Z. angew. Math. Phys. 57 (2006) 1–11
0044-2275/06/010001-11
DOI 10.1007/s00033-005-0019-4

Zeitschrift für angewandte Mathematik und Physik ZAMP

Existence of 2π-periodic solutions for the non-dissipative Duffing equation under asymptotic behaviors of potential function

W. B. Liu and Y. Li

Abstract. In this paper,we give the existence conditions of periodic solutions for non-dissipative forced Duffing equations by asymptotic behaviors of potential under crossing resonant points, improving some known results to some extent.

Mathematics Subject Classification (2000). 34C25.

Keywords. Duffing equation, periodic solution, Fučik spectrum, asymptotic behavior of potential function.

1. Introduction

In this paper, we study the existence of 2π-periodic solutions for the non-dissipative Duffing equation

$$x'' + g(x) = p(t), \tag{1.1}$$

where $g(x) \in C(R), p(t) = p(t + 2\pi) \in C(R^+)$.

There are many papers devoted to the study of the existence of periodic solutions for Eq. (1.1) (see [1–13] and the references therein). In 1980, Dancer [7] and Fučik [8] established the conception being called "Fučik spectrum" respectively, here we call (μ, ν) is the Fučik spectrum if equation

$$x'' + \mu x^+ - \nu x^- = 0,$$

has a nontrivial 2π-periodic solution, where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}$, $\mu, \nu > 0$. It is easy to obtain that the Fučik spectrum (μ, ν) satisfies the equality

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{n},$$

where n is a positive integer. Under the frame of the Fučik spectrum, many scholars studied the existence of periodic solutions for Eq. (1.1) using the asymptotic behavior of the ratio $\frac{g(x)}{x}$ as $|x| \to \infty$. Such as Fučik [8] proved that if

$$\lim_{x\to+\infty} \frac{g(x)}{x} = \mu, \qquad \lim_{x\to-\infty} \frac{g(x)}{x} = \nu, \tag{1.2}$$

and (μ, ν) is not the Fučik spectrum, then Eq. (1.1) has a 2π-periodic solution; Drabek and Invernizzi [11], Gossez and Omari [12] improved the condition (1.2) into

$$p \le \liminf_{x\to+\infty} \frac{g(x)}{x} \le \limsup_{x\to+\infty} \frac{g(x)}{x} \le q, r \le \liminf_{x\to-\infty} \frac{g(x)}{x} \le \limsup_{x\to-\infty} \frac{g(x)}{x} \le s, \tag{1.3}$$

where p, q, r, s are positive numbers and satisfy

$$\frac{2}{n+1} < \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{s}} \le \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{p}} < \frac{2}{n}, \tag{1.4}$$

n is a positive integer. Obviously, if $p = q, r = s$, the conditions (1.3)–(1.4) are identical with (1.2). In 1991, Ding and Zanolin [9], using time-map

$$\tau(c) = 2\left|\int_0^c \frac{d\xi}{\sqrt{2(G(c) - G(\xi))}}\right|$$

where

$$G(c) = \int_0^c g(u)du, \;\; g(u) : \lim_{|u|\to\infty} g(u)sign(u) = +\infty,$$

proved that the Eq. (1.1) has a 2π-periodic solution [9, Theorem 1] provided that

$$\left[J_- + J_+, J^- + J^+\right] \cap \left\{\frac{2\pi}{n} : n \in N\right\} = \emptyset \tag{1.5}$$

where

$$\begin{aligned} J_- &= \liminf_{c\to-\infty} \tau(c), \quad J_+ = \liminf_{c\to+\infty} \tau(c) \\ J^- &= \limsup_{c\to-\infty} \tau(c), \quad J^+ = \limsup_{c\to+\infty} \tau(c). \end{aligned} \tag{1.6}$$

In [9], Ding and Zanolin also gave the following two special existence conditions of 2π-periodic solutions for Eq. (1.1) respectively:

$$[\frac{\pi}{\sqrt{q_+}} + \frac{\pi}{\sqrt{q_-}}, \frac{\pi}{\sqrt{\gamma_-}} + \frac{\pi}{\sqrt{\gamma_+}}] \cap \{\frac{2\pi}{n} : n \in N\} = \emptyset \tag{1.7}$$

where

$$\gamma_- = \liminf_{x\to-\infty} \frac{g(x)}{x} \le \lim_{x\to-\infty} \frac{2G(x)}{x^2} = q_-, \gamma_+ = \liminf_{x\to+\infty} \frac{g(x)}{x} \le \lim_{x\to+\infty} \frac{2G(x)}{x^2} = q_+. \tag{1.8}$$

and

$$\lim_{x\to+\infty} \frac{2G(x)}{x^2} = \mu, \lim_{x\to-\infty} \frac{2G(x)}{x^2} = \nu, \tag{1.9}$$

where $\mu, \nu > 0$ and (μ, ν) is not the Fučik spectrum.

