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Symmetrization of the growth deformation and velocity gradients in residually stressed biomaterials

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Abstract. Some fundamental issues in the kinematic and kinetic analysis of the stress-modulated growth of residually stressed biological materials are addressed within the context of the multiplicative decomposition of deformation gradient into its elastic and growth parts. The symmetrizations of the growth part of the deformation gradient and the growth part of the velocity gradient are derived for isotropic pseudoelastic soft tissues. The significance of results in the formulation of the biomechanic constitutive theory is discussed.

Keywords. Biomaterials, residual stress, mass growth, deformation gradient, multiplicative decomposition, symmetrization.

1. Introduction

The analysis of the stress-modulated growth of biological materials such as blood vessels and other soft tissues has received a great deal of attention in bioengineering community. A reference to earlier work can be found in the book by Fung [1] and review articles by Taber [2] and Humphrey [3]. Recently, there has been a considerable effort devoted to the formulation of the analysis in the framework of large deformation continuum mechanics based on the multiplicative decomposition of the deformation gradient into its elastic and growth parts. This decomposition was first used in the context of biomechanics by Rodrigez *et al.* [4]. A similar decomposition of elastoplastic deformation gradient has long been known and successfully applied to problems of polycrystalline and single crystal plasticity [5–7]. In the previous work on the multiplicative decomposition, for both elastoplastic and biomaterials [8–11], the assumption was commonly made that the initial configuration of material sample is stress free. The intermediate configuration is then defined by a complete elastic distressing of the currently deformed configuration to zero stress. In this paper, we extend the elastic analysis of residually stressed bodies [12, 13] to materials with a growing mass. We assume that the initial configuration of material sample is not stress-free, but characterized by a self-equilibrating distribution of residual stress. This configuration evolves by a

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non-uniform volumetric deposition of mass, without any applied external load. The newly deformed configuration is characterized by a new distribution of residual stress. The intermediate configuration is then obtained by reducing the current residual stress to the initial state of residual stress. The elastic and growth parts of the deformation gradient are defined relative to this intermediate configuration. In general, the growth part of the deformation gradient is a non-symmetric tensor, involving all nine independent components. The procedure for its symmetrization is constructed, and the relationship between the corresponding symmetrized and non-symmetrized quantities is established. The symmetrization of the growth velocity gradient relative to partially and completely relaxed intermediate configuration is derived. The significance of the results for the constitutive analysis of the stress-modulated growth of pseudoelastic soft tissues is then discussed.

2. Elastic stress response of a growing tissue

Consider a material sample in its initial configuration $\mathcal{B}_\mathrm{o}$, which is characterized by an internal distribution of self-equilibrating residual stress $\boldsymbol{\sigma}_\mathrm{o}$. This stress can be relieved by dissecting the material sample into sufficiently small pieces, which are uniformly stressed in the limit. When each piece is allowed to relax, a virtual initial configuration $\hat{\mathcal{B}}_\mathrm{o}$ is obtained, which is incompatible but stress free. Denote by $\mathbf{F}_\mathrm{o}$ the local deformation gradient between $\hat{\mathcal{B}}_\mathrm{o}$ and $\mathcal{B}_\mathrm{o}$. Restricting considerations to isotropic hyperelastic materials, the residual stress in $\mathcal{B}_\mathrm{o}$ can be expressed as [14]

$$\boldsymbol{\sigma}_\mathrm{o} = \mathbf{f}(\mathbf{B}_\mathrm{o})\,, \quad \mathbf{f}(\mathbf{B}_\mathrm{o}) = \frac{1}{(\det \mathbf{B}_\mathrm{o})^{1/2}} \left(\mathbf{B}_\mathrm{o} \cdot \frac{\partial \psi_\mathrm{o}}{\partial \mathbf{B}_\mathrm{o}} + \frac{\partial \psi_\mathrm{o}}{\partial \mathbf{B}_\mathrm{o}} \cdot \mathbf{B}_\mathrm{o} \right). \tag{1}$$

The left Cauchy–Green deformation tensor is $\mathbf{B}_\mathrm{o} = \mathbf{F}_\mathrm{o} \cdot \mathbf{F}_\mathrm{o}^T = \mathbf{V}_\mathrm{o}^2$, and $\mathbf{V}_\mathrm{o}$ is the left stretch tensor. The elastic strain energy per unit unstressed volume is $\psi_\mathrm{o} = \psi_\mathrm{o}(\mathbf{B}_\mathrm{o})$. The stress response does not depend on any rotation superposed to $\hat{\mathcal{B}}_\mathrm{o}$, since the material is assumed to be isotropic. If $\mathbf{f}$ is invertible, the stretch tensor $\mathbf{V}_\mathrm{o}$, corresponding to the residual stress $\boldsymbol{\sigma}_\mathrm{o}$, can be calculated from

$$\mathbf{V}_\mathrm{o} = \mathbf{B}_\mathrm{o}^{1/2}\,, \quad \mathbf{B}_\mathrm{o} = \mathbf{f}^{-1}(\boldsymbol{\sigma}_\mathrm{o})\,. \tag{2}$$

Suppose that the material sample deforms due to non-uniform volumetric mass growth, without externally applied loads. After some time the configuration $\mathcal{B}$ is reached which supports a self-equilibriating stress field $\boldsymbol{\sigma}$. Denote the deformation gradient between $\mathcal{B}_\mathrm{o}$ and $\mathcal{B}$ by $\mathbf{F}$. We assume that material points are everywhere dense during the volumetric mass growth, so that in any small neighborhood around the particle there are always points that existed before the growth. This assumption enables us to treat the problems of volumetric mass growth by using the usual continuum mechanics concepts, such as deformation gradient and strain tensors. The internal stress $\boldsymbol{\sigma}$ can be relieved by dissecting the material sample into sufficiently small pieces and by allowing them to relax to zero stress.

