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On the existence of periodic solutions of Rayleigh equations

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Abstract. In this paper, we study the existence of periodic solutions of Rayleigh equation

$$x'' + f(x') + g(x) = p(t)$$

where f, g are continuous functions and p is a continuous and 2π-periodic function. We prove that the given equation has at least one 2π-periodic solution provided that $f(x)$ is sublinear and the time map of equation $x'' + g(x) = 0$ satisfies some nonresonant conditions. We also prove that this equation has at least one 2π-periodic solution provided that $g(x)$ satisfies $\lim_{|x|\to+\infty} sgn(x)g(x) = +\infty$ and $f(x)$ satisfies $sgn(x)(f(x) - p(t)) \geq c$, for $t \in \mathbf{R}$, $|x| \geq d$ with c, d being positive constants.

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1. Introduction

We are concerned with the existence of 2π-periodic solutions of Rayleigh equation

$$x'' + f(x') + g(x) = p(t), \tag{1.1}$$

where $f, g : \mathbf{R} \to \mathbf{R}$ are continuous, $p : \mathbf{R} \to \mathbf{R}$ is continuous and 2π-periodic.

Arising from nonlinear oscillations, Eq.(1.1) has been studied by many authors (see [1-4, 9, 14] and the references therein). In [14], using the method of upper and lower solutions, Habets and Torres studied the existence and multiplicity of 2π-periodic solutions of Eq.(1.1) by assuming that $g = g(t, x, x')$ is bounded (or bounded from below) and other conditions hold.

When $f(x) \equiv 0$, Eq.(1.1) is a conservative system. It is well known that time map plays a crucial role in dealing with the existence and multiplicity of periodic solutions of equation $x'' + g(x) = p(t)$. Assume that $g(x)$ satisfies the following

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condition,

$$(H_1) \qquad \lim_{|x|\to+\infty} sgn(x)g(x) = +\infty.$$

Let $G(x) = \int_0^x g(u)du$. Define a function $\tau(c)$ as follows,

$$\tau(c) = \sqrt{2}|\int_0^c \frac{du}{\sqrt{G(c) - G(u)}}|,$$

which is usually called time map and was first introduced in [13]. From condition (H_1) we know that $\tau(c)$ is continuous for $|c|$ large enough. Now, we introduce notations,

$$\tau_+ = \liminf_{c\to+\infty} \tau(c), \qquad \tau^+ = \limsup_{c\to+\infty} \tau(c),$$

$$\tau_- = \liminf_{c\to-\infty} \tau(c), \qquad \tau^- = \limsup_{c\to-\infty} \tau(c).$$

By the asymptotic property of $\tau(c)$, Ding and Zanolin [5] proved that equation $x'' + g(x) = p(t)$ has at least one 2π-periodic solution provided that condition (H_1) holds and there is an integer $n > 0$ such that

$$(H_2) \qquad \frac{2\pi}{n+1} < \tau_- + \tau_+ \leq \tau^- + \tau^+ < \frac{2\pi}{n}.$$

Wang [8] generalized this result to Liénard equation $x'' + f(x)x' + g(x) = p(t)$. He proved that this Liénard equation has at least one 2π-periodic solution provided that conditions (H_1), (H_2) hold and $F(x)$ is bounded, where $F(x) = \int_0^x f(u)du$.

One aim of this paper is to prove the existence of 2π-periodic solutions of Eq.(1.1) provided that conditons (H_1), (H_2) hold and $f(x)$ is sublinear. Assume that $f(x)$ satisfies

$$(H_3) \qquad \lim_{|x|\to+\infty} f(x)/x = 0.$$

Since $f(x)$ maybe unbounded under condition (H_3), the method in [8] is invalid under present situation. By developing a different estimate method, we overcome the difficulty in estimating time owing to the possible unboundedness of $f(x)$. We obtain

Theorem 1. *Assume that conditions (H_1), (H_2) and (H_3) hold. Then Eq.(1.1) has at least one 2π-periodic solution.*

If the condition (H_2) is replaced by a condition as follows,

$$(H_4) \qquad \tau_- + \tau_+ > 2\pi,$$

then we also have

Theorem 2. *Assume that conditions (H_1), (H_3) and (H_4) hold. Then Eq.(1.1) has at least one 2π-periodic solution.*

Another aim of this paper is to prove the existence of periodic solutions of Eq.(1.1) provided that $f(x)$ satisfies sign condition. It was proved in [11] that Liénard equation

$$x'' + f(x)x' + g(x) = p(t)$$

has at least one 2π-periodic solution provided that the following conditions hold,
(i) $f(x)$ is continuous and $\lim_{|x|\to+\infty} sgn(x)F(x) = +\infty$ (or $-\infty$),
(ii) $g(x)$ is locally Lipschitz continuous and $sgn(x)g(x) \geq 0$, $|x| \geq c_0$, where c_0 is a positive constant and $\int_0^{2\pi} p(t)dt = 0$.

Obviously, the same conclusion still holds if the condition (ii) is replaced by the condition as follows,
(ii)$'$ $g(x)$ is locally Lipschitz continuous and $sgn(x)(g(x) - \bar{p}) \geq 0$, $|x| \geq c_0$, where c_0 is a positive constant and $\bar{p} = (1/2\pi)\int_0^{2\pi} p(t)dt$.

For Rayleigh equation, we have a similar result. Assume that $f(x)$ satisfies the following sign condition,

$$(H_5) \qquad sgn(x)(f(x) - p(t)) \geq c, \quad \forall t \in \mathbf{R}, \quad |x| \geq d$$

with c, d being arbitrary positive constants. We prove

Theorem 3. *Assume that f, g are locally Lipschitz continuous and conditions (H_1), (H_5) hold. Then Eq.(1.1) has at least one 2π-periodic solution.*

Throughout this paper, we always use $\mathbf{R}$ to denote the whole real number set.