Because

$$\begin{aligned}
&\liminf_{x\to-\infty} \tfrac{g(x)}{x} \leq \liminf_{x\to-\infty} \tfrac{2G(x)}{x^2} \\
&\leq \limsup_{x\to-\infty} \tfrac{2G(x)}{x^2} \\
&\leq \limsup_{x\to-\infty} \tfrac{g(x)}{x}, \\
&\liminf_{x\to+\infty} \tfrac{g(x)}{x} \leq \liminf_{x\to+\infty} \tfrac{2G(x)}{x^2} \\
&\leq \limsup_{x\to+\infty} \tfrac{2G(x)}{x^2} \\
&\leq \limsup_{x\to+\infty} \tfrac{g(x)}{x},
\end{aligned} \tag{1.10}$$

the condition (1.7) is more precise than (1.4) if the limits $\lim_{x\to\pm\infty} \frac{2G(x)}{x^2}$ exist. Just as reason, the estimate by asymptotic behavior of $\dfrac{2G(x)}{x^2}$ instead of $\dfrac{g(x)}{x}$ may become trend (see[9], [6], [13]). However, the question is whether or not there is a 2π-periodic solution for Eq. (1.1) if the limits $\lim_{x\to+\infty} \frac{2G(x)}{x^2}$ and $\lim_{x\to-\infty} \frac{2G(x)}{x^2}$ don't exist. The aim of this paper is to solve this question, i.e., to establish the existence of 2π-periodic solutions for Eq. (1.1) by using $\limsup_{x\to+\infty} \frac{2G(x)}{x^2}$ and $\liminf_{x\to-\infty} \frac{2G(x)}{x^2}$ instead of $\lim_{x\to\pm\infty} \frac{2G(x)}{x^2}$ under the frame of the Fučik spectrum. Our conditions on $g(x)$ are more general and more easily checked than those of some known results to some extent (Remark 2, 3).

It needs mentioning that the conditions relative to $\dfrac{2G(x)}{x^2}$ may lead to the oscillation crossing resonant points, just as which, it is difficult to deal with.

2. Main results

It is easily verified that if $x(t)$ is a 2π-periodic solution of Eq. (1.1), then $x(t)$ satisfies the periodic boundary condition

$$x(0) = x(2\pi),\ x'(0) = x'(2\pi). \tag{2.1}$$

In this paper, we often use (2.1) and throughout suppose that

I_1) there exist positive real numbers c_0, d_0, d, such that

$$d_0 \geq \frac{g(x)}{x} \geq c_0 > 0, \quad \text{when } |x| \geq d,$$

I_2) there exist positive real numbers p, q, r, s, such that

$$p \leq \lim_{x\to+\infty} \inf \frac{2G(x)}{x^2} \leq \lim_{x\to+\infty} \sup \frac{2G(x)}{x^2} \leq q, \tag{2.2}$$

$$r \leq \lim_{x\to-\infty} \inf \frac{2G(x)}{x^2} \leq \lim_{x\to-\infty} \sup \frac{2G(x)}{x^2} \leq s, \tag{2.3}$$

and
$$\frac{2}{n+1} < \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{s}} \le \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{r}} < \frac{2}{n}, \tag{2.4}$$
where n is a positive integer,
$$G(x) = \int_0^x g(s)ds.$$

Lemma 2.1. [14] *Equation*
$$x'' + u_1(t)x^+ - u_2(t)x^- = 0,$$
has only a trivial 2π-periodic solution provided that $u_1(t), u_2(t) \in L^2[0, 2\pi]$ and
$$p \le u_1(t) \le q, \quad r \le u_2(t) \le s,$$
where p, q, r, s satisfy the condition (2.4).