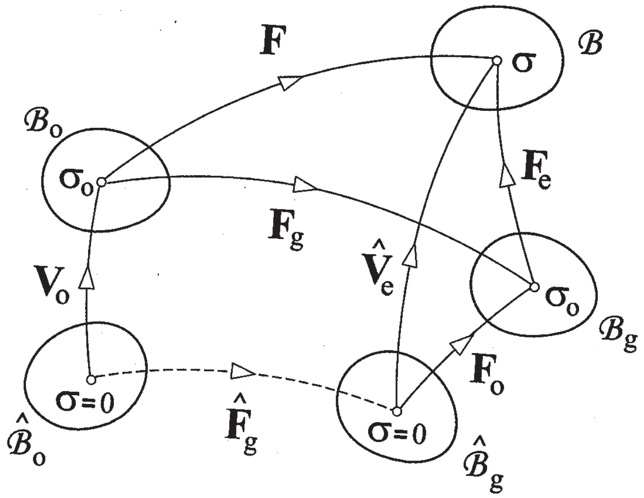

Figure 1. The initial configuration $\mathcal{B}_o$ supports the residual stress $\boldsymbol{\sigma}_o$. Upon the mass growth the configuration $\mathcal{B}$ is reached, which supports the residual stress $\boldsymbol{\sigma}$. The corresponding unstressed configurations are $\hat{\mathcal{B}}_o$ and $\hat{\mathcal{B}}_g$. For elastically isotropic material these are obtained by distressing without rotation. The intermediate configuration $\mathcal{B}_g$ supports the initial residual stress $\boldsymbol{\sigma}_o$.r

The corresponding incompatible stress free configuration is $\hat{\mathcal{B}}_g$ (Fig. 1). If material remains elastically isotropic during the growth, the stress $\boldsymbol{\sigma}$ depends only on the left stretch tensor $\hat{\mathbf{V}}_e$ of the elastic deformation $\hat{\mathbf{F}}_e$ from $\hat{\mathcal{B}}_g$ to $\mathcal{B}$. A particular intermediate configuration $\hat{\mathcal{B}}_g$ can be defined by requiring that the removal of stress from $\mathcal{B}$ to $\hat{\mathcal{B}}_g$ is performed along the principal directions of $\boldsymbol{\sigma}$. This is shown in Fig. 1. Assuming that the material remains isotropic and that it preserves its elastic properties during the growth process, we can write

$$\boldsymbol{\sigma} = \mathbf{f}\,(\hat{\mathbf{B}}_e)\,. \tag{3}$$

Thus, the elastic stretch $\hat{\mathbf{V}}_e$ corresponding to the stress tensor $\boldsymbol{\sigma}$ is

$$\hat{\mathbf{V}}_e = \hat{\mathbf{B}}_e^{1/2}\,, \quad \hat{\mathbf{B}}_e = \mathbf{f}^{-1}\,(\boldsymbol{\sigma})\,. \tag{4}$$

Introduce next the intermediate configuration $\mathcal{B}_g$ by reducing the self equilibrating stress $\boldsymbol{\sigma}$ in the current configuration $\mathcal{B}$ to the initial state of the residual stress $\boldsymbol{\sigma}_o$. The configuration $\mathcal{B}_g$ is in general incompatible, because a non-uniform mass growth during the deformation from $\mathcal{B}_o$ to $\mathcal{B}$ produces a non-uniform change of the initial pattern of residual stress. In addition, the configuration $\mathcal{B}_g$ is non-unique. For example, one specification of $\mathcal{B}_g$ is obtained by reducing the principal stresses of $\boldsymbol{\sigma}$ along their principal directions to the principal values of $\boldsymbol{\sigma}_o$, followed by the rotation that carries the principal directions of $\boldsymbol{\sigma}$ to the principal directions of $\boldsymbol{\sigma}_{\rm o}$. Other specifications are also possible, as discussed in the sequel. Denoting by $\mathbf{F}_{\rm o} = \mathbf{V}_{\rm o} \cdot \mathbf{R}_{\rm o}$ the elastic deformation gradient from $\hat{\mathcal{B}}_{\rm g}$ to $\mathcal{B}_{\rm g}$, and by $\mathbf{F}_{\rm e} = \mathbf{V}_{\rm e} \cdot \mathbf{R}_{\rm e}$ the elastic deformation gradient from $\mathcal{B}_g$ to $\mathcal{B}$, the multiplicative decomposition holds

$$\hat{\mathbf{V}}_{\rm e} = \mathbf{F}_{\rm e} \cdot \mathbf{F}_{\rm o} = \mathbf{F}_{\rm e} \cdot \mathbf{V}_{\rm o} \cdot \mathbf{R}_{\rm o} \,. \tag{5}$$

Since $\mathbf{V}_{\rm o}$ and $\hat{\mathbf{V}}_{\rm e}$ are both unique, by Eqs. (2) and (4), it is clear from Eq. (5) that $\mathbf{F}_{\rm e}$ is not unique, but changes with the change of the rotation $\mathbf{R}_{\rm o}$. The relationship between $\mathbf{F}_{\rm e}$ and $\mathbf{R}_{\rm o}$ has to be such that the product on the right-hand side of Eq. (5) is a symmetric tensor, i.e.,

$$\mathbf{F}_{\rm e} \cdot \mathbf{V}_{\rm o} \cdot \mathbf{R}_{\rm o} = \mathbf{R}_{\rm o}^T \cdot \mathbf{V}_{\rm o} \cdot \mathbf{F}_{\rm e}^T \,. \tag{6}$$

The rotation $\mathbf{R}_{\rm o}$ is arbitrary since for elastically isotropic material the stress $\boldsymbol{\sigma}_{\rm o}$ in the configuration $\mathcal{B}_{\rm g}$ is independent of the rotation $\mathbf{R}_{\rm o}$, superposed to $\hat{\mathcal{B}}_{\rm g}$ prior to the stretching $\mathbf{V}_{\rm o}$. For example, if $\mathbf{R}_{\rm o} = \mathbf{I}$ (identity tensor), the elastic deformation gradient is

$$\mathbf{F}_{\rm e} = \hat{\mathbf{V}}_{\rm e} \cdot \mathbf{V}_{\rm o}^{-1} \,. \tag{7}$$

