supported by W^* without body force.

This result was first proved by Shield [1] and a simpler proof was given later by Carlson and Shield [2]. The theorem generalized the earlier result of Adkins [7] for plane deformations. The theorem applies for materials with or without internal constraints, such as incompressibility. The proofs [1,2] do not invoke isotropy of W, however, in view of the discussion of the objectivity of W^* in the previous section, we will confine further discussion to the isotropic case.

Incidentally, (3.8) furnishes a proof of the Inverse Deformation Theorem. Suppose that W supports $\mathbf{F}^{-1}$ without body force, so that

$$\partial/\partial x_k(\partial W/\partial X_{C,k}) = (\partial^2 W/\partial X_{C,k}\partial X_{D,l})X_{D,kl} = 0. \tag{4.3}$$

Then it follows from (3.8) and (4.3) that

$$\begin{aligned}\partial/\partial X_A(\partial W^*/\partial x_{i,A}) &= (\partial^2 W^*/\partial x_{i,A}\partial x_{j,B})x_{j,AB} \\ &= -J\partial^2 W/\partial X_{C,k}\partial_{D,l}X_{C,i}X_{D,kl} = 0. \end{aligned}\tag{4.4}$$

For incompressible materials, the image of a strain energy $W(i_1, i_2)$ is the strain energy $W(i_2, i_1)$. Similarly, the image of a strain energy $W(I_1, I_2)$ is a strain energy $W(I_2, I_1)$, I_1 and I_2 being the first two principal invariants of the deformation tensor $\mathbf{C}$. In particular, if a deformation is supported without body force in a particular Mooney–Rivlin material, then the inverse deformation is supported withour body force in another Mooney–Rivlin material.

An immediate corollary of the Inverse Deformation Theorem is the following:

If a deformation (4.1) is a controllable deformation for a class of strain energies, so that it is supported without body force for every strain energy in the class, then its inverse (4.2) is a controllable deformation for the image class of strain energies under the Shield transformation.

The inverse of a very known controllable deformation for incompressible isotropic materials is also a controllable deformation (for example, bending of a rectangular block into a circular cylindrical sector and straightening of a sector into a block). Indeed, this is a manifestation of the Inverse Deformation Theorem. It follows that use of the theorem does not lead to any new controllable deformations for incompressible materials.

The situation is quite different in the compressible case. Ericksen [9] proved that the only deformations that are controllable for the full class of compressible isotropic materials are homogeneous deformations. (A short proof of this result was given by Shield [10].) However, there are three known families of compressible isotropic materials – one being harmonic materials – that have controllable deformations (Carroll [4, 5]). The Inverse Deformation Theorem implies that the images of these classes under the Shield transformation will also afford controllable deformations. As stated above, the Shield transformation of the class of harmonic materials yield the class of coharmonic materials. Transformation of the second class of strain energies (called Class II in [4, 5]) also yields a new class (Class II*). However, the third class (Class III) is such that the Shield transformation of any particular strain energy in the class yields another energy in the class.

5. Two new classes of compressible isotropic strain energies

We begin by listing five classes of compressible isotropic strain energies:
Class I (harmonic materials):

$$W = f(i_1) - 2\mu\{\zeta i_2 + (1-\zeta)i_3 - 1 - 2\zeta\} \tag{5.1}$$

Class I* (coharmonic materials):

$$W = i_3 f(i_2/i_3) - 2\mu\{\zeta i_1 - (1+2\zeta)i_3 + 1 - \zeta\} \tag{5.2}$$

Class II:

$$W = g(i_2) + \mu\{(1-\zeta)i_1 - (3+\zeta)i_3 + 4\zeta\} \tag{5.3}$$

Class II*:

$$W = i_3 g(i_1/i_3) + \mu\{(1-\zeta)i_2 + 4\zeta i_3 - 3 - \zeta\} \tag{5.4}$$

Class III:

$$W = h(i_3) + 2\mu\{\zeta i_1 + (1-\zeta)i_2 - 3\}. \tag{5.5}$$

Classes I, II and III were introduced previously by John [3] and Carroll [4]. Classes I* and II* are new and they are the images of Classes I and II under the Shield transformation. They might also be written as

Class I*:

$$W = 1/j_3\{f(j_1) - 2\mu(\zeta j_2 + (1-\zeta)j_3 - 1 - 2\zeta\} \tag{5.6}$$

Class II*:

$$W = 1/j_3\{g(j_2) + \mu\{(1-\zeta)j_1 - (3+\zeta)j_3 + 4\zeta\}, \tag{5.7}$$

where j_1, j_2 and j_3 are the principal invariants of the inverse stretch tensor $\mathbf{V}^{-1}$, i.e.,

$$j_1 = 1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3; \quad j_2 = 1/\lambda_2\lambda_3 + 1/\lambda_3\lambda_1 + 1/\lambda_1\lambda_2; \quad j_3 = 1/\lambda_1\lambda_2\lambda_3. \tag{5.8}$$