Lemma 2.2. [15] *Suppose that L is a Fredholm mapping of index zero, $N : \bar{\Omega} \to Z$ is L-compact, $A : X \to Z$ is L-completely continuous in $\bar{\Omega}$, and*

i) $Ker(L - A) = \{0\}$;
ii) $\forall (x, \lambda) \in (D(L) \cap \partial\Omega) \times (0, 1),\ Lx - (1 - \lambda)Ax - \lambda Nx \ne 0.$

Then equation
$$Lx = Nx$$
has at least one solution in $D(L) \cap \bar{\Omega}$, where X is normed space, $0 \in \Omega \subset X$ open bounded.

Theorem 1. *Let the conditions I_1) and I_2) hold. Then the Eq. (1.1) has at least one 2π-periodic solution.*

Proof. Consider homotopy equation with (1.1)
$$x'' + \lambda g(x) + (1 - \lambda)(ax^+ - bx^-) = \lambda e(t), 0 \le \lambda \le 1, \tag{2.5}$$
where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}, a$ and b are constants satisfying
$$p \le a \le q, \ \ r \le b \le s. \tag{2.6}$$
First of all, we prove that there exists a positive number M independent of λ, such that $\|x\|_{C^1} \le M$ for all possible $2\pi-$periodic solution $x(t)$ of (2.5), where $\|\cdot\|_{C^1}$ denotes the norm in $C^1[0, 2\pi]$ by
$$\|x\|_{C^1} = \max(\max_{t\in[0,2\pi]} |x(t)|, \max_{t\in[0,2\pi]} |x'(t)|).$$

We assume on the contrary that there exists a sequence of $2\pi-$periodic solutions $\{x_m(t)\}_1^\infty$ of (2.5) corresponding to sequence $\{\lambda_m\}_1^\infty \subset [0, 1]$ such that
$$\|x_m\|_{C^1} \to \infty, \quad \text{as} \quad m \to \infty \tag{2.7}$$

Dividing both sides of (2.5) by $\|x_m\|_{C^1}$ and writing

$$Z_m(t) = \frac{x_m(t)}{\|x_m\|_{C^1}},$$

we have

$$Z_m'' + \frac{\lambda_m}{\|x_m\|_{C^1}} g(x_m) + (1-\lambda_m)(aZ_m^+ - bZ_m^-) = \frac{\lambda_m}{\|x_m\|_{C^1}} e(t). \tag{2.8}$$

In view of the condition I_1), there exists a positive number b_1 such that

$$\left| \frac{g(x_m)}{\|x_m\|_{C^1}} \right| \le b_1,$$

moreover, there exists $M_0 > 0$ independent of λ,such that

$$\left| Z_m''(t) \right| \le M_0 \quad t \in [0, 2\pi].$$

This shows that the sequences $\{Z_m(t)\}$ and $\{Z_m'(t)\}$ are uniformly bounded and equicontinuous on $[0, 2\pi]$. Thus we know by the Arzela–Ascoli theorem that there are uniformly convergent subsequences on $[0, 2\pi]$ for $\{Z_m(t)\}$ and $\{Z_m'(t)\}$ respectively, which may be taken as themselves without loss of generality, such that

$$\lim_{m\to\infty} Z_m(t) = Z(t), \lim_{m\to\infty} Z_m'(t) = Z'(t). \tag{2.9}$$

and

$$\|Z\|_{C^1} = 1 \tag{2.10}$$

from the definition of $Z_m(t)$. Noting that

$$x_m''(t) + \lambda_m g(x_m(t)) + (1-\lambda_m)[ax_m^+(t) - bx_m^-(t)] = \lambda_m e(t)$$

and integrating both sides of above equation on $[0, 2\pi]$, we have

$$\begin{aligned} &\lambda_m \textstyle\int_0^{2\pi} g(x_m(t))dt + (1-\lambda_m) \int_0^{2\pi} [ax_m^+(t) - bx_m^-(t)]dt \\ &\quad = \lambda_m \textstyle\int_0^{2\pi} e(t)dt. \end{aligned} \tag{2.11}$$

We claim that for each m, there exists $t_m \in [0, 2\pi]$ and $d_1 > 0$ independent of λ_m, such that

$$|x_m(t_m)| \le d_1. \tag{2.12}$$

For example, we might as well take d_1 satisfying

$$d_1 \ge d, \text{ and } ad_1, bd_1, c_0 d_1 > \frac{1}{2\pi} \int_0^{2\pi} |e(t)|dt.$$

If the claim is false, then $|x_m(t)| \ge d_1$ for each $t \in [0, 2\pi]$. However, we observe from I_1) that when $x_m(t) \ge d_1$,

$$\lambda_m \int_0^{2\pi} g(x_m(t))dt + (1-\lambda_m) \int_0^{2\pi} [ax_m^+(t) - bx_m^-(t)]dt$$

$$\begin{aligned}&\geq \lambda_m 2\pi c_0 d_1 + (1-\lambda_m)2\pi a d_1\\&> \int_0^{2\pi} |e(t)|dt\\&\geq \lambda_m \int_0^{2\pi} e(t)dt,\end{aligned}$$

which yields a contradiction with (2.11). Similarly, we may get a contradiction if $x_m(t) \leq -d_1,\ t \in [0, 2\pi]$. The claim is complete. Consider the boundedness of the sequence $\{t_m\}$ and $\{\lambda_m\}$ bounded, we deduce without loss of generality that

$$\lim_{m\to\infty} t_m = t_0, \qquad \lim_{m\to\infty} \lambda_m = \lambda_0. \tag{2.13}$$

Multiplying both sides of (2.8) by $Z'_m(t)$ and integrating from t_m to t, we have

$$\begin{aligned}&[Z'_m(t)]^2 - [Z'_m(t_m)]^2 + \lambda_m \frac{2G(x_m(t))}{x_m^2(t)} \cdot \frac{x_m^2(t)}{\|x_m\|_{C^1}^2} - \lambda_m \frac{2G(x_m(t_m))}{\|x_m\|_{C^1}^2}\\&+(1-\lambda_m)[a(Z_m^+(t))^2 + b(Z_m^-(t))^2 - (a(Z_m^+(t_m))^2 + b(Z_m^-(t_m))^2)]\\&= \lambda_m \frac{2}{\|x_m\|_{C^1}^2} \int_{t_m}^t e(s)Z_m'(s)ds. \end{aligned}\tag{2.14}$$

Taking a superior limit in (2.14) ($m \to \infty$) and combing (2.7), (2.9), (2.12), (2.13) as well as

$$\lim_{m\to\infty} Z_m(t_m) = \lim_{m\to\infty} \frac{x_m(t_m)}{\|x_m\|_{C^1}} = 0,$$

we get

$$\begin{aligned}&[Z'(t)]^2 - [Z'(t_0)]^2 + \lambda_0 \lim_{m\to\infty} \sup \tfrac{2G(x_m(t))}{x_m^2(t)} \cdot [Z(t)]^2\\&+(1-\lambda_0)[a(Z^+(t))^2 + b(Z^-(t))^2]\\&= 0.\end{aligned}$$

From I_2) and the definition of a and b, we obtain

$$(Z'(t))^2 - (Z'(t_0))^2 + q(Z(t))^2 \geq 0, \quad \text{when } \ Z(t) > 0.$$

Analogously, we also obtain

$$\begin{aligned}&[Z'(t)]^2 - [Z'(t_0)]^2 + p[Z(t)]^2 \leq 0, \quad \text{when } \ Z(t) > 0,\\&[Z'(t)]^2 - [Z'(t_0)]^2 + s[Z(t)]^2 \geq 0, \quad \text{when } \ Z(t) < 0,\\&[Z'(t)]^2 - [Z'(t_0)]^2 + r[Z(t)]^2 \leq 0, \quad \text{when } \ Z(t) < 0.\end{aligned}$$

According to the continuity of $Z(t)$, we may rewrite above inequalities as following equivalent forms:

$$\begin{aligned}&-q[Z(t)]^2 \leq [Z'(t)]^2 - [Z'(t_0)]^2 \leq -p[Z(t)]^2, \quad Z(t) \geq 0,\\&-s[Z(t)]^2 \leq [Z'(t)]^2 - [Z'(t_0)]^2 \leq -r[Z(t)]^2, \quad Z(t) \leq 0.\end{aligned}\tag{2.15}$$

It follows that $Z'(t_0) \neq 0$. If not, it is easily verified from (2.15) that $Z(t) = 0, Z'(t) = 0, t \in [0, 2\pi]$, which contradicts with $\|Z\|_{C^1} = 1$. Next, we shall prove that $Z'(t)$ has only limited zero points on $[0, 2\pi]$.