On the other hand, if $\mathbf{R}_{\rm o} = \mathbf{Q}^T$, where $\mathbf{Q}$ is the known rotation tensor that carries the principal directions of $\boldsymbol{\sigma}_{\rm o}$ to principal directions of $\boldsymbol{\sigma}$, we have

$$\mathbf{F}_{\rm e} = \mathbf{V}_{\rm e} \cdot \mathbf{Q} \,, \quad \mathbf{F}_{\rm o} = \mathbf{V}_{\rm o} \cdot \mathbf{Q}^T \,, \tag{8}$$

and

$$\hat{\mathbf{V}}_{\rm e} = \mathbf{V}_{\rm e} \cdot (\mathbf{Q} \cdot \mathbf{V}_{\rm o} \cdot \mathbf{Q}^T) \,. \tag{9}$$

For isotropic material the principal directions of $\mathbf{V}_{\rm o}$ are parallel to those of $\boldsymbol{\sigma}_{\rm o}$, while the principal directions of $\mathbf{Q} \cdot \mathbf{V}_{\rm o} \cdot \mathbf{Q}^T$ are parallel to those of $\hat{\mathbf{V}}_{\rm e}$ and $\boldsymbol{\sigma}$. It consequently follows from Eq. (9) that $\mathbf{V}_{\rm e}$ is coaxial with $\boldsymbol{\sigma}$. Thus, the choice $\mathbf{R}_{\rm o} = \mathbf{Q}^T$ corresponds to previously discussed unloading scenario in which the principal stresses of $\boldsymbol{\sigma}$ are reduced along their principal directions to principal stresses of $\boldsymbol{\sigma}_{\rm o}$, followed by the rotation $\mathbf{Q}^T$.

Whatever the specification of $\mathbf{R}_{\rm o}$, the stress response from $\hat{\mathcal{B}}_{\rm g}$ to $\mathcal{B}$ can be expressed from Eq. (3) as

$$\boldsymbol{\sigma} = \mathbf{f}\left(\mathbf{F}_{\rm e} \cdot \mathbf{B}_{\rm o} \cdot \mathbf{F}_{\rm e}^T\right) \,, \tag{10}$$

having regard to

$$\hat{\mathbf{B}}_{\rm e} = \mathbf{F}_{\rm e} \cdot \mathbf{B}_{\rm o} \cdot \mathbf{F}_{\rm e}^T \,, \quad \mathbf{B}_{\rm o} = \mathbf{V}_{\rm o}^2 \,. \tag{11}$$

Although the elastic deformation gradient $\mathbf{F}_{\rm e}$ is not unique, the product $\mathbf{F}_{\rm e} \cdot \mathbf{B}_{\rm o} \cdot \mathbf{F}_{\rm e}^T$ is a unique tensor. Indeed, from Eq. (5),

$$\mathbf{F}_{\rm e} = \hat{\mathbf{V}}_{\rm e} \cdot \mathbf{R}_{\rm o}^T \cdot \mathbf{V}_{\rm o}^{-1} \,, \tag{12}$$

and the substitution into Eq. (11) gives $\hat{\mathbf{B}}_{\rm e} = \hat{\mathbf{V}}_{\rm e}^2$.

3. Symmetrization of the growth part of deformation gradient

The growth part of the deformation gradient $\mathbf{F}_\mathrm{g}$, corresponding to deformation gradients $\mathbf{F}$ and $\mathbf{F}_\mathrm{e}$, is defined by the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}_\mathrm{e} \cdot \mathbf{F}_\mathrm{g}\,. \tag{13}$$

The transition (growth) from the initial configuration $\mathcal{B}_\mathrm{o}$ to the intermediate configuration $\mathcal{B}_\mathrm{g}$ is considered to take place at the constant state of initial residual stress $\boldsymbol{\sigma}_\mathrm{o}$. Since $\mathbf{F}_\mathrm{e}$ is not unique, the growth part of the deformation gradient $\mathbf{F}_\mathrm{g}$ is not unique either. Indeed, the substitution of Eq. (12) into Eq. (13) gives

$$\mathbf{F}_\mathrm{g} = \mathbf{F}_\mathrm{e}^{-1} \cdot \mathbf{F} = \mathbf{V}_\mathrm{o} \cdot \mathbf{R}_\mathrm{o} \cdot \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F}\,, \tag{14}$$

which demonstrates the dependence of $\mathbf{F}_\mathrm{g}$ on the rotation $\mathbf{R}_\mathrm{o}$.

In general, the growth deformation tensor $\mathbf{F}_\mathrm{g}$ is a non-symmetric tensor, involving nine independent components. The question of practical importance arises if there is a choice of the rotation $\mathcal{R}_\mathrm{o}$ which makes $\mathbf{F}_\mathrm{g}$ symmetric (denoted in the sequel by $\mathcal{F}_\mathrm{g}$), so that

$$\mathcal{F}_\mathrm{g} = \mathbf{V}_\mathrm{o} \cdot \mathcal{R}_\mathrm{o} \cdot \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F} = \mathbf{F}^T \cdot \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathcal{R}_\mathrm{o}^T \cdot \mathbf{V}_\mathrm{o} = \mathcal{F}_\mathrm{g}^T\,. \tag{15}$$

The answer is affirmative. The pre- and post-multiplication of Eq. (15) with $\mathbf{V}_\mathrm{o}^{-1}$ gives

$$\mathcal{R}_\mathrm{o} \cdot \left(\hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F} \cdot \mathbf{V}_\mathrm{o}^{-1}\right) = \left(\hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F} \cdot \mathbf{V}_\mathrm{o}^{-1}\right)^T \cdot \mathcal{R}_\mathrm{o}^T\,. \tag{16}$$

This equation, in conjunction with the identity $\mathcal{R}_\mathrm{o} \cdot \mathcal{R}_\mathrm{o}^T = \mathbf{I}$, can be solved for $\mathcal{R}_\mathrm{o}$ to give

$$\mathcal{R}_\mathrm{o} = (\mathbf{Z}^T \cdot \mathbf{Z})^{1/2} \cdot \mathbf{Z}^{-1}\,, \tag{17}$$

where

$$\mathbf{Z} = \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F} \cdot \mathbf{V}_\mathrm{o}^{-1}\,. \tag{18}$$

Details of the derivation can be found in the Appendix. Equation (17) defines the rotation tensor $\mathcal{R}_\mathrm{o}$ which symmetrizes the growth part of the deformation gradient in Eq. (14), according to Eq. (15).