The image of a particular Class III strain energy (5.5) is

$$\begin{aligned} W^* = 1/j_3\{h(j_3) + 2\mu\{\zeta j_1 + (1-\zeta)j_2 - 3\} \\ = i_3 h(1/i_3) + 2\mu\{\zeta i_2 + (1-\zeta)i_1 - 3i_3\}, \end{aligned} \tag{5.9}$$

which is a different strain energy in Class III.

The constants in (5.1)–(5.5) are chosen so that the strain energies and the stresses vanish in the undeformed state, provided

$$f(3) = g(3) = h(1) = 0 \tag{5.10}$$

and

$$f'(3) = 2\mu(1+\zeta), \quad g'(3) = \mu(1+\zeta), \quad h'(1) = 2\mu(\zeta - 2). \tag{5.11}$$

In each case, the constant $\mu(>0)$ is the shear modulus for infinitesimal deformation and the bulk modulus $K(>0)$ is given by

$$f''(3) = 4g''(3) = 3h''(1) = K + 4/3\mu. \tag{5.12}$$

(The Shield transformation preserves the response in infinitesimal deformation.) It was pointed out in [4] that a strain energy that is linear in all three invariants i_1, i_2 and i_3 does not describe realistic material response and this is evident from (5.12). The image of such a strain energy under the Shield transformation is another such strain energy.

The fact that the Shield transformation preserves strong ellipticity, Hadamard stability and the Baker–Ericksen condition suggests that if the constitutive functions $f(\,), g(\,)$ and $h(\,)$ are such that the strain energies (5.1), (5.3) and (5.5) model realistic material response, then so too will the image energies (5.2), (5.4) and (5.9). The Baker–Ericksen conditions are

$$f'(i_1) - 2\mu\zeta\lambda > 0, \quad \lambda g'(i_2) + \mu(1-\zeta) > 0, \quad \lambda < \zeta/(\zeta - 1), \tag{5.13}$$

for each principal stretch λ. Observe from $(5.13)_3$ that materials of Class III can not meet the Baker–Ericksen condition for all stretch states, except in the case $\zeta = 1$.

6. Examples: Harmonic and coharmonic materials

By way of example, we present two new deformations that are controllable for harmonic materials and we verify that the inverse deformations are controllable for coharmonic materials.

For the class of harmonic materials (5.1), the Cauchy stress is given by (2.14) as

$$\mathbf{T} = -2\mu(1-\zeta)\mathbf{1} + f'(i_1)\mathbf{V}/i_3 - 2\mu\zeta\{(\mathrm{tr}\mathbf{V}^{-1})\mathbf{1} - \mathbf{V}^{-1}\}, \tag{6.1}$$

so that the equation of equilibrium $(2.9)_2$ reduces to

$$f''(i_1)(\mathbf{V}/i_3)\mathrm{grad}i_1 + f'(i_1)\mathrm{div}(\mathbf{V}/i_3) - 2\mu\zeta\mathrm{div}\{(\mathrm{tr}\mathbf{V}^{-1})\mathbf{1} - \mathbf{V}^{-1}\} = \mathbf{o}. \tag{6.2}$$

It was shown in [5] that

$$\mathrm{div}(\mathbf{V}/i_3) = \mathbf{o} \text{ and } \mathrm{div}\{(\mathrm{tr}\mathbf{V}^{-1})\mathbf{1} = \mathbf{V}^{-1}\} = \mathbf{o} \tag{6.3}$$

for any irrotational (potential) deformation†

$$\mathbf{x} = \mathrm{Grad}\Psi(\mathbf{X}). \tag{6.4}$$

The first result $(6.3)_1$ is an immediate consequence of the indentity

$$i_3\mathrm{div}(\mathbf{V}/i_3) = \mathrm{Div}\mathbf{R}. \tag{6.5}$$

† All irrotational deformations are potential deformations but the opposite is not true.