We suppose, on the contrary, that $Z'(t)$ has unlimited zero points $\{\xi_i\}_1^\infty \subset [0, 2\pi]$ and $\lim_{i\to\infty} \xi_i = \xi_0$. Letting $t = \xi_i$ in (2.15) and taking limit as $i \to \infty$, we immediately get that $Z(\xi_0) \neq 0$ and might let as well $Z(\xi_0) > 0$. Because of the continuity of $Z(t)$ there exist $k, \delta > 0$, such that for all $t \in [\xi_0 - \delta,\ \xi_0 + \delta]$, $Z(t) > k > 0$. Furthermore, for sufficient large m,

$$Z_m(t) \geq k, \quad t \in [t_0 - \delta, t_0 + \delta]$$

this shows $Z_m^-(t) = 0, t \in [t_0 - \delta, t_0 + \delta]$. Since ξ_0 is the limiting point of $\{\xi_i\}$, we may take two zero points $\xi_*, \xi^*(\xi_* < \xi^*)$ of $Z'(t)$ in $[\xi_0 - \delta, \xi_0 + \delta]$ and integrate (2.8) from ξ_* to ξ^*, so that

$$Z'_m(\xi^*) - Z'_m(\xi_*) + \lambda_m \frac{1}{\|x_m\|_{C^1}} \int_{\xi_*}^{\xi^*} g(x_m(s))ds + (1 - \lambda_m) \int_{\xi_*}^{\xi^*} aZ_m(s)ds$$
$$= \lambda_m \frac{1}{\|x_m\|_{C^1}} \int_{\xi_*}^{\xi^*} e(t)dt. \tag{2.16}$$

Notice that when m is sufficiently large and $t \in [\xi_*, \xi^*]$

$$x_m(t) = Z_m(t)\|x_m\|_{C^1} \geq \|x_m\|_{C^1} \cdot k > d_0.$$

Therefore

$$\frac{g(x_m(t))}{\|x_m\|_{C^1}} = \frac{g(x_m(t))}{x_m(t)} \frac{x_m(t)}{\|x_m\|_{C^1}} = \frac{g(x_m(t))}{x_m(t)} \cdot Z_m(t) \geq c_0 \cdot k \qquad t \in [\xi_*, \xi^*].$$

Combining (2.16) and the above inequalities we get that

$$\begin{aligned} &Z'_m(\xi^*) - Z'_m(\xi_*) + \lambda_m c_0 k(\xi^* - \xi_*) + (1 - \lambda_m)ak(\xi^* - \xi_*) \\ &\leq \lambda_m \tfrac{1}{\|x_m\|_{C^1}} \textstyle\int_{\xi_*}^{\xi^*} e(s)ds, \end{aligned}$$

which, letting $m \to \infty$, gives the contradictory inequality

$$[\lambda_0 c_0 k + (1 - \lambda_0)ak](\xi^* - \xi_*) \leq 0.$$

Hence $Z'(t)$ has limited zero points. For proving (2.7) is impossible now, we deduce a contrary conclusion to (2.10) that the inequality (2.15) has only a trivial 2π-periodic solution.

We suppose on the contrary that (2.15) has a non-trivial 2π-periodic solution $\bar{Z}(t)$ and let for convenience that $t_0 = 0$ and $\bar{Z}'(0) > 0$. If $Z_1(t)$ and $Z_2(t)$ are solutions of the following equations, respectively

$$[Z'(t)]^2 - [Z'(0)]^2 = -q[Z(t)]^2, [Z'(t)]^2 - [Z'(0)]^2 = -p[Z(t)]^2, Z(t) \geq 0, \tag{2.17}$$

with $\bar{Z}(0) = Z_1(0) = Z_2(0)$ and $Z'_1(0) \leq \bar{Z}'(0) \leq Z'_2(0)$, then we have

$$Z_1(t) \leq \bar{Z}(t) \leq Z_2(t), t \in [0, t_1],$$

here t_1 is the first zero point of $\bar{Z}(t)$ in $(0, 2\pi]$. Similarly, if $Z_1(t)$ and $Z_2(t)$ are solutions of the following equations, respectively

$$[Z'(t)]^2 - [Z'(0)]^2 = -s[Z(t)]^2, [Z'(t)]^2 - [Z'(0)]^2 = -r[Z(t)]^2, Z < 0 \tag{2.18}$$

with $\bar{Z}(t_1) = Z_1(t_1) = Z_2(t_1)$ and $Z_1'(t_1) \le \bar{Z}'(t_1) \le Z_2'(t_1)$, then we get

$$Z_1(t) \le \bar{Z}(t) \le Z_2(t), t \in [t_1, t_2],$$

here t_2 is the first zero point of $\bar{Z}(t)$ in $(t_1, 2\pi]$. As $Z'(t)(Z(t)$ is the solution of (2.17) or (2.18)) has limited zero points in $[0, 2\pi]$, the Eqs. (2.17) and (2.18) are thus equivalent to the following equations

$$\begin{array}{ll} Z'' = -qZ, & Z'' = -p\ Z, \text{for} Z \ge 0, \\ Z'' = -sZ, & Z'' = -r\ Z, \text{for} Z < 0 \end{array}$$

respectively. Thus there are positive constants A, B, C, D, such that

$$A \sin \sqrt{q} t \le \bar{Z}(t) \le B \sin \sqrt{p} t, \quad 0 < t \le t_1,$$
$$-C \sin \sqrt{s}(t - t_1) \le \bar{Z}(t) \le -D \sin \sqrt{r}(t - t_1), \quad t_1 < t \le t_2,$$

where t_1 and t_2 are the same as above. It follows that

$$\frac{\pi}{\sqrt{q}} \le t_1 \le \frac{\pi}{\sqrt{p}}, \frac{\pi}{\sqrt{s}} + \frac{\pi}{\sqrt{q}} \le t_2 \le \frac{\pi}{\sqrt{r}} + \frac{\pi}{\sqrt{p}}. \tag{2.19}$$

From periodicity of $\bar{Z}(t)$ and above discussions, we conclude that there exists a positive integer m, such that

$$\frac{2m\pi}{n+1} < \frac{m\pi}{\sqrt{s}} + \frac{m\pi}{\sqrt{q}} \le t_{2m} = 2\pi \le \frac{m\pi}{\sqrt{r}} + \frac{m\pi}{\sqrt{p}} < \frac{2m\pi}{n}. \tag{2.20}$$

Clearly this is impossible. Therefore the assumption (2.7) is false. That is, there exists a constant $M > 0$ independent of λ such that

$$\|x\|_{C^1} \le M. \tag{2.21}$$

Let

$$\begin{array}{l} Lx = x'',\ \ D(L) = \{x(t) \in C^2[0, 2\pi], x(0) = x(2\pi), x'(0) = x'(2\pi)\}, \\ Nx = g(x) - e(t),\ \ A(x) = ax^+ - bx^-, \\ \bar{\Omega} = \{x(t) \in C^1[0, 2\pi] :\ \ \|x\|_{C^1} \le M + 1\}. \end{array}$$

It is easy to see that $N(\cdot)$ is L-compact, $A(\cdot)$ completely continuous in $\bar{\Omega}$ and

$$Lx + (1 - \lambda)Ax + \lambda Nx \ne 0,$$

for all $(x, \lambda) \in (D(L) \cap \partial\Omega) \times (0, 1)$. From Lemma 2.1 we know that

$$Ker(L + A) = \{0\}.$$

According to Lemma 2.2, the operator equation

$$Lx + Nx = 0$$

has at least one solution in $D(L) \cap \bar{\Omega}$, that is, the equation (1.1) has at least one 2π-periodic solution.

Corollary 1. Under the conditions I_1), I_2), if $p = q \triangleq q_+$, $\quad r = s \triangleq q_-$, that is

$$\lim_{x \to +\infty} \frac{2G(x)}{x^2} = q_+,$$
$$\lim_{x \to -\infty} \frac{2G(x)}{x^2} = q_-,$$

with

$$\frac{2}{n+1} < \frac{1}{\sqrt{q_+}} + \frac{1}{\sqrt{q_-}} < \frac{2}{n},$$

where n is a positive integer, then the Eq. (1.1) has a 2π-periodic solution.

Corollary 2. If $p = q = r = s$ in the conditions $I_1), I_2)$,that is

$$\lim_{|x| \to \infty} \frac{2G(x)}{x^2} \neq k^2 \ (k = \pm 1, \pm 2, \cdots)$$

then the Eq. (1.1) has a 2π-periodic solution.

Remark 1. Corollaries 1, 2 are the same as Corollaries 2, 3 in [9].