2. Periodic solutions via continuation theorem

In this section, we deal with the existence of periodic solutions of (1.1) under conditions (H_1), (H_2) and (H_3) or (H_1), (H_3) and (H_4). Consider an equivalent system of Eq.(1.1),

$$x' = y, \quad y' = -g(x) - f(y) + p(t). \tag{2.1}$$

In order to use the Continuation Theorem [6, Theorem 2], we embed (2.1) into a system family with one parameter $\lambda \in [0, 1]$,

$$x' = y, \quad y' = -g(x) - \lambda f(y) + \lambda p(t). \tag{2.2}$$

Let $(x_\lambda(t), y_\lambda(t)) = (x(t, x_0, y_0, \lambda), y(t, x_0, y_0, \lambda))$ be a solution of (2.2) satisfying the initial value condition $(x_\lambda(0), y_\lambda(0)) = (x_0, y_0)$. In what follows, for simplicity, we always use $(x(t), y(t))$ to denote $(x_\lambda(t), y_\lambda(t))$. We have the following lemma.

Lemma 1. *Assume that conditions (H_1), (H_3) hold. Then every solution $(x(t), y(t))$ of (2.2) exists on the whole t-axis.*

Proof. Define a function

$$V(x, y) = \frac{1}{2}y^2 + G(x).$$

Set

$$v(t) = V(x(t), y(t)) = \frac{1}{2}y(t)^2 + G(x(t)).$$

Then we have

$$v'(t) = -\lambda[f(y(t)) - p(t)]y(t). \tag{2.3}$$

Since $\lim_{|x|\to\infty} f(x)/x = 0$, there exists a constant $a > 0$ such that

$$|f(x)x| \le x^2, \quad |x| \ge a.$$

Furthermore, there exists a positive constant b such that

$$|f(x)x| \le x^2 + b, \quad x \in \mathbf{R}. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$|v'(t)| \le y(t)^2 + b + \frac{1}{2}y(t)^2 + \frac{1}{2}p(t)^2 \le \frac{3}{2}y(t)^2 + M, \tag{2.5}$$

with $M = b + M_p^2/2$, $M_p = max\{|p(t)| : t \in \mathbf{R}\}$. From (H_1) we know that there exists a positive constant M_0 such that

$$G(x) + M_0 \ge 0, \quad x \in \mathbf{R},$$

which, together with (2.5), implies that

$$|v'(t)| \le \frac{3}{2}y(t)^2 + 3G(x(t)) + \bar{M},$$

with $\bar{M} = M + 3M_0$. Thus, we have that $|v'(t)| \le 3v(t) + \bar{M}$. Hence,

$$v'(t) \le 3v(t) + \bar{M}.$$

Multiplying both sides of this inequality by e^{-3t} and integrating over any bounded interval $[0, T_0)$ $(T_0 > 0)$ we have that

$$v(t) \le v(0)e^{3T_0} + \frac{1}{3}\bar{M}(e^{3T_0} - 1), \quad t \in [0, T_0).$$

Therefore, there is no blow-up for solution $(x(t), y(t))$ on any bounded interval. Furthermore, every solution $(x(t), y(t))$ of (2.2) exists on the whole t-axis.

On the basis of Lemma 1, we have

Lemma 2. *Under conditions (H_1), (H_3). Then there is a nondecreasing function $\sigma : \mathbf{R}^+ \to \mathbf{R}^+$, with $\sigma(s) \ge s$, for all $s > 0$, such that for any $\lambda \in [0, 1]$, $r > 0$ and each solution $(x(t), y(t))$ of (2.2), the following conclusion holds,*

(i) *If $(x_0^2 + y_0^2)^{1/2} \le r$, then $(x(t)^2 + y(t)^2)^{1/2} \le \sigma(r)$, for $t \in [0, 2\pi]$.*
(ii) *If $(x_0^2 + y_0^2)^{1/2} \ge \sigma(r)$, then $(x(t)^2 + y(t)^2)^{1/2} \ge r$, for $t \in [0, 2\pi]$.*

This lemma can be proved by standard methods [7, 12].

Lemma 3. *Assume that (H_1), (H_3) hold. Then there exists a constant $R_0 > 0$ such that if $(x(t), y(t))$ is a 2π-periodic solution of (2.2) with $x_0^2 + y_0^2 \ge R_0^2$ and*

$x_1 = x(t_1)$ is a local maximum of $x(t)$ and $x_2 = x(t_2)$ is a local minimum of $x(t)$, then

$$x_1 > 0, \quad x_2 < 0.$$

Proof. We only give the proof of $x_1 > 0$. The other case can be treated similarly. Assume by contradiction that $x_1 \le 0$. Let $A > 0$ be a constant satisfying

$$|f(0)| + |p(t)| + 1 \le A, \quad t \in \mathbf{R}.$$

It follows from (H_1) that there exists a constant $a > 0$ such that

$$sgn(x)g(x) \ge A, \quad |x| \ge a. \tag{2.6}$$

Define $R_0 = \sigma(a)$, where σ is defined in Lemma 2. Since $x_1 = x(t_1)$ is a local maximum of $x(t)$, we know that $y(t_1) = x'(t_1) = 0$. Therefore,

$$y'(t_1) = -g(x(t_1)) - \lambda f(0) - \lambda p(t_1).$$

From Lemma 2 we have that if $x_0^2 + y_0^2 \ge R_0^2$, then

$$x(t)^2 + y(t)^2 \ge a^2, \quad t \in [0, 2\pi].$$

Therefore,

$$x(t_1)^2 \ge a^2,$$

which implies that $x(t_1) \ge a$ or $x(t_1) \le -a$. By the hypothesis $x_1 \le 0$ we have that $x(t_1) \le -a$, which, together with (2.6), implies that $y'(t_1) > 0$. Then we have $x''(t_1) = y'(t_1) > 0$. From the continuity of $x''(t)$ we know that there exists an interval (α, β) such that $t_1 \in (\alpha, \beta)$ and $x''(t) > 0$, for $t \in (\alpha, \beta)$. Since $x'(t_1) = 0$, we get that

$$x'(t) < 0, t \in (\alpha, t_1); \quad x'(t) > 0, t \in (t_1, \beta).$$

Thus we obtain that

$$x(t_1) < x(t), t \in (\alpha, t_1); \quad x(t_1) < x(t), t \in (t_1, \beta).$$

This contradicts with the fact that x_1 is a local maximum of $x(t)$.

