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Positive solutions of a Schrödinger equation with critical nonlinearity

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Abstract. We study the nonlinear Schrödinger equation

$$-\Delta u + \lambda a(x)u = \mu u + u^{2^*-1},\ u \in \mathbb{R}^N,$$

with critical exponent $2^* = 2N/(N-2)$, $N \geq 4$, where $a \geq 0$ has a potential well. Using variational methods we establish existence and multiplicity of positive solutions which localize near the potential well for μ small and λ large.

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1. Introduction and statement of results

In recent years much attention has been paid to the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + a(x)\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N \tag{S}$$

where $\hbar$ is the Planck constant. When looking for stationary waves of the form $\psi(t,x) = e^{-i\mu(\hbar t)}\varphi(x)$ with $\mu \in \mathbb{R}$, one is lead to considering an elliptic equation in $\mathbb{R}^N$, namely, replacing $\hbar$ by ε one sees that φ must satisfy

$$-\varepsilon^2\Delta\varphi + a(x)\varphi = \varