Z. angew. Math. Phys. 56 (2005) 148–182
0044-2275/05/010148-35
DOI 10.1007/s00033-004-4065-0

Zeitschrift für angewandte Mathematik und Physik ZAMP

A geometrically exact viscoplastic membrane-shell with viscoelastic transverse shear resistance avoiding degeneracy in the thin-shell limit. Part I: The viscoelastic membrane-plate

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Abstract. We reduce a viscoelastic finite-strain continuum model to a two-dimensional membrane-plate. The reduction is based on assumed kinematics, analytical integration through the thickness and physically motivated simplifications. The resulting formulation is observer-invariant and accounts for thickness stretch and finite rotations.

The membrane energy is a quadratic, uniformly Legendre-Hadamard elliptic, first order energy in contrast to classical membrane models and the corresponding system of balance equations remains of second order. An evolution equation for some independent rotation is appended (already in the bulk-model) introducing viscoelastic transverse shear resistance. It can be shown that this reduced membrane formulation is locally well-posed. Use is made of a dimensionally reduced version of an extended Korn's first inequality.

In the equilibrium relaxation limit an intrinsic membrane-plate formulation is obtained similar to the proposal of Fox/Simo, which is, however, non-elliptic. Nevertheless, the linearization of this last equilibrium model coincides with the classical linear membrane-plate model. In this sense, the new viscoelastic membrane-plate model regularizes the occurring loss of ellipticity in classical finite-strain membrane models.

Mathematics Subject Classification (2000). 74K15, 74K20, 74G65.

Keywords. Membranes, plates, shells, thin films, energy minimization, viscoelasticity.

1. Introduction

1.1. The underlying finite-strain viscoelastic-plastic 3D-model

In [41] a model of finite-strain elasto-plasticity has been introduced, based on the multiplicative decomposition of the deformation gradient $F = F_e\, F_p$, incorporating viscoelastic effects due to **grain boundary relaxation**. The model preserves **observer-invariance** and is invariant with respect to superposed spatially constant rotations of the so called intermediate configuration induced by F_p. The model is **geometrically nonlinear** and allows for finite elastic rotations, finite plastic deformations and overall finite deformations but remains a truly "**physically linear**" theory in the sense that simple uniaxial tension is modelled as linear and without viscosity.

We need to mention, however, that the new model is **intrinsically rate-dependent**, i.e., it is not possible to "freeze" the "**viscoelastic**" rotations and obtain a frame-indifferent reduced plasticity model. In other words, the used elastic free energy W is not expressible as a reduced function of $C = F^T F$, nevertheless, the model is observer-invariant[1] and the common wisdom that observer-invariance implies a representation in C or the stretch U applies as such only to intrinsically non-dissipative problems [23, p.203]. In general, form-invariance under superposed time-dependent rigid rotations (frame-indifference) implies observer-invariance but is not identical to it. For this subtle point compare also to the lucid discussion in [25, p.269] and [30, p.159] together with [52, 8, 35].[2]

To begin with let us first introduce the considered 3D-model which we have modified compared to [41, 37] to include also in a consistent manner "compressible" plasticity, i.e., $\det[F_p] \neq 1$. In the quasi-static setting appropriate for slow loading, where we neglect consistently inertia terms, we are led to study the following coupled minimization and evolution problem for the finite deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$, the plastic variable $F_p : [0,T] \times \overline{\Omega} \mapsto \mathrm{GL}^+(3,\mathbb{R})$ and the independent local viscoelastic rotation $\overline{R}_e : [0,T] \times \overline{\Omega} \mapsto \mathrm{SO}(3)$ on Ω

$$\begin{aligned} &\int_\Omega W(F_e,\overline{R}_e) \det[F_p] - \langle f, \varphi\rangle \det[F_p] \, \mathrm{dV} \\ &\qquad - \int_{\Gamma_S} \langle N, \varphi\rangle \, \| \operatorname{Cof} F_p . \vec{n}_{\partial\Omega} \| \, \mathrm{dS} \mapsto \min . \text{ w.r.t. } \varphi \text{ at fixed } (\overline{R}_e, F_p)\,, \end{aligned} \tag{1.1}$$

with prescribed Dirichlet boundary conditions $\varphi_{|\Gamma} = g_{\mathrm{d}}(t)$ on $\Gamma \subset \partial\Omega$. The constitutive assumption on the density is

$$\begin{aligned} W(F_e,\overline{R}_e) &= \frac{\mu}{4} \, \|F_e^T \overline{R}_e + \overline{R}_e^T F_e - 2 1\!\!1\|^2 + \frac{\lambda}{8} \operatorname{tr}\left[F_e^T \overline{R}_e + \overline{R}_e^T F_e - 2 1\!\!1\right]^2 , \\ &= \mu \, \| \operatorname{sym}(\overline{U}_e - 1\!\!1)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\overline{U}_e - 1\!\!1\right]^2 , \quad \overline{U}_e = \overline{R}_e^T F_e\,, \qquad (1.2)\\ F_e &= \nabla\varphi \cdot F_p^{-1}, \ S_1(F_e,\overline{R}_e) \\ &= \overline{R}_e \left[\mu(F_e^T \overline{R}_e + \overline{R}_e^T F_e - 2 1\!\!1) + \lambda \operatorname{tr}\left[F_e^T \overline{R}_e - 1\!\!1\right] 1\!\!1\right] F_p^{-T} , \end{aligned}$$

where $S_1 = D_F \left[W(F_e,\overline{R}_e)\right]$ denotes the first Piola-Kirchhoff stress tensor and $\mu, \lambda > 0$ are the classical Lamé constants of isotropic elasticity. The coupled

[1] observer-invariant means that material properties do not depend on the choice of representation tools used to portray them.

[2] And the undisputed physical principle is observer-invariance and not directly frame-indifference (form-invariance under rigid rotations). The strengthening of form-invariance of the equations under superposed rigid rotations to form-invariance under the group of all diffeomorphisms is called **covariance** [23]. We understand that form-invariance and covariance are additional constitutive assumptions.

plastic and viscoelastic evolution is defined by

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] \in -F_p^{-1}\cdot \partial\chi(\Sigma_E),\\
&\qquad \Sigma_E = F_e^T D_{F_e} W(F_e,\overline{R}_e)\det[F_p] - W(F_e,\overline{R}_e)\det[F_p]1\!\!1\,, \qquad (1.3)\\
&\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}\overline{R}_e(t) = \nu^+\ \mathrm{skew}(B)\cdot\overline{R}_e(t),\ B = B_{\mathrm{mech}} \text{ or } \quad B_{\mathrm{tc}},\ \nu^+ = \nu^+(F_e,\overline{R}_e)\in\mathbb{R}^+\,,\\
&\quad B_{\mathrm{mech}} = \mu\, F_e\overline{R}_e^T,\quad B_{\mathrm{tc}} = \left[\mu(2\,1\!\!1 - F_e\overline{R}_e^T) + \lambda\,[3 - \langle F_e\overline{R}_e^T, 1\!\!1\rangle]\right] F_e\overline{R}_e^T\,,\\
&\quad F_p^{-1}(0) = F_{p_0}^{-1},\quad F_{p_0}\in \mathrm{GL}(3,\mathbb{R}),\\
&\quad \overline{R}_e(0) = \overline{R}_e^0,\qquad \overline{R}_e^0 \in \mathrm{SO}(3), \overline{R}_e^0 = 1\!\!1 \text{ if } F_{p_0} = \nabla\Theta,
\end{aligned}
$$

where the **flow potential** $\chi : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ governs the plastic evolution (here associated plasticity for simplicity only) and which is motivated through the principle of maximal dissipation sufficient for the thermodynamical consistency of the model. B_{mech} or B_{tc} are alternative constitutive choices. The dead load body force and the boundary tractions on $\Gamma_S \subset \partial\Omega$ are denoted by f, N, respectively and defined w.r.t. the intermediate plastic configuration F_p and $\vec{n}_{\partial\Omega}$ is the unit outward normal to $\partial\Omega$. Corresponding natural boundary conditions apply.

Here Σ_E denotes the **elastic Eshelby stress tensor** (the driving force behind evolving inhomogeneities in the reference configuration [32]) which may be reduced to $\Sigma_M = F_e^T D_{F_e} W(F_e,\overline{R}_e)$, the **elastic Mandel stress tensor** in case of a deviatoric flow rule which preserves the incompressibility constraint $\det[F_p] = 1$.

By $\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}$ we mean the **observer-invariant (corotated) time derivative** on $\mathrm{SO}(3,\mathbb{R})$

$$
\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}[R(t)] := \frac{\mathrm{d}}{\mathrm{dt}}[R(t)] - \hat\omega(t)\cdot R(t)\,,\quad \hat\omega := \frac{\mathrm{d}}{\mathrm{dt}}[Q(t)]\cdot Q(t)^T\,, \qquad (1.4)
$$

where $Q(t)\in \mathrm{SO}(3,\mathbb{R})$ is the instantaneous rotation of the current frame with respect to the inertial frame and $\hat\omega$ is the corresponding angular velocity. Without loss of generality, we confine attention to the inertial frame, i.e. $\hat\omega \equiv 0$ and $\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}$.

The term $\nu^+ := \frac{1}{\eta_e}\nu^+(F_e,\overline{R}_e)$ represents a scalar valued function introducing elastic viscosity within the elastic domain and η_e plays the role of a relaxation time with units $[\eta_e] = $ sec. $F_{p_0}^{-1}$ and $\overline{R}_e^0$ are the initial conditions for the plastic variable and viscoelastic rotation part, respectively. The choice $B = B_{\mathrm{tc}}$ is thermodynamically consistent whereas the simpler choice $B = B_{\mathrm{mech}}$ is (only) mechanically consistent in the sense that various invariance requirements are met. Due to the underlying isotropy the resulting model (1.1) with $B = B_{\mathrm{mech}}$ approaches in the (vanishing elastic viscosity = zero relaxation limit $\eta_e \to 0$ viz. for arbitrary slow processes) equilibrium limit $\nu^+ \to \infty$ formally the coupled problem

$$\int_\Omega W_\infty(F_e)\det[F_p] - \langle f,\varphi\rangle \det[F_p]\,\mathrm{dV}$$
$$- \int_{\Gamma_S} \langle N,\varphi\rangle \,\|\operatorname{Cof} F_p.\vec{n}_{\partial\Omega}\|\ \mathrm{dS} \mapsto \text{stat .w.r.t. } \varphi \text{ at fixed } F_p\,,$$
$$W_\infty(F_e) := \mu\,\|U_e - 1\!\!1\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[U_e - 1\!\!1\right]^2, \quad F_e = \nabla\varphi F_p^{-1}\,, \tag{1.5}$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right](t) \in -F_p^{-1}(t)\cdot\partial\chi(\Sigma_{E,\infty})\,,$$
$$\Sigma_{E,\infty} := U_e D_{U_e} W_\infty(U_e)\det[F_p] - W_\infty(U_e)\det[F_p]\,1\!\!1\,,$$

with $U_e = (F_e^T F_e)^{\frac{1}{2}}$ the classical **symmetric** elastic stretch, $U_e - 1\!\!1$ the elastic Biot strain tensor and $W_\infty(U_e)$ the **non-elliptic** equilibrium energy. The system (1.5) is an exact equilibrium model for small elastic strains and finite plastic deformations in the classical sense with no extra internal dissipation. The transition from (1.1) to (1.5) is not entirely trivial since it is not just the replacement of the independent rotation $\overline{R}_e$ by the continuum rotation $\overline{R}_e \to R_e = \operatorname{polar}(F_e)$ and note the subtle change from **global minimization** to a **stationarity** requirement only. Observe as well that $\mu\,\|U - 1\!\!1\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[U - 1\!\!1\right]^2$ leads to a linear stress response in uniaxial tension/compression while e.g. $\mu\,\|E\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[E\right]^2$, $E = \frac{1}{2}(F^T F - 1\!\!1)$ would lead to a nonlinear, unphysical non-monotone stress response in uniaxial tension/compression.

In the companion paper [41] the implications, predictions and physical relevance of the new model have been investigated in great detail. It is shown that the additional degrees of freedom inherent through the independent local **viscoelastic** rotations $\overline{R}_e$ can be interpreted in the framework of a material with a polycrystalline substructure where the individual rotations of the grains may deviate from the continuum rotation. Then, in the presence of plasticity, $\overline{R}_e$ represents a reversible, "viscoelastic" part of the total rotation of the grains and leads to **texture** effects (deformation induced anisotropy). The evolution equation for $\overline{R}_e$ introduces hysteresis effects into the model already within the elastic region, i.e. immediately for arbitrary small stress levels. The physical reality of this behaviour for polycrystalline material is well documented and it is shown that the new model (1.1) allows a qualitative and in parts quantitative description of such effects which are ascribed to internal friction at the grain boundaries. In [41] it has also been motivated that the elastic viscosity is larger for larger internal surfaces, i.e. the smaller the grain size, while single crystals behave nearly rate-independent for that matter.

In [42] the local well-posedness of (1.1) under Dirichlet conditions has been shown, while such a result is not yet known for (1.5). The general applicability of the model (1.1) in the three-dimensional case has been investigated numerically in [45]. This is our motivation to extend the model to a reduced membrane formulation. It is planned to investigate in a sequel the full dimensional reduction problem for the viscoelastic-viscoplastic problem (1.1). Here, we concentrate on the viscoelastic formulation.

1.2. The finite-strain viscoelastic 3D-model

Before we proceed to the dimensional reduction, the need has been felt to further motivate this model (1.1) since it departs considerably from classical viscoelastic models to which the reader might be aquainted. Let us therefore look at the purely viscoelastic version of (1.1) with $B = B_{\mathrm{mech}}$, $\nu^+ \in \mathbb{R}$ but without surface tractions. The problem reads

$$\int_\Omega W(F,\overline{R}) - \langle f, \varphi \rangle \, \mathrm{dV} \mapsto \min . \text{ w.r.t. } \varphi \text{ at fixed } \overline{R}\,, \quad \varphi_{|\partial\Omega} = g_{\mathrm{d}}(t)\,, \tag{1.6}$$

$$W(F,\overline{R}) = \frac{\mu}{4} \, \|F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\|^2 + \frac{\lambda}{8} \, \mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right]^2, \quad F = \nabla\varphi\,,$$

with coupled viscoelastic evolution

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t,x) = \nu^+ \, \mathrm{skew}(F(t,x)\overline{R}(t,x)^T) \cdot \overline{R}(t,x)\,, \quad \overline{R}(0,x) = \overline{R}_0(x)\,. \tag{1.7}$$

The minimization at fixed $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ in (1.6) is in fact strictly equivalent to the balance of linear momentum equation

$$-\,\mathrm{Div}_x \, D_F[W(\nabla\varphi(t,x), \overline{R}(t,x))] = f(t,x)\,, \quad \varphi_{|\partial\Omega}(t,x) = g_{\mathrm{d}}(t,x)\,, \tag{1.8}$$

as long as $\partial\Omega$ and g_{d} are sufficiently smooth. The local evolution equation for $\overline{R}$ introduces the viscoelastic effects. In contrast to a more traditional Cosserat approach, **the rotations are not determined by simultaneous minimization of some augmented elastic energy (which would include curvature terms $\mathrm{D_x}\overline{R}$) w.r.t. both φ and $\overline{R}$.**

In order to appreciate the relaxation properties of (1.7) already hinted at, assume now that we are given a deformation history $F \in C^1(\mathbb{R}^+, \mathrm{GL}^+(3,\mathbb{R}))$ for a specific point $x_0 \in \Omega$. Then

Theorem 1.1 (Dynamic polar decomposition and relaxation). *The viscoelastic evolution problem* (1.7) *admits a unique global in time solution* $\overline{R} \in C^1(\mathbb{R}^+, \mathrm{SO}(3,\mathbb{R}))$. *Moreover,*

1. *if F is constant in time and $\|\overline{R} - \mathrm{polar}(F)\|^2 < 8$, then we have the asymptotic behaviour $\overline{R}(t) \to \mathrm{polar}(F)$ for $t \to \infty$.*
2. $\forall\, t \in \mathbb{R}^+: \quad \|\,\mathrm{skew}(F(t)\overline{R}^T(t))\|^2 \leq \frac{M^+}{\nu^+}(1 - e^{-\nu^+ t}) + \|F(0)^T\overline{R}(0) - \mathbb{1}\|^2 \, e^{-\nu^+ t}$, *where* $M^+ = (\|F\|_\infty + \sqrt{3}) \, \|F'\|_\infty$ *is independent of* ν^+.

Proof. The right hand side in (1.7) is globally Lipschitz as a function of $\overline{R}$, hence there exists a unique global solution $\overline{R} \in C^1(\mathbb{R}^+, \mathrm{SO}(3,\mathbb{R}))$.

Part i.) is proved in [41, p.173]. The proof of part ii.) will be given in the appendix. □

Since $\|\,\mathrm{skew}(F(t)\overline{R}(t)^T)\|$ is a measure for the difference between $\overline{R}(t)$ and the continuum rotation $\mathrm{polar}(F(t))$ c.f. Lemma 6.4, we see that by choosing ν^+ appropriately large (low viscosity) this difference can be effectively controlled. In the limit $\nu^+ \to \infty$ we determine the constraint rotation $\overline{R}(t) = \mathrm{polar}(F(t))$.

1.3. Dimensionally reduced kinematics

The dimensional reduction of a given model is already an old and mature subject and it has seen many "solutions". The different approaches toward elastic shell theory proposed in the literature and relevant references thereof are, therefore, too numerous to list here. In any case our proposal falls within the so called **derivation approach**, i.e., reducing a given three-dimensional model via (physically) reasonable constitutive assumptions to a two-dimensional model as opposed to either the **intrinsic** approach which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **asymptotic methods** which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small parameter. The intrinsic approach is closely related to the **direct** approach which takes the shell to be a directed medium in the sense of a **restricted Cosserat-theory** [9].[3] A detailed presentation of the classical shell theories can be found in [36]. A thorough mathematical analysis of linear, infinitesimal shell theory, based on asymptotic methods is to be found in [12] and the extensive references therein, see also [11, 13, 1, 20, 16]. Excellent reviews and insightful discussions of the modelling and finite element implementation may be found in [51, 49, 50, 23, 24, 3, 5] and in the series of papers [55, 58, 59, 61, 60, 56, 14]. Properly invariant elastic plate theories for membrane and bending are derived by formal asymptotic methods in [21] and extended to the case of curvilinear coordinates in [34, 31].

The mathematical analysis establishing the wellposedness of all the infinitesimal linearized models is fairly well established and will not be our concern.

In the finite-strain, geometrically exact elastic case, mostly based on the Saint Venant-Kirchhoff free energy density $\mu\,\|E\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[E]^2$, the formal asymptotic methods are still successful in that they identify again leading membrane and bending terms. As far as the occurring membrane contribution is concerned, it is the form (6.9) which is given e.g. in [22, 21, 34]. However, variational methods based on Γ-convergence [17] suggest a fundamentally different membrane term which leads to a non-resistance of the membrane plate/shell in compression.[4] The non-resistance to compression in this analysis is related to the use of the quasiconvex hull[5] QW_0 of a dimensionally reduced St.Venant Kirchhoff energy, see (6.10).

[3] Restricted, since no material length scale enters the direct approach, only the relative thickness h appears.

[4] They remark [18, p.550]: "...then the corresponding nonlinear membranes offer no resistance to crumpling. This is an empirical fact, witnessed by anyone who ever played with a deflated balloon."

[5] "... the fact that this function is not quasiconvex already implied that it had to be relaxed in

This quasiconvex hull, surprisingly enough, can be given in closed form [19, 27] and shows to be in general positive but zero in the compression range.

The classical linear models proposed in the literature lead to effective numerical schemes only if the thickness h of the structure is still appreciable, i.e. classical bending terms are present and regularize the computation. However, there is an abundance of new applications where very thin structures are used, e.g. very thin metal layers on a substrate (in computer hardware, for the characteristic non dimensional relative thickness $h \leq 5 \cdot 10^{-4}$). See [4] for an application to thin films.

Since locally rotating the thin structure is energetically "cheap" compared to stretching, we are forced to consider models including finite rotations in an objective manner. But the proposed finite-strain membrane terms found in the literature are either **non-elliptic** and the remaining (minimization) problem is not well-posed or they lead to the aforementioned non-resistance in compression.

1.4. Outline and scope of this contribution

In order to improve on this unsatisfactory state of the art for finite-strain membrane plate formulations we propose here a new membrane-plate model for very thin almost rigid[6], viscoelastic materials which is non-degenerate in the thin shell limit without addition of bending terms and which in principle allows to describe the detailed geometry of deformation in a finely wrinkled plate. This might be contrasted with the variational approach in [17] and tension field theory which describes the approximate stress distribution in the membrane but determines the deformation only to within a probability measure. Strictly speaking, the use of the quasiconvex hull leads to a so called **tension field theory** [62]. Steigmann [62, p.143] notes "A question then arises concerning the validity of tension filed theory as an approximation to a theory of shells with bending stiffness that is small in some sense. Evidently, the deformation is not well described, though the theory delivers solutions that approximate the average of the deformation observed in a real membrane containing many wrinkles. We conjecture that the stress is accurately predicted, however."

Our contribution is organized as follows. After this introductory part we consider the finite-strain purely viscoelastic model (1.1) on an absolutely thin domain. Using a quadratic kinematical ansatz through the thickness, which is consistent with the appearance of independent rotations in the three-dimensional theory and subsequent analytical integration through the thickness together with certain simplifications we formally reduce the equilibrium energy.[7] For the viscoelastic evolu-

order to give rise to a well posed problem." [18, p.575].

[6] almost rigid: a material with high Lamé moduli $\mu, \lambda \gg 1$[MPa] such that $F \approx \mathrm{SO}(3,\mathbb{R})$ whenever kinematically possible.

[7] One should not confuse this approach with energy projection on a reduced ansatz space, since we do not introduce additional fields in the process of dimensional reduction.

tion equation we obtain the dimensional reduction by averaging the generator on the Lie-algebra of the flow through the thickness. Consistently reducing the boundary conditions and putting the results together defines formally the viscoelastic membrane-plate model (3.1).

The new viscoelastic model is shown to remain observer-invariant and its membrane equilibrium energy density satisfies a uniform Legendre-Hadamard ellipticity condition (3.7) while it is not uniformly convex.

Then the elastic equilibrium limit for vanishing viscosity ($\nu^+ \to \infty$) is investigated. It is shown that the formal limit exhibits a **non-elliptic** membrane strain energy density (3.20), similar to the membrane model of Fox/Simo (6.9). We close with a local existence and uniqueness result for the obtained viscoelastic membrane-plate.

The notation will be found in the appendix as well as the dimensional reduction of the external loads. Finally, we present two alternative propositions from the literature for the computation of membrane dominated problems.

A different formulation of elastic plate models with independent rotations leading to a true, geometrically exact Cosserat theory of plates has been given in [40, 44].

2. The formal dimensional reduction in the viscoelastic case

2.1. The three-dimensional finite-strain viscoelastic problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably "exact" 3D-theory to 2D. We assume from now on small elastic strains (almost rigidity) and no plasticity (i.e., $F_p = 1\!\!1$ in (1.1) and $\overline{R}_e = \overline{R}$). We will adapt the bulk problem to a plate like theory. Let us assume that we are given a three-dimensional **absolutely thin domain**

$$\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2 , \tag{2.1}$$

with **transverse boundary** $\partial\Omega_h^{\text{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **lateral boundary** $\partial\Omega_h^{\text{lat}} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the thickness, and a deformation φ^{3d} and **rotation** $\overline{R}^{3d}$

$$\varphi^{3d}: \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 , \qquad \overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3, \mathbb{R}) , \tag{2.2}$$

solving the following coupled minimization and evolution problem on Ω_h:

$$\int_{\Omega_h} W(\overline{U}) - \langle f, \varphi \rangle \, \mathrm{dV} - \int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} \langle N, \varphi \rangle \, \mathrm{dS} \mapsto \min . \text{ w.r.t. } \varphi \text{ at fixed } \overline{R},$$
$$\overline{U} = \overline{R}^T F, \quad \varphi_{|_{\Gamma_0^h}} = g_{\mathrm{d}}, \quad \Gamma_0^h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}], \quad \gamma_0 \subset \partial\omega, \ \gamma_s \cap \gamma_0 = \emptyset ,$$

$$W(\overline{U}) = \mu \, \| \operatorname{sym}(\overline{U} - 1\!\!1) \|^2 + \frac{\lambda}{2} \operatorname{tr} \left[\operatorname{sym}(\overline{U} - 1\!\!1) \right]^2 , \tag{2.3}$$

$$\frac{\mathrm{d}_{\hat{\omega}}}{\mathrm{dt}} \overline{R}(t) = \nu^+ \ \operatorname{skew}(B) \cdot \overline{R}(t), \quad B = B_{\mathrm{mech}} \quad \text{or} \quad B_{\mathrm{tc}}, \quad \nu^+ = \nu^+(F, \overline{R}) \in \mathbb{R}^+ ,$$

$$B_{\mathrm{mech}} = \mu \, F \overline{R}^T , \, B_{\mathrm{tc}} = \left[\mu (2 \, 1\!\!1 - F \overline{R}^T) + \lambda \, [3 - \langle F \overline{R}^T , 1\!\!1 \rangle] \right] F \overline{R}^T , \, \overline{R}(0) \in \mathrm{SO}(3) ,$$

where $\overline{U} = \overline{R}^T F$ is **not** necessarily **symmetric**. $\overline{U}$ is known as the **first Cosserat deformation tensor**. We want to find a reasonable approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi^{3d}, \overline{R}^{3d})$ involving only two-dimensional quantities. The reduction is based on assumed kinematics and analytical integration through the thickness.