An independent derivation of this result is instructive. From Eq. (14) and the symmetrization condition (15), we have

$$\mathbf{V}_\mathrm{o}^{-1} \cdot \mathcal{F}_\mathrm{g} = \mathcal{R}_\mathrm{o} \cdot \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F}\,, \quad \mathcal{F}_\mathrm{g} \cdot \mathbf{V}_\mathrm{o}^{-1} = \mathbf{F}^T \cdot \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathcal{R}_\mathrm{o}^T\,. \tag{19}$$

Upon the multiplication of these two expressions to eliminate the rotation tensor $\mathcal{R}_\mathrm{o}$, there follows

$$\mathcal{F}_\mathrm{g} \cdot \mathbf{V}_\mathrm{o}^{-2} \cdot \mathcal{F}_\mathrm{g} = \mathbf{F}^T \cdot \hat{\mathbf{V}}_\mathrm{e}^{-2} \cdot \mathbf{F}\,. \tag{20}$$

The pre- and post-multiplication with $\mathbf{V}_\mathrm{o}^{-1}$ then yields

$$\left(\mathbf{V}_\mathrm{o}^{-1} \cdot \mathcal{F}_\mathrm{g} \cdot \mathbf{V}_\mathrm{o}^{-1}\right)^2 = \mathbf{Z}^T \cdot \mathbf{Z}\,. \tag{21}$$

This can be solved for the symmetrized growth part of the deformation gradient as

$$\mathcal{F}_{\mathrm{g}} = \mathbf{V}_{\mathrm{o}} \cdot \left(\mathbf{Z}^T \cdot \mathbf{Z}\right)^{1/2} \cdot \mathbf{V}_{\mathrm{o}} \,. \tag{22}$$

On the other hand, Eq. (14) gives

$$\mathcal{R}_{\mathrm{o}} = \mathbf{V}_{\mathrm{o}}^{-1} \cdot \mathcal{F}_{\mathrm{g}} \cdot \mathbf{F}^{-1} \cdot \hat{\mathbf{V}}_{\mathrm{e}} = \left(\mathbf{V}_{\mathrm{o}}^{-1} \cdot \mathcal{F}_{\mathrm{g}} \cdot \mathbf{V}_{\mathrm{o}}^{-1}\right) \cdot \mathbf{Z}^{-1} \,. \tag{23}$$

The substitution of Eq. (22) into Eq. (23) confirms the result (17). The orthogonality property of $\mathcal{R}_{\mathrm{o}}$ can be verified by inspection.

In the applications involving a symmetric growth of isotropic material (e.g., cylindrically symmetric blood vessels), it may happen that the stress tensor $\boldsymbol{\sigma}$ and the deformation gradient $\mathbf{F} = \mathbf{V}$ are coaxial with $\boldsymbol{\sigma}_{\mathrm{o}}$. In this case it readily follows that $\mathbf{Z}$ is a symmetric tensor and that $\mathcal{R}_{\mathrm{o}} = \mathbf{I}$, so that the growth deformation tensor becomes

$$\mathcal{F}_{\mathrm{g}} = \mathbf{V}_{\mathrm{o}} \cdot \mathbf{Z} \cdot \mathbf{V}_{\mathrm{o}} \,, \quad \mathbf{Z} = \hat{\mathbf{V}}_{\mathrm{e}}^{-1} \cdot \mathbf{V} \cdot \mathbf{V}_{\mathrm{o}}^{-1} \,. \tag{24}$$

This result can be easily verified directly by combining the multiplicative decompositions $\mathbf{V} = \mathbf{V}_{\mathrm{e}} \cdot \mathcal{F}_{\mathrm{g}}$ and $\hat{\mathbf{V}}_{\mathrm{e}} = \mathbf{V}_{\mathrm{e}} \cdot \mathbf{V}_{\mathrm{o}}$.

3.1. Symmetrization of the growth deformation gradient in the absence of residual stress

If there was no initial residual stress ($\boldsymbol{\sigma}_{\mathrm{o}} = \mathbf{0}$), then $\mathbf{V}_{\mathrm{o}} = \mathbf{I}$, $\hat{\mathbf{V}}_{\mathrm{e}} = \mathbf{V}_{\mathrm{e}}$, $\mathbf{R}_{\mathrm{o}} = \mathbf{R}_{\mathrm{e}}^T$, and

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{g}} = \mathbf{V}_{\mathrm{e}} \cdot \hat{\mathbf{F}}_{\mathrm{g}} \,, \quad \hat{\mathbf{F}}_{\mathrm{g}} = \mathbf{R}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{g}} \,. \tag{25}$$

The construction from Fig. 1 reduces to that shown in Fig. 2. In this case, it readily follows that the growth deformation gradient is symmetrized if the rotation $\mathcal{R}_{\mathrm{e}}$ is chosen such that

$$\mathcal{R}_{\mathrm{e}}^T = (\mathbf{Z}^T \cdot \mathbf{Z})^{1/2} \cdot \mathbf{Z}^{-1} \,, \tag{26}$$

where

$$\mathbf{Z} = \hat{\mathbf{V}}_{\mathrm{e}}^{-1} \cdot \mathbf{F} = \hat{\mathbf{F}}_{\mathrm{g}} \,. \tag{27}$$

The corresponding symmetrized growth deformation gradient is

$$\mathcal{F}_{\mathrm{g}} = \left(\mathbf{Z}^T \cdot \mathbf{Z}\right)^{1/2} = \mathbf{U}_{\mathrm{g}} \,, \tag{28}$$

where $\mathbf{U}_{\mathrm{g}}$ is the right stretch tensor appearing in the polar decomposition

$$\hat{\mathbf{F}}_{\mathrm{g}} = \hat{\mathbf{R}}_{\mathrm{g}} \cdot \mathbf{U}_{\mathrm{g}} \,, \quad \mathbf{F} = \mathbf{V}_{\mathrm{e}} \cdot \hat{\mathbf{R}}_{\mathrm{g}} \cdot \mathbf{U}_{\mathrm{g}} \,. \tag{29}$$

Note that all constituents of the latter decomposition of the total deformation gradient, i.e., $\mathbf{V}_{\mathrm{e}}$, $\hat{\mathbf{R}}_{\mathrm{g}}$ and $\mathbf{U}_{\mathrm{g}}$, are uniquely defined.

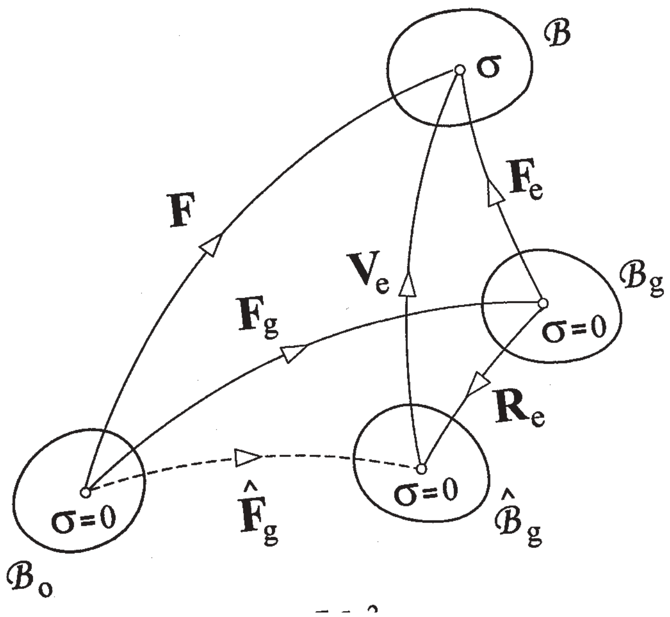

Figure 2. The initial unstressed configuration $\mathcal{B}_o$, and the deformed configuration $\mathcal{B}$ under the stress $\boldsymbol{\sigma}$. The intermediate configuration $\hat{\mathcal{B}}_g$ is obtained from $\mathcal{B}$ by elastic distressing to zero stress, without rotation. An arbitrary intermediate configuration $\mathcal{B}_g$ differs from $\hat{\mathcal{B}}_g$ by the rotation $\mathbf{R}_e$. The elastic stretch tensor is $\mathbf{V}_e$, so that $\mathbf{F}_e = \mathbf{V}_e \cdot \mathbf{R}_e$.

4. Relationship between symmetrized and non-symmetrized growth deformation gradients

Denote the symmetrized growth deformation tensor of Eq. (22) by $\mathcal{F}_g$, and the corresponding elastic deformation gradient by $\mathcal{F}_e$, such that

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_g = \mathcal{F}_e \cdot \mathcal{F}_g \,. \tag{30}$$

An arbitrary, non-symmetrized growth deformation tensor is denoted by $\mathbf{F}_g$ and its corresponding elastic deformation gradient by $\mathbf{F}_e$. Since $\hat{\mathbf{B}}_e$ in Eq. (11) is independent of the choice of $\mathbf{F}_e$, we can write

$$\mathbf{F}_e \cdot \mathbf{B}_o \cdot \mathbf{F}_e^T = \mathcal{F}_e \cdot \mathbf{B}_o \cdot \mathcal{F}_e^T \,. \tag{31}$$

Expressing from Eq. (30) the elastic deformation gradients in terms of the growth deformation gradients as

$$\mathbf{F}_e = \mathbf{F} \cdot \mathbf{F}_g^{-1} \,, \quad \mathcal{F}_e = \mathbf{F} \cdot \mathcal{F}_g^{-1} \,, \tag{32}$$

the substitution into Eq. (31) gives

$$\mathbf{F}_g^{-1} \cdot \mathbf{B}_o \cdot \mathbf{F}_g^{-T} = \mathcal{F}_g^{-1} \cdot \mathbf{B}_o \cdot \mathcal{F}_g^{-1} \,. \tag{33}$$

By the pre- and post-multiplication with $\mathbf{V}_\mathrm{o}$, this equation can be rewritten as

$$\mathbf{V}_\mathrm{o} \cdot \left(\mathbf{F}_\mathrm{g}^{-1} \cdot \mathbf{B}_\mathrm{o} \cdot \mathbf{F}_\mathrm{g}^{-T}\right) \cdot \mathbf{V}_\mathrm{o} = \left(\mathbf{V}_\mathrm{o} \cdot \mathcal{F}_\mathrm{g}^{-1} \cdot \mathbf{V}_\mathrm{o}\right)^2 . \tag{34}$$

Thus,

$$\mathcal{F}_\mathrm{g} = \mathbf{V}_\mathrm{o} \cdot \left(\mathbf{V}_\mathrm{o}^{-1} \cdot \mathbf{F}_\mathrm{g}^{T} \cdot \mathbf{B}_\mathrm{o}^{-1} \cdot \mathbf{F}_\mathrm{g} \cdot \mathbf{V}_\mathrm{o}^{-1}\right)^{1/2} \cdot \mathbf{V}_\mathrm{o} \,, \tag{35}$$

which establishes a desired relationship between the symmetrized and non symmetrized growth deformation gradients $\mathcal{F}_\mathrm{g}$ and $\mathbf{F}_\mathrm{g}$. It is also noted that the incorporation of Eq. (14) into Eq. (35) recovers the symmetric representation (22).

If there was no initial residual stress, Eq. (35) reduces to

$$\mathcal{F}_\mathrm{g} = \left(\mathbf{F}_\mathrm{g}^{T} \cdot \mathbf{F}_\mathrm{g}\right)^{1/2} = \mathbf{C}_\mathrm{g}^{1/2} = \mathbf{U}_\mathbf{g} \,, \tag{36}$$

in accordance with Eq. (28).