The second result $(6.3)_2$ follows from the fact that the inverse deformation is also irrotational, so that

$$\mathbf{X} = \text{grad}\psi(\mathbf{x}), \tag{6.6}$$

and hence

$$\mathbf{V}^{-1} = \mathbf{F}^{-1} = \text{gradgrad}\psi(\mathbf{x}). \tag{6.7}$$

It follows from (6.2) and (6.3) that any irrotational deformation (6.4) is controllable for the class of harmonic materials provided i_i is constant, i.e., provided

$$\nabla^2\Psi = \nu \text{ (constant)}. \tag{6.8}$$

This implies [5] that every harmonic scalar ($\Phi = \Psi - 1/2\nu\mathbf{X}\cdot\mathbf{X}$) gives a deformation that is controllable for harmonic materials.

For coharmonic materials, the stress is

$$\begin{aligned}\mathbf{T} = \{f(i_2/i_3) - i_2/i_3 f'(i_2/i_3) + 2\mu(1+2\zeta)\}\mathbf{1} - 2\mu\zeta\mathbf{V}/i_3 \\ + f'(i_2/i_3)\{(\text{tr}\mathbf{V}^{-1})\mathbf{1} - \mathbf{V}^{-1}\}.\end{aligned} \tag{6.9}$$

In view of (6.3), the equation of equilibrium for any potential deformation (6.4) reduces to the condition that $j_1 = i_2/i_3$ be constant. This class of controllable deformations is clearly the inverse of the class of controllable deformations for harmonic materials.

Consider a potential

$$\Psi = 1/2(\alpha X^2 + \beta Y^2 + \gamma Z^2) + \kappa \ln R \quad (R^2 = X^2 + Y^2;\ \alpha > 0, \beta > 0, \gamma > 0). \tag{6.10}$$

which meets the condition (6.8) with $\zeta = \alpha + \beta + \gamma$. The corresponding deformation

$$x = (\alpha + \kappa/R^2)X, \quad y = (\beta + \kappa/R^2)Y, \quad z = \gamma Z \tag{6.11}$$

is controllable for harmonic materials. It follows from (6.11) that

$$x^2/(\alpha R + \kappa/R)^2 + y^2/(\beta R + \kappa/R)^2 = 1. \tag{6.12}$$

so that this deformation carries material circular cylinders $R = R_0$ into elliptical cylinders. The stretch tensor is

$$\mathbf{V} = \mathbf{F} = \begin{pmatrix} \alpha + \kappa/R^2 - 2\kappa X^2/R^4 & -2\kappa XY/R^4 & 0 \\ -2\kappa XY/R^4 & \beta + \kappa/R^2 - 2\kappa Y^2/R^4 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{6.13}$$

and the stress tensor is given by (6.1) and (6.13).

The inverse deformation is given implicitly as

$$X = (\alpha + \kappa/r^2)x, \quad Y = (\beta + \kappa/r^2)y, \quad Z = \gamma z. \tag{6.14}$$

This is controllable for harmonic materials and it carries material elliptical cylinders into circular cylinders $r = r_0$. The inverse stretch is

$$\mathbf{V}^{-1} = \mathbf{F}^{-1} = \begin{pmatrix} \alpha + \kappa/r^2 - 2\kappa x^2/r^4 & -2\kappa xy/r^4 & 0 \\ -2\kappa xy/r^4 & \beta + \kappa/r^2 - 2\kappa y^2/r^4 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{6.15}$$

and the stress is given by (6.9) and (6.15), with $i_2/i_3 = \alpha + \beta + \gamma$.

As a second example, consider the potential

$$\Psi = 1/2(\alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2) + \kappa/R \qquad (R^2 = X_i X_i,\ \alpha_i > 0), \tag{6.16}$$

which meets the condition (6.8) with $\zeta = \alpha_1 + \alpha_2 + \alpha_3$. The corresponding deformation

$$x_i(\alpha_i - \kappa/R^3)X_i \quad \text{(no summation)} \tag{6.17}$$

is controllable for harmonic materials. It follows from (6.17) that

$$x_1^2/(\alpha_1 R - \kappa/R^2)^2 + x_2^2/(\alpha_2 R - \kappa/R^2)^2 + x_3^2/(\alpha_3 R - \kappa/R^2)^2 = 1, \tag{6.18}$$

so that this deformation carries material spheres $R = R_0$ into ellipsoids. The components of the stretch tensor are

$$V_{ij} = (\alpha_i - \kappa/R^3)\delta_{ij} - 3\kappa X_i X_j/R^5 \quad \text{(no summation)} \tag{6.19}$$

and the stress tensor is given by (6.1) and (6.19).

The inverse deformation, given implicitly as

$$X_i = (\alpha_i - \kappa/r^3)x_i \quad \text{(no summation)}, \tag{6.20}$$

is controllable for coharmonic materials and it carries material ellipsoids into spheres $r = r_0$. The inverse stretch components are

$$V_{ij}^{-1} = (\alpha_i - \kappa/r^3)\delta_{ij} - 3\kappa x_i x_j/r^5 \tag{6.21}$$

and the stresses are given by (6.9) and (6.21).