2.2. Enriched quadratic kinematics

In order to characterize the shell deformation, let us assume that the deformation φ^{3d} can be represented by a converging function expansion in thickness direction, i.e.

$$\varphi^{3d}(x, y, z) = \sum_{i=0}^{\infty} \vec{\alpha}_i(x, y) \cdot v_i(z), \quad \vec{\alpha}_i : \omega \mapsto \mathbb{R}^3, \ v_i : [-h/2, h/2] \mapsto \mathbb{R} , \tag{2.4}$$

with linearly independent functions v_i. Without loss of generality, we may take $v_i(z) = z^i$.

In the engineering shell community it is well known [10, 54, 46] that the ansatz through the thickness should at least be quadratic in order to avoid the **Poisson thickness-locking**[8] and to fully capture the three-dimensional kinematics without artificial modification of the material laws if applying projection methods. See the detailed discussion of this point in [7] and compare with [5, 2, 48, 6, 53].

For the three-dimensional theory with small elastic strains which captures shells with large in-plane rigidity and high transverse flexibility we **truncate** (2.4) and **assume** the **quadratic ansatz** in the thickness direction[9] for the **reconstructed** finite deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ of the shell-like structure

$$\varphi_s(x, y, z) = m(x, y) + \left(z \, \varrho_m(x, y) + \frac{z^2}{2} \varrho_b(x, y) \right) \cdot \vec{d}(x, y) , \tag{2.5}$$

where $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ takes on the role of the deformation of the **midsurface** of the shell viewed as a parametrized surface and the **independent unit director** of the shell $\vec{d} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$. The yet indeterminate scalar functions $\varrho_m, \varrho_b : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$ allow in principal for **symmetric thickness stretch** ($\varrho_m \neq 1$) and **asymmetric thickness stretch** ($\varrho_b \neq 0$) about the midsurface. For $\vec{d} \neq \vec{n}_m$ ($\vec{n}_m$ the outer unit normal to m) **transverse shear** occurs.

[8] The bending stiffness of the reduced theory would tend to ∞ as the Poisson-number $\nu \to \frac{1}{2}$.

[9] Identify $\vec{\alpha}_0 = m, \ \vec{\alpha}_1 = \varrho_m \vec{d}, \ \vec{\alpha}_2 = \varrho_b \vec{d}$.

This leads at first glance to a 10 “dof” constraint theory: 3 components of the membrane deformation, 3 degrees of freedom for the bulk microrotations $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$, including naturally one **drilling degree** of freedom for in-plane rotations, 2 degrees of freedom for the unit director $\vec{d} \in \mathbb{S}^2$ and 2 degrees of freedom ϱ_m, ϱ_b over the thickness. However, the director $\vec{d}$ will be specialized and the two thickness coefficients ϱ_m, ϱ_b will be eliminated analytically, leaving us finally with 6 six degrees of freedom and the rotations $\overline{R}$ remain locally coupled to the deformation gradient through viscoelasticity. Already in the classical elasticity context the beneficial influence of drill rotations for the numerical implementation has been investigated in the linear case in [26] and in the finite-strain case in [57].

The (reconstructed) rotations $\overline{R}^s : \Omega_h \mapsto \mathrm{SO}(3, \mathbb{R})$ in the thin shell are assumed not to depend on the thickness variable z

$$\overline{R}^s(x, y, z) = \overline{R}(x, y)\,, \tag{2.6}$$

in line with the assumed thinness and material homogeneity of the structure. This is now a kind of plate formulation, since for the moment the unstressed reference configuration ω was assumed to lie in the plane. We immediately replace the independent unit director $\vec{d}$ in the ansatz (2.5) by specializing

$$\vec{d}(x, y) := \overline{R}^s(x, y, 0).e_3 =: \overline{R}_3\,, \tag{2.7}$$

including now also **drill-rotations**. This implies for the (reconstructed) deformation gradient of the shell (plate)

$$F_s = \nabla\varphi_s(x, y, z) = \underbrace{(\nabla m|\, \varrho_m\, \overline{R}_3)}_{A_m} + z\, \underbrace{(\nabla(\varrho_m\, \overline{R}_3)|\varrho_b\, \overline{R}_3)}_{\tilde{A}_r} + \frac{z^2}{2}\, \underbrace{(\nabla(\varrho_b\, \overline{R}_3)|0)}_{\tilde{\tilde{A}}_r}\,. \tag{2.8}$$

It should be noted that the augmented quadratic ansatz already changes the term which is linear in the transverse direction. The stress field through the thickness $\overline{R}^{s,T} S_1(\nabla\varphi_s(x, y, z), \overline{R}^s).e_3$ is at least linear in the transverse variable z and not constant, as would be the case in a first order (linear) ansatz for the deformation.

Invertibility of the reconstructed shell deformation (as a physical requirement) entails

$$\forall\, z \in [-h/2, h/2] :\ \det[\nabla\varphi_s(x, y, z)] > 0 \Rightarrow \varrho_m(x, y) > 0\,, \tag{2.9}$$

and we should guarantee that $\varrho_m : \omega \mapsto \mathbb{R}^+$. The three-dimensional local part of the elastic free energy in (2.3) has the form

$$W(F, \overline{R}) = \frac{\mu}{4}\, \|\overline{R}^T F + F^T \overline{R} - 2 1\!\!1\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[\overline{R}^T F + F^T \overline{R} - 2 1\!\!1\right]^2. \tag{2.10}$$

The equilibrium equations ensuing from (2.3) show that on the transverse boundary (upper and lower face of the plate) the Neumann condition (3D-exact)

$$S_1^{3d}(\nabla\varphi^{3d}(x, y, \pm h/2), \overline{R}^{3d}(x, y, \pm h/2)).(\pm e_3) = N^{\mathrm{trans}}(x, y, \pm h/2)\,, \tag{2.11}$$

holds. N^{trans} are the prescribed tractions $N[\mathrm{N/m^2}]$ on the transverse boundary given globally in the basis (e_1, e_2, e_3). This implies (3D-exact, multiplication with $\overline{R}^{3d,T}$)

$$\begin{aligned}\overline{R}^{3d}(x,y,\pm h/2)^T S_1^{3d}(\nabla\varphi^{3d}(x,y,\pm h/2),\overline{R}^{3d}(x,y,\pm h/2)).(\pm e_3) = \\ \overline{R}^{3d}(x,y,\pm h/2)^T N^{\text{trans}}(x,y,\pm h/2)\,.\end{aligned} \tag{2.12}$$

As a consequence of (2.11) we have (3D-exact)

$$\begin{aligned}\langle\overline{R}^{3d}(x,y,\pm h/2)^T S_1^{3d}(\nabla\varphi^{3d}(x,y,\pm h/2),\overline{R}^{3d}(x,y,\pm h/2)).e_3, e_3\rangle = \\ \pm\,\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}^{3d}(x,y,\pm h/2).e_3\rangle\,.\end{aligned} \tag{2.13}$$

We determine the coefficients ϱ_m, ϱ_b from the corresponding requirement in terms of the assumed kinematics $(\varphi_s, \overline{R}^s)$, yielding

$$\begin{aligned}&\langle\overline{R}^{s,T}(x,y,\pm h/2)S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}^s).e_3,e_3\rangle\\ &= \pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}^s(x,y,\pm h/2).e_3\rangle \quad \Rightarrow\\ &\langle\overline{R}^T S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}).e_3,e_3\rangle = \pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle\,,\end{aligned} \tag{2.14}$$

which condition reduces to **zero normal tractions on the transverse free boundary** (in the absence of transverse tractions N^{trans}) in the classical, non-polar continuum limit of $\overline{R} \to R = \text{polar}(\nabla\varphi)$. The physical motivation for this condition is simple: if the transverse surface of the plate is free of loads and if we take the plate to be a thin three-dimensional structure made of a regular array of springs, the springs will not be elongated in normal direction. Since from (2.10)

$$\begin{aligned}D_F W_{\text{mp}}(F,\overline{R}) &= S_1(F,\overline{R})\\ &= \overline{R}\left[\mu\left(F^T\overline{R}+\overline{R}^T F - 2\mathbb{1}\right) + \frac{\lambda}{2}\operatorname{tr}\left[F^T\overline{R}+\overline{R}^T F-2\mathbb{1}\right]\mathbb{1}\right]\,,\end{aligned} \tag{2.15}$$

the requirement (2.14) turns for $z = \pm h/2$ into the **local condition**

$$\begin{aligned}\pm\,\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m-1)+2z\,\varrho_b)\\ &+\lambda\Big(\langle\overline{R}^T(\nabla m|0),\mathbb{1}\rangle+\varrho_m+z\,\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},\mathbb{1}\rangle\\ &+z\,\varrho_b-3+\frac{z^2}{2}\,\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),\mathbb{1}\rangle\Big)\,,\end{aligned} \tag{2.16}$$

surprisingly **without spatial derivatives** of ϱ_m, ϱ_b appearing, which would have been the case did we not assume (2.7). Define now $N_{\text{res}}, N_{\text{diff}}: \omega \mapsto \mathbb{R}^3$ by

$$\begin{aligned}N_{\text{res}}(x,y) &:= \left[N^{\text{trans}}(x,y,+h/2)+N^{\text{trans}}(x,y,-h/2)\right]\,,\\ N_{\text{diff}}(x,y) &:= \frac{1}{2}\left[N^{\text{trans}}(x,y,+h/2)-N^{\text{trans}}(x,y,-h/2)\right]\,.\end{aligned} \tag{2.17}$$

In terms of (2.17) the local statement (2.16) yields two **linear** equations in ϱ_m, ϱ_b[10] the **exact solution** of which is given by

$$\begin{aligned}\begin{pmatrix}\varrho_m^{\mathrm{ex}}\\ \varrho_b^{\mathrm{ex}}\end{pmatrix} &= \frac{1}{(2\mu+\lambda)^2\,h - \underbrace{\frac{\lambda^2\,h^3}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle^2}_{\text{small of higher order}}}\\
&\quad\times\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda\,h^2}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle & (2\mu+\lambda)\end{pmatrix}\\
&\qquad\times\begin{pmatrix}\langle N_{\mathrm{diff}},\overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]\\ \langle N_{\mathrm{res}},\overline{R}_3\rangle\end{pmatrix}. \qquad (2.18)\end{aligned}$$

Skipping the indicated term of higher order we obtain the approximation

$$\begin{aligned}\varrho_m^{\mathrm{ex}} &\approx 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}\\
&\quad - \underbrace{\frac{\lambda\,h}{8(2\mu+\lambda)^2}}_{\text{small for }\lambda\gg 1}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\,\langle N_{\mathrm{res}},\overline{R}_3\rangle\,,\\
\varrho_b^{\mathrm{ex}} &\approx -\frac{\lambda}{2\mu+\lambda}\underbrace{\langle(\nabla\overline{R}_3|0),\overline{R}\rangle}_{\sharp} + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h} \qquad (2.19)\\
&\quad - \underbrace{\frac{\lambda}{2(2\mu+\lambda)^2}}_{\text{small for }\lambda\gg 1}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\langle N_{\mathrm{diff}},\overline{R}_3\rangle\\
&\qquad + \frac{\lambda^2}{(2\mu+\lambda)^2}\underbrace{\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]}_{\text{small for small elongational strain, compared to }\sharp}.\end{aligned}$$

For an almost rigid material with $\mu,\lambda\gg 1$ we have $\frac{\lambda}{(2\mu+\lambda)^2}\ll 1$, which motivates to neglect these terms. The term $\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right]$ represents a **nonlinear coupling** between midsurface in-plane (membrane) strain and normal curvature, a result of the derivation not present in the underlying three-dimensional theory where only products of deformation gradient and rotations occur. Since we have in mind a small strain situation, this product is one order smaller than $\langle(\nabla\overline{R}_3|0),\overline{R}\rangle$. Therefore, we neglect this term as well. Thus we set

[10] ϱ_m, ϱ_b have different units. ϱ_m is dimensionless, whereas $[\varrho_b] = \mathrm{m}^{-1}$.

finally

$$\begin{aligned}\varrho_m &:= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}\,, \quad \text{mainly } \textbf{membrane related}\,,\\ \varrho_b &:= \quad -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\,, \qquad \text{mainly } \textbf{bending related}\,.\end{aligned} \tag{2.20}$$

Note that the possibility to determine ϱ_m, ϱ_b exactly in (2.18) is predicated on the isotropy of the underlying model and the choice (2.7).

