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Some problems in nonlinear magnetoelasticity

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Abstract. In this paper we examine the influence of magnetic fields on the static response of magnetoelastic materials, such as magneto-sensitive elastomers, that are capable of large deformations. The analysis is based on a simple formulation of the mechanical equilibrium equations and constitutive law for such materials developed recently by the authors, coupled with the governing magnetic field equations. The equations are applied in the solution of some simple representative and illustrative problems, with the focus on incompressible materials. First, we consider the pure homogeneous deformation of a slab of material in the presence of a magnetic field normal to its faces. This is followed by a review of the problem of simple shear of the slab in the presence of the same magnetic field. Next we examine a problem involving non-homogeneous deformations, namely the extension and inflation of a circular cylindrical tube. In this problem the magnetic field is taken to be either axial (a uniform field) or circumferential. For each problem we give a general formulation for the case of an isotropic magnetoelastic constitutive law, and then, for illustration, specific results are derived for a prototype constitutive law. We emphasize that in general there are significant differences in the results for formulations in which the magnetic field or the magnetic induction is taken as the independent magnetic variable. This is demonstrated for one particular problem, in which restrictions are placed on the admissible class of constitutive laws if the magnetic induction is the independent variable but no restrictions if the magnetic field is the independent variable.

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1. Introduction

In some recent papers [1, 2, 3] we have discussed alternative formulations of the governing equilibrium equations for nonlinear magnetoelastic deformations of magneto-sensitive (MS) solids, and have applied the theory in the solution of several illustrative boundary-value problems. In the most recent of these papers [3] we have provided a rather elegant and simple formulation of the equations based on the use of a modified free-energy function with the referential magnetic induction vector as the independent magnetic variable. We also provided an alternative formulation with the magnetic field itself as the independent variable.

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The differences arising from use of one or other of these formulations are quite significant. As indicated in [3], for certain problems involving non-homogeneous deformations, restrictions are placed on the class of constitutive laws admitting particular magnetic (or magnetic induction) fields. These restrictions are generally more severe when the magnetic induction is used as the independent magnetic variable. A case in point is highlighted in the present paper for the problem of extension and inflation of a tube with an azimuthal magnetic field.

The motivation for this work, which is discussed in detail in the papers cited above and not therefore repeated here, lies in the increasing use of these materials, in particular MS-elastomers, as 'smart materials' in electromechanical devices. Other recent related theoretical works are those by Brigadnov and Dorfmann [4], Steigmann [5] and Kankanala and Triantafyllidis [6], while the theoretical foundations for a continuum deforming in the presence of an electromagnetic field have been provided by, for example, Truesdell and Toupin [7]. Formulations of the magnetoelastic equations, based in part on the work by Brown [8], Hutter [9, 10] and Pao [11], are summarized in the monographs by Hutter [12] and Eringen and Maugin [13], for example; see also the more recent book by Kovetz [14].

In the present paper we continue the development in [3] by applying the equations to some simple deformations in order to illustrate the influence of the magnetic field on the mechanical response. In Section 2 we summarize the basic equations in general form following the formulation given in [3], together with the appropriate constitutive law for a compressible or an incompressible material. Equations based on either the magnetic induction vector or the magnetic field vector as the independent magnetic variable are included. Problems involving homogeneous deformations are considered in Section 3. First, pure homogeneous deformation of a slab of material with a magnetic field normal to its faces is discussed and this is followed by an analysis of the problem of simple shear of a slab with a magnetic field initially normal to the direction of shear and in the plane of shear. A variant of the latter problem was considered in [4] for a particular form of constitutive law but here we revisit this problem in the general setting of the constitutive framework discussed in Section 2.

In Section 4 we consider the problem of inflation and extension of a circular cylindrical tube under internal pressure and axial load. It is appropriate in this problem to consider a (uniform) axial magnetic field or a circumferential magnetic field. In the latter case, with circular symmetry preserved, the field component is inversely proportional to the radial coordinate so that a singularity arises along the tube axis unless this is excluded from the domain of interest by introducing a cylindrical core concentric with the tube. The field is generated by a steady current along this core. The results in the latter problem highlight the differences in the two distinct (but in principle equivalent) formulations. If the magnetic field vector is taken as the independent magnetic variable then no restrictions are imposed by the field equations on the form of constitutive law for the considered combination of deformation and magnetic field to be admissible. By contrast, when the magnetic induction vector is used as the independent variable the problem admits a solution only for a very restricted class of constitutive laws.

In these problems we restrict attention to *incompressible isotropic* magnetoelastic materials, and each problem is formulated without restriction otherwise on the form of the constitutive law, but with the proviso that the restriction mentioned in the above paragraph arises naturally in one particular problem. In each case the results are illustrated by use of a prototype material model. At present there is very little experimental data available for the considered materials, in particular data that (a) show in detail how the mechanical response in different tests changes with the magnetic field and/or (b) characterize the magnetization of the material at different strain levels. Part of the objective of considering the simple problems discussed here is therefore to provide a theoretical basis for the analysis of experimental data and to encourage the generation of experimental results for comparison with the theory.

2. Basic equations

2.1. Kinematics

We consider a magnetoelastic solid that is subject to a magnetic field and occupies a region $\mathcal{B}_0$ in a three-dimensional Euclidean space. Let the magnetic field be denoted $\mathbf{H}_0$, the associated magnetic induction vector $\mathbf{B}_0$ and the magnetization vector $\mathbf{M}_0$. Application of the magnetic field will in general have generated an initial deformation of the body from its natural (stress-free) configuration in which it is subject to neither magnetic fields nor *mechanical* body forces or surface tractions. This initial deformation, achieved purely by the application of the magnetic field, is the effect referred to as *magnetostriction*, and in general it will be associated with a (residual) stress distribution in $\mathcal{B}_0$ dependent on $\mathbf{B}_0$. For any given magnetic field we take the geometrical configuration $\mathcal{B}_0$ to be the reference configuration from which subsequent deformation generated by the application of mechanical loads is measured. The geometry of $\mathcal{B}_0$ can be maintained, if required, when $\mathbf{B}_0$ is changed if appropriate mechanical loads are applied.

Let a typical point of the solid be labelled by its position vector $\mathbf{X}$ in the reference configuration $\mathcal{B}_0$ relative to an arbitrarily chosen origin. When the body is deformed the point $\mathbf{X}$ has a new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the resulting deformed configuration, which we denote by $\mathcal{B}$. Time dependence is not considered here. The vector field $\boldsymbol{\chi}$ describes the deformation of the body and is defined for $\mathbf{X} \in \mathcal{B}_0 \cup \partial\mathcal{B}_0$, where the boundary of $\mathcal{B}_0$ is denoted by $\partial\mathcal{B}_0$. The deformation gradient tensor $\mathbf{F}$ relative to $\mathcal{B}_0$, and its determinant, are

$$\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}, \qquad J = \det\mathbf{F} > 0, \tag{1}$$

respectively, where Grad denotes the gradient operator with respect to $\mathbf{X}$ and wherein the notation J is defined.

2.2. Magnetic balance equations

In the absence of deformation $\mathbf{F} = \mathbf{I}$ and the relevant magnetic vector fields are $\mathbf{B}_0$, $\mathbf{H}_0$ and $\mathbf{M}_0$, which are related by the standard formula

$$\mathbf{B}_0 = \mu_0(\mathbf{H}_0 + \mathbf{M}_0), \tag{2}$$

where the constant μ_0 is the magnetic permeability *in vacuo*. In the absence of material we note that $\mathbf{M}_0 = \mathbf{0}$ and (2) reduces accordingly to $\mathbf{B}_0 = \mu_0\mathbf{H}_0$. In the reference configuration, the vectors $\mathbf{B}_0$ and $\mathbf{H}_0$ satisfy the forms of Maxwell's equations specialized to the considered static situation as

$$\mathrm{Div}\,\mathbf{B}_0 = 0, \quad \mathrm{Curl}\,\mathbf{H}_0 = \mathbf{0}, \tag{3}$$

where, respectively, Div and Curl are the divergence and curl operators with respect to $\mathbf{X}$. A second connection between the vector fields $\mathbf{B}_0$, $\mathbf{H}_0$ and $\mathbf{M}_0$ will be provided by an appropriate form of constitutive law, which will be discussed in a general setting in Section 2.5.

When the material is deformed with deformation gradient $\mathbf{F}$ the initial vector fields $\mathbf{B}_0$, $\mathbf{H}_0$ and $\mathbf{M}_0$ (which are functions of $\mathbf{X}$) change, and we denote the corresponding fields in the deformed configuration $\mathcal{B}$ by $\mathbf{B}_f$ (the magnetic induction), $\mathbf{H}_f$ (the magnetic field) and $\mathbf{M}_f$ (the magnetization), respectively. These are Eulerian vector fields and may be regarded as 'push forward' versions of the initial fields from $\mathcal{B}_0$ to $\mathcal{B}$. This accounts for the presence of the subscript f, signifying 'forward'. We have the connections

$$\mathbf{B}_f = J^{-1}\mathbf{F}\mathbf{B}_0, \quad \mathbf{H}_f = \mathbf{F}^{-T}\mathbf{H}_0, \tag{4}$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T$ and ^{T} denotes the transpose of a second-order tensor. These arise naturally through the standard identities

$$\mathrm{Div}\,\mathbf{B}_0 = J\mathrm{div}\,\mathbf{B}_f, \quad \mathrm{Curl}\,\mathbf{H}_0 = J\mathbf{F}^{-1}\mathrm{curl}\,\mathbf{H}_f \tag{5}$$

(or their integral versions), which require that $\boldsymbol{\chi}$ be suitably well behaved. It follows that the counterparts of (3) hold in the deformed configuration, i.e.

$$\mathrm{div}\,\mathbf{B}_f = 0, \quad \mathrm{curl}\,\mathbf{H}_f = \mathbf{0}, \tag{6}$$

where div and curl are the usual divergence and curl operators with respect to $\mathbf{x}$.

Analogously to $\mathbf{H}_f$ we define $\mathbf{M}_f = \mathbf{F}^{-T}\mathbf{M}_0$ and then, in terms of the Eulerian vectors, equation (2) becomes

$$\mathbf{B}_f = \mu_0 J^{-1}\mathbf{b}(\mathbf{H}_f + \mathbf{M}_f), \tag{7}$$

where $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green deformation tensor. We point out that (7) is somewhat different in form from (2) in that it involves the multiplier $J^{-1}\mathbf{b}$ on the right-hand side, which highlights the fact that the form of equation (2) is not preserved under general deformations.

Now consider the situation in which the body is in the deformed configuration $\mathcal{B}$ and subject to a magnetic field $\mathbf{H}$ with magnetic induction $\mathbf{B}$ and magnetization

$\mathbf{M}$. These fields are related by the counterpart of equation (2), namely

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{8}$$

Equation (8) defines the third vector field when one vector is used as an independent variable and the other is given by a constitutive equation. Note that for a given deformation there is in general no immediate connection between $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{B}_f$, $\mathbf{H}_f$, $\mathbf{M}_f$ since, for example, $\mathbf{B}$ and $\mathbf{B}_f$ (equivalently, $\mathbf{B}_0$) may be chosen independently. However, when there is no deformation (8) reduces to (2) and $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$ *may* then be equated to the initial fields $\mathbf{B}_0$, $\mathbf{H}_0$, $\mathbf{M}_0$.

In this Eulerian description, the relevant Maxwell field equations to be satisfied by $\mathbf{B}$ and $\mathbf{H}$ are

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{H} = \mathbf{0}, \tag{9}$$

the counterparts of (3).

Just as the initial fields may be pushed forward, the fields $\mathbf{B}$ and $\mathbf{H}$ may be pulled back from $\mathcal{B}$ to $\mathcal{B}_0$ to their Lagrangian forms, denoted $\mathbf{B}_l$ and $\mathbf{H}_l$ respectively, by the identifications

$$\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \quad \mathbf{H}_l = \mathbf{F}^T\mathbf{H}, \tag{10}$$

as in [3]. Since $\mathbf{H}$ and $\mathbf{M}$ occur as a sum in (8), it is appropriate to define, similarly to $(10)_2$, a Lagrangian form of $\mathbf{M}$, denoted $\mathbf{M}_l$, by

$$\mathbf{M}_l = \mathbf{F}^T\mathbf{M}. \tag{11}$$

On use of (10) and (11) in (8) we obtain

$$J^{-1}\mathbf{c}\mathbf{B}_l = \mu_0(\mathbf{H}_l + \mathbf{M}_l), \tag{12}$$

where $\mathbf{c}$ is the right Cauchy-Green deformation tensor defined by $\mathbf{c} = \mathbf{F}^T\mathbf{F}$. This is another manifestation of the lack of form invariance of the relation (8). In general, there is no immediate connection between $\mathbf{B}_l, \mathbf{H}_l, \mathbf{M}_l$ and $\mathbf{B}_0, \mathbf{H}_0, \mathbf{M}_0$.

Suppose, however, that a magnetic field is applied prior to deformation and we take $\mathbf{B}_0$ as the independent magnetic variable. Then, $\mathbf{H}_0$ and $\mathbf{M}_0$ are determined in terms of $\mathbf{B}_0$ by a constitutive law and (2) together. After deformation $\mathbf{B}_0$ becomes $\mathbf{B}_f$, as defined by $(4)_1$. If we *define* $\mathbf{B}$ to be $\mathbf{B}_f$, this amounts to applying the field $\mathbf{B}$ after deformation (without changing the deformation), and then $\mathbf{B}_l = \mathbf{B}_0$. The corresponding values of $\mathbf{H}$ and $\mathbf{M}$ are obtained by means of a constitutive law together with (8). In general they are not the same as $\mathbf{H}_f$ and $\mathbf{M}_f$. We shall elaborate on this point in connection with the constitutive laws discussed in Section 2.5.

2.3. Mechanical balance equations

Let ρ_0 and ρ denote the mass densities of the material in the reference and deformed configurations, $\mathcal{B}_0$ and $\mathcal{B}$ respectively. Then, for our purposes, the conservation of mass equation is written conveniently in the form

$$J\rho = \rho_0, \tag{13}$$

where, we recall, $J = \det \mathbf{F}$.

In the absence of *mechanical* body forces, the equilibrium equation for a magnetoelastic solid may be written in the Eulerian form

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \tag{14}$$

where $\boldsymbol{\tau}$ is the *total* stress tensor, which, according to the balance of angular momentum, is *symmetric* (for detailed discussion of different stress tensors in the present context see Steigmann [5]). We note, in particular, that the influence of the magnetic field on the deforming continuum is thereby incorporated through the stress tensor and not through body force terms (which, in any case, can be written as the divergence of a second-order tensor).

With $\boldsymbol{\tau}$ we may associate a total 'nominal' stress tensor, analogous to that used in elasticity theory [15], here denoted $\mathbf{T}$ and given by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{15}$$

Then, the equilibrium equation (14) is expressed simply in Lagrangian form as

$$\operatorname{Div} \mathbf{T} = \mathbf{0}. \tag{16}$$

2.4. Boundary conditions

In order to formulate boundary-value problems we need material constitutive laws in which $\boldsymbol{\tau}$ and $\mathbf{H}$ (or $\mathbf{M}$) are given in terms of the basic variables $\mathbf{F}$ and $\mathbf{B}$. These will be discussed in the following section. In addition, appropriate boundary conditions must be satisfied by the fields $\mathbf{B}, \mathbf{H}, \boldsymbol{\tau}$ and $\boldsymbol{\chi}$. These are summarized here.

Across a surface of discontinuity within the material or across a surface bounding the considered material in the deformed configuration the vector fields $\mathbf{B}$ and $\mathbf{H}$ satisfy the standard jump conditions

$$\mathbf{n} \cdot [\mathbf{B}] = 0, \quad \mathbf{n} \times [\mathbf{H}] = \mathbf{0}, \tag{17}$$

where the square brackets indicate a discontinuity across the surface and $\mathbf{n}$ is the unit normal to the surface, which, at the material boundary, is taken as the outward pointing normal. The Lagrangian counterparts of the equations in (17) are

$$\mathbf{N} \cdot [\mathbf{B}_l] = 0, \quad \mathbf{N} \times [\mathbf{H}_l] = \mathbf{0}, \tag{18}$$

where $\mathbf{N}$ is the unit normal to the surface in the reference configuration that deforms into the surface with unit normal $\mathbf{n}$. Provided $\mathbf{F}$ is continuous equations (17) and (18) are equivalent. For the reference fields $\mathbf{B}_0$ and $\mathbf{H}_0$ equations (18) also apply with the subscript l replaced by 0, and the push forward fields $\mathbf{B}_f$ and $\mathbf{H}_f$ satisfy (17) with the subscripts f attached.

Across any surface within the material the traction calculated from the total stress tensor must be continuous. This continuity condition is, in its Eulerian and Lagrangian forms,

$$[\boldsymbol{\tau}]\mathbf{n} = \mathbf{0}, \quad [\mathbf{T}]\mathbf{N} = \mathbf{0}. \tag{19}$$

At an external boundary of the considered body the latter, for example, may be replaced by specification of the traction in the form

$$\mathbf{T}\mathbf{N} = \mathbf{t}, \tag{20}$$

where $\mathbf{t}$ in general includes both magnetic and mechanical contributions. The mechanical part is a prescribed function of position that may depend on the deformation, as is the case for a pressure loading for example. The magnetic part is determined by the expression for the Maxwell stress exterior of the body (if the exterior is a vacuum, for example), which itself is determined by continuity of the relevant magnetic field components across the boundary. Another possible mechanical boundary condition involves spatial constraint of (part of) the boundary of the body, in which case the relevant boundary condition requires specification of $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ as a function of $\mathbf{X}$ on that part of the boundary.

2.5. Constitutive equations

For isothermal deformations the free energy may be treated as a function of $\mathbf{F}$ and one of the magnetic vector field variables. In [1, 2] the variables $\mathbf{F}$ and $\mathbf{B}$ were taken as the independent variables (as in, for example, Kovetz [14]), and the free energy was denoted $\Psi = \Psi(\mathbf{F}, \mathbf{B})$. The connection $(10)_1$ between $\mathbf{B}$ and $\mathbf{B}_l$ allows $\Psi(\mathbf{F}, \mathbf{B})$ to be regarded equally as a function of $\mathbf{F}$ and $\mathbf{B}_l$, and then, following [2, 3], we write

$$\Phi(\mathbf{F}, \mathbf{B}_l) \equiv \Psi(\mathbf{F}, J^{-1}\mathbf{F}\mathbf{B}_l). \tag{21}$$

Note that since $\mathbf{B}_l$ is a Lagrangian vector it is indifferent to observer transformations in the deformed configuration, while the deformation gradient $\mathbf{F}$ changes to $\mathbf{QF}$, where $\mathbf{Q}$ is the rotation tensor associated with the transformation. For Φ to be frame indifferent (objective) we must have

$$\Phi(\mathbf{QF}, \mathbf{B}_l) = \Phi(\mathbf{F}, \mathbf{B}_l) \tag{22}$$

for all proper orthogonal $\mathbf{Q}$. This requirement enables Φ to be treated as a function of $\mathbf{c} = \mathbf{F}^T\mathbf{F}$ (instead of $\mathbf{F}$), and $\mathbf{B}_l$.