5. Relationship between $\mathbf{F}_\mathrm{g}$ and $\hat{\mathbf{F}}_\mathrm{g}$

The growth part of the deformation gradient $\mathbf{F}_\mathrm{g}$ was defined in the previous sections relative to the initial residually stressed configuration $\mathcal{B}_\mathrm{o}$. The intermediate configuration $\mathcal{B}_\mathrm{g}$ was defined as the reference configuration which supports the same state of the residual stress $\boldsymbol{\sigma}_\mathrm{o}$. The total deformation gradient $\mathbf{F}$ was the gradient of deformation from the initial configuration $\mathcal{B}_\mathrm{o}$ to the current configuration $\mathcal{B}$. It is of interest to compare these and related kinematic quantities to the corresponding quantities defined relative to completely relaxed (stress free) reference configurations $\hat{\mathcal{B}}_\mathrm{o}$ and $\hat{\mathcal{B}}_\mathrm{g}$ (Fig. 1). Let $\hat{\mathbf{F}}$ be the total deformation gradient from $\hat{\mathcal{B}}_\mathrm{o}$ to $\mathcal{B}$, and let $\hat{\mathbf{F}}_\mathrm{g}$ be the growth deformation gradient from $\hat{\mathcal{B}}_\mathrm{o}$ to $\hat{\mathcal{B}}_\mathrm{g}$, such that

$$\hat{\mathbf{F}} = \hat{\mathbf{V}}_\mathrm{e} \cdot \hat{\mathbf{F}}_\mathrm{g} \,. \tag{37}$$

Since the intermediate configuration $\hat{\mathcal{B}}_\mathrm{g}$ is defined by elastic distressing without rotation, the growth tensor $\hat{\mathbf{F}}_\mathrm{g}$ is in general non-symmetric. It readily follows that

$$\mathbf{F} \cdot \mathbf{V}_\mathrm{o} = \hat{\mathbf{V}}_\mathrm{e} \cdot \hat{\mathbf{F}}_\mathrm{g} \,, \tag{38}$$

and

$$\mathbf{F}_\mathrm{g} \cdot \mathbf{V}_\mathrm{o} = \mathbf{F}_\mathrm{o} \cdot \hat{\mathbf{F}}_\mathrm{g} \,. \tag{39}$$

Consequently,

$$\hat{\mathbf{F}}_\mathrm{g} = \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F} \cdot \mathbf{V}_\mathrm{o} \,, \tag{40}$$

and

$$\mathbf{F}_\mathrm{g} = \mathbf{F}_\mathrm{o} \cdot \hat{\mathbf{F}}_\mathrm{g} \cdot \mathbf{V}_\mathrm{o}^{-1} \,. \tag{41}$$

The latter equation represents a desired relationship between the growth deformation gradients $\mathbf{F}_\mathrm{g}$ and $\hat{\mathbf{F}}_\mathrm{g}$, defined relative to partially and completely relaxed intermediate configurations. The substitution of Eq. (40) into Eq. (41) yields

$$\mathbf{F}_\mathrm{g} = \mathbf{F}_\mathrm{o} \cdot \hat{\mathbf{V}}_\mathrm{e}^{-1} \cdot \mathbf{F} \,, \tag{42}$$

in accord with the previously derived expression (14).

6. Symmetrization of the growth velocity gradient

The relationship between the growth velocity gradients in the partially and completely relaxed intermediate configurations is obtained by incorporating Eq. (34) into

$$\dot{\mathbf{F}}_{\mathrm{g}} \cdot \mathbf{F}_{\mathrm{g}}^{-1} = \dot{\mathbf{F}}_{\mathrm{o}} \cdot \mathbf{F}_{\mathrm{o}}^{-1} + \mathbf{F}_{\mathrm{o}} \cdot \left(\dot{\hat{\mathbf{F}}}_{\mathrm{g}} \cdot \hat{\mathbf{F}}_{\mathrm{g}}^{-\mathbf{1}} \right) \cdot \mathbf{F}_{\mathrm{o}}^{-1} \,. \tag{43}$$

Since $\mathbf{F}_{\mathrm{o}} = \mathbf{V}_{\mathrm{o}} \cdot \mathbf{R}_{\mathrm{o}}$ and $\dot{\mathbf{V}}_{\mathrm{o}} = \mathbf{0}$, we also have

$$\dot{\mathbf{F}}_{\mathrm{o}} \cdot \mathbf{F}_{\mathrm{o}}^{-1} = \mathbf{V}_{\mathrm{o}} \cdot \left(\dot{\mathbf{R}}_{\mathrm{o}} \cdot \mathbf{R}_{\mathrm{o}}^{-1} \right) \cdot \mathbf{V}_{\mathrm{o}}^{-1} \,. \tag{44}$$

The growth part of the velocity gradient in the relaxed configuration can be expressed in terms of the total velocity gradient as

$$\dot{\hat{\mathbf{F}}}_{\mathrm{g}} \cdot \hat{\mathbf{F}}_{\mathrm{g}}^{-\mathbf{1}} = \hat{\mathbf{V}}_{\mathrm{e}}^{-\mathbf{1}} \cdot \left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-\mathbf{1}} - \dot{\hat{\mathbf{V}}}_{\mathrm{e}} \cdot \hat{\mathbf{V}}_{\mathrm{e}}^{-\mathbf{1}} \right) \cdot \hat{\mathbf{V}}_{\mathrm{e}} \,. \tag{45}$$