It follows from Lemma 2 that if $x_0^2 + y_0^2$ is large enough, then we can introduce the polar coordinates. Set $x = r\cos\theta$, $y = r\sin\theta$. Under this transformation, (2.2) becomes

$$\begin{cases} \dfrac{dr}{dt} = & r\sin\theta\cos\theta - g(r\cos\theta)\sin\theta - \lambda f(r\sin\theta)\sin\theta + \lambda p(t)\sin\theta \\ \dfrac{d\theta}{dt} = & -\sin^2\theta - \dfrac{g(r\cos\theta)cos\theta}{r} - \dfrac{\lambda f(r\sin\theta)cos\theta}{r} + \dfrac{\lambda p(t)cos\theta}{r}. \end{cases}$$

Denote by $(r(t), \theta(t)) = (r(t, r_0, \theta_0, \lambda), \theta(t, r_0, \theta_0, \lambda))$ the solution of above system through the initial point (r_0, θ_0).

Lemma 4. *Assume that* (H_1), (H_3) *hold and* $\delta (0 < \delta < 1)$ *is a given constant. Then there exists* $R_\delta > 0$ *such that for* $r_0 \geq R_\delta$ *and* $(r(t)\cos\theta(t), r(t)\sin\theta(t)) \in \{(x,y) : |y| \geq \delta|x|\}$, *the following inequality holds,*

$$\frac{d\theta(t)}{dt} < 0, \quad t \in [0, 2\pi].$$

Proof. It follows from (H_1) that there exists a constant $a_0 > 0$ such that

$$(g(x) - \lambda p(t))x \geq 0, \quad |x| \geq a_0, t \in \mathbf{R}, \lambda \in [0,1].$$

Therefore,

$$\frac{(g(r\cos\theta) - \lambda p(t))\cos\theta}{r} \geq 0, \quad |r\cos\theta| \geq a_0, t \in \mathbf{R}, \lambda \in [0,1]. \tag{2.7}$$

On the other hand, since $g(x)$ is bounded in interval $[-a_0, a_0]$, we have

$$-\frac{\epsilon_\delta}{4} \leq \frac{(g(r\cos\theta) - \lambda p(t))\cos\theta}{r} \leq \frac{\epsilon_\delta}{4}, \quad |r\cos\theta| \leq a_0, t \in \mathbf{R}, \lambda \in [0,1] \tag{2.8}$$

for $r > 0$ large enough, where $\epsilon_\delta = \sin^2(\arctan\delta) > 0$. From (H_3) we have that

$$-\frac{\epsilon_\delta}{4} \leq \frac{f(r\sin\theta)\cos\theta}{r} \leq \frac{\epsilon_\delta}{4}, \quad \theta \in \mathbf{R} \tag{2.9}$$

for $r > 0$ large enough. If $(r(t)\cos\theta(t), r(t)\sin\theta(t)) \in \{(x,y) : |y| \geq \delta|x|\}$ and $|r(t)\cos\theta(t)| \geq a_0$, $t \in [0, 2\pi]$, then it follows from Lemma 2 and (2.7), (2.9) that

$$\frac{d\theta(t)}{dt} \leq -\epsilon_\delta + \frac{\epsilon_\delta}{4} < 0$$

with r_0 large enough. If $(r(t)\cos\theta(t), r(t)\sin\theta(t)) \in \{(x,y) : |y| \geq \delta|x|\}$ and $|r(t)\cos\theta(t)| \leq a_0$, $t \in [0, 2\pi]$, then it follows from Lemma 2 and (2.8), (2.9) that

$$\frac{d\theta(t)}{dt} \leq -\epsilon_\delta + \frac{\epsilon_\delta}{2} < 0$$

with r_0 large enough. Thus, we have reached the conclusion.

Let $(x(t), y(t))$ be a 2π-periodic solution of (2.2) with $r_0 = \sqrt{x_0^2 + y_0^2} \geq \max\{R_0, R_\delta\}$, which has polar coordinates expression $(r(t), \theta(t))$. Then we can define the rotation number as follows,

$$n(x,y) = \frac{\theta(0) - \theta(2\pi)}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} \frac{x'(t)y(t) - x(t)y'(t)}{x^2(t) + y^2(t)} dt.$$

Assume that (H_1), (H_3) hold. From Lemma 3 that there exists some $t_0 \in [0, 2\pi]$ such that $x(t_0) = 0$. According to Lemma 4, if $(x(t), y(t)) \in \{(x,y) : |y| \geq \delta|x|\}$ and $t \in [0, 2\pi]$, then $\theta(t)$ decreases strictly. Therefore, there exists $t_1 \in [t_0, t_0 + 2\pi]$ such that $y(t_1) = \delta x(t_1)$, $x(t_1) > 0$ and $y(t) \geq \delta x(t)$, $t \in [t_0, t_1]$. Since $(x(t), y(t))$ is 2π-periodic, the solution $(x(t), y(t))$ must leave the region $\{(x,y) : |y| \leq \delta|x|\}$

after it enters this region. Hence, there exists some $t_2 \in [t_0, t_0 + 2\pi]$ such that $y(t_2) = -\delta x(t_2)$ and $(x(t), y(t)) \in \{(x, y) : |y| \le \delta|x|, x > 0\}$, $t \in [t_1, t_2]$. Thus, we have that

$$\theta(t_1) - \theta(t_2) = 2\arctan\delta > 0.$$

Every time when the solution $(x(t), y(t))$ goes through the region $\{(x, y) : |y| \le \delta|x|\}$, we have the same result. Recalling that $\theta(t)$ decreases strictly when $(x(t), y(t)) \in \{(x, y) : |y| \ge \delta|x|\}$, we obtain that $\theta(0) - \theta(2\pi) > 0$, which implies

$$n(x, y) \in \mathbf{N}$$

for $r_0 = \sqrt{x_0^2 + y_0^2}$ large enough.

Lemma 5. *Assume that conditions (H_1), (H_2) and (H_3) hold. Let $k_0 > 0$ be a fixed integer. Suppose that there exists a sequence of 2π-periodic solutions $\{(x_j(t), y_j(t))\}_{j=1}^{\infty}$ of (2.2), with rotation numbers $n(x_j, y_j) = k_0$, $j = 1, 2, \cdots$, such that*

$$\lim_{j\to+\infty}(x_j^2(t) + y_j^2(t)) = +\infty,$$

then

$$k_0(\tau_+ + \tau_-) \le 2\pi.$$

Proof. For simplicity, we assume that

$$sgn(x)g(x) > 0, \quad x \in \mathbf{R}, x \neq 0.$$

Let $(x(t), y(t))$ be any one of $(x_j(t), y_j(t))$ with j large enough. Then there exist constants $t_1^1 < t_2^1 < t_3^1 = t_1^2 < t_2^2 < t_3^2 = \cdots = t_1^{k_0} < t_2^{k_0} < t_3^{k_0} = t_1^1 + 2\pi$ such that

$$x(t_1^i) = 0; \quad x(t_2^i) = 0; \quad x(t_3^i) = 0; \quad i = 1, 2, \cdots, k_0$$

and

$$x(t) \ge 0, t \in [t_1^i, t_2^i]; \quad x(t) \le 0, t \in [t_2^i, t_3^i], \quad i = 1, 2, \cdots, k_0.$$

For simplicity, let (α, β) $(\alpha < \beta)$ denote any couple of (t_1^i, t_2^i) $(i = 1, 2, \cdots, k_0)$. Set $x_* = x(t_*) = \max\{x(t) : \alpha \le x \le \beta\}$. In what follows, we shall estimate $t_* - \alpha$ and $\beta - t_*$, respectively. At first, we estimate the former one. It can be inferred from the first equation of (2.2) that $y(t) \ge 0$, for $t \in [\alpha, t_*]$. From condition (H_3) we know that for any sufficiently small $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that

$$|f(x)| \le \varepsilon|x|, \quad |x| \ge a_\varepsilon,$$

which implies that

$$|f(x)x| \le \varepsilon x^2, \quad |x| \ge a_\varepsilon.$$

Thus, there exists a constant $b_\varepsilon > 0$ such that

$$|f(x)x| \le \varepsilon x^2 + b_\varepsilon, \quad x \in \mathbf{R}. \tag{2.10}$$

Multiplying both sides of $y' = -g(x) - \lambda f(y) + \lambda p(t)$ by $y(t)$ and applying $x'(t) = y(t)$, we have

$$y(t)y'(t) = -g(x(t))x'(t) - \lambda f(y(t))y(t) + \lambda p(t)y(t). \tag{2.11}$$

Integrating both sides (2.11) over interval $[t, t_*]$ with $\alpha \le t \le t_*$ yields

$$\int_t^{t_*} y(\tau)y'(\tau)d\tau = -\int_t^{t_*} g(x(\tau))x'(\tau)d\tau - \lambda \int_t^{t_*} f(y(\tau))y(\tau)dt + \lambda \int_t^{t_*} p(\tau)y(\tau)d\tau.$$

Since $y(t_*) = x'(t_*) = 0$, we have that, for $\alpha \le t \le t_*$,

$$y^2(t) = 2(G(x(t_*)) - G(x(t))) + 2\lambda \int_t^{t_*} f(y(\tau))y(\tau)d\tau - 2\lambda \int_t^{t_*} p(\tau)y(\tau)d\tau. \tag{2.12}$$

Combining (2.10) and (2.12) we get that

$$y^2(t) \le 2(G(x(t_*)) - G(x(t))) + 2\varepsilon \int_t^{t_*} y^2(\tau)d\tau + 2M_p \int_t^{t_*} y(\tau)d\tau + 4b_\varepsilon \pi. \tag{2.13}$$

with $M_p = \max\{|p(t)| : t \in \mathbf{R}\}$. Write

$$\Phi(t) = \int_t^{t_*} y^2(\tau)d\tau.$$

Then

$$\Phi'(t) = -y^2(t).$$

Therefore, we have that

$$-\Phi'(t) - 2\varepsilon\Phi(t) \le 2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4b_\varepsilon \pi.$$

Multiplying both sides of above inequality by $e^{2\varepsilon t}$ and integrating over interval $[t, t_*]$ yields

$$-\int_t^{t_*} [\Phi(\tau)e^{2\varepsilon\tau}]' d\tau \le \int_t^{t_*} [2(G(x(t_*)) - G(x(\tau))) + 2M_p(x(t_*) - x(\tau)) + 4b_\varepsilon \pi] e^{2\varepsilon\tau} d\tau.$$

Since $\Phi(t_*) = 0$, we have

$$\Phi(t)e^{2\varepsilon t} \le \int_t^{t_*} [2(G(x(t_*)) - G(x(\tau))) + 2M_p(x(t_*) - x(\tau)) + 4b_\varepsilon \pi] e^{2\varepsilon\tau} d\tau.$$

On the other hand, from $x'(t) = y(t) \ge 0$ we know that $x(\tau)$ is increasing on the interval $[t, t_*]$. Consequently,

$$\Phi(t)e^{2\varepsilon t} \le e^{2\varepsilon t_*} \int_t^{t_*} [2(G(x(t_*)) - G(x(\tau))) + 2M_p(x(t_*) - x(\tau)) + 4b_\varepsilon \pi] d\tau.$$

Furthermore, for $t \in [\alpha, t_*]$,

$$\Phi(t) \le 2\pi e^{4\pi\varepsilon}[2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4b_\varepsilon \pi]. \tag{2.14}$$

It follows from (2.13) and (2.14) that, for $t \in [\alpha, t_*]$,

$$\begin{aligned} y^2(t) \leq\ & 2(G(x(t_*)) - G(x(t))) \\ & +4\pi\varepsilon e^{4\pi\varepsilon}[2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4b_\varepsilon\pi] \\ & +2M_p(x(t_*) - x(t)) + 4b_\varepsilon\pi. \end{aligned} \tag{2.15}$$

Set $\eta = 4\pi\varepsilon e^{4\pi\varepsilon}$. Obviously, $\eta \to 0$ as $\varepsilon \to 0$. By (2.15) we have that

$$y^2(t) \leq (1+\eta)[2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4b_\varepsilon\pi].$$

Recalling $x'(t) = y(t)$, we get that, for $t \in [\alpha, t_*]$,

$$x'(t) \leq \sqrt{1+\eta}\sqrt{2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4b_\varepsilon\pi}. \tag{2.16}$$

According to (2.16), we have

$$\frac{x'(t)}{\sqrt{1+\eta}\sqrt{2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4b_\varepsilon\pi}} \leq 1.$$

Integrating both sides of this inequality over $[\alpha, t_*]$, we obtain that

$$\frac{1}{\sqrt{1+\eta}} \int_0^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x) + 4b_\varepsilon\pi}} \leq t_* - \alpha, \tag{2.17}$$

where $x_* = x(t_*)$. Take a constant $L > 0$ and write

$$\int_0^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x) + 4b_\varepsilon\pi}} = I_1 + I_2$$

with

$$I_1 = \int_0^{x_*-L} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x) + 4b_\varepsilon\pi}},$$

$$I_2 = \int_{x_*-L}^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x) + 4b_\varepsilon\pi}}.$$

If $x \in [0, x_* - L]$, then

$$G(x_*) - G(x) + M_p(x_* - x) \geq G(x_*) - G(x_* - L) + LM_p = [g(\xi) + M_p]L \tag{2.18}$$

with $\xi \in [x_* - L, x_*]$. From (2.18) and (H_1) we know that

$$\lim_{x_* \to +\infty} [G(x_*) - G(x) + M_p(x_* - x)] = +\infty.$$

Therefore, I_1 can be expressed in the form

$$I_1 = \int_0^{x_*-L} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x)[1 + o(1)]}}$$

for $x_* \to \infty$. From (H_2) we know that $\tau(e)$ is bounded. Thus we obtain

$$I_1 = \int_0^{x_*-L} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x)}} + o(1). \tag{2.19}$$

On the other hand, if $x \in [x_* - L, x_*]$, then

$$G(x_*) - G(x) + M_p(x_* - x) \geq [\mu(x_*) + M_p](x_* - x)$$

with $\mu(x_*) = \min\{g(x) : x \in [x_* - L, x_*]\}$. Consequently,

$$I_2 \leq \int_{x_*-L}^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x)}} \leq \frac{\sqrt{2L}}{\sqrt{\mu(x_*) + M_p}} = o(1)$$

for $x_* \to \infty$. Furthermore,

$$I_2 = \int_{x_*-L}^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x)}} + o(1). \tag{2.20}$$

It follows from (2.17), (2.19) and (2.20) that

$$\frac{1}{\sqrt{1+\eta}} \int_0^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_p(x_* - x)}} + o(1) \leq t_* - \alpha$$

for $x_* \to \infty$. Applying a Lemma in [5, 13] we have that

$$\frac{1}{\sqrt{1+\eta}} \int_0^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x))}} + o(1) \leq t_* - \alpha,$$

which implies that, for $x_* \to \infty$,

$$\frac{1}{2} \frac{1}{\sqrt{1+\eta}} \tau_+ + o(1) \leq t_* - \alpha. \tag{2.21}$$

Similarly, we have that

$$\frac{1}{2} \frac{1}{\sqrt{1+\eta}} \tau_+ + o(1) \leq \beta - t_*. \tag{2.22}$$

Combining (2.21) and (2.22), we get

$$\frac{1}{\sqrt{1+\eta}} \tau_+ + o(1) \leq \beta - \alpha,$$

for $x_* \to \infty$. Sine (α, β) denote any couple of (t_1^i, t_2^i), $i = 1, 2, \cdots, k_0$, we obtain that, for $x_* \to \infty$,

$$\frac{k_0}{\sqrt{1+\eta}} \tau_+ + o(1) \leq \sum_{i=1}^{i=k_0} (t_2^i - t_1^i). \tag{2.23}$$

Using the same methods, we can derive that

$$\frac{k_0}{\sqrt{1+\eta}} \tau_- + o(1) \leq \sum_{i=1}^{i=k_0} (t_3^i - t_2^i). \tag{2.24}$$

From (2.23) and (2.24) we have that

$$\frac{k_0}{\sqrt{1+\eta}} (\tau_+ + \tau_-) + o(1) \leq \sum_{i=1}^{i=k_0} (t_2^i - t_1^i) + \sum_{i=1}^{i=k_0} (t_3^i - t_2^i) = 2\pi.$$

Since $\eta \to 0$ as $\varepsilon \to 0$, we get that

$$k_0(\tau_+ + \tau_-) \le 2\pi.$$

Lemma 6. *Under the same conditions of Lemma 5. The following conclusion holds,*

$$k_0(\tau^+ + \tau^-) \ge 2\pi.$$

Proof. We use the same notations as in Lemma 5. From (2.10) and (2.12) we know that

$$y^2(t) \ge 2(G(x(t_*)) - G(x(t))) - 2\varepsilon \int_t^{t_*} y^2(\tau)d\tau - 2M_p \int_t^{t_*} y(\tau)d\tau - 4\pi b_\varepsilon,$$

which, together with (2.14), yields

$$\begin{aligned} y^2(t) \ge \quad & 2(G(x(t_*)) - G(x(t))) \\ & -4\pi\varepsilon e^{4\pi\varepsilon}[2(G(x(t_*)) - G(x(t))) + 2M_p(x(t_*) - x(t)) + 4\pi b_\varepsilon] \\ & -2M_p(x(t_*) - x(t)) - 4b_\varepsilon\pi. \end{aligned}$$

Therefore, we have that, for $t \in [\alpha, \beta]$,

$$y^2(t) \ge 2(1-\eta)(G(x(t_*)) - G(x(t))) - 2(1+\eta)M_p(x(t_*) - x(t)) - 4(1+\eta)\pi b_\varepsilon,$$

where $\eta = 4\pi\varepsilon e^{4\pi\varepsilon}$. Let $L_0 > 0$ be a constant. If $x(t) \in [0, x(t_*) - L_0]$, then we have

$$\begin{aligned} & [(1-\eta)G(x(t_*)) - (1+\eta)M_p x(t_*)] - [(1-\eta)G(x(t)) - (1+\eta)M_p x(t)] \\ & \ge [(1-\eta)G(x(t_*)) - (1+\eta)M_p x(t_*)] \\ & \quad -[(1-\eta)G(x(t_*) - L_0) - (1+\eta)M_p(x(t_*) - L_0)] \\ & = (1-\eta)g(\xi_*)L_0 - (1+\eta)M_p L_0, \quad \xi_* \in [x(t_*) - L_0, x(t_*)]. \end{aligned}$$

Hence, if $x(t_*)$ is large enough and $x(t) \in [0, x(t_*) - L_0]$, then

$$y^2(t) \ge 2(1-\eta)(G(x(t_*)) - G(x(t))) - 2(1+\eta)M_p(x(t_*) - x(t)) - 4(1+\eta)\pi b_\varepsilon > 0.$$

Let $\bar{t}_* \in [\alpha, t_*]$ such that $x(\bar{t}_*) = x(t_*) - L_0$. If $t \in [\alpha, \bar{t}_*]$, then

$$x'(t) \ge \sqrt{2(1-\eta)(G(x(t_*)) - G(x(t))) - 2(1+\eta)M_p(x(t_*) - x(t)) - 4(1+\eta)\pi b_\varepsilon}.$$

Consequently,

$$\frac{x'(t)}{\sqrt{2(1-\eta)(G(x(t_*)) - G(x(t))) - 2(1+\eta)M_p(x(t_*) - x(t)) - 4(1+\eta)\pi b_\varepsilon}} \ge 1.$$

Integrating both sides of this inequality over $[\alpha, \bar{t}_*]$ results in

$$\int_0^{x_*-L_0} \frac{dx}{\sqrt{2(1-\eta)(G(x_*)-G(x))-2(1+\eta)M_p(x_*-x)-4(1+\eta)\pi b_\varepsilon}} \geq \bar{t}_* - \alpha$$

with $x_* = x(t_*)$. Applying the same methods as in Lemma 5, we have that

$$\begin{aligned} &\int_0^{x_*-L_0} \frac{dx}{\sqrt{2(1-\eta)(G(x_*)-G(x))-2(1+\eta)M_p(x_*-x)-4(1+\eta)\pi b_\varepsilon}} \\ &= \int_0^{x_*-L_0} \frac{dx}{\sqrt{2(1-\eta)(G(x_*)-G(x))}} + o(1) \end{aligned} \tag{2.25}$$

for $x_* \to \infty$. On the other hand, it is easy to check that

$$\int_{x_*-L_0}^{x_*} \frac{dx}{\sqrt{2(1-\eta)(G(x_*)-G(x))}} = \frac{1}{\sqrt{1-\eta}} \int_{x_*-L_0}^{x_*} \frac{dx}{\sqrt{2(G(x_*)-G(x))}} = o(1). \tag{2.26}$$

From (2.25) and (2.26) we know that, for $x_* \to \infty$,

$$\frac{1}{\sqrt{1-\eta}} \int_0^{x_*} \frac{dx}{\sqrt{2(G(x_*)-G(x))}} + o(1) \geq \bar{t}_* - \alpha. \tag{2.27}$$

Next, we estimate $t_* - \bar{t}_*$. Since $x'(t) = y(t) \geq 0$, $t \in [\bar{t}_*, t_*]$, we know that $x(t) \in [x(t_*) - L_0, x(t_*)]$, for $t \in [\bar{t}_*, t_*]$. From $x'(t) = y(t)$ we have that

$$\int_{\bar{t}_*}^{t_*} y(t)dt = \int_{\bar{t}_*}^{t_*} x'(t)dt = x(t_*) - x(\bar{t}_*) = L_0.$$

According to condition (H_3), for any sufficiently small $\varepsilon > 0$, there exists a c_ε such that

$$|f(x)| \leq \varepsilon|x| + c_\varepsilon, \quad x \in \mathbf{R}. \tag{2.28}$$

By $y'(t) = -g(x(t)) - \lambda f(y(t)) + \lambda p(t)$ and (2.28) we get that, for $t \in [\bar{t}_*, t_*]$,

$$y'(t) \leq -g(x(t)) + \varepsilon y(t) + c_\varepsilon + M_p. \tag{2.29}$$

Integrating both sides of (2.29) over interval $[t, t_*]$, with $t \in [\bar{t}_*, t_*]$, we obtain

$$\int_t^{t_*} y'(\tau)d\tau \leq -\int_t^{t_*} g(x(t))dt + \varepsilon \int_t^{t_*} y(t)dt + 2\pi(c_\varepsilon + M_p).$$

Therefore, if $t \in [\bar{t}_*, t_*]$, then

$$y(t) \geq \int_t^{t_*} g(x(t))dt - \varepsilon L_0 - 2\pi(c_\varepsilon + M_p). \tag{2.30}$$

Define $\nu(x_*) = \min\{g(x) : x \in [x(t_*) - L_0, x(t_*)]\}$. By condition (H_1) we know that $\nu(x_*) \to +\infty$, as $x(t_*) \to +\infty$. From (2.30) we derive that, for $t \in [\bar{t}_*, t_*]$,

$$y(t) \geq \nu(x_*)(t_* - t) - \varepsilon L_0 - 2\pi(c_\varepsilon + M_p). \tag{2.31}$$

Integrating both sides of (2.31) over $[\bar{t}_*, t_*]$ yields

$$\int_{\bar{t}_*}^{t_*} y(t)dt \geq \frac{1}{2}\nu(x_*)(t_* - \bar{t}_*)^2 - [\varepsilon L_0 + 2\pi(c_\varepsilon + M_p)](t_* - \bar{t}_*).$$

Hence, we get that

$$L_0 \geq \frac{1}{2}\nu(x_*)(t_* - \bar{t}_*)^2 - [\varepsilon L_0 + 2\pi(c_\varepsilon + M_p)](t_* - \bar{t}_*),$$

which implies that

$$t_* - \bar{t}_* = o(1), \tag{2.32}$$

for $x_* \to \infty$. Combining (2.27) and (2.32) we obtain

$$\frac{1}{\sqrt{1-\eta}} \int_0^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x))}} + o(1) \geq t_* - \alpha.$$

Furthermore,

$$\frac{\tau^+}{2\sqrt{1-\eta}} + o(1) \geq t_* - \alpha. \tag{2.33}$$

Similarly, we have

$$\frac{\tau^+}{2\sqrt{1-\eta}} + o(1) \geq \beta - t_*. \tag{2.34}$$

It can be inferred from (2.33) and (2.34) that, for $x_* \to \infty$,

$$\frac{\tau^+}{\sqrt{1-\eta}} + o(1) \geq \beta - \alpha.$$

Since (α, β) denotes any one of (t_1^i, t_2^i), we reach that

$$\frac{k_0\tau^+}{\sqrt{1-\eta}} + o(1) \geq \sum_{i=1}^{i=k_0} (t_2^i - t_1^i).$$

Similarly, we have that

$$\frac{k_0\tau^-}{\sqrt{1-\eta}} + o(1) \geq \sum_{i=1}^{i=k_0} (t_3^i - t_2^i).$$

Therefore, we obtain that

$$\frac{k_0(\tau^+ + \tau^-)}{\sqrt{1-\eta}} + o(1) \geq \sum_{i=1}^{i=k_0} (t_2^i - t_1^i) + \sum_{i=1}^{i=k_0} (t_3^i - t_2^i) = 2\pi.$$

Recalling that $\eta \to 0$, as $\varepsilon \to 0$, we know that

$$k_0(\tau^+ + \tau^-) \geq 2\pi.$$

Lemma 7. *Assume that conditions (H_1), (H_2) and (H_3) hold. Let $k > 0$ be an arbitrary integer. Then there exists a constant $R_k > 0$ such that for any*

2π-periodic solution $(x(t), y(t))$ of (2.2), with rotation number $n(x, y) = k$, the following conclusion holds,

$$x(t)^2 + y(t)^2 \leq R_k^2, \quad t \in \mathbf{R}.$$

Proof. Assume by contradiction that there exist an integer $k_0 > 0$ and a sequence of 2π-periodic solutions $(x_j(t), y_j(t))$ of (2.2), with the rotation number $n(x_j, y_j) = k_0$ $(j = 1, 2, \cdots)$, such that

$$\lim_{j \to +\infty} (x_j^2(t) + y_j^2(t)) = +\infty$$

uniformly for $t \in \mathbf{R}$. From Lemma 5 and Lemma 6 we know that

$$\tau_+ + \tau_- \leq \frac{2\pi}{k_0} \leq \tau^+ + \tau^-.$$

This contradicts with condition (H_2).

Proof of Theorem 1. In order to use the Continuation Theorem [6] to prove the existence of 2π-periodic solution of (2.1), we shall check that all conditions of the Continuation Theorem are satisfied. From [8] we know that

(i) There exists $B > 0$ such that every 2π-periodic solution $(x(t), y(t))$ of system $x' = y$, $y' = -g(x)$ satisfies $|x(t)| + |y(t)| \leq B$, $t \in [0, 2\pi]$.

(ii) Define $h(x, y) = (y, -g(x))$. Then the Brouwer degree $d(h, B(0, r), 0) = 1$, with r large enough, $B(0, r) = \{(x, y) : x^2 + y^2 \leq r^2\}$.

From Lemma 2 we have that

(iii) For any $r_1 > 0$, there exists $r_2 > 0$ such that, for each 2π-periodic solution of (2.2), we have

$$\min_{[0,2\pi]} (x^2(t) + y^2(t)) \leq r_1^2 \Longrightarrow \max_{[0,2\pi]} (x^2(t) + y^2(t)) \leq r_2^2.$$

From Lemma 7 we know that

(iv) For any integer $k > 0$, there exists $R_k > 0$ such that, for each 2π-periodic solution of (2.2), we have

$$n(x, y) = k \Longrightarrow \min_{[0,2\pi]} (x^2(t) + y^2(t)) \leq R_k^2.$$

Thus, all conditions of the Continuation Theorem are satisfied. Therefore, (2.1) has at least one 2π-periodic solution.

The proof of Theorem 2 can be handled similarly. Indeed, under conditions of Theorem 2, the conditions (i), (ii) and (iii) in proof of Theorem 1 are still satisfied. From Lemma 5 and condition (H_4) we know that (iv) still holds. Therefore, all conditions of the Continuation Theorem are satisfied. Hence, (2.1) has at least one 2π-periodic solution.

3. Periodic solutions via Lyapunov function

From Massera's theorem [10] we know that if every Cauchy problem of the system

$$x' = h_1(x, y, t), \qquad y' = h_2(x, y, t)$$
$$x(0) = x_0, \qquad y(0) = y_0$$

exists uniquely and is positively bounded, then this system has at least one 2π-periodic solution, where $h_i \in C(\mathbf{R}, \mathbf{R}, \mathbf{R})$ and $h_i(x, y, t + 2\pi) = h_i(x, y, t)$, for $x, y, t \in \mathbf{R}$, $i = 1, 2$. In this section, by means of Lyapunov function, we will show that all solutions of (2.1) are positively bounded under conditions (H_1) and (H_5) and hence (2.1) possesses at least one 2π-periodic solution. Let us recall that condition (H_5) refers to

$$sgn(x)(f(x) - p(t)) \geq c, \quad \forall t \in \mathbf{R}, \quad |x| \geq d$$

with c, d being positive constants.

Proof of Theorem 3. We follow an argument in [11]. Since f, g are locally Lipschitz continuous, every solution $(x(t), y(t))$ of (2.1) satisfying the initial value condition $(x(0), y(0)) = (x_0, y_0)$ exists uniquely. Define a potential function V as in Lemma 1,

$$V(x, y) = \frac{1}{2}y^2 + G(x).$$

Set

$$v(t) = V(x(t), y(t)) = \frac{1}{2}y(t)^2 + G(x(t)).$$

Then

$$v'(t) = -y(t)(f(y(t)) - p(t)). \tag{3.1}$$

Write

$$m_1 = \max\{|f(y)| : -d \leq y \leq d\}, \quad m_2 = \max\{|p(t)| : t \in \mathbf{R}\}.$$

If $|y| \leq d$, then

$$|(1 - y)(f(y) - p(t))| \leq m_3, \quad |(1 + y)(f(y) - p(t))| \leq m_3, \forall t \in \mathbf{R} \tag{3.2}$$

with $m_3 = (1 + d)(m_1 + m_2)$. Take a constant $k > 0$ sufficiently large such that

$$|g(x)| \geq m_3, \quad |x| \geq k \quad \text{and} \quad 2d/k \leq c. \tag{3.3}$$

Define a Lyapunov function $W(x, y)$ as follows,

$$W(x,y) = \begin{cases} V(x,y), & |x| < +\infty, \quad y \geq d, \\ V(x,y) - y + d, & x \leq -k, \quad |y| \leq d, \\ V(x,y) + 2d, & x \leq -k, \quad y \leq -d, \\ V(x,y) + y - d, & x \geq k, \quad |y| \leq d, \\ V(x,y) - 2d, & x \geq k, \quad y \leq -d, \\ V(x,y) - \frac{2d}{k}x, & |x| \leq k, \quad y \leq -d. \end{cases}$$

It is easy to check that $W(x, y)$ is continuous and locally Lipschitz with respect to $(x, y) \in \{(x, y) : |x| \geq k, |y| \geq d\}$. Moreover, $W(x, y)$ tends to infinity uniformly for $x \in \mathbf{R}$ as $|y| \to +\infty$. Set $\Gamma(x, y) = V(x, y) + 2dx/k + y + 2d$. Then $\Gamma(x, y)$ is continuous and $W(x, y) \leq \Gamma(|x|, |y|)$. By using (3.1)-(3.3) and the expression of $W(x, y)$, we have that the derivative $W'(x(t), y(t))$ of $W(x(t), y(t))$ with respect to t satisfies

$$W'(x(t), y(t)) \leq 0.$$

On the other hand, let $l > 0$ be a constant. Then there exists a constant $\mathcal{L} > 0$ such that, for $|y| \leq l$,

$$|y - f(y) + p(t)| \leq \mathcal{L}, \quad \forall t \in \mathbf{R}.$$

Take a constant $r > 0$ such that

$$|g(x)| \geq \mathcal{L}, \quad |x| \geq r.$$

Define another Lyapunov function

$$U(x, y) = \begin{cases} x + y, & x \geq r, \quad |y| \leq l, \\ -x - y, & x \leq -r, \quad |y| \leq l. \end{cases}$$

Obviously, $U(x, y)$ satisfies the following conclusions.

(1)$U(x, y)$ tends to infinity uniformly for $|y| \leq l$ as $|x|$ tends to infinity.

(2)$U(x, y) \leq |x| + l$, for $|y| \leq l$.

(3)if $x \geq r$, $|y| \leq l$, then $U'(x(t), y(t)) = y(t) - g(x(t)) - f(y(t)) + p(t) \leq 0$ and if $x \leq -r$, $|y| \leq l$, then $U'(x(t), y(t)) = -y(t) + g(x(t)) + f(y(t)) - p(t) \leq 0$.

Therefore, all conditions of Theorem 8.9 in [11] are satisfied. Furthermore, all solutions of (2.1) are positively bounded. It follows from Corollary 15.1 in [11] that Eq.(2.1) has at least one 2π-periodic solution.

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