2.5.1. Compressible materials

If there is no internal mechanical constraint then standard thermomechanical arguments based on use of Φ lead to

$$\boldsymbol{\tau} = \rho \mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} + \mu_0^{-1}[\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}], \tag{23}$$

where $\mathbf{I}$ is the identity tensor, and the symmetry requirement on $\boldsymbol{\tau}$ is

$$\mathbf{F} \frac{\partial \Phi}{\partial \mathbf{F}} \quad \text{is symmetric}, \tag{24}$$

which follows automatically from the objectivity condition (22).

From (23) and the definition (15) an expression may be given for the associated (total) nominal stress $\mathbf{T}$, and the corresponding Lagrangian form of the magnetization is given by

$$\mathbf{M}_l = -\rho_0 \frac{\partial \Phi}{\partial \mathbf{B}_l}. \tag{25}$$

In the absence of material $\Phi = 0$, $\mathbf{M}_l = \mathbf{0}$ and equation (23) reduces to

$$\boldsymbol{\tau} = \mu_0^{-1}[\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}], \tag{26}$$

which is the expression for the Maxwell stress *in vacuo*. Thus, in (23) the contributions to the stress from the interaction between the material and the magnetic field and that due to the magnetic field in the absence of material are conveniently separated out. At a boundary between the material and a vacuum it is the traction calculated from the stress (26) that is needed in any traction boundary condition (along with applied mechanical tractions) to balance the traction calculation from (23), but, of course, the vector $\mathbf{B}$ will in general be different on the two sides of the boundary.

An alternative and compact formulation that incorporates the Maxwell stress contribution into a modified free energy, denoted $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$ and defined by

$$\Omega(\mathbf{F}, \mathbf{B}_l) = \rho_0 \Phi + \frac{1}{2} \mu_0^{-1} J^{-1} \mathbf{B}_l \cdot (\mathbf{c} \mathbf{B}_l), \tag{27}$$

was introduced in [3]. This allows $\mathbf{T}$ and $\boldsymbol{\tau}$ to be written in the simple forms

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \boldsymbol{\tau} = J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}}, \tag{28}$$

which can then be used in the equilibrium equations (16) and (14).

We also have, on use of (12),

$$\mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}, \tag{29}$$

and, for a given $\mathbf{B}_l$, $\mathbf{M}_l$ is then determined from (12).

Noting that in the reference configuration we set $\mathbf{B}_l = \mathbf{B}_0$, it is convenient here to define

$$\Omega_0(\mathbf{B}_0) = \Omega(\mathbf{I}, \mathbf{B}_0), \tag{30}$$

and then we have

$$\mathbf{H}_0 = \frac{\partial \Omega_0}{\partial \mathbf{B}_0}, \tag{31}$$

and $\mathbf{M}_0$ is given by (2).

2.5.2. Incompressible materials

For an incompressible magnetoelastic solid the constraint

$$\det \mathbf{F} \equiv 1 \tag{32}$$

must be satisfied for all deformations. The modified free energy (27) then simplifies to

$$\Omega = \rho_0 \Phi + \frac{1}{2}\mu_0^{-1} \mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l), \tag{33}$$

and the total stress $(28)_2$ is replaced by

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \tag{34}$$

where p is a Lagrange multiplier associated with the constraint (32). The other relevant Eulerian quantities are given by

$$\mathbf{B} = \mathbf{F}\mathbf{B}_l, \quad \mathbf{H} = \mathbf{F}^{-T}\frac{\partial \Omega}{\partial \mathbf{B}_l}, \quad \mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}. \tag{35}$$

The corresponding Lagrangian expressions are

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1} \tag{36}$$

and

$$\mathbf{B}_l, \quad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}, \quad \mathbf{M}_l = \mu_0^{-1}\mathbf{c}\mathbf{B}_l - \mathbf{H}_l. \tag{37}$$

Note that for an incompressible material part of the standard Maxwell stress is incorporated into the 'arbitrary hydrostatic pressure' term but that outside the material (in vacuum) the Maxwell stress is given by (26).

2.6. Isotropic magnetoelastic materials

We now restrict attention to isotropic magnetoelastic materials, with Φ, or equivalently Ω, regarded as a function of $\mathbf{c}$ and $\mathbf{B}_l$. For such materials the material symmetry is similar to that associated with a transversely isotropic elastic material, for which there is a preferred direction in the reference configuration analogous to $\mathbf{B}_l$. Here, however, $\mathbf{B}_l$ is not a unit vector so the theory involves one more invariant than is the case for transverse isotropy.

The magnetoelastic material considered here is said to be *isotropic* if Φ (or Ω) is an isotropic function of the two tensors $\mathbf{c}$ and $\mathbf{B}_l \otimes \mathbf{B}_l$. In this case the dependence of Φ (or Ω) is reduced to dependence on the six invariants

$$I_1(\mathbf{c}) = \operatorname{tr}\mathbf{c}, \quad I_2(\mathbf{c}) = \frac{1}{2}\left[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}(\mathbf{c}^2)\right], \quad I_3(\mathbf{c}) = \det\mathbf{c} = J^2, \tag{38}$$

$$I_4 = |\mathbf{B}_l|^2, \quad I_5 = (\mathbf{c}\mathbf{B}_l)\cdot\mathbf{B}_l, \quad I_6 = (\mathbf{c}^2\mathbf{B}_l)\cdot\mathbf{B}_l, \tag{39}$$

where tr is the trace of a second-order tensor. Note that the invariants (39) are unaffected by a reversal in the sign of $\mathbf{B}_l$. A slightly more general theory could accommodate direction dependence of $\mathbf{B}_l$, but here we restrict attention to the simpler case.

Henceforth we concentrate on incompressible materials, so that $I_3 \equiv 1$, and we write $\Omega = \Omega(I_1, I_2, I_4, I_5, I_6)$. In what follows the subscripts $1, 2, 4, 5, 6$ on Φ or Ω signify differentiation with respect to I_1, I_2, I_4, I_5, I_6, respectively. Then, $\boldsymbol{\tau}$, for example, is given explicitly in terms of Ω as

$$\begin{aligned}\boldsymbol{\tau} = {} & 2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} \\ & + 2\Omega_5\mathbf{B}\otimes\mathbf{B} + 2\Omega_6(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B}),\end{aligned} \tag{40}$$

and

$$\mathbf{H} = 2(\Omega_4\mathbf{b}^{-1}\mathbf{B} + \Omega_5\mathbf{B} + \Omega_6\mathbf{b}\mathbf{B}) = \mu_0^{-1}\mathbf{B} - \mathbf{M}. \tag{41}$$

Note that (33) may be written

$$\Omega = \rho_0\Phi + \frac{1}{2}\mu_0^{-1}I_5, \tag{42}$$

the final term of which is associated with the Maxwell stress, as discussed in Section 2.5.1.

2.7. The undeformed configuration

An important consideration for all subsequent analysis is the status of any (residual) magnetic field in the undeformed configuration $\mathcal{B}_0$, in the absence of applied mechanical loading (or in the presence of loading applied to maintain the configuration $\mathcal{B}_0$ when $\mathbf{B}_0$ is changed). Suppose the material is subject to a magnetic (induction) field $\mathbf{B}_0$ in this configuration. Here this is taken to be associated with an applied external field rather than with a field 'frozen-in' during the curing process of the host material. Then, $\mathbf{F} = \mathbf{I}$ and hence

$$I_1 = I_2 = 3, \quad I_4 = I_5 = I_6 = \mathbf{B}_0\cdot\mathbf{B}_0. \tag{43}$$

Let $\boldsymbol{\tau}_0$ and $\mathbf{H}_0$ be the values of $\boldsymbol{\tau}$ and $\mathbf{H}$ in this configuration, calculated from the appropriate specializations of equations (34) and (35) with (33) since the material is incompressible. Then, within the material, we have

$$\boldsymbol{\tau}_0 = [-p + 2(\Omega_1 + 2\Omega_2)]\mathbf{I} + 2(\Omega_5 + 2\Omega_6)\mathbf{B}_0\otimes\mathbf{B}_0, \tag{44}$$

with $\Omega_1, \Omega_2, \Omega_5, \Omega_6$ evaluated for the invariants given by (43).

It is convenient here to adapt the notation Ω_0 defined in (30) to the present situation by defining

$$\Omega_0(I_4) = \Omega(3, 3, I_4, I_4, I_4). \tag{45}$$

Then, we have simply

$$\mathbf{H}_0 = 2\Omega_0'\mathbf{B}_0, \quad \mathbf{M}_0 = \mu_0^{-1}\mathbf{B}_0 - \mathbf{H}_0, \tag{46}$$

where the prime signifies differentiation with respect to I_4. This expression for $\mathbf{H}_0$ needs to be accounted for when satisfying equations (3). In particular, equation $(3)_2$ in general places a restriction on $\mathbf{B}_0$ for any given form of Ω. The fact that $\mathbf{H}_0$ is parallel to $\mathbf{B}_0$ is a consequence of isotropy.

Outside the material (in vacuum) the (Maxwell) stress is

$$\boldsymbol{\tau}_0 = \mu_0^{-1}[\mathbf{B}_0 \otimes \mathbf{B}_0 - \frac{1}{2}(\mathbf{B}_0 \cdot \mathbf{B}_0)\mathbf{I}], \tag{47}$$

subject to the continuity conditions (18) and (19).

The stress $\boldsymbol{\tau}_0$ may be regarded as a residual stress in the material in configuration $\mathcal{B}_0$ induced by the magnetic (induction) field $\mathbf{B}_0$. The equilibrium equation (14) reduces to

$$\mathrm{Div}\,\boldsymbol{\tau}_0 = \mathbf{0}, \tag{48}$$

with $\boldsymbol{\tau}_0$ given by (44).

If there is both a 'frozen-in' field and an externally applied field then there would in general be two 'preferred directions' in $\mathcal{B}_0$, the material would no longer be isotropic in the sense described above and the list of invariants would need to be extended to account for the second field, thereby considerably complicating the constitutive description of the material. This possibility is not considered here.

In the following sections we apply the equations summarized in Section 2 for incompressible materials to some representative boundary-value problems. In each case we begin with a magnetic induction field $\mathbf{B}_0$ given in the reference configuration and we set its push forward $\mathbf{F}\mathbf{B}_0$ under the deformation to be the field $\mathbf{B}$. In other words, we choose the independent Lagrangian variable $\mathbf{B}_l$ to be the initial field $\mathbf{B}_0$. The associated fields $\mathbf{H}$ and $\mathbf{M}$ are then calculated as described above.

3. Homogeneous deformations

3.1. Pure homogeneous deformation

Here we consider a slab of material of uniform finite thickness, with (plane) faces normal to the X_2 direction and of infinite extent in the X_1 and X_3 directions, where (X_1, X_2, X_3) are Cartesian coordinates in the reference configuration $\mathcal{B}_0$. The material is subject to an initial magnetic induction field $\mathbf{B}_0$, which is taken to be uniform and in the X_2 direction with component B_{02}. Deformation associated with $\mathbf{B}_0$ is prevented by the application of an appropriate traction on the faces of the slab, so that $\mathcal{B}_0$ is the reference configuration from which the subsequent deformation is measured. Suppose that the slab is now subject to the pure homogeneous deformation

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{49}$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants. The deformation gradient is diagonal with components $(\lambda_1, \lambda_2, \lambda_3)$, and $\mathbf{B}_f$ has just an x_2 component, namely $B_{f2} = \lambda_2 B_{02}$. We set $\mathbf{B} = \mathbf{B}_f$, so that $B_2 = B_{f2}$.

The associated invariants are calculated in terms of two independent principal stretches, say λ_1 and λ_2, as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2, \tag{50}$$
$$I_4 = B_{02}^2, \quad I_5 = \lambda_2^2 I_4, \quad I_6 = \lambda_2^4 I_4, \tag{51}$$

which involve three independent quantities in all, namely λ_1 and λ_2 and I_4. We may therefore regard the (modified) free energy as a function of $\lambda_1, \lambda_2, I_4$, and we write

$$\omega(\lambda_1, \lambda_2, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6). \tag{52}$$

Then, on specializing (40) and (41), we obtain the component expressions

$$\tau_{11} - \tau_{33} = \lambda_1 \omega_1, \quad \tau_{22} - \tau_{33} = \lambda_2 \omega_2, \tag{53}$$

and

$$H_2 = 2\lambda_2^{-2}\omega_4 B_2, \quad M_2 = \mu_0^{-1} B_2 - H_2, \tag{54}$$

for the stress differences, magnetic field and magnetization within the slab, where the subscripts $1, 2, 4$ on ω indicate differentiation with respect to $\lambda_1, \lambda_2, I_4$ respectively. Note that the push forward of $\mathbf{H}_0$ has component $H_{f2} = \lambda_2^{-1} H_{02}$ and that $H_{02} = 2\omega_0' B_{02}$, where $\omega_0(I_4) \equiv \omega(1, 1, I_4)$ and the prime again denotes differentiation with respect to I_4. Clearly, H_2 and H_{f2} match only if $\omega_4 = \omega_0'$, i.e. only if ω_4 is independent of the deformation (but this is not a requirement). Since the considered deformation is homogeneous and the magnetic field uniform all the governing differential equations are satisfied trivially.

Outside the slab, assumed to be a vacuum, the magnetic induction is, by continuity, the same as that within the slab, while, from (26) and $(51)_2$, the Maxwell stress has components

$$\tau_{11} = \tau_{33} = -\frac{1}{2}\mu_0^{-1} I_5 = -\tau_{22}. \tag{55}$$

If no mechanical traction is supplied to the plane faces of the slab then the normal stress τ_{22} is continuous, and hence (53) and (55) yield

$$\tau_{33} = \frac{1}{2}\mu_0^{-1}\lambda_2^2 I_4 - \lambda_2\omega_2, \quad \tau_{11} = \tau_{33} + \lambda_1\omega_1, \tag{56}$$

which provide expressions for the stresses τ_{11} and τ_{33} required to produce the deformation in the presence of the magnetic field. More particularly, if we specialize to the case of equibiaxial deformation, with $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_1^{-2}$, we obtain

$$\omega_1(\lambda_1, \lambda_1^{-2}, I_4) = 0, \quad \tau_{11} = \frac{1}{2}\mu_0^{-1}\lambda_1^{-4}I_4 - \lambda_1^{-2}\omega_2(\lambda_1, \lambda_1^{-2}, I_4), \tag{57}$$

where the latter provides an expression for the stress τ_{11} as a function of λ_1 and I_4. In the reference configuration

$$\tau_{11} = \frac{1}{2}\mu_0^{-1}I_4 - \omega_2(1, 1, I_4), \tag{58}$$

which is the initial lateral stress needed to prevent deformation due to the magnetic field.

3.1.1. Illustration

For purposes of illustration we now choose a specific form of ω, namely

$$\omega(\lambda_1, \lambda_2, I_4) = \frac{1}{2}\mu(I_4)(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) + \nu(I_4) + \frac{1}{4}\mu_0^{-1}I_5, \tag{59}$$

where μ and ν are functions of I_4. Note that for simplicity the factor 1/4 in the final term has been chosen so that τ_{11} in (58) vanishes, but this can be modified as necessary. Equally, I_4 may be replaced by I_5 in the argument of the function ν to provide a slightly different model. In the absence of a magnetic field $I_4 = 0$, and then $\mu(0)$ is just the shear modulus of the material in the reference configuration and (59) is the classical neo-Hookean strain-energy function provided, for compatibility, we set $\nu(0) = 0$. The term in ν, combined with the final term, accounts for the magnetic energy in the material in the reference configuration.

This yields

$$\lambda_1\omega_1 = \mu(I_4)(\lambda_1^2 - \lambda_3^2), \quad \lambda_2\omega_2 = \mu(I_4)(\lambda_2^2 - \lambda_3^2) + \frac{1}{2}\mu_0^{-1}I_5, \tag{60}$$

and

$$\omega_4 = \frac{1}{2}\mu'(I_4)(I_1 - 3) + \nu'(I_4) + \frac{1}{4}\mu_0^{-1}\lambda_2^2. \tag{61}$$

For the considered equibiaxial stress we then have, from $(57)_2$,

$$\tau_{11} = \mu(I_4)(\lambda_1^2 - \lambda_1^{-4}). \tag{62}$$

The corresponding expression for the magnetization is

$$M_2 = \frac{1}{2}[\mu_0^{-1}\lambda_2^2 - 2\mu'(I_4)(I_1 - 3) - 4\nu'(I_4)]\lambda_2^{-1}B_{02}. \tag{63}$$

For definiteness we choose $\mu(I_4) = \mu_1 + \epsilon\mu_0^{-1}I_4$, where $\mu_1 = \mu(0)$ and ϵ is a dimensionless material parameter. Then (62) reduces (in dimensionless form) to

$$\tau_{11}^* \equiv \tau_{11}/\mu_1 = (1 + m\epsilon)(\lambda_1^2 - \lambda_1^{-4}), \tag{64}$$

where m is another dimensionless parameter, defined by $m = \mu_0^{-1} I_4/\mu_1$ and reflecting the magnitude of the initial magnetic (induction) field. Note that this expression is not affected by the function ν. The corresponding dimensionless magnetization is defined and given by

$$M_2^* \equiv M_2 B_{02}/\mu_1 = \frac{1}{2} m\lambda_1^{-2} - m\epsilon(2\lambda_1^4 + \lambda_1^{-2} - 3\lambda_1^2) - 2\mu_0 m\nu'(I_4)\lambda_1^2. \qquad (65)$$

In the reference configuration this reduces to

$$M_{02}^* = \frac{1}{2} m - 2\mu_0 m\nu'(I_4). \qquad (66)$$

Thus, we can interpret the function $\nu'(I_4)$ as characterizing the magnetization in the absence of deformation.

In Figure 1 we plot τ_{11}^* against λ_1 for different values of the parameter m with ϵ taken to be 0.2. Clearly, the stiffness of the material increases with the value of the magnetic field strength in the small strain region. We emphasize that this result is for a very simple prototype example of constitutive law but it does nevertheless reflect the limited available data for MS-elastomers (for references and discussion, see [1, 2]). There is some limited evidence to suggest [16] that the stiffness of the material reaches a maximum as the magnetic field strength increases, but, in using a linear form for $\mu(I_4)$, we have not attempted to model this possible effect although it could easily be accommodated if more comprehensive data indicate that this is indeed an important characteristic of the considered materials.

Although data for magnetization of MS-elastomers under deformation do not appear to be available it is of interest to examine the predictions of the magnetization in respect of the same simple model. Therefore, in Figure 2, we show the dimensionless magnetization (65) plotted against λ_1 for a series of values of the dimensionless parameter m (in this case $0.5, 1, 2, 3, 3.5$), again for $\epsilon = 0.2$. We choose a form of $\nu'(I_4)$ that leads to magnetic saturation in the reference configuration. Typically, this means that $|\mathbf{B}_0|$, and hence I_4, tends to a finite limit as $|H_{02}|$ increases. This behaviour is accommodated by taking

$$\mu_0\nu'(I_4) = \frac{\alpha}{2m} \tanh^{-1}\left(\frac{m}{\beta}\right) - \frac{1}{4}, \qquad (67)$$

where α and β are dimensionless material constants. Then, for $\lambda_1 = 1$ we have $H_{02}^* \equiv H_{02}B_{02}/\mu_1 = \alpha \tanh^{-1}(m/\beta)$. The limiting value of m is β. In the absence of the applied magnetic field ($m = 0$) there is, of course, no magnetization. For illustrative purposes, we have taken $\alpha = 2.8, \beta = 4$.

Note that the sign of the magnetization reverses with that of B_{02} but that the dimensionless magnetization defined here does not similarly reverse sign since it depends on B_{02} only through I_4. The (dimensionless) magnetization is positive at $\lambda_1 = 1$ for the smaller values of $m > 0$ but negative for larger values (m greater than approximately 3.3) and in each case it decreases as λ_1 increases. Of course, for different forms of $\nu'(I_4)$ very different behaviour can be predicted, as is also the case for different values of α and β in (67). Note that because of the connection

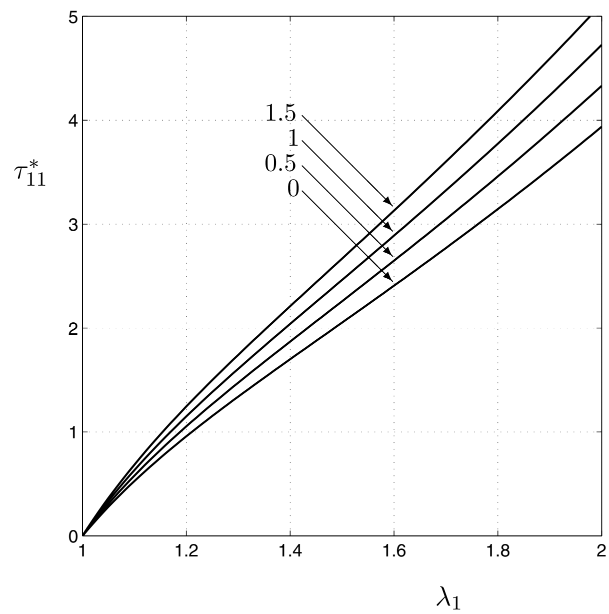

Figure 1. Plot of the dimensionless stress τ^*_{11} from equation (64) against stretch λ_1 for the following values of the dimensionless parameter m: $0, 0.5, 1, 1.5$. The gradient increases with m.

$B_2 = \lambda_1^{-2} B_{02}$, for a given B_{02}, the magnetic induction B_2 becomes smaller and smaller as λ_1 increases, but for the simple model considered here the corresponding magnetic field behaves as $2\lambda_1^2 \omega_4 B_{02}$, which, in dimensionless form, tends to $2m\epsilon\lambda_1^4$ as $\lambda_1 \to \infty$. The equibiaxial extension of a thin plate in the presence of a magnetic field normal to its plane provides a possible means of testing the above predictions experimentally and informing the construction of more realistic material models.

3.2. Simple shear

Consider the slab of material discussed in Section 3.1 but now, instead of pure homogeneous deformation, subject to a simple shear deformation in the X_1 direction in the (X_1, X_2) plane with amount of shear γ (which is uniform). The matrix of Cartesian components F of the deformation gradient tensor $\mathbf{F}$ is

$$\mathsf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{68}$$

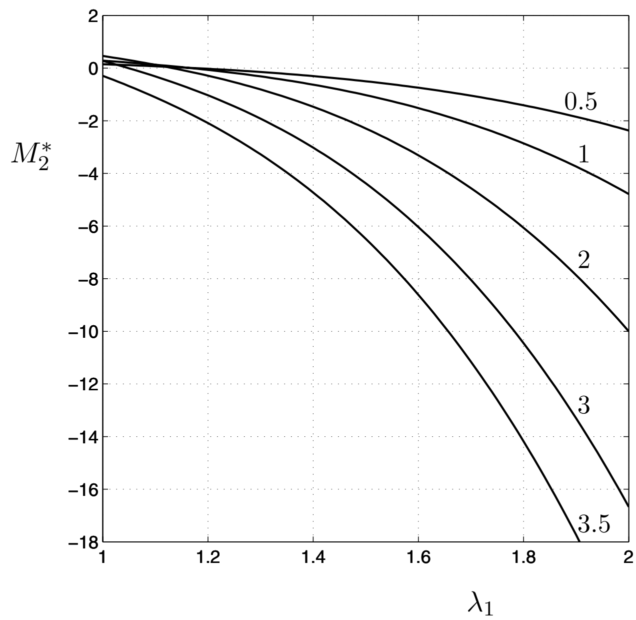

Figure 2. Plot of the dimensionless magnetization M_2^* from equation (65) against stretch λ_1 for the following values of the dimensionless parameter m: $0.5, 1, 2, 3, 3.5$. For $m = 0$ we have $M_2^* \equiv 0$. For the considered model M_2^* is positive at $\lambda_1 = 1$ for the smaller values of m but negative for m larger than about 3.3. As λ_1 increases M_2^* decreases in each case.

The corresponding matrices of the left and right Cauchy-Green deformation tensors $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{c} = \mathbf{F}^T\mathbf{F}$, written b and c, are

$$\mathsf{b} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{69}$$

while the components of $\mathbf{b}^2$ and $\mathbf{b}^{-1}$, required in equations (40) and (41), are given by the matrices

$$\mathsf{b}^2 = \begin{pmatrix} 1+3\gamma^2+\gamma^4 & \gamma(2+\gamma^2) & 0 \\ \gamma(2+\gamma^2) & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{b}^{-1} = \begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{70}$$

The principal invariants I_1, I_2 defined in (38) simplify to

$$I_1 = I_2 = 3 + \gamma^2. \tag{71}$$

For this problem we again take $\mathbf{B}_0$ to be in the X_2 direction with component B_{02}. The corresponding magnetic field $\mathbf{H}_0$ similarly has a single component H_{02}. Then, the components (B_{f1}, B_{f2}, B_{f3}) of $\mathbf{B}_f$ in the deformed configuration follow from the component form of $(35)_1$ with $\mathbf{B}_f = \mathbf{B} = \mathbf{F}\mathbf{B}_l = \mathbf{F}\mathbf{B}_0$ on use of (68). Thus,

$$B_{f1} = \gamma B_{02}, \quad B_{f2} = B_{02}, \quad B_{f3} = 0, \tag{72}$$

which shows that the magnetic induction vector changes direction as the amount of shear changes. From (39) we calculate, for the considered deformation,

$$I_4 = B_{02}^2, \quad I_5 = I_4(1+\gamma^2), \quad I_6 = I_4(1+3\gamma^2+\gamma^4). \tag{73}$$

We now set $\mathbf{B} = \mathbf{B}_f$. The components of $\mathbf{bB}$, which are required in the expressions (40) for $\boldsymbol{\tau}$ and (41) for $\mathbf{H}$, are given by

$$\mathsf{bB} = \left[2\gamma+\gamma^3, 1+\gamma^2, 0\right]^T B_{02}, \tag{74}$$

and $\mathbf{b}^{-1}\mathbf{B}$, which is needed to determine $\mathbf{H}$, has the component form

$$\mathsf{b}^{-1}\mathsf{B} = [0,1,0]^T B_{02}. \tag{75}$$

The resulting components of $\boldsymbol{\tau}$, obtained from equation (40), are

$$\begin{aligned}
\tau_{11} &= -p + 2\Omega_1(1+\gamma^2) + 2\Omega_2(2+\gamma^2) + 2I_4\gamma^2\left[\Omega_5 + 2\Omega_6(2+\gamma^2)\right], \\
\tau_{22} &= -p + 2\Omega_1 + 4\Omega_2 + 2I_4\left[\Omega_5 + 2\Omega_6(1+\gamma^2)\right], \qquad (76)\\
\tau_{33} &= -p + 2\Omega_1 + 2\Omega_2(2+\gamma^2), \\
\tau_{12} &= 2\gamma\left[\Omega_1 + \Omega_2 + I_4[\Omega_5 + \Omega_6(3+2\gamma^2)]\right], \qquad (77)
\end{aligned}$$

and $\tau_{13} = \tau_{23} = 0$. The components of the magnetic field $\mathbf{H}$ and magnetization vector $\mathbf{M}$ are, from (41),

$$H_1 = 2\gamma B_{02}\left[\Omega_5 + \Omega_6(2+\gamma^2)\right], \quad M_1 = \gamma\mu_0^{-1}B_{02} - H_1, \tag{78}$$

$$H_2 = 2B_{02}\left[\Omega_4 + \Omega_5 + \Omega_6(1+\gamma^2)\right], \quad M_2 = \mu_0^{-1}B_{02} - H_2, \tag{79}$$

with $H_3 = M_3 = 0$. Note that for this problem $\mathbf{H}_f = \mathbf{F}^{-T}\mathbf{H}_0 = \mathbf{H}_0$.