The last formula (2.20) has a clear **physical significance**:

1. to first order: transverse fibers will be symmetrically elongated by opposite transverse tractions and symmetrically shortened through in-plane stretch.
2. to second order: the midsurface will be asymmetrically shifted through bending, moderated through resulting transverse tractions.
3. in pure bending there is only a shift of the midsurface.

Having obtained a physically reasonable form of the relevant coefficients ϱ_m, ϱ_b, it is expedient to base the expansion and subsequent integration of the three-dimensional elastic energy on a further simplified expression. We take $\overline{F}_s$, where

$$F_s = \nabla\varphi_s(x,y,z) \approx \underbrace{(\nabla m|\,\varrho_m\,\overline{R}_3)}_{A_m} + z\,\underbrace{(\nabla\overline{R}_3|\varrho_b\,\overline{R}_3)}_{A_r} =: A_m + z\,A_r =: \overline{F}_s\,, \tag{2.21}$$

motivated by the form of the deformation gradient $F_s^{\mathrm{lin}} = (\nabla m|\overline{R}_3) + z(\nabla\overline{R}_3|0)$, based on a naive linear Reissner-Mindlin (1|1|0)-ansatz $\varphi_s^{\mathrm{lin}} = m + z\cdot\overline{R}_3$. Note that the **"assumed gradient"** $\overline{F}_s$ is in general **not** a **gradient** of some form of reconstructed deformation any more. It should be observed that by using (2.21) we are consistent with John's general result [28, 29] that the stress distribution through the thickness is approximately linear for a thin shell.

A simple but tedious calculation reveals now that (reminder $A_r := (\nabla\overline{R}_3|\varrho_b\overline{R}_3)$)

$$\begin{aligned}&\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R}\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 = \mu\|\,\mathrm{sym}(\overline{R}^T A_r)\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T A_r)\right]^2\\ &\quad = \mu\|\,\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\,.\end{aligned} \tag{2.22}$$

Exactly the same computations as for the bending term allows us to conclude that (reminder $A_m := (\nabla m|\varrho_m\overline{R}_3)$)

$$\frac{\mu}{4}\|\overline{R}^T A_m + A_m^T\overline{R} - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[\overline{R}^T A_m + A_m^T\overline{R} - 2\,\mathbb{1}\right]^2 \tag{2.23}$$

$$= \mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\right]^2 + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)}\,.$$

2.3. Dimensionally reduced energy: analytical integration through the thickness

Now we perform the analytical integration through the thickness in terms of the reduced kinematics. We insert the assumed expression $\overline{F}_s$ (2.21) and $\overline{R}_s$ instead of F and $\overline{R}^{3d}$ into the bulk energy (2.3). Since

$$\begin{aligned}\|\operatorname{sym}(\overline{R}_s^T\overline{F}_s)-1\!\!1\|^2 &= \frac{1}{4}\|A_m^T\overline{R}+\overline{R}^TA_m+z\,A_r^T\overline{R}+z\,\overline{R}^TA_r-21\!\!1\|^2\\ &=\frac{1}{4}\|A_m^T\overline{R}+\overline{R}^TA_m-21\!\!1\|^2+z\,\langle A_m^T\overline{R}+\overline{R}^TA_m-21\!\!1,A_r^T\overline{R}\rangle\\ &\quad+\frac{z^2}{4}\|A_r^T\overline{R}+\overline{R}^TA_r\|^2\,.\end{aligned}\tag{2.24}$$

and a similar result for $\operatorname{tr}\left[\operatorname{sym}(\overline{R}_s^T\overline{F}_s)-1\!\!1\right]^2$, we obtain by explicitly integrating over the (absolutely thin plate like referential) domain $\Omega_h=\omega\times[-\frac{h}{2},\frac{h}{2}]$, using (2.23) and (2.22)

$$\begin{aligned}&\int_\omega\int_{-\frac{h}{2}}^{\frac{h}{2}}W_{\mathrm{mp}}(\overline{F}_s,\overline{R}_s)\,\mathrm{dV}\\ &\quad=\int_\omega h\Big(\mu\|\operatorname{sym}(\overline{R}^T(\nabla m|\overline{R}_3))-1\!\!1\|^2+\frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla m|\overline{R}_3))-1\!\!1\right]^2\\ &\qquad+\frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)}+\frac{\langle N_{\mathrm{res}},\overline{R}_3\rangle^2}{24\,(2\mu+\lambda)}\Big)\,\mathrm{d}\omega\\ &\qquad+\int_\omega\frac{h^3}{12}\left(\mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2+\frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\right)\mathrm{d}\omega\,,\end{aligned}\tag{2.25}$$

and we call the factor of h the **membrane part** and the factor of h^3 the **bending part**, in line with the classical terminology. The result (2.25) shows the characteristic apparent change of the Lamé moduli for the two-dimensional structure in membrane and bending as well as the additive decoupling of these effects. Such a decoupling would also have been obtained by formal energy projection based on the naive linear Reissner-Mindlin ansatz $\varphi_s=m+z\,\overline{R}_3$. The final energy expression (2.25), however, cannot be obtained by energy projection.

2.4. Effective evolution of rotations: averaged generator on $\mathfrak{so}(3,\mathbb{R})$

It remains to reduce the three-dimensional evolution equation for the rotations of the thin structure into an evolution equation for some effective rotation defined over the midsurface ω only. Now consider the evolution equation for the viscoelastic rotations in (1.1) with $B=B_{\mathrm{mech}}$ first. If we insert F_s (2.8) instead of F we can completely reconstruct the three-dimensional evolution equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t,x,y,z) = \nu^+ \ \mathrm{skew}\left(F_s\overline{R}(t,x,y,z)^T\right) \cdot \overline{R}(t,x,y,z)\,. \tag{2.26}$$

In order to get some **effective equation** for rotations $\overline{R}$ which are defined over the two-dimensional referential domain ω only, we consider

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(x,y,z) &= \nu^+ \ \mathrm{skew}(B^{\mathrm{res,h}}_{\mathrm{mech}}) \cdot \overline{R}(x,y,z)\,,\\ B^{\mathrm{res,h}}_{\mathrm{mech}}(x,y) &:= \frac{1}{h}\int_{\frac{-h}{2}}^{\frac{h}{2}} \overline{F}_s\overline{R}(x,y,z)^T \ \mathrm{dz}\,, \end{aligned} \tag{2.27}$$

where $B^{\mathrm{res,h}}_{\mathrm{mech}}$ is the **thickness averaged generator** on the Lie-algebra of the evolution and we use $\overline{F}_s$ instead of F_s to be consistent with the simplification (2.21).[11] In addition we assume that the rotations do not depend on the transverse variable z, i.e. $\overline{R}(x,y,z) = \overline{R}(x,y,0)$. This leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(x,y,z) &= \nu^+ \ \mathrm{skew}\!\left(\frac{1}{h}\int_{\frac{-h}{2}}^{\frac{h}{2}} (A_m(x,y) + z\ A_r(x,y))\,\overline{R}(x,y,0)^T \ \mathrm{dz}\right) \cdot \overline{R}(x,y,z)\\ &= \nu^+ \ \mathrm{skew}\left(A_m(x,y)\overline{R}(x,y,0)^T\right)\cdot \overline{R}(x,y,z)\,. \end{aligned} \tag{2.28}$$

Hence an effective equation based on $B = B_{\mathrm{mech}}$ is

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(x,y) = \nu^+ \ \mathrm{skew}(B^{\mathrm{res,h}}_{\mathrm{mech}})\,\overline{R}(x,y)\,, \quad B^{\mathrm{res,h}}_{\mathrm{mech}} = A_m\overline{R}^T\,, \tag{2.29}$$

independent of the thickness h. This result could have been obtained by setting $z = 0$ in (2.26) incidentally. The derivation for $B = B_{\mathrm{tc}}$ proceeds similarly. We need the averaged quantity $B^{\mathrm{res,h}}_{\mathrm{tc}}(x,y) := \frac{1}{h}\int_{\frac{-h}{2}}^{\frac{h}{2}} B_{\mathrm{tc}}(x,y,z)\,\mathrm{dz}$. A small calculation reveals

$$\begin{aligned} B^{\mathrm{res,h}}_{\mathrm{tc}} &= \left(\mu\,(2\mathbb{1} - A_m\overline{R}^T) + \lambda\,[3 - \langle A_m\overline{R}^T, \mathbb{1}\rangle]\right)\ A_m\overline{R}^T\\ &\quad - \frac{h^2}{12}\left(\mu A_r\overline{R}^T + \lambda\langle A_r\overline{R}^T, \mathbb{1}\rangle\right)\ A_r\overline{R}^T\,, \end{aligned} \tag{2.30}$$

i.e., the three-dimensional thermodynamically consistent evolution equation automatically furnishes a certain "bending" like influence in the viscoelastic flow, while the mechanically consistent evolution equation alone does not.

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[11] Rotations live on the nonlinear manifold $\mathrm{SO}(3,\mathbb{R})$ and cannot be averaged over the thickness, since the average might cease to be a rotation.

2.5. Deduction of the boundary conditions

Taking the Dirichlet boundary conditions for φ into account and the quadratic kinematical ansatz, we should have $\varphi_s(x,y,z)_{|_{\Gamma_0^h}} = g_{\mathrm{d}}(x,y,z)$ and

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\,\varrho_b(x,y) \right) \cdot \overline{R}_{s,3}(x,y,0)\,, \tag{2.31}$$

Evaluating for $\pm h/2$ yields two vector equations:

$$g_{\mathrm{d}}(x,y,\pm h/2) = m(x,y) + \left(\pm h/2\,\varrho_m(x,y) + \frac{h^2}{8}\,\varrho_b(x,y) \right) \cdot \overline{R}_{s,3}(x,y,0)\,. \tag{2.32}$$

Adding and subtracting shows

$$\begin{aligned} g_{\mathrm{d}}(x,y,+h/2) + g_{\mathrm{d}}(x,y,-h/2) &= 2\,m(x,y) + \frac{h^2}{4}\,\varrho_b(x,y) \cdot \overline{R}_{s,3}(x,y,0) \\ g_{\mathrm{d}}(x,y,+h/2) - g_{\mathrm{d}}(x,y,-h/2) &= h\,\varrho_m(x,y)\,\overline{R}_{s,3}(x,y,0) \Rightarrow \\ \nabla g_{\mathrm{d}}(x,y,0).e_3 &= \varrho_m(x,y)\,\overline{R}_{s,3}(x,y,0) + o(h)\,. \end{aligned} \tag{2.33}$$

This implies to first order $m(x,y) = \frac{1}{2}\left(g_{\mathrm{d}}(x,y,+h/2) + g_{\mathrm{d}}(x,y,-h/2)\right) \approx g_{\mathrm{d}}(x,y,0)$, which we take as reduced boundary condition for simple support. It is also suggested that one should take $\overline{R}_{s,3}(x,y,0) = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}$. However, for a membrane plate it is not possible to specify higher order boundary conditions corresponding to some sort of clamping.

3. The reduced viscoelastic membrane-plate model

Since in the underlying three-dimensional model (1.1) we minimized the elastic energy at fixed rotations $\overline{R}$ we are led to minimizing (2.25) with respect to the deformation of the midsurface m at fixed reduced rotation $\overline{R} : \omega \mapsto \mathrm{SO}(3)$. We observe that the term $h\left(\frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)} + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{24\,(2\mu+\lambda)} \right)$ in the membrane part of the reduced energy (2.25) does not contribute to the minimization w.r.t. the membrane deformation m. The same is true for the complete h^3-bending expression in (2.25).