In general, this is a non-symmetric tensor. The velocity gradient in Eq. (43) is also non-symmetric, even when $\mathbf{R}_{\mathrm{o}}$ is selected so that $\mathbf{F}_{\mathrm{g}}$ is symmetric. In the rate-type analysis [11, 15, 16], it may be of interest to symmetrize the growth velocity gradient, rather than the growth deformation gradient. This can also be accomplished by an appropriate selection of the rotation $\mathbf{R}_{\mathrm{o}}$, i.e., the spin tensor $\mathbf{\Omega}_{\mathrm{o}} = \dot{\mathbf{R}}_{\mathrm{o}} \cdot \mathbf{R}_{\mathrm{o}}^{-1}$. The growth velocity gradient will be symmetric if its antisymmetric part is equal to zero. This gives

$$\mathbf{V}_{\mathrm{o}}^{2} \cdot \left(\mathbf{V}_{\mathrm{o}}^{-1} \cdot \mathbf{\Omega}_{\mathrm{o}} \cdot \mathbf{V}_{\mathrm{o}}^{-1} \right) + \left(\mathbf{V}_{\mathrm{o}}^{-1} \cdot \mathbf{\Omega}_{\mathrm{o}} \cdot \mathbf{V}_{\mathrm{o}}^{-1} \right) \cdot \mathbf{V}_{\mathrm{o}}^{2} = \hat{\mathbf{\Omega}} \,, \tag{46}$$

where the spin tensor $\hat{\mathbf{\Omega}}$ is defined by

$$\hat{\mathbf{\Omega}} = -2 \left[\mathbf{V}_{\mathrm{o}} \cdot \mathbf{R}_{\mathrm{o}} \cdot \left(\dot{\hat{\mathbf{F}}}_{\mathrm{g}} \cdot \hat{\mathbf{F}}_{\mathrm{g}}^{-\mathbf{1}} \right) \cdot \mathbf{R}_{\mathrm{o}}^{T} \cdot \mathbf{V}_{\mathrm{o}}^{-1} \right]_{\mathrm{a}} \,. \tag{47}$$

The subscript (a) stands for the antisymmetric part. The solution of Eq. (46) can be obtained by using the procedure outlined in [17], with the end result

$$\mathbf{V}_{\mathrm{o}}^{-1} \cdot \mathbf{\Omega}_{\mathrm{o}} \cdot \mathbf{V}_{\mathrm{o}}^{-1} = k_2 \, \hat{\mathbf{\Omega}} - \left(k_1 \, \mathbf{I} - \mathbf{V}_{\mathrm{o}}^{2} \right)^{-1} \cdot \hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}} \cdot \left(k_1 \, \mathbf{I} - \mathbf{V}_{\mathrm{o}}^{2} \right)^{-1} \,, \tag{48}$$

where

$$k_1 = \operatorname{tr} \left(\mathbf{V}_{\mathrm{o}}^{2} \right) , \quad k_2 = \operatorname{tr} \left(k_1 \, \mathbf{I} - \mathbf{V}_{\mathrm{o}}^{2} \right)^{-1} \,. \tag{49}$$

The pre- and post-multiplication of Eq. (48) with $\mathbf{V}_{\mathrm{o}}$ yields an explicit expression for the spin $\mathbf{\Omega}_{\mathrm{o}}$, and thus the rate of rotation $\dot{\mathbf{R}}_{\mathrm{o}} = \mathbf{\Omega}_{\mathrm{o}} \cdot \mathbf{R}_{\mathrm{o}}$, which symmetrizes the growth part of the velocity gradient.

If the principal directions of the growth deformation gradient remain fixed during the mass growth, the symmetrizations of the growth deformation gradient and the growth velocity gradient are simultaneous. For example, suppose that the growth deformation gradient allows a spectral representation

$$\mathbf{F}_{\rm g} = \sum_{i=1}^{3} \vartheta_i \, \mathbf{N}_i \otimes \mathbf{N}_i \,, \tag{50}$$

relative to a triad of orthogonal unit vectors $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$, whose orientation remain fixed during the growth process. The corresponding growth velocity gradient is

$$\dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1} = \sum_{i=1}^{3} \frac{\dot{\vartheta}_i}{\vartheta_i} \, \mathbf{N}_i \otimes \mathbf{N}_i \,. \tag{51}$$

Both tensors are clearly symmetric tensors. If the mass growth is isotropic, they become

$$\mathbf{F}_{\rm g} = \vartheta \, \mathbf{I} \,, \quad \dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1} = \frac{\dot{\vartheta}}{\vartheta} \, \mathbf{I} \,. \tag{52}$$

6.1. Symmetrization of the growth velocity gradient in the absence of residual stress

If there was no initial residual stress ($\boldsymbol{\sigma}_{\rm o} = \mathbf{0}$), Eq. (43) reduces to

$$\dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1} = \mathbf{R}_{\rm e}^{T} \cdot \left(\dot{\hat{\mathbf{F}}}_{\rm g} \cdot \hat{\mathbf{F}}_{\rm g}^{-1} - \mathbf{\Omega}_{\rm e} \right) \cdot \mathbf{R}_{\rm e} \,, \tag{53}$$

where

$$\mathbf{\Omega}_{\rm e} = \dot{\mathbf{R}}_{\rm e} \cdot \mathbf{R}_{\rm e}^{-1} \tag{54}$$

is the spin tensor associated with a time dependent rotation $\mathbf{R}_{\rm e}$. Clearly, if the growth velocity gradient in Eq. (53) is to be symmetric, we must have

$$\mathbf{\Omega}_{\rm e} = \left(\dot{\hat{\mathbf{F}}}_{\rm g} \cdot \hat{\mathbf{F}}_{\rm g}^{-1} \right)_{\rm a} \,, \tag{55}$$

since then

$$\dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1} = \mathbf{R}_{\rm e}^{T} \cdot \left(\dot{\hat{\mathbf{F}}}_{\rm g} \cdot \hat{\mathbf{F}}_{\rm g}^{-1} \right)_{\rm s} \cdot \mathbf{R}_{\rm e} = \left(\dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1} \right)^{T} \,. \tag{56}$$

The subscript (s) stands for the symmetric part. Alternatively, in view of $\mathbf{F} = \mathbf{V}_{\rm e} \cdot \hat{\mathbf{F}}_{\rm g}$ and

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \dot{\mathbf{V}}_{\rm e} \cdot \mathbf{V}_{\rm e}^{-1} + \mathbf{V}_{\rm e} \cdot \left(\dot{\hat{\mathbf{F}}}_{\rm g} \cdot \hat{\mathbf{F}}_{\rm g}^{-1} \right) \cdot \mathbf{V}_{\rm e}^{-1} \,, \tag{57}$$

we can write, instead of (55),

$$\mathbf{\Omega}_{\rm e} = \left[\mathbf{V}_{\rm e}^{-1} \cdot \left(\mathbf{L} - \dot{\mathbf{V}}_{\rm e} \cdot \mathbf{V}_{\rm e}^{-1} \right) \cdot \mathbf{V}_{\rm e} \right]_{\rm a} \,. \tag{58}$$