In view of (71) and (73) there remain two independent variables, namely γ and I_4. It is again convenient to use a reduced form of the modified energy Ω, as a function of these two variables only. We define the appropriate specialization, denoted ω, by

$$\omega(\gamma, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6), \tag{80}$$

with (71) and (73).

It follows that

$$\begin{aligned}
\omega_\gamma &= 2\gamma\left[\Omega_1 + \Omega_2 + I_4(\Omega_5 + \Omega_6(3+2\gamma^2))\right], \\
\omega_4 &= \Omega_4 + \Omega_5(1+\gamma^2) + \Omega_6(1+3\gamma^2+\gamma^4),
\end{aligned} \tag{81}$$

where the subscripts γ and 4 on ω indicate partial differentiation with respect to γ and I_4.

This allows, in particular, the shear stress to be expressed in a very simple form, namely

$$\tau_{12} = \omega_\gamma, \tag{82}$$

just as in standard nonlinear elasticity theory.

The components of the magnetic field in the deformed configuration can be combined simply in the forms

$$\gamma H_1 + H_2 = 2\omega_4 B_{02}, \quad H_1 - \gamma H_2 = -2\gamma(\Omega_4 - \Omega_6)B_{02}, \tag{83}$$

the first of which reduces to

$$H_{02} = 2\omega_0' B_{02} \tag{84}$$

in the undeformed configuration.

Outside the material the relevant in-plane components of the Maxwell stress are calculated by using the continuity of B_2 and H_1. Inside the material, H_1 is given by $(78)_1$ and outside the material $B_1 = \mu_0 H_1$. The components of the Maxwell stress are then calculated from (26) as

$$\tau_{22} = -\tau_{11} = \frac{1}{2}\mu_0^{-1} I_4 - 2\mu_0\gamma^2 I_4[\Omega_5 + (2+\gamma^2)\Omega_6]^2 \tag{85}$$

and

$$\tau_{12} = 2\gamma I_4[\Omega_5 + (2+\gamma^2)\Omega_6]. \tag{86}$$

The differences between the components of τ_{22} and τ_{12} inside and outside the material give the mechanical traction components required to achieve the considered deformation.

3.2.1. Illustration

For the model used in Section 3.1.1 and for the considered deformation, we have

$$\omega = \frac{1}{2}\mu(I_4)\gamma^2 + \nu(I_4) + \frac{1}{4}\mu_0^{-1} I_4(1+\gamma^2), \tag{87}$$

and hence, from (82),

$$\tau_{12} = [\mu(I_4) + \frac{1}{2}\mu_0^{-1} I_4]\gamma. \tag{88}$$

This is linear in γ with a gradient that is a function of the magnetic field strength through I_4. The linearity is a consequence of the adoption of the neo-Hookean model for the underlying elastic response. The precise details of the function $\mu(I_4)$ remain to be characterized when sufficient data become available. Note that this expression is independent of $\nu(I_4)$, as in the corresponding stresses for the example considered in Section 3.1.1. The components $(78)_2$ and $(79)_2$ of the magnetization are given by

$$M_1 = \frac{1}{2}\mu_0^{-1}\gamma B_{02}, \quad M_2 = \frac{1}{2}\mu_0^{-1} B_{02} - [\mu'(I_4)\gamma^2 + 2\nu'(I_4)]B_{02}, \tag{89}$$

and we note that M_2 does depend on $\nu(I_4)$ and exhibits quadratic dependence on γ.

We observe that while $\mu(I_4)$ characterizes the dependence of the (mechanical) stiffness on the magnetic field the function $\nu(I_4)$, through its derivative, characterizes the magnetization M_{02} in the reference configuration (where $\gamma = 0$), as discussed in Section 3.1.1.

In the simple shear problem examined by Brigadnov and Dorfmann [4] a magnetic field was applied normal to the faces of the slab after deformation. If we do this here then the results are similar to those obtained above with a few differences. If we take B_2 as the only component of the magnetic induction then the invariants defined by (39) are given by $I_4 = (1+\gamma^2)I_5, I_5 = I_6 = B_2^2$. Then, defining ω by $\omega(\gamma, I_5) = \Omega(I_1, I_2, I_4, I_5, I_6)$ with (71), we obtain, instead of $(78)_1$ and $(79)_1$,

$$H_1 = 2\gamma(\Omega_6 - \Omega_4)B_2, \quad H_2 = 2\omega_5 B_2. \tag{90}$$

If, instead of using $\mathbf{B}$ as the independent variable, $\mathbf{H}$ is used, with an appropriate recasting of the constitutive law, as described in [3] and in Section 4.4 of the present paper, the results are very similar again. In particular, if $\mathbf{H}$ has only a component H_2 then $\mathbf{B}$ will have components B_1 and B_2.

4. Extension and inflation of a tube

For any problem involving non-uniform fields the relevant magnetic governing equations to be solved for the initial magnetic vectors are

$$\operatorname{Div}\mathbf{B}_0 = 0, \quad \operatorname{Curl}\mathbf{H}_0 = \mathbf{0}, \tag{91}$$

together with

$$\mathbf{H}_0 = 2\Omega_0' \mathbf{B}_0, \tag{92}$$

in which the factor on the right-hand side depends on $I_4 = \mathbf{B}_0 \cdot \mathbf{B}_0$. Equations (91) guarantee that the corresponding equations for $\mathbf{B}_f = \mathbf{F}\mathbf{B}_0$ and $\mathbf{H}_f = \mathbf{F}^{-T}\mathbf{H}_0$ in the deformed configuration are automatically satisfied. However, if we take $\mathbf{B} = \mathbf{B}_f$ and calculate $\mathbf{H}$ from (41) with $\mathbf{B}_l = \mathbf{B}_0$ then $\operatorname{curl}\mathbf{H} = \mathbf{0}$ does not in general follow and needs to be checked. In general, this equation will impose restrictions on the class of constitutive laws for which the considered combination of deformation and magnetic induction is admissible. This is the case whether or not $\mathbf{B}$ is chosen as above since, for any given $\mathbf{B}$, $\mathbf{H}$ has to be calculated from the constitutive law (41).

The analysis is based on equations (40) and (41) for $\boldsymbol{\tau}$ and $\mathbf{H}$ for an incompressible material, with Ω given by (42) as a function of the invariants I_1, I_2, I_4, I_5, I_6 defined in (38) and (39), together with the equilibrium equation

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0} \tag{93}$$

in the absence of mechanical body forces, coupled with suitable boundary conditions.

4.1. Kinematics

We now apply these equations to the problem of an infinitely long circular cylindrical tube whose reference cross-sectional geometry is defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi \tag{94}$$

with respect to cylindrical polar coordinates (R, Θ, Z). Note that a tube of finite length would present difficulties associated with compatibility of the magnetic boundary conditions on the ends of the tube and on the lateral surfaces and would not admit the simple nature of the fields arising for a tube of infinite length.

The tube is deformed by combined uniform axial extension and radial expansion so that the deformed geometrical cross-section is described by the equations

$$a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi \tag{95}$$

in terms of cylindrical polar coordinates (r, θ, z). Since the material is incompressible the deformation has the form

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{96}$$

where λ_z (constant) is the axial stretch.

The deformation gradient is diagonal with respect to the cylindrical coordinate axes and the associated principal stretches are written

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \frac{r}{R} \equiv \lambda, \quad \lambda_3 = \lambda_z, \tag{97}$$

wherein the notation λ is defined. We regard λ and λ_z as the two independent stretches.

4.2. The magnetic field

In cylindrical polar coordinates (R, Θ, Z) equations (91) have the explicit forms

$$\frac{\partial B_{0R}}{\partial R} + \frac{1}{R}B_{0R} + \frac{1}{R}\frac{\partial B_{0\Theta}}{\partial \Theta} + \frac{\partial B_{0Z}}{\partial Z} = 0, \tag{98}$$

and

$$\frac{1}{R}\frac{\partial H_{0Z}}{\partial \Theta} - \frac{\partial H_{0\Theta}}{\partial Z} = 0, \quad \frac{\partial H_{0R}}{\partial Z} - \frac{\partial H_{0Z}}{\partial R} = 0, \quad \frac{1}{R}\frac{\partial (RH_{0\Theta})}{\partial R} - \frac{1}{R}\frac{\partial H_{0R}}{\partial \Theta} = 0, \tag{99}$$

where $(B_{0R}, B_{0\Theta}, B_{0Z})$ are the components of $\mathbf{B}_0$ and $(H_{0R}, H_{0\Theta}, H_{0Z})$ those of $\mathbf{H}_0$. We consider only magnetic fields that preserve the circular cylindrical symmetry of the problem, so that these components are functions of R only and equations (98) and (99) simplify accordingly.

If the field is purely radial, then we deduce that

$$B_{0R} = \frac{C}{R}, \quad H_{0R} = 2\omega_0' B_{0R}, \tag{100}$$

where C is a constant, $I_4 = B_{0R}^2$ and $\omega_0 = \omega(1, 1, I_4)$ is defined similarly to the corresponding expression in Section 3.1. (For the present deformation ω is defined below.) However, since B_{0R} must be continuous across a circular cylindrical surface its singularity at $R = 0$ cannot be avoided and we conclude that $C = 0$. (We remark that in [1, 2] a radial field was used and the singularity was overlooked.) Thus, it suffices to consider either an axial or an azimuthal field.