3.1. The two-dimensional membrane-plate

Collecting all the former results we postulate the following coupled problem for the deformation of the midsurface of the membrane plate $m : [0,T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the independent local viscoelastic rotation $\overline{R} : [0,T] \times \overline{\omega} \mapsto \mathrm{SO}(3,\mathbb{R})$ on ω

$$\int_\omega h\,W(F,\overline{R})\,\mathrm{d}\omega - \Pi(m,\overline{R}_3) \mapsto \min.\ \text{w.r.t. } m \text{ at fixed } \overline{R}\,, \tag{3.1}$$

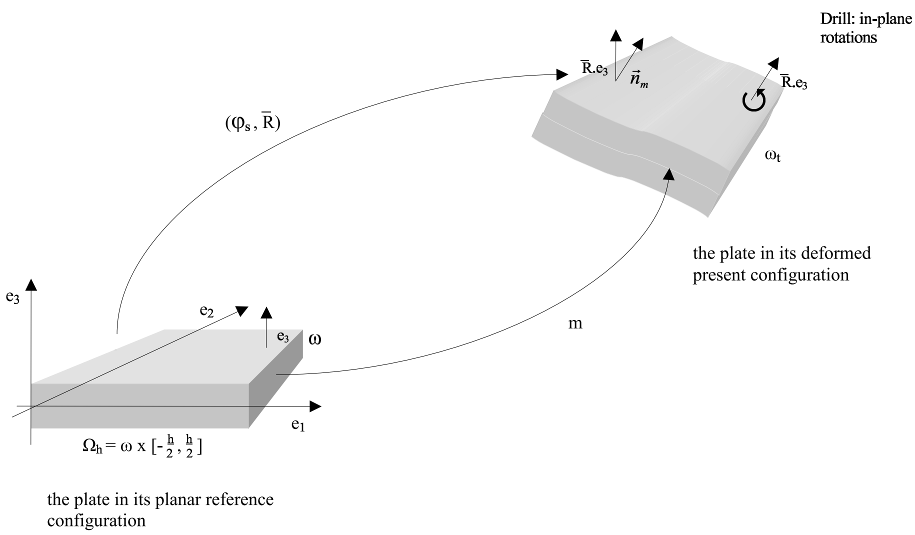

Figure 1. The assumed membrane-plate kinematics incorporating viscoelastic transverse shear resistance ($\overline{R}_3 \neq \vec{n}_m$), instantaneous thickness stretch ($\varrho_m \neq 1$) and viscoelastic drill-rotations. Reconstructed three-dimensional deformation
$\varphi_s : \Omega_h \mapsto \mathbb{R}^3$, $\varphi_s(x,y,z) = m(x,y) + z\,\varrho_m(x,y)\,\overline{R}_3(x,y)$, midsurface deformation $m : \omega \mapsto \mathbb{R}^3$, independent viscoelastic rotation $\overline{R} : \omega \mapsto \mathrm{SO}(3,\mathbb{R})$.

with prescribed Dirichlet boundary conditions for simple support $m_{|_{\gamma_0}}(t,x,y) = g_{\mathrm{d}}(t,x,y,0)$, $\gamma_0 \subset \partial\omega$. The constitutive assumptions on the reduced density are

$$W(F,\overline{R}) := \frac{\mu}{4}\,\|F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}\right]^2 \tag{3.2}$$
$$F = (\nabla m|\varrho_m\,\overline{R}_3), \quad \varrho_m = 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\mathrm{diff}},\overline{R}_3\rangle}{2\mu+\lambda}\,.$$

The effective viscoelastic evolution for the "moving orthonormal three-frame" $\overline{R}(t,x,y) \in \mathrm{SO}(3,\mathbb{R})$ is given by

$$\begin{aligned}
\frac{\mathrm{d}_{\hat{\omega}}}{\mathrm{d}t}\overline{R}(t) &= \nu^+ \cdot \mathrm{skew}\,(B^{\mathrm{res}}) \cdot \overline{R}(t)\,, \\
B^{\mathrm{res}} &= B^{\mathrm{res,h}}_{\mathrm{mech}} \quad \text{or} \quad B^{\mathrm{res,h}}_{\mathrm{tc}}\,, \\
B^{\mathrm{res,h}}_{\mathrm{tc}} &= \left[\mu(2\,\mathbb{1} - F\overline{R}^T) + \lambda\left[3 - \langle F\overline{R}^T, \mathbb{1}\rangle\right]\right] F\overline{R}^T \\
&\quad - \frac{h^2}{12}\left(\mu A_r\overline{R}^T + \lambda\langle A_r\overline{R}^T, \mathbb{1}\rangle\right) A_r\overline{R}^T\,,
\end{aligned} \tag{3.3}$$

where

$$A_r = (\nabla \overline{R}_3 | \varrho_b \, \overline{R}_3) \,, \quad \varrho_b = -\frac{\lambda}{2\mu+\lambda} \langle (\nabla \overline{R}_3 | 0), \overline{R} \rangle + \frac{\langle N_{\mathrm{res}}, \overline{R}_3 \rangle}{(2\mu+\lambda)\, h} \,. \tag{3.4}$$

The effective viscoelastic evolution (3.3) is a local, nonlinear ordinary differential equation for $B^{\mathrm{res}} = B^{\mathrm{res,h}}_{\mathrm{mech}}$, but turns into a nonlocal, nonlinear first order partial differential system for $\overline{R}$ in case of $B^{\mathrm{res}} = B^{\mathrm{res,h}}_{\mathrm{tc}}$ if $h > 0$. Subsequently, we restrict attention to the simpler local choice $B^{\mathrm{res}} = B^{\mathrm{res,h}}_{\mathrm{mech}}$.

Here, Π is the linear functional of resultant external loading, cf.(6.8). We have already observed (2.23) that for $\widehat{F} = (\nabla m | \overline{R}_3)$ and $N_{\mathrm{diff}} = 0$ in fact

$$\begin{aligned} W(F, \overline{R}) &= \mu \, \| \operatorname{sym} \left(F^T \overline{R} - 1\!\!1 \right) \|^2 + \frac{\lambda}{2} \operatorname{tr} \left[\operatorname{sym} \left(F^T \overline{R} - 1\!\!1 \right) \right]^2 \\ &= \mu \, \| \operatorname{sym} \left(\widehat{F}^T \overline{R} - 1\!\!1 \right) \|^2 + \frac{\mu\lambda}{(2\mu+\lambda)} \operatorname{tr} \left[\operatorname{sym} \left(\widehat{F}^T \overline{R} - 1\!\!1 \right) \right]^2 , \end{aligned} \tag{3.5}$$

showing the characteristic apparent change of the Lamé moduli for the two-dimensional structure.[12]Observe that $\frac{\mu\lambda}{(2\mu+\lambda)} = \frac{2}{\frac{1}{\mu}+\frac{2}{\lambda}}$ is half the **harmonic mean** of μ and $\frac{\lambda}{2}$.

3.2. Uniform Legendre-Hadamard ellipticity

Let us consider the membrane contribution to the elastic free energy (for simplicity take $\mu > 0$, $\lambda = 0$)

$$\begin{aligned} &\int_\omega h W(F, \overline{R}) \, \mathrm{d}\omega \\ &\quad = h \int_\omega \frac{\mu}{4} \| (\nabla m | \varrho_m(\nabla m, \overline{R}) \, \overline{R}_3)^T \overline{R} + \overline{R}^T (\nabla m | \varrho_m(\nabla m, \overline{R}) \, \overline{R}_3) - 2 1\!\!1 \|^2 \, \mathrm{d}\omega \,. \end{aligned} \tag{3.6}$$

It is easy to see that this remaining membrane energy density is **uniformly Legendre-Hadamard elliptic** at frozen $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$ with ellipticity constant μ **independent** of $\overline{R}(x, y)$, since its second differential with respect to m verifies (reminder $F = (\nabla m | \varrho_m \overline{R}_3)$ and (3.5))

$$\begin{aligned} &\forall \, H \in \mathbb{M}^{2\times 3} \qquad D^2_{\nabla m} W(F, \overline{R}).(H, H) \geq \frac{\mu}{2} \| (H|0)^T \overline{R} + \overline{R}^T (H|0) \|^2 \;\Rightarrow \\ &\forall \, \xi \otimes \eta \in \mathbb{M}^{2\times 3} \quad D^2_{\nabla m} W(F, \overline{R}).(\xi \otimes \eta, \xi \otimes \eta) \geq \mu \, \|\xi\|^2_{\mathbb{R}^3} \cdot \|\eta\|^2_{\mathbb{R}^2} \,. \end{aligned} \tag{3.7}$$

Moreover, the membrane energy is a convex functional in ∇m at frozen $\overline{R}$, later we will see that it is indeed uniformly convex if integrated over ω also for nonconstant rotations $\overline{R}$ if $\overline{R}$ satisfies some additional smoothness requirements. This is precisely the property which can be exploited to our advantage in a subsequent mathematical analysis.

[12] It is not expedient to use $\widehat{F}$ in general in (3.1) since it is F which appears in the local evolution.

Remark 3.1. A possible advantage of the resulting model (3.1) is the fact that the membrane part alone is not degenerate. This has to be paid with the additional internal viscoelastic relaxation which is, however, only a local problem and does not involve additional field equations. Spatially discontinuous rotations $\overline{R}(x,y)$ are no major numerical concern, since they are only local quantities. Usually, the implementational burden associated with either a fourth order system coming from the classical Kirchoff-Love ansatz or the additional field equations for the rotations $\overline{R}$ in a Reissner-Mindlin (restricted Cosserat-surface) type theory counterbalances the gain of the dimensional reduction. A particular appealing feature of the model (3.1) is the absence of a C^1-continuity requirement and the absence of additional field equations. For the membrane equilibrium part any standard 2D-H^1-finite element might be suitable.

3.3. Observer-invariance of the reduced viscoelastic model

Observer-invariance amounts to the requirement of invariance of the stresses in model (3.1) with respect to superposed rigid body rotations $Q \in \mathrm{SO}(3,\mathbb{R})$ in the sense that

$$\forall\, Q \in \mathrm{SO}(3): \quad QS_1(F,\overline{R}) = S_1(QF, Q\overline{R})\,, \tag{3.8}$$

where S_1 is the first Piola-Kirchhoff stress tensor. In our context we check invariance of the model under the transformation $(m,\overline{R}) \mapsto (Q.m, Q\overline{R})$. Now,

$$\begin{aligned} W((\nabla Q.m|((Q\overline{R})_3), Q\overline{R})) &= W((Q\nabla m|((Q\overline{R})_3), Q\overline{R}) = W(Q(\nabla m|\overline{R}_3), Q\overline{R}) \\ &= W(QF, Q\overline{R}) = W(F,\overline{R}) = W((\nabla m|\overline{R}_3),\overline{R})\,, \end{aligned} \tag{3.9}$$

by frame-indifference of the 3D-strain energy density. The evolution equation for the rotations is also observer-invariant due to the use of the **corotated time derivative** $\frac{\mathrm{d}_{\hat\omega}}{\mathrm{dt}}$. Thus the invariance of the reduced thin plate viscoelastic model under $m \mapsto Q.m,\ \overline{R} \mapsto Q\overline{R}$ is guaranteed. However, unlike classical theories based on just one hyperelastic free energy formulation and Hamilton's principle, where frame-indifference of the energy implies balance of external angular momentum, this is not true in the case (3.1) due to a viscoelastic dissipative nonsymmetric stress contribution coming from the evolution equation for $\overline{R}$.

3.4. Thin membrane-plate non-elliptic relaxation limit

If the viscosity is related to friction occuring at internal surfaces, it is reasonable to assume that the viscosity for the plate should scale like $\nu^+ \sim \frac{1}{h^3}$ with h the plate thickness. Hence, the (vanishing elastic viscosity) limit $\nu^+ \to \infty$ corresponds to the interesting limit of vanishing thickness $h \to 0$.

Assume now that for a sequence of vanishing viscosity $\nu_k^+ \to \infty$ we obtain a corresponding sequence $m_k, \overline{R}_k$ as solutions to the problem (3.1) with thickness stretch $\varrho_m \equiv 1$ (for simplicity only) and which converges to $\widehat{m} \in C^1(\mathbb{R}^+, H^1(\omega, \mathbb{R}^3))$ and $\widehat{\overline{R}} \in C^1(\mathbb{R}^+, L^\infty(\omega, \mathrm{SO}(3, \mathbb{R})))$, respectively. Then the limit membrane deformation $\widehat{m}$ and rotation $\widehat{\overline{R}}$ satisfy forall times (note that $\mathrm{skew}(F\overline{R}^T) = 0 \sim \overline{R} = \mathrm{polar}(F)$, c.f. Lemma 6.3 and recall Theorem 1.1)

$$\begin{aligned}&\int_\omega h\, W(F, \overline{R})\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min . \text{ w.r.t. } m \text{ at fixed } \overline{R}\,, \\ &F = (\nabla m|\overline{R}_3), \quad \overline{R} = \mathrm{polar}(F) = \mathrm{polar}\left((\nabla m|\overline{R}_3)\right)\,,\end{aligned} \tag{3.10}$$

and the computed equilibrium energy level at a given time is

$$\begin{aligned}W(F, \overline{R}) &= \frac{\mu}{4}\, \|F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\right]^2 \\ &= \mu\, \|U - 1\!\!1\|^2 + \frac{\lambda}{2}\, \mathrm{tr}\, [U - 1\!\!1]^2 =: W_\infty(U)\,,\end{aligned} \tag{3.11}$$

with $U = (F^TF)^{\frac{1}{2}}$ the classical **symmetric** elastic stretch and $U - 1\!\!1$ the elastic Biot strain tensor. Remark, however, that it is not W_∞ which underlies the variational problem (3.10).