The deformations (6.11) and (6.17) are treated here merely as illustrative examples. They are somewhat artificial in that there are not enough adjustable parameters for them to provide solutions of realistic boundary value problems. For example, if we attempt to use the controllable deformation (6.11) to solve the plane strain ($\gamma = 1$) problem of a hollow circular cylinder with inner and outer radii R_0 and R_1, threaded on to an elliptical spindle with principal radii a and b, the boundary conditions at $R = R_0$ are

$$\alpha R_0 + \kappa/R_0 = a, \quad \beta R_0 + \kappa/R_0 = b. \tag{6.22}$$

This leaves just one adjustable parameter to meet boundary conditions at $R = R_1$. Similar remarks apply for the deformation (6.17). More realistic solutions for harmonic and coharmonic materials will be treated subsequently.

7. Radial cylindrical deformations for materials in Class II*

As a final example, we treat a radial cylindrical expansion or compaction that is controllable for materials in Class II*. We first describe briefly the corresponding deformation for materials in Class II, first treated in [4].

For a Class II strain energy (5.3), the Cauchy stress is given by (2.14) as

$$\mathbf{T} = -\mu(3+\zeta)\mathbf{1} + \mu(1-\zeta)\mathbf{V}/i_3 + g'(i_2)\{(\mathrm{tr}\mathbf{V}^{-1})\mathbf{1} - \mathbf{V}^{-1}\}. \tag{7.1}$$

For irrotational deformations (6.4), the equations of equilibrium reduce to the condition that i_2 be constant. For the Class II* strain energy (5.4), the Cauchy stress is

$$\mathbf{T} = \{g(i_1/i_3) - i_1/i_3 g'(i_1/i_3) + 4\mu\zeta\}\mathbf{1} + g'(i_1/i_3)\mathbf{V}/i_3 + \mu(1-\zeta)\{(\mathrm{tr}\mathbf{V}^{-1})\mathbf{1} - \mathbf{V}^{-1}\}. \tag{7.2}$$

For irrotational deformations, the equations of equilibrium reduce to the condition that $j_2 = i_1/i_3$ be constant.

An irrotational deformation with potential

$$\Psi = F(R) + 1/2\lambda Z^2; \quad (R^2 = X^2 + Y^2,\ F'(R) > 0,\ F''(R) > 0,\ \lambda > 0) \tag{7.3}$$

generates a radial cylindrical deformation

$$r = r(R), \quad \theta = \Theta, \quad z = \lambda Z. \tag{7.4}$$

This radial expansion or compaction has principal stretches dr/dR, r/R and λ and the condition that i_2 be constant

$$r/R\, dr/dR + \lambda(dr/dR + r/R) = \text{constant} \tag{7.5}$$

integrates to

$$r = (\alpha R^2 + \beta)^{1/2} - \lambda R. \tag{7.6}$$

The deformation described by (7.4) and (7.6) is controllable for materials in Class II.

The inverse deformation is given implicitly by

$$R = (\alpha r^2 + \beta)^{1/2} - \lambda r, \qquad \Theta = \theta \qquad Z = \lambda z \tag{7.7}$$

and explicitly by

$$r = \lambda R/(\alpha - \lambda^2)\,[1 + \{1 + (\alpha/\lambda^2 - 1)(\beta/R^2 - 1)\}^{1/2}], \quad \theta = \Theta, \quad z = Z/\lambda. \tag{7.8}$$

The invariant $j_2 = i_1/i_3 = \alpha - \lambda^2$ and the deformation (7.8) is controllable for materials in Class II*. The inverse stretch tensor is found from (7.7):

$$\mathbf{V}^{-1} = \{\alpha r/(\alpha r^2 + \beta)^{1/2} - \lambda\}\mathbf{e}_r \otimes \mathbf{e}_r + \{(\alpha r^2 + \beta)^{1/2}/r - \lambda\}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda \mathbf{e}_z \otimes \mathbf{e}_z \tag{7.9}$$

and the stress is found from (7.2) and (7.9). In particular, the radial stress is

$$T_{rr} = g(\alpha - \lambda^2) - \alpha g'(\alpha - \lambda^2) + 4\mu\zeta + \{\lambda g'(\alpha - \lambda^2) + \mu(1-\zeta)\}(\alpha + \beta/r^2)^{1/2}. \tag{7.10}$$

Solutions of boundary value problems of interest are given by (7.8) and (7.10) for any particular response function $g(\,)$ and constants μ and ζ.

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