If the magnetic field is purely axial then we deduce from (99) that H_{0Z} (and hence B_{0Z}) is constant and we have

$$H_{0Z} = C, \quad H_{0Z} = 2\omega_0' B_{0Z}, \tag{101}$$

where C is again a constant, and

$$I_4 = B_{0Z}^2, \quad I_5 = \lambda_z^2 I_4, \quad I_6 = \lambda_z^4 I_4. \tag{102}$$

The deformed fields are given by

$$H_{fz} = \lambda_z^{-1} H_{0Z}, \quad B_z = B_{fz} = \lambda_z B_{0Z}. \tag{103}$$

If the magnetic field is purely circumferential then the solution of equations (98) and (99) is given by

$$H_{0\Theta} = \frac{C}{R}, \quad H_{0\Theta} = 2\omega_0' B_{0\Theta}, \tag{104}$$

C again being a constant and

$$I_4 = B_{0\Theta}^2, \quad I_5 = \lambda^2 I_4, \quad I_6 = \lambda^4 I_4. \tag{105}$$

The deformed fields are given by

$$H_{f\theta} = \lambda^{-1} H_{0\Theta}, \quad B_\theta = B_{f\theta} = \lambda B_{0\Theta}. \tag{106}$$

In the case of $H_{0\Theta}$ the singularity at $R = 0$ can be excluded by use of a concentric circular cylindrical core across the boundary of which $H_{0\Theta}$ satisfies an appropriate jump condition associated with an axial steady current (such a jump condition requires a generalization of the continuity condition $(18)_2$).

The deformation is locally a pure homogeneous strain and we may take the (modified) free energy to depend on only λ, λ_z, I_4, and, whichever of the axial or azimuthal magnetic fields is used, write

$$\omega(\lambda, \lambda_z, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6). \tag{107}$$

This is similar to the definition (52), except for the notation used for the stretches and the fact that here there is dependence on the radial coordinate.

Whether the magnetic field is axial or azimuthal the stress differences are given by the simple formulas

$$\tau_{\theta\theta} - \tau_{rr} = \lambda \omega_\lambda, \quad \tau_{zz} - \tau_{rr} = \lambda_z \omega_{\lambda_z}. \tag{108}$$

The magnetic field $\mathbf{H}$ and magnetization $\mathbf{M}$ are axial in the first case with components

$$H_z = 2\lambda_z^{-1} \omega_4 B_{0Z}, \quad M_z = \mu_0^{-1} \lambda_z B_{0Z} - H_z. \tag{109}$$

In the second case they are circumferential with

$$H_\theta = 2\lambda^{-1}\omega_4 B_{0\Theta}, \quad M_\theta = \mu_0^{-1}\lambda B_{0\Theta} - H_\theta. \tag{110}$$

In the above the subscripts $\lambda, \lambda_z, 4$ signify partial derivatives with respect to λ, λ_z, I_4, respectively.

In the reference configuration equations (108), (109) and (110) reduce to, respectively,

$$\tau_{0\Theta\Theta} - \tau_{0RR} = \omega_\lambda(1,1,I_4), \quad \tau_{0ZZ} - \tau_{0RR} = \omega_{\lambda_z}(1,1,I_4), \tag{111}$$

$$H_{0Z} = 2\omega_0' B_{0Z}, \quad M_{0Z} = \mu_0^{-1} B_{0Z} - H_{0Z}, \tag{112}$$

$$H_{0\Theta} = 2\omega_0' B_{0\Theta}, \quad M_{0\Theta} = \mu_0^{-1} B_{0\Theta} - H_{0\Theta}. \tag{113}$$

In either the problem with an axial or a circumferential magnetic field the only component of the equilibrium equation not satisfied trivially is the radial equation

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0 \tag{114}$$

in the deformed configuration. The expression for the stress difference in (114) is obtained from $(108)_1$, and this equation then serves to determine τ_{rr} (or equivalently p) subject to appropriate boundary conditions. In the reference configuration the (residual) stresses must satisfy the equation

$$\frac{d\tau_{0RR}}{dR} + \frac{1}{R}(\tau_{0RR} - \tau_{0\Theta\Theta}) = 0, \tag{115}$$

in which $(111)_1$ should be used.

4.3. Solutions

4.3.1. Axial magnetic field

In this case B_{0Z} is constant and connected to (constant) H_{0Z} through $(101)_2$. The corresponding value of H_{fz} is $\lambda_z^{-1} H_{0Z}$. Since, in the absence of surface currents, the tangential component of $\mathbf{H}$ must be continuous across an interface it follows that H_z is continuous across the surfaces $r = a$ and $r = b$ and is therefore spatially uniform in the present problem. Outside the material (assumed to be vacuum) the magnetic induction is given by $\mu_0 H_z$, and there the components of the Maxwell stress are given by

$$\tau_{rr} = \tau_{\theta\theta} = -\frac{1}{2}\mu_0 H_z^2 = -\tau_{zz}. \tag{116}$$

In particular, τ_{rr} contributes to the (radial) traction on the boundaries $r = a$ and $r = b$, but has the same value on each boundary.

Suppose that inflation of the tube is achieved by application of an internal pressure P with no mechanical load applied on the exterior boundary. Then we can apply boundary conditions in the form

$$\tau_{rr} = -P - \frac{1}{2}\mu_0 H_z^2 \quad \text{on} \quad r = a, \qquad \tau_{rr} = -\frac{1}{2}\mu_0 H_z^2 \quad \text{on} \quad r = b. \tag{117}$$

Integration of equation (114), use of the relation $(108)_1$ and application of the boundary conditions (117) leads to

$$P = \int_a^b \lambda\omega_\lambda \frac{dr}{r}, \tag{118}$$

and we note, in particular, that this does not involve the exterior Maxwell stress. The corresponding expression for the resultant axial load $\mathcal{N}$ on a cross-section of the cylinder is calculated from

$$\mathcal{N} = 2\pi \int_a^b \tau_{zz} r dr, \tag{119}$$

using (108), (114) and (118), as

$$\mathcal{N} = \pi \int_a^b (2\lambda_z\omega_{\lambda_z} - \lambda\omega_\lambda) r dr + \pi a^2 P - \frac{1}{2}\pi(b^2 - a^2)\mu_0 H_z^2. \tag{120}$$

The expression for P is identical in form to the corresponding equation in the purely elastic theory, while that for $\mathcal{N}$ differs by the inclusion of the final term involving H_z (see, for example, Ogden [15], pp. 289–291), but now ω includes the influence of an axial magnetic field through I_4. Note that H_z is given by $(109)_1$.

For the model used in Section 3.1.1 the expression for P takes the form

$$P = \mu(I_4) \int_a^b (\lambda^2 - \lambda^{-2}\lambda_z^{-2}) \frac{dr}{r}. \tag{121}$$

This differs from the corresponding expression in the purely elastic case (with the neo-Hookean strain-energy function) only by virtue of the dependence of μ on I_4. Thus, if, for example, μ is an increasing function of I_4 then the pressure-radius response stiffens as a result of the presence of the axial magnetic field.

A similar comment applies to $\mathcal{N}$ in that it is influenced by the dependence of μ on I_4, but there is also an additional contribution from the 'Maxwell' term in the energy function. Thus, we can write

$$\mathcal{N} = \mathcal{N}_{\mathrm{nh}} + \frac{1}{2}\pi(b^2 - a^2)(\mu_0^{-1}\lambda_z^2 I_4 - \mu_0 H_z^2), \tag{122}$$

where $\mathcal{N}_{\mathrm{nh}}$ is the expression for $\mathcal{N}$ obtained for the neo-Hookean material but with μ dependent on I_4.

Consider next the residual stress field governed by (115), which is now written

$$\frac{d\tau_{0RR}}{dR} = \frac{1}{R}\omega_\lambda(1, 1, I_4). \tag{123}$$

Since I_4 is constant this may be integrated immediately. Use of the boundary conditions (117) specialized to the reference configuration with $P = 0$ leads to the solution τ_{0RR} = constant, the constant being equal to the value of the Maxwell stress on each of the boundaries $R = A$ and $R = B$, i.e. $-\mu_0 H_{0Z}^2/2$, with H_{0Z} given by (104). We also deduce that $\omega_\lambda(1, 1, I_4) = 0$, which is consistent with $P = 0$ in equation (118) for this specialization. Then, from (111), we have, within the material,

$$\tau_{0ZZ} = \omega_{\lambda_z}(1, 1, I_4) - 2\mu_0 I_4 (\omega_0')^2, \tag{124}$$

which is independent of R. The sign of this term therefore determines whether the load associated with the axial magnetic field required to maintain the initial geometry is tensile or compressive. Depending on the form of $\omega(\lambda, \lambda_z, I_4)$ this can be positive or negative.

4.3.2. Circumferential magnetic field

This is similar to the above except that in ω the influence of the circumferential magnetic field is incorporated through $I_4 = B_{0\Theta}^2$, which depends on the radial coordinate. We also have $H_{f\theta} = \lambda^{-1} H_{0\Theta}$. Here we must have continuity of H_θ at the boundaries $r = a$ and $r = b$, with

$$H_\theta = 2\lambda^{-1}\omega_4 B_{0\Theta}, \quad H_{0\Theta} = \frac{C}{R} = 2\omega_0' B_{0\Theta} \tag{125}$$

within the material. Suppose that there is a central core of radius $c < a$ carrying a steady current I. Then $C = I/2\pi$. This core is excluded from consideration. In principle, the second equation in (125) determines I_4 as a function of R when I is given. In order to satisfy the equation $\operatorname{curl}\mathbf{H} = \mathbf{0}$ it is necessary (under the assumed cylindrical symmetry) that rH_θ is constant. When taken together with the first equation in (125) this leads to the conclusion that ω_4 must depend only on I_4 for this deformation and therefore puts a restriction on the class of material models admitting extension and inflation in the presence of an azimuthal magnetic field. Indeed, this restriction is quite severe since it implies that ω is decoupled in the form $\omega^{(1)}(\lambda, \lambda_z) + \omega^{(2)}(I_4)$, where $\omega^{(1)}$ depends only on the deformation and $\omega^{(2)}$ only on the initial magnetic field, through I_4.

Outside the material (in the regions $c < r < a$ and $r > b$) the components of the Maxwell stress are

$$\tau_{rr} = \tau_{zz} = -\tau_{\theta\theta} = -\frac{1}{2}\mu_0 H_\theta^2 \equiv -\tau_m(r), \tag{126}$$

wherein the notation $\tau_m(r)$ is defined. Since this depends on the radius the radial components $\tau_m(a)$ and $\tau_m(b)$ now influence the pressure-radius relation, which becomes

$$P + \tau_m(a) - \tau_m(b) = \int_a^b \lambda\omega_\lambda \frac{dr}{r}. \tag{127}$$

The corresponding expression for $\mathcal{N}$ in this case is

$$\mathcal{N} = \pi \int_a^b (2\lambda_z\omega_{\lambda_z} - \lambda\omega_\lambda) r dr + \pi a^2 P - \pi[b^2\tau_m(b) - a^2\tau_m(a)]. \tag{128}$$

However, since we must have $rH_\theta =$ constant, the term $b^2\tau_m(b)-a^2\tau_m(a)$ vanishes.

In the absence of internal pressure in the reference configuration, equation (128) reduces to

$$\mathcal{N} = \pi \int_A^B [2\omega_{\lambda_z}(1,1,I_4) - \omega_\lambda(1,1,I_4)]RdR, \tag{129}$$

which gives the axial load required to prevent deformation due to the applied magnetic field. For the specific model (59) this reduces further to

$$\mathcal{N} = -\frac{1}{2}\pi\mu_0^{-1}\int_A^B I_4 RdR, \tag{130}$$

which is negative. Thus, for this model, initial application of the circumferential magnetic field requires a compressive axial load to prevent axial extension of the tube.

4.4. An alternative formulation

In view of the restriction on the class of constitutive laws imposed by the problem considered in Section 4.3 in the case of an azimuthal magnetic induction field we now consider an alternative formulation of the governing equations based on use of $\mathbf{H}$ (or, equivalently, $\mathbf{H}_l$) as the independent magnetic variable. We make use of the Legendre transformation

$$\Omega^*(\mathbf{F},\mathbf{H}_l) = \Omega(\mathbf{F},\mathbf{B}_l) - \mathbf{H}_l\cdot\mathbf{B}_l, \tag{131}$$

which defines $\Omega^*(\mathbf{F},\mathbf{H}_l)$ as a function of $\mathbf{F}$ and $\mathbf{H}_l$. Then, it follows that

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial\Omega^*}{\partial\mathbf{H}_l}, \tag{132}$$

with $\mathbf{M}_l$ given by

$$\mathbf{M}_l = \mu_0^{-1}\mathbf{c}\mathbf{B}_l - \mathbf{H}_l. \tag{133}$$

Alternatively, instead of defining $\Omega^*(\mathbf{F},\mathbf{H}_l)$ via (131), one could begin with an energy defined as a function of $\mathbf{F}$ and $\mathbf{H}$ or $\mathbf{F}$ and $\mathbf{H}_l$ and construct the constitutive law *ab initio* in terms of $\mathbf{H}_l$, as was done in Section 2.5 for $\mathbf{B}_l$.

In the same way as we may choose $\mathbf{B}_l = \mathbf{B}_0$, we may now set, instead, $\mathbf{H}_l = \mathbf{H}_0$, but then, in general, $\mathbf{B}_l \neq \mathbf{B}_0$. Here, we use the notation K_4, K_5, K_6 in respect of an isotropic material for the counterparts of I_4, I_5, I_6. These are defined by

$$K_1 = |\mathbf{H}_0|^2, \quad K_5 = (\mathbf{c}\mathbf{H}_0)\cdot\mathbf{H}_0, \quad K_6 = (\mathbf{c}^2\mathbf{H}_0)\cdot\mathbf{H}_0. \tag{134}$$

For an incompressible material, for example, the total stress is

$$\begin{aligned}\boldsymbol{\tau} = {} & 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} \\ & + 2\Omega_5^*\mathbf{b}\mathbf{H}\otimes\mathbf{b}\mathbf{H} + 2\Omega_6^*(\mathbf{b}\mathbf{H}\otimes\mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H}\otimes\mathbf{b}\mathbf{H}),\end{aligned} \tag{135}$$

and the magnetic induction

$$\mathbf{B} = -2(\Omega_4^*\mathbf{b}\mathbf{H} + \Omega_5^*\mathbf{b}^2\mathbf{H} + \Omega_6^*\mathbf{b}^3\mathbf{H}), \tag{136}$$

where now $\Omega^* = \Omega^*(I_1, I_2, K_4, K_5, K_6)$ and Ω_i^* is defined as $\partial\Omega^*/\partial I_i$ for $i = 1, 2$, and $\partial\Omega^*/\partial K_i$ for $i = 4, 5, 6$.

In the reference configuration, we have

$$I_1 = I_2 = 3, \quad K_4 = K_5 = K_6 = |\mathbf{H}_0|^2, \tag{137}$$

and we define

$$\Omega_0^*(K_4) = \Omega^*(3, 3, K_4, K_4, K_4). \tag{138}$$

Then,

$$\boldsymbol{\tau}_0 = [-p + 2(\Omega_1^* + 2\Omega_2^*)]\mathbf{I} + 2(\Omega_5^* + 2\Omega_6^*)\mathbf{H}_0 \otimes \mathbf{H}_0, \tag{139}$$

and

$$\mathbf{B}_0 = -2(\Omega_4^* + \Omega_5^* + \Omega_6^*)\mathbf{H}_0 \equiv -2\Omega_0^{*\prime}(K_4)\mathbf{H}_0, \tag{140}$$

wherein the prime indicates differentiation with respect to K_4 and Ω_i^*, $i = 1, 2, 4, 5, 6$, is evaluated for (137).

4.4.1. The circumferential magnetic field revisited

We now review briefly the problem considered in Section 4.3.2 on the basis of this alternative formulation. In order to satisfy $\operatorname{curl}\mathbf{H} = \mathbf{0}$ the azimuthal field is given by

$$H_{0\Theta} = \frac{C}{R}, \quad H_\theta = H_{f\theta} = \frac{C}{r} = \lambda^{-1}H_{0\Theta}, \tag{141}$$

where C is a constant, and it follows that

$$K_4 = H_{0\Theta}^2, \quad K_5 = \lambda^2 K_4, \quad K_6 = \lambda^4 K_4. \tag{142}$$

Analogously to (107) we define

$$\omega^*(\lambda, \lambda_z, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6), \tag{143}$$

with $\omega_0^*(K_4) = \omega^*(1, 1, K_4)$. Then, equations (108) are replaced by

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\omega_\lambda^*, \quad \tau_{zz} - \tau_{rr} = \lambda_z\omega_{\lambda_z}^*, \tag{144}$$

and (110) by

$$B_\theta = -2\lambda\omega_4^* H_{0\Theta}. \tag{145}$$

Equations (127) and (128) are unchanged except that ω is replaced by ω^*.

With the present formulation the equation $\operatorname{div}\mathbf{B} = 0$ is automatically satisfied and places no restriction on the form of constitutive law, unlike the situation in Section 4.3.2.

5. Conclusions

In this paper we have analyzed some basic problems in the theory of nonlinear isotropic magnetoelasticity in order to provide a framework within which the theory can be compared with experiment data for, in particular, magneto-sensitive elastomers. This requires development of experimental tests that can provide appropriate data since extensive data of the required type are not currently available. In particular, more information on the change in material constants with magnetic field is needed and on the deformation dependence of the magnetization of these materials. Once such data become available, a key objective will be to characterize these materials through use of specific forms of constitutive law based on a free energy function or a variant of this function. As a starting point for this process the simple prototype model introduced in Section 3.1.1 includes two constitutive functions, one related to the material stiffness and the other to the magnetization.

An important consideration is whether it is appropriate to use the magnetic field $\mathbf{H}$ or the magnetic induction $\mathbf{B}$ as the independent variable. While, in principle, the associated formulations of the constitutive law and the governing field equations are equivalent, in practice, at least for the problem considered in Section 4.3.2 and for several problems considered in [3], more restrictions are placed on the admissible class of constitutive laws if $\mathbf{B}$ is used rather than $\mathbf{H}$. This problem would be compounded if the magnetization $\mathbf{M}$ were used as the independent variable since the fields $\mathbf{H}$ and $\mathbf{B}$ then derived via the constitutive law and the connection $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ have to satisfy $\mathrm{curl}\,\mathbf{H} = \mathbf{0}$ and $\mathrm{div}\,\mathbf{B} = \mathbf{0}$, respectively.

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