Let us investigate in more detail this limit equilibrium system in the viscoelastic case without external loads and without loss of generality only based on the simplified energy expression ($\mu = 1, \lambda = 0$)

$$W(\nabla m, \overline{R}) = \frac{1}{4}\|(\nabla m|\overline{R}_3)^T\overline{R} + \overline{R}^T(\nabla m|\overline{R}_3) - 2 1\!\!1\|^2. \tag{3.12}$$

Since $\widehat{m}$ minimizes (3.10) with respect to m at fixed $\widehat{\overline{R}} \in \mathrm{SO}(3, \mathbb{R})$, we have necessarily for the relaxation limit $\widehat{m}, \widehat{\overline{R}}$

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{dt}}_{|t=0} \int_\omega W((\nabla\widehat{m} + t\nabla\phi|\widehat{\overline{R}}_3), \widehat{\overline{R}})\ \mathrm{d}\omega \\ &= \frac{1}{2}\langle(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T\widehat{\overline{R}} + \widehat{\overline{R}}^T(\nabla\widehat{m}|\widehat{\overline{R}}_3) - 2 1\!\!1, (\nabla\phi|0)^T\widehat{\overline{R}} + \widehat{\overline{R}}^T(\nabla\phi|0)\rangle_\omega \\ &= \langle(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T\widehat{\overline{R}} + \widehat{\overline{R}}^T(\nabla\widehat{m}|\widehat{\overline{R}}_3) - 2 1\!\!1, (\nabla\phi|0)^T\widehat{\overline{R}}\rangle_\omega, \quad \forall\, \phi \in H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0)\,.\end{aligned} \tag{3.13}$$

Now based on the identity $\mathrm{polar}(X)^T \cdot \mathrm{polar}(X) = 1\!\!1$ for $X \in \mathrm{GL}(3, \mathbb{R})$ the (pointwise) expansion

$$\begin{aligned}\mathrm{polar}((\nabla\widehat{m} + H|\widehat{\overline{R}}_3)) &= \mathrm{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3) + (H|0)) \\ &= \mathrm{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)) + D\,\mathrm{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0) + \ldots\end{aligned} \tag{3.14}$$

with $H \in \mathbb{M}^{2\times 3}$ implies that

$$\mathrm{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3))^T \cdot D\,\mathrm{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(\nabla\phi|0) \in \mathfrak{so}(3). \tag{3.15}$$

Taking $U = \frac{1}{2}(F^T\widehat{\overline{R}} + \widehat{\overline{R}}^T F)$ if $\widehat{\overline{R}} = \operatorname{polar}(F)$ into account and computing the variation with respect to $\widehat{m}$ at fixed column $\widehat{\overline{R}}_3$ of

$$\begin{aligned}\|U((\nabla\widehat{m}|\widehat{\overline{R}}_3)) - 1\!\!1\|^2_\omega = \frac{1}{4}\|(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T \operatorname{polar}(\nabla\widehat{m}|\widehat{\overline{R}}_3)\\ + \operatorname{polar}(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T(\nabla\widehat{m}|\widehat{\overline{R}}_3) - 2 1\!\!1\|^2_\omega\,,\end{aligned} \tag{3.16}$$

we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}_{|t=0}\|U((\nabla(\widehat{m}+t\phi)|\widehat{\overline{R}}_3)) - 1\!\!1\|^2_\omega = \qquad (3.17)\\
&= \frac{1}{2}\langle(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T \operatorname{polar}(\nabla\widehat{m}|\widehat{\overline{R}}_3) + \operatorname{polar}(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T(\nabla\widehat{m}|\widehat{\overline{R}}_3) - 2 1\!\!1, \rangle\\
&(\nabla\phi|0)^T\widehat{\overline{R}} + \widehat{\overline{R}}^T(\nabla\phi|0)_\omega +\\
&\quad\langle U - 1\!\!1(\nabla\widehat{m}|\widehat{\overline{R}}_3)^T D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0)\\
&+ D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0)^T(\nabla\widehat{m}|\widehat{\overline{R}}_3)\rangle_\omega\\
&\overset{(3.13)}{=} 0 + \langle U - 1\!\!1, (\nabla\widehat{m}|\widehat{\overline{R}}_3)^T\widehat{\overline{R}}\,\widehat{\overline{R}}^T D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0) +\\
&\qquad\qquad D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0)^T\widehat{\overline{R}}\,\widehat{\overline{R}}^T(\nabla\widehat{m}|\widehat{\overline{R}}_3)\rangle_\omega\\
&= \langle U - 1\!\!1, U\widehat{\overline{R}}^T D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0) + D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0)^T\widehat{\overline{R}}U\rangle_\omega\\
&\overset{(3.15)}{=} \langle U^2 - U, \widehat{\overline{R}}^T D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0) + D\operatorname{polar}((\nabla\widehat{m}|\widehat{\overline{R}}_3)).(H|0)^T\widehat{\overline{R}}\rangle_\omega = 0\,,
\end{aligned}$$

by (3.13) and (3.15), since U is symmetric. Thus we have proved that if the equilibrium relaxation limit exists in fact

$$\begin{aligned}
\int_\omega h\,W_\infty(U((\nabla m|\overline{R}_3))\,\mathrm{d}\omega - \Pi(m,\overline{R}_3) \;&\mapsto \text{stat .w.r.t. } m \text{ at fixed } \overline{R}_3\,,\\
\overline{R} &= \operatorname{polar}\left((\nabla m|\overline{R}_3)\right)\,,\\
W_\infty(U) = \mu\,\|U - 1\!\!1\|^2 &+ \frac{\lambda}{2}\operatorname{tr}[U - 1\!\!1]^2\,,
\end{aligned} \tag{3.18}$$

is solved by $\widehat{m}, \widehat{\overline{R}}$. This means that at fixed viscoelastic "director" $\overline{R}.e_3$, the membrane energy is **stationary**, but no claim as respects minimality of this solution can be made and indeed it is very likely to end up in a **metastable** state by which we mean a local but not a global minimum.

In this relaxation limit the true **Cauchy-stresses** $\sigma = \frac{1}{\det[F]}S_1F^T$ turn out to be **symmetric** upon inspection of (3.1), which means that classical balance of external angular momentum constrains the theory to its relaxed version. It might be worth remembering, however, that in continuum mechanics balance of external angular momentum is an additional hypotheses, independent of balance of linear momentum and frame-indifference [23, p.137].

We push the analysis of the elastic case further: the holonomic constraint $\overline{R} = \text{polar}\big((\nabla m|\overline{R}_3)\big)$ in (3.18) is essentially a generalization of the normality condition for the unit outward normal to the surface m, the director $\vec{n}_m$ in a Kirchhoff-Love model. To see this we note that the condition $\overline{R} = \text{polar}(\nabla m|\overline{R}_3)$ implies already $\overline{R}_3 = \vec{n}_m$. Thus $\overline{R}_3$ coincides with the unit normal on the midsurface $\vec{n}_m$. This is a welcome feature of the theory since **normality has not been imposed** yet anywhere. Since

$$U^2 = C = F^T F = (\nabla m|\vec{n}_m)^T(\nabla m|\vec{n}_m) = \begin{pmatrix} \|m_x\|^2 & \langle m_x, m_y\rangle & 0 \\ \langle m_x, m_y\rangle & \|m_y\|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.19)$$

we understand that $U = U((\nabla m|\vec{n}_m))$ is in fact independent of $\vec{n}_m$, such that in the **elastic relaxation** equilibrium limit we have actually solved the **intrinsic, purely elastic**[13] problem ($\varrho_m \neq 1$)

$$\int_\omega h\, W_\infty(U((\nabla m|\vec{n}_m))\,\mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \text{ stat . w.r.t. } m\,,$$
$$W_\infty(U) := \mu\, \|U - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\text{tr}\,[U - 1\!\!1]^2\,. \quad (3.20)$$

Note that $W_\infty(U)$ is a **non-quasiconvex, non-elliptic** elastic energy w.r.t. ∇m but convex in U, ensuring in fact the **Baker-Ericksen inequalities**.[14]Currently there are no mathematical theorems available establishing the existence of minimizers or stationary points based directly on W_∞. In this sense, the viscoelastic formulation (3.1) provides a physical **regularization** of the occurring loss of ellipticity. The **linearization** of (3.20) **coincides with** the classical, rigourously justified **linearized membrane plate**, cf. [15].

To sum up, we have motivated that **normality** of the **director $\overline{R}_3$ is an asymptotic feature** of our model **for vanishing absolute thickness** or **the absolutely thinner the shell the less transverse shear** is possible.

4. Local existence and uniqueness

In this part we sketch the methods and mathematical tools which allow us to establish a local existence and uniqueness result. Since the formal structure of energy projection does not obtain for our membrane model, it is not possible to simply transfer the three-dimensional existence and uniqueness result [42] to a reduced ansatz space. However, the ideas used in [42] still apply.

[13] intrinsic: depending only on the first fundamental form $I_m = \nabla m^T \nabla m \in \mathbb{M}^{2\times 2}$ of the surface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$.

[14] One version of the **BE**-inequalities for membranes can be stated as follows: for $\lambda_i^2 \geq 0$, $i = 1, 2\,, \lambda_3^2 = 1$ the (generalized) principal stretches (here λ_i^2 are the eigenvalues of $(\nabla m|\vec{n})^T(\nabla m|\vec{n})$), the free energy $\Phi(\lambda_1, \lambda_2, 1) := \hat{W}(\nabla m^T \nabla m) = W_\infty(U)$ is **separately convex** in λ_i. No mathematical existence results based only on BE are known. Note also that BE is enough to effectively exclude phase-transformations, modelled with multi-well potentials.

At frozen viscoelastic rotations $\overline{R}$ the equilibrium system corresponding to (3.1) proves to be a linear, second order, strictly Legendre-Hadamard elliptic boundary value problem with nonconstant coefficients set by $\overline{R}(t,x,y)$. This system has variational structure in the sense that the equilibrium part of (3.1) is formally equivalent to the minimization problem

$$
\begin{aligned}
\forall\, t \in [0,T]: \quad & I(m(t), \overline{R}(t)) \mapsto \min . \,\text{w.r.t. } m, \quad m(t) \in g_{\mathrm{d}}(t) + H^{1,2}_{\circ}(\omega, \mathbb{R}^3; \gamma_0)\,, \\
I(m, \overline{R}) &:= \int_\omega h\, W(F, \overline{R})\, \mathrm{d}\omega - \Pi(m, \overline{R}_3)\,, \qquad (4.1)\\
W(F, \overline{R}) &:= \frac{\mu}{4}\, \|F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\right]^2, \\
F &= (\nabla m | \varrho_m \overline{R}_3), \quad \varrho_m = \varrho_m(\nabla m, \overline{R})\,.
\end{aligned}
$$

The main task in proving that (3.1) is well posed consists of showing uniform estimates for solutions of elliptic systems whose coefficients are time dependent and do not induce a pointwise uniformly positive bilinear form. Thus we are first concerned with the static situation where $\overline{R}$ is assumed to be known. We prove the existence, uniqueness and regularity of solutions to the two-dimensional boundary value problem corresponding to (4.1). In addition we elucidate in which manner these solutions depend on the rotations $\overline{R}$. Decisive use is made of the following new two-dimensional coercivity inequality:

Theorem 4.1 (Improved Korn's inequality for rigid plates and shells). *Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\gamma_0 \subset \partial\omega$ be a part of the boundary with non vanishing 1-dimensional Hausdorff measure. Define $H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0) := \{\phi \in H^{1,2}(\omega) \mid \phi_{|\gamma_0} = 0\}$ and let $F_p, F_p^{-1} \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$. Then*

$$
\begin{aligned}
&\exists\, c^+ > 0\ \forall\, \phi \in H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0): \\
&\qquad \|(\nabla\phi|0)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\omega)}\,, \qquad (4.2)
\end{aligned}
$$

and the constant is bounded away from zero for F_p, F_p^{-1} bounded in $W^{1,2+\delta}(\overline{\omega}$, $\mathrm{GL}(3,\mathbb{R}))$.

Proof. The proof is based on a generalized three-dimensional Korn's first inequality [39, 47] and subsequent dimensional reduction; it can be found in [40, 44]. $\square$

We have not yet specified the form of ν^+. One possible choice is to take ν^+ scaled with the thickness of the plate h (not necessary) and set formally similar to a viscoplastic Norton-Hoff formulation

$$
\nu^+ := \frac{[1\mathrm{m}]^3}{h^3\,\eta} \left(1 + \left[\frac{\|\,\mathrm{skew}(\mu\, F\overline{R}^T)\| - 0}{\bar{\sigma}_0}\right]_+^{r+1}\right)^k \cdot \left[\frac{\|\,\mathrm{skew}\left(\mu\, F\overline{R}^T\right)\| - 0}{\bar{\sigma}_0}\right]_+^{r-1}, \qquad (4.3)
$$

with η a **relaxation time**, $\bar{\sigma}_0 = 1[\mathrm{MPa}]$ and positive parameters r, k.