7. Discussion

We have demonstrated in this paper that within the framework of the multiplicative decomposition of the deformation gradient, applied to study the volumetric mass growth of isotropic pseudoelastic soft tissues, either the growth part of deformation gradient or the growth part of velocity gradient can be symmetrized. This was possible to achieve because the elastic stress response does not depend on the rotation superposed to the reference configuration of an isotropic material. The symmetrization was derived for both residually stressed and unstressed reference configuration, which may be useful in the constitutive analysis of the biomechanic material response. If the constitutive expression for the growth part of the deformation gradient is being constructed directly, its representation involving only six independent components is simpler and more appealing to relate to experimental data. This is also the case in the rate-type biomechanic theory. If the growth part of the velocity gradient is symmetrized, a constitutive expression for the symmetric tensor $\dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1}$, in conjunction with the rate-type constitutive expression for the elastic rate of deformation $\left(\dot{\mathbf{F}}_{\rm e} \cdot \mathbf{F}_{\rm e}^{-1}\right)_{\rm s}$, yields the overall constitutive structure for the total rate of deformation

$$\mathbf{D} = \left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}\right)_{\rm s} = \left(\dot{\mathbf{F}}_{\rm e} \cdot \mathbf{F}_{\rm e}^{-1}\right)_{\rm s} + \left[\mathbf{F}_{\rm e} \cdot \left(\dot{\mathbf{F}}_{\rm g} \cdot \mathbf{F}_{\rm g}^{-1}\right) \cdot \mathbf{F}_{\rm e}^{-1}\right]_{\rm s}. \tag{59}$$

The analysis presented in this paper was concerned with isotropic materials, for which there is an arbitrariness in the choice of the intermediate configuration to within a rigid-body rotation, due to independence of the stress response on the prior rotation of the reference configuration. This motivated our search for the most convenient choice of the intermediate configuration in the constitutive analysis, leading us to consider the intermediate configuration that symmetrizes either the growth deformation gradient or the growth velocity gradient. In the case of anisotropic biomaterials, the stress response does depend on the rotation of the intermediate configuration, which constrains the choice of the intermediate configuration. Interestingly enough, however, the symmetric form of the growth deformation appears to be an appealing candidate for both transversely isotropic and orthotropic biomaterials (soft tissues with a longitudinal or an orthogonal network of biofibers). For example, for transversely isotropic materials the intermediate configuration is uniquley specified by requiring that the fibers are oriented relative to the material as in the initial undeformed configuration. If $\mathbf{m}^0$ is the unit vector parallel to the fibers, we require that the growth deformation gradient $\mathbf{F}_{\rm g}$ has $\mathbf{m}^0$ as one of its eigendirections, i.e., $\mathbf{F}_{\rm g} \cdot \mathbf{m}^0 = \eta_{\rm g}\, \mathbf{m}^0$, where $\eta_{\rm g}$ is the stretch ratio in the fibers direction. This condition is fulfilled by the symmetric form of the growth deformation gradient

$$\mathbf{F}_{\rm g} = \vartheta_{\rm g}\, \mathbf{I} + (\eta_{\rm g} - \vartheta_{\rm g})\, \mathbf{m}^0 \otimes \mathbf{m}^0\,. \tag{60}$$

The stretch ratio in any direction orthogonal to $\mathbf{m}^0$ is $\vartheta_{\rm g}$, and $\otimes$ stands for the dyadic product. The constitutive development based on Eq. (60) is presented in [11].

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Appendix

Consider a matrix equation for the orthogonal matrix $\mathbf{Q}$, given by

$$\mathbf{Q} \cdot \mathbf{A} = \mathbf{A}^T \cdot \mathbf{Q}^T \,, \tag{61}$$

where $\mathbf{A}$ is a nonsingular matrix. The solution of this equation is

$$\mathbf{Q} = \left(\mathbf{A}^T \cdot \mathbf{A}\right)^{1/2} \cdot \mathbf{A}^{-1} \,, \tag{62}$$

as can be checked by inspection. It remains to prove that $\mathbf{Q}$, given by Eq. (62), is indeed orthogonal, i.e., that $\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I}$. This can be easily verified. For example,

$$\mathbf{Q} \cdot \mathbf{Q}^T = \left(\mathbf{A}^T \cdot \mathbf{A}\right)^{1/2} \cdot \left(\mathbf{A}^T \cdot \mathbf{A}\right)^{-1} \cdot \left(\mathbf{A}^T \cdot \mathbf{A}\right)^{1/2} \,, \tag{63}$$

which is equal to identity matrix, because $\mathbf{A}^T \cdot \mathbf{A}$ is symmetric and thus $(\mathbf{A}^T \cdot \mathbf{A})^{-1}$ and $(\mathbf{A}^T \cdot \mathbf{A})^{1/2}$ are commutative matrices. The square root of the matrix appearing in Eq. (62) can be evaluated by using the recipe of Hoger and Carlson [18]. It readily follows that

$$\left(\mathbf{A}^T \cdot \mathbf{A}\right)^{1/2} = \frac{1}{I_1 I_2 + I_3} \left[\left(\mathbf{A}^T \cdot \mathbf{A}\right)^2 - (I_1^2 + I_2)\left(\mathbf{A}^T \cdot \mathbf{A}\right) - I_1 I_3 \,\mathbf{I}\right]. \tag{64}$$

The invariants

$$I_1 = \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} \,, \tag{65}$$

$$I_2 = -\left(\sqrt{\lambda_1 \lambda_2} + \sqrt{\lambda_2 \lambda_3} + \sqrt{\lambda_3 \lambda_1}\right), \tag{66}$$

$$I_3 = \sqrt{\lambda_1 \lambda_2 \lambda_3} \tag{67}$$

are expressed in terms of the principal values λ_1, λ_2, and λ_3 of the symmetric matrix $\mathbf{A}^T \cdot \mathbf{A}$.

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