Remark 2. If adding a condition on $g(x)$ aside from the conditions I_1), I_2), we may conclude the following theorem similar to [9].

Theorem 2. Assume that the conditions in Theorem 1 hold, and for any $a > 0$

$$\lim_{x \to \infty} \frac{g(ax)}{g(x)} = a.$$

Then

$$[J_- + J_+, J^- + J^+] = \left[\frac{\pi}{\sqrt{q}} + \frac{\pi}{\sqrt{s}}, \frac{\pi}{\sqrt{p}} + \frac{\pi}{\sqrt{r}} \right],$$

where J_-, J_+, J^-, J^+ are the same as (1.6),

$$p = \lim_{x \to \infty} \inf \frac{2G(x)}{x^2}, \quad q = \lim_{x \to \infty} \sup \frac{2G(x)}{x^2},$$
$$r = \lim_{x \to -\infty} \inf \frac{2G(x)}{x^2}, \quad s = \lim_{x \to -\infty} \sup \frac{2G(x)}{x^2},$$

Proof. According to the proof of Theorem 1 in [9], we have

$$\frac{\sqrt{2G(c)}}{2c} \tau(c) = \frac{\sqrt{2G(c)}}{c} \int_0^c \frac{du}{\sqrt{G(c) - G(u)}} = \int_0^1 \frac{d\xi}{\sqrt{1 - (G(c\xi)/G(c))}}$$
$$= \int_0^\varepsilon \frac{d\xi}{\sqrt{1 - (G(c\xi)/G(c))}} + \int_\varepsilon^{1-\varepsilon} \frac{d\xi}{\sqrt{1 - (G(c\xi)/G(c))}} + \int_{1-\varepsilon}^1 \frac{d\xi}{\sqrt{1 - (G(c\xi)/G(c))}}$$
$$:= T_1(c) + T_2(c) + T_3(c)$$

where $\varepsilon : 0 < \varepsilon < \frac{1}{4}$, $\tau(c), J_\pm, J^\pm$ as above, and $T_1(c), T_2(c), T_3(c)$:

$$0 \le \liminf_{c\to+\infty} T_1(c) \le \limsup_{c\to+\infty} T_1(c) \le \sqrt{\varepsilon},$$

$$\lim_{c\to+\infty} T_2(c) = \int_\varepsilon^{1-\varepsilon} \frac{d\xi}{\sqrt{1-\xi^2}},$$

$$0 \le \liminf_{c\to+\infty} T_3(c) \le \limsup_{c\to+\infty} T_3(c) \le \frac{L}{c_0}\sqrt{\varepsilon}.$$

Thus, letting $\varepsilon \to 0^+$, we get that for $c \gg 1$

$$\tau(c) = \left(\frac{1}{2\pi} + \alpha(c)\right) \frac{1}{\sqrt{2G(c)}/(2c)}$$

where $\alpha(c) \to 0$ as $c \to +\infty$. So

$$\begin{aligned} J_+ &= \liminf_{c\to+\infty} \tau(c) = \tfrac{1}{2\pi} \liminf_{c\to+\infty} \tfrac{1}{\frac{\sqrt{2G(c)}}{2c}} \\ &= \tfrac{\pi}{2} \tfrac{1}{\limsup\limits_{c\to+\infty} \frac{\sqrt{2G(c)}}{2c}} \\ &= \tfrac{\pi}{\sqrt{q}.} \end{aligned}$$

Similarly

$$J_- = \liminf_{c\to-\infty} \tau(c) = \frac{\pi}{\sqrt{s}}, J^+ = \limsup_{c\to+\infty} \tau(c) = \frac{\pi}{\sqrt{p}}, J^- = \limsup_{c\to-\infty} \tau(c) = \frac{\pi}{\sqrt{r}}.$$

Thus

$$[J_- + J_+, J^- + J^+] = \left[\frac{\pi}{\sqrt{q}} + \frac{\pi}{\sqrt{s}}, \frac{\pi}{\sqrt{p}} + \frac{\pi}{\sqrt{r}}\right].$$

Remark 3. The conditions of Theorem 1 are general to some extent, this is because we don't need $\lim\limits_{x\to\pm\infty} \frac{2G(x)}{x^2}$ to exist; on the other hand, even if these limits exist, because

$$\begin{aligned} \gamma_- &= \lim_{x\to-\infty} \inf \tfrac{g(x)}{x} \le \lim_{x\to-\infty} \inf \tfrac{2G(x)}{x^2} \\ &\le \lim_{x\to-\infty} \sup \tfrac{2G(x)}{x^2} \\ &\le \lim_{x\to-\infty} \sup \tfrac{g(x)}{x} = \Gamma_-, \\ \gamma_+ &= \lim_{x\to+\infty} \inf \tfrac{g(x)}{x} \le \lim_{x\to+\infty} \inf \tfrac{2G(x)}{x^2} \\ &\le \lim_{x\to+\infty} \sup \tfrac{2G(x)}{x^2} \\ &\le \lim_{x\to+\infty} \sup \tfrac{g(x)}{x} = \Gamma_+, \end{aligned} \tag{2.22}$$

$$\left[\frac{1}{\sqrt{q}} + \frac{1}{\sqrt{s}}, \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{p}}\right] \subset \left[\frac{1}{\sqrt{q_+}} + \frac{1}{\sqrt{q_-}}, \frac{1}{\sqrt{\gamma_+}} + \frac{1}{\sqrt{\gamma_-}}\right],$$

where $\gamma_\pm, q_\pm$ are the same as (1.8). Theorem 1 also improves some relative results in [8, 11].

Acknowledgement

The authors are grateful to the referee for his helpful suggestions.

References

[1] A. C. Lazer, On Schauder's fixed point theorem and forced second-order nonlinear oscillations, *J. Math. Anal. Appl.* **21** (1968), 421–425.

[2] J. Mawhin, J. R. Ward, Nonuniform non-resonance conditions at the two first eigenvalues for periodic solutions of forced Lienard and Duffing equations, *Rocky Mountain J. Math.* **12** (1982), 643–654.

[3] C. W. Wang, Multiplicity of periodic solutions for Duffing equations under nonuniform non-resonance conditions, *Proc. Amer.Math. Soc.* **126** (1998), 1725–1732.

[4] H. Jacobowitz, Periodic solutions of $x'' + f(t, x) = 0$ via Poincaré–Birkhoff theorem, *J. Diff. Equ.* **20** (1976), 37–52.

[5] P. Omari and F. Zanolin, A note on nonlinear oscillations at resonance, *Acta. Math. Sinica. (N.S.)* **3** (1087), 351–361.

[6] A. Fonda, On the existence of periodic solutions for scalar second order differential equations when only the asymptotic behavior of the potential is known, *Proc. Amer. Math. Soc.*, **(119)** (1993), 439–445.

[7] E. N. Dancer, Boundary value problems for weakly nonlinear ordinary differential equations, *Bull. Austral. Math. Soc.* **15** (1976), 321–328.

[8] S. Fučik, *Solvability of Nonlinear Equations and Boundary Value Problems*, Reidel, Dordrecht 1980.

[9] T. Ding and F. Zanolin, Time-maps for the solvability of periodically perturbed nonlinear Duffing equations, *Nonlinear Analysis* **7** (1991), 635–654.

[10] H. Wang and Y. Li, Existence and uniqueness of periodic solutions for Duffing equations across many points of resonance, *J. Diff. Equ.* **108** (1994), 152–169.

[11] P. Drabek and S. Invernizzi, On the periodic boundary problem for forced Duffing equations with jumping non-linearity. *Nonlinear Anal.* **10** (1986), 643–650.

[12] J. P. Gossez and P. Omari, Non-resonance with respect to the Fučik spectrum for periodic solutions of second ordinary differential equations. *Nonlinear Analysis* **14** (1990), 1079–1104.

[13] A. Capietto, J. Mawhin and F. Zanolin, A continuation theorem for periodic boundary value problems with oscillatory nonlinearities, *NoDEA* **2** (1995), 133–163.

[14] A. Fonda and P. Habets, Periodic solutions of asymptotically positively homogeneous differential equations, *J. Diff. Equs.* **81** (1989), 68–97.

[15] R. E. Gaines, J. L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, in *Lecture Notes in Mathematics,* **568**, Springer-Verlag 1977.

W. B. Liu
Department of Mathematics
China University of Mining and Technology
Xuzhou 221008
P. R. China
e-mail:liuwenbin-xz@163.com

Y. Li
Department of Mathematics
Jilin University
Chongchun 130012
P. R. China

(Received: November 12, 2003; revised: August 19, 2004)