The conceptual idea to treat the evolution problem is then straightforward: we write the ordinary differential equation (3.3) in the following form

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) = \mathfrak{f}(\nabla_x m(\overline{R}), \overline{R}) \cdot \overline{R}\,, \tag{4.4}$$

with some "nice" function $\mathfrak{f} : \mathbb{M}^{2\times3} \times \mathbb{M}^{3\times3} \mapsto \mathfrak{so}(3,\mathbb{R})$ and where $m = m(\overline{R})$ is the unique solution of the elliptic second order two-dimensional boundary value problem corresponding to (4.1) at fixed rotations $\overline{R}$.

It remains to show that the right hand side of (4.4) as a function of $\overline{R}$ is locally Lipschitz,[15] allowing to apply the standard local existence and uniqueness theorem. With appropriate changes this program can be carried out similar to [38, 42], but will be presented in detail elsewhere [43]. Thus we are in a position to announce the following result for the case of the everywhere ($\gamma_0 = \partial\omega$) simply supported finite-strain viscoelastic membrane-plate:

Theorem 4.2 (Local existence and uniqueness for the viscoelastic membrane-plate). *Let $\omega \subset \mathbb{R}^2$ be a bounded smooth domain and suppose for the displacement boundary data $g_{\mathrm{d}} \in C^1(\mathbb{R}, H^{3,2}(\omega,\mathbb{R}^3))$. Moreover, assume for the resultant body force $\overline{f} \in L^2(\omega,\mathbb{R}^3)$, see* (6.8). *Assume for the initial condition on the rotation $\overline{R}^0 \in H^{2,2}(\omega, \mathrm{SO}(3)))$. Then there exists a time $t_1 > 0$ such that the initial boundary value problem* (3.1) *with $B^{\mathrm{res}} = B^{\mathrm{res,h}}_{\mathrm{mech}}$ and ν^+ according to* (4.3) *together with $\gamma_0 = \partial\omega$ admits a unique solution*

$$(m, \overline{R}) \in C([0,t_1], H^{3,2}(\omega,\mathbb{R}^3)) \times C^1([0,t_1], H^{2,2}(\omega, \mathrm{SO}(3))). \qquad \square$$

Remark 4.3. The level of smoothness required and the kind of boundary conditions are due to technical details pending on the use of refined elliptic regularity.

5. Discussion and concluding remarks

In this contribution we have formally derived membrane-plate equations for viscoelastic materials at small elastic strains starting from a given three-dimensional formulation. The ensuing theory is neither a Kirchhoff-Love nor a Reissner-Mindlin (restricted Cosserat surface) type theory, but combines elements of both theories together with the use of the specific strain measure $\mathrm{sym}\,\overline{R}^T F - 1\!\!1$ and a nonstandard treatment of finite rotations. The derivation turns out to be straight forward in the elastic case once the correct corresponding kinematical assumption for small elastic strains on the underlying finite deformation of the plate is made. The resulting equations in the thin plate limit, where the possibility of bending-like influence in the viscoelastic evolution problem has been neglected, retain a

[15] This is more than a simple requirement on $\mathfrak{f}$; precise estimates of the non-local solution operator $\overline{R} \mapsto m(\overline{R})$ are involved.

particular simple form. The dimensionally reduced system inherits in a natural way the observer-invariance of the three-dimensional formulation which is a basic requirement in continuum mechanics.

A special feature of the new system (3.1) is that the remaining membrane part at frozen viscoelastic rotations $\overline{R}$ is uniformly Legendre-Hadamard elliptic and indeed non-degenerate due to a novel extended Korn's first inequality applicable to thin plates (and shells). This structure of the resulting plate model allows to prove a local existence and uniqueness result following the ideas which made the treatment of the three-dimensional system possible [38, 42]. The model is locally in time well-posed independent of the thickness $h > 0$. And it is again this structure which should prove its worth when doing numerical calculations: only a standard 2D-H^1-finite element is in principal required in refreshing contrast to the ubiquitous C^1-smoothness requirement for Kirchhoff-Love shells. The numerical treatment of the evolution equations may follow merely standard practice in finite-strain elasto-plasticity (exponential-update for the rotations and consistent tangent). An extension of the model (3.1) and the announced mathematical results to finite-strain viscoelastic-viscoplastic membrane plates and shells is already known to the author but will be detailed in a future contribution.

In conclusion it can be seen that the general assumption of small elastic strains (almost rigidity) in conjunction with a non-standard treatment of finite rotations represents a refreshing departure from more traditional degenerate approaches. It opens a rich and as yet mostly unexplored structure linking the well established infinitesimal, linear theories to the at present analytically difficult two-dimensional, geometrically exact finite-strain problems.

Acknowledgements

This paper was conceived in the academic year 2001/2002 while the author held a visiting faculty position under the ASCI program in Michael Ortiz group at the California Institute of Technology, Graduate Aeronautical Laboratories, Pasadena. The author would like to thank Fehmi Cirak and Kerstin Weinberg for helpful discussions and Michael Ortiz for first directing his attention to thin structures.

The author is also grateful for several discussions with Alexander Mielke, Friedrich Gruttmann and Jörg Schröder which were of great help in clarifying important parts of the present work.

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6. Appendix

6.1. Notation

Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively.

We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \,|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \,|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^T X = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \,|X^T X = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \,|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \,|\mathrm{tr}\,[X] = 0\}$ of traceless tensors. We set $\mathrm{sym}(X) = \frac{1}{2}(X^T + X)$ and $\mathrm{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$.

We write the polar decomposition in the form $F = RU = \mathrm{polar}(F)\,U$ with $R = \mathrm{polar}(F)$ the orthogonal part of F and U the symmetric stretch. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Furthermore, $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{dt}}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H,H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega} \|X(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we define $\mathrm{Curl}\,A(x)$ as the operation curl applied row wise. We define $H^{1,2}_\circ(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$.

Notation for shells

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The thickness of the plate is taken to be $h > 0$ with dimension length (contrary to Ciarlet's definition of the thickness to be 2ε, which difference leads to various different constants in the resulting formulas). We denote by $\mathbb{M}^{n\times m}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{2\times 3}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3\times 3}$ to denote the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. The identity tensor on $\mathbb{M}^{2\times 2}$ will be denoted by $1\!\!1_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, $\nabla m = (m_x|m_y)$ is the corresponding deformation gradient with $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$. The standard volume element is written $\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = \mathrm{dV} = \mathrm{d}\omega\,\mathrm{dz}$.

6.2. The treatment of external loads

Dead load body forces for the thin plate

In the three-dimensional theory the dead load body forces $f(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term $\int_{\Omega_h} f(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dV}$. Inserting the quadratic ansatz for the reconstructed deformation φ_s results in the approximation

$$\begin{aligned}
\int_{\Omega_h} f(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dV} &\approx \int_{\Omega_h} f(x,y,z) \cdot \left[m(x,y) + z\,\varrho_m \overline{R}_3 + \frac{z^2}{2}\varrho_b\,\overline{R}_3 \right] \mathrm{dV} \\
= \int_\omega h\,\hat{f}(x,y) \cdot m(x,y)\,\mathrm{d}\omega &+ \int_\omega \left(\int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz} \right) \varrho_m \overline{R}_3\,\mathrm{d}\omega \\
+ \int_\omega \left(\int_{-h/2}^{h/2} \frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz} \right) \varrho_b \overline{R}_3\,\mathrm{d}\omega & \qquad (6.1)
\end{aligned}$$

Let us define

$$\begin{aligned}
\hat{f}_0(x,y) &:= \int_{-h/2}^{h/2} f(x,y,z)\,\mathrm{dz}\,, \\
\hat{f}_1(x,y) &:= \int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz}\,, \\
\hat{f}_2(x,y) &:= \int_{-h/2}^{h/2} \frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz}\,, \qquad (6.2)
\end{aligned}$$

such that $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$ are the zero, first, second moment of f in thickness direction. This implies

$$\begin{aligned}
\int_{\Omega_h} f(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dV} &\approx \int_\omega \hat{f}_0(x,y) \cdot m(x,y)\,\mathrm{d}\omega \\
&+ \int_\omega \hat{f}_1(x,y)\varrho_m \overline{R}_3\,\mathrm{d}\omega + \int_\omega \hat{f}_2(x,y)\varrho_b \overline{R}_3\,\mathrm{d}\omega\,. \qquad (6.3)
\end{aligned}$$

Traction boundary conditions for the thin plate

In the three-dimensional theory the traction boundary forces $N(x,y,z) \in \mathbb{R}^3$, $[N] = \frac{[\text{Newt.}]}{[\text{m}]^2}$ were simply included by appending the potential with the term $\int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dS}$. Inserting our quadratic ansatz for the reconstructed deformation φ_s results in the approximation

$$\begin{aligned}
\int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} & N(x,y,z) \cdot \varphi(x,y,z)\ \mathrm{dS} \\
&\approx \int_{\omega \times \{-\frac{h}{2},\frac{h}{2}\}} N(x,y,z) \cdot \left[m(x,y) + z\varrho_m \overline{R}_3 + \frac{z^2}{2}\varrho_b \overline{R}_3 \right] \mathrm{dS}
\end{aligned}$$

$$+ \int_{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]} N(x,y,z) \cdot \left[m(x,y) + z \varrho_m \overline{R}_3 + \frac{z^2}{2} \varrho_b \overline{R}_3 \right] \mathrm{dS}.$$

Let us define on γ_s

$$\begin{aligned} \hat{N}_{\mathrm{lat},0}(x,y) &:= \int_{-h/2}^{h/2} N(x,y,z)\,\mathrm{dz}\,, \\ \hat{N}_{\mathrm{lat},1}(x,y) &:= \int_{-h/2}^{h/2} z\, N(x,y,z)\,\mathrm{dz}\,, \\ \hat{N}_{\mathrm{lat},2}(x,y) &:= \int_{-h/2}^{h/2} \frac{z^2}{2}\, N(x,y,z)\,\mathrm{dz}\,, \end{aligned} \tag{6.4}$$

such that $\hat{N}_{\mathrm{lat},0}$, $\hat{N}_{\mathrm{lat},1}$, $\hat{N}_{\mathrm{lat},2}$ are the zero, first, second moment of the tractions N at the lateral boundary in thickness direction. Hence

$$\begin{aligned} &\int_{\partial\Omega_h} N(x,y,z) \cdot \varphi(x,y,z)\,\mathrm{dS} \approx \int_\omega N_{\mathrm{res}}(x,y) \cdot m(x,y)\,\mathrm{d}\omega \\ &+ \int_\omega h\, N_{\mathrm{diff}}(x,y) \varrho_m \overline{R}_3\,\mathrm{d}\omega + \int_\omega \frac{h^2}{8} N_{\mathrm{res}} \varrho_b \overline{R}_3\,\mathrm{d}\omega \\ &+ \int_{\gamma_s} \hat{N}_{\mathrm{lat},0}(x,y) \cdot m(x,y)\,\mathrm{ds} + \int_{\gamma_s} \hat{N}_{\mathrm{lat},1}(x,y)\, \varrho_m \overline{R}_3\,\mathrm{ds} + \int_{\gamma_s} \hat{N}_{\mathrm{lat},2}(x,y)\, \varrho_b \overline{R}_3\,\mathrm{ds}\,, \end{aligned} \tag{6.5}$$

with

$$N_{\mathrm{res}} := [N(x,y,\frac{h}{2}) + N(x,y,-\frac{h}{2})]\,, \quad N_{\mathrm{diff}} := \frac{1}{2}[N(x,y,\frac{h}{2}) - N(x,y,-\frac{h}{2})]\,. \tag{6.6}$$

The external loading functional

Let us gather all influences of the external loading terms. In view of a reasonable simplification for membrane-plates we consider only those terms, which would have appeared, if we had made the restricted linear ansatz without thickness stretch $\varphi_s = m + z\,\overline{R}_3$. To leading order we have

$$\begin{aligned} \overline{f} &= \hat{f}_0 + N_{\mathrm{res}}\,, && \textbf{resultant body force} \\ \overline{M} &= \hat{f}_1 + h\,N_{\mathrm{diff}}\,, && \textbf{resultant body couple} \\ \overline{N} &= \hat{N}_{\mathrm{lat},0}\,, && \textbf{resultant surface traction} \\ \overline{M}_c &= \hat{N}_{\mathrm{lat},1}\,, && \textbf{resultant surface couple}. \end{aligned} \tag{6.7}$$

The **resultant loading functional** Π is given by

$$\Pi(m, \overline{R}_3) = \int_\omega \langle \overline{f}, m \rangle + \langle \overline{M}, \overline{R}_3 \rangle\,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, m \rangle + \langle \overline{M}_c, \overline{R}_3 \rangle\,\mathrm{ds}\,. \tag{6.8}$$

If we denote the dependence of Π on the loads of the underlying three-dimensional problem as $\Pi(f, N;\, m, \overline{R}_3)$, then it is easily seen that frame-indifference of the external loading functional is satisfied in the sense that $\Pi(Q.f, Q.N;\, Q.m, Q.\overline{R}_3) = \Pi(f, N;\, m, \overline{R}_3)$ for all rigid rotations $Q \in \mathrm{SO}(3,\mathbb{R})$. Since $\overline{R}$ is only a passive parameter in the static minimization problem (3.1) of the viscoelastic plate, the dependence in the resulting loading functional Π on $\overline{R}$ can be dropped.

6.3. The finite-strain membrane model of Fox/Simo

In [21] the following geometrically exact, frame-indifferent membrane model has been derived by formal asymptotic analysis based on the St. Venant-Kirchhoff energy. In a variational form the model can be written in our notation in the form of a minimization problem for the deformation of the midsurface of the membrane $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ on ω:

$$\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{C})\,\mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min.\ \text{w.r.t. } m\,, \quad m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0)\\
&\overline{C} = \widehat{F}^T\widehat{F}, \quad \widehat{F} = (\nabla m|\vec{n}_m), \quad F_s = (\nabla m|\varrho_m\,\vec{n}_m)\,, \qquad\qquad (6.9)\\
&\varrho_m = \frac{\langle N_{\mathrm{diff}}, \vec{n}_m\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\mathrm{tr}\left[\overline{C} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \vec{n}_m\rangle^2}{(2\mu+\lambda)^2}}\,,\\
&\qquad\qquad \text{first order thickness stretch}\,,\\
&W_{\mathrm{mp}}(\overline{C}) = \frac{\mu}{4}\,\|\overline{C} - 1\!\!1\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\mathrm{tr}\left[\overline{C} - 1\!\!1\right]^2\\
&\quad = \frac{\mu}{4}\,\|\nabla m^T\nabla m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\mathrm{tr}\left[\nabla m^T\nabla m - 1\!\!1_2\right]^2,\\
&\quad = \frac{\mu}{4}\,\|I_m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\mathrm{tr}\left[I_m - 1\!\!1_2\right]^2,\\
&\qquad\qquad I_m = \nabla m^T\nabla m\text{: } \textbf{first fundamental form}\,.
\end{aligned}$$

The reconstructed membrane deformation $\varphi_s(x,y,z) = m(x,y) + z\varrho_m\,\vec{n}_m$ yields the plane stress condition $S_1(\nabla\varphi_s(x,y,0).e_3 = 0$, which is only consistent with three-dimensional equilibrium if there are no normal tractions at the transverse boundary and indeed, in [21, p.176] it is assumed that $N_{\mathrm{diff}} \equiv 0$, for otherwise, formal asymptotic expansion is impossible.

It is easily seen that the resultant membrane strain energy $W_{\mathrm{mp}}(\overline{C})$ is neither quasiconvex nor Legendre-Hadamard elliptic. Moreover, the resultant membrane strain energy density **does not satisfy the Baker-Ericksen inequalities** in contrast to the equilibrium model (3.20).

6.4. The finite-strain, quasiconvex membrane model of Le Dret/Raoult

By means of Γ-convergence arguments based on the St. Venant-Kirchhoff energy LeDret and Raoult [18] derive the following quasiconvex geometrically exact, frame-indifferent minimization problem which is, however, degenerate in compression. The membrane deformation $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies on ω:

$$\int_\omega h\, QW_0(\nabla m)\, \mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min . \text{ w.r.t. } m\,, \quad m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0)\,, \tag{6.10}$$

$$W_0(\nabla m) := \inf_{\eta \in \mathbb{R}^3} W((\nabla m|\eta)^T(\nabla m|\eta))\,, \quad W(C) = \frac{\mu}{4}\,\|C - 1\!\!1\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\,[C - 1\!\!1]^2\,,$$

$$\widehat{\varrho}_m := \begin{cases} \varrho_m & 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] \geq 0\,, \quad (\nabla m|\widehat{\varrho}_m\vec{n}) \in \mathrm{GL}^+(3,\mathbb{R}) \\ 0 & 1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] < 0\,, \quad (\nabla m|\widehat{\varrho}_m\vec{n}) \notin \mathrm{GL}^+(3,\mathbb{R}) \end{cases} \Rightarrow$$

$$W_0(\nabla m) = W((\nabla m|\widehat{\varrho}_m\vec{n})^T(\nabla m|\widehat{\varrho}_m\vec{n})) = W_{\mathrm{mp}}(\overline{C}) \quad \text{if} \quad \widehat{\varrho}_m = \varrho_m$$

with the definition of $\overline{C}$, ϱ_m and W_{mp} given in (6.9). QW_0 denotes the quasiconvex hull of W_0 which can be determined analytically showing the degenerate feature that $QW_0 = 0$ in uniform compression. In compression, this model can only predict the stresses in the membrane appropriately while the geometry of deformation cannot be accounted for.

6.5. The viscoelastic evolution

Here we provide the missing proofs for the properties of the viscoelastic evolution in Theorem 1.1.

Lemma 6.1. *Assume that for positive constants $A^+, M^+, \nu^+ > 0$ it holds that*

$$\forall\, t > 0: \quad u^2(t) + \nu^+ \int_0^t u^2(s)\,\mathrm{ds} \leq A^+ + M^+\,t\,. \tag{6.11}$$

Then we have the estimate

$$\forall\, t > 0: \quad u^2(t) \leq A^+\,e^{-\nu^+\,t} + \frac{M^+}{\nu^+}\left(1 - e^{-\nu^+\,t}\right)\,. \tag{6.12}$$

Proof. We can easily find a smooth function $g : \mathbb{R}^+ \mapsto \mathbb{R}$, which satisfies

$$g(t) + \nu^+ \int_0^t g(s)\,\mathrm{ds} = A^+ + M^+\,t\,. \tag{6.13}$$

This implies $g(0) = A^+$. Differentiation yields the equation

$$g'(t) + \nu^+\,g(t) = M^+\,. \tag{6.14}$$

The unique solution is given by

$$g(t) = A^+\,e^{-\nu^+\,t} + \frac{M^+}{\nu^+}\left(1 - e^{-\nu^+\,t}\right)\,. \tag{6.15}$$

Now we consider the difference $u^2(t) - g(t)$. Substracting the equality for g from the inequality for u^2 we obtain the differential inequality

$$[u^2(t) - g(t)] + \nu^+ \int_0^t [u^2(s) - g(s)]\,\mathrm{ds} \le 0. \tag{6.16}$$

Define $h(t) = \int_0^t [u^2(s) - g(s)]$. This implies $h(0) = 0$ and the differential inequality

$$h' + \nu^+ h(t) \le 0. \tag{6.17}$$

Muliplication with $e^{\nu^+ t}$ and integration shows that $e^{\nu^+ t}\, h(t) \le 0$, hence $u^2(t) \le g(t)$. $\square$

Lemma 6.2. *Assume that $F \in C^1(\mathbb{R}^+, \mathrm{GL}^+(3,\mathbb{R}))$ is given and consider the ordinary differential equation for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$:*

$$\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) = \nu^+ \ \mathrm{skew}(F(t)\overline{R}^T(t)) \cdot \overline{R}(t)\,, \quad \overline{R}(0) = \overline{R}_0\,. \tag{6.18}$$

Then the unique global solution satisfies for all times $t \in \mathbb{R}^+$

$$\begin{aligned}\|\,\mathrm{skew}(F(t)\overline{R}^T(t))\|^2 &\le -2\nu^+ \int_0^t \|\,\mathrm{skew}(F(s)\overline{R}^T(s))\|^2\,\mathrm{ds}\\ &\quad + 2\int_0^t \left(\|F(s)\| + \|\overline{R}(s)\|\right)\,\|F'(s)\|\,\mathrm{ds} + \|F^T(0)\overline{R}(0) - 1\!\!1\|^2\,.\end{aligned} \tag{6.19}$$

Proof. Consider

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|F^T\overline{R} - 1\!\!1\|^2\right) &= \langle F^T\overline{R} - 1\!\!1, F^T\frac{\mathrm{d}}{\mathrm{dt}}\overline{R}(t) + [F'(t)]^T\overline{R}\rangle\\ &= \langle F^T\overline{R} - 1\!\!1, \nu^+ F^T\,\mathrm{skew}(F\overline{R}^T)\overline{R} + [F'(t)]^T\overline{R}\rangle\\ &= \nu^+\langle FF^T - F\overline{R}^T, \mathrm{skew}(F\overline{R}^T)\rangle + \langle F^T\overline{R} - 1\!\!1, [F'(t)]^T\overline{R}\rangle\\ &= -\nu^+\|\,\mathrm{skew}(F\overline{R}^T)\|^2 + \langle F - \overline{R}, F'(t)\rangle\\ &\le -\nu^+\|\,\mathrm{skew}(F\overline{R}^T)\|^2 + \|F'(t)\|\,(\|F\| - \sqrt{3})\,.\end{aligned} \tag{6.20}$$

Integration yields

$$\begin{aligned}\frac{1}{2}\|F^T(t)\overline{R}(t) - 1\!\!1\|^2 &\le -\nu^+ \int_0^t \|\,\mathrm{skew}(F(s)\overline{R}^T(s))\|^2\,\mathrm{ds}\\ &\quad + \int_0^t \|F'(2)\|\,(\|F(s)\| - \sqrt{3})\,\mathrm{ds} + \frac{1}{2}\|F^T(0)\overline{R}(0) - 1\!\!1\|^2\,.\end{aligned} \tag{6.21}$$

We use finally that

$$\|\,\mathrm{skew}(F\overline{R}^T)\| = \|\,\mathrm{skew}(F\overline{R}^T - 1\!\!1)\| \le \|F\overline{R}^T - 1\!\!1\| = \|\overline{R}F^T - 1\!\!1\|$$

$$= \|\overline{R}^T(\overline{R}F^T - 1\!\!1)\overline{R}\| = \|F^T\overline{R} - 1\!\!1\|\,. \tag{6.22}$$

This shows the desired integral inequality. □

The proof of Theorem 1.1, part ii.) is achieved by identifying $u^2(t) = \|\operatorname{skew}(F(t)\overline{R}^T(t))\|^2$ and using Lemma 6.4 and Lemma 6.1.

Lemma 6.3 (The rotation constraint). *Let* $F \in \mathrm{GL}^+(3,\mathbb{R})$ *and* $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$. *Then*

$$\operatorname{skew}(F\overline{R}^T) = 0 \;\Leftrightarrow\; \overline{R}^T \operatorname{polar}(F) \in$$
$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Proof. The proof is based on the polar decomposition of F and can be found in [41, p. 175]. □

Lemma 6.4. *Let* $F \in \mathrm{GL}^+(3,\mathbb{R})$ *be given, then*

$$\begin{aligned} &\forall\, \overline{R} \in \mathrm{SO}(3,\mathbb{R}) : \|\overline{R} - \operatorname{polar}(F)\|^2 < 8 : \\ &\qquad \exists\, c^+ > 0 : \quad \|\operatorname{skew}(F\overline{R}^T)\|^2 \ge c^+\, \|\overline{R} - \operatorname{polar}(F)\|^2\,. \end{aligned} \tag{6.23}$$

Proof. We proceed by contradiction and a compactness argument. Assume to the contrary that the inequality does not hold good. Then we can find a sequence of rotations $\overline{R}_k \in \mathrm{SO}(3,\mathbb{R})$ with $\|\overline{R}_k - \operatorname{polar}(F)\|^2 < 8$ such that $\|\operatorname{skew}(F\overline{R}_k^T)\| \to 0$ but $\|\overline{R}_k - \operatorname{polar}(F)\| \ge a^+ > 0$. Since $\mathrm{SO}(3,\mathbb{R})$ is compact, by Bolzano-Weierstrass we can extract a subsequence $\overline{R}_{k_j}$, converging to some $\widehat{\overline{R}}$ with $\|\widehat{\overline{R}} - \operatorname{polar}(F)\|^2 < 8$, $\|\operatorname{skew}(F\widehat{\overline{R}}^T)\| = 0$ and $\|\widehat{\overline{R}} - \operatorname{polar}(F)\| \ge a^+ > 0$. This is a contradiction due to Lemma 6.3.

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(Received: May 17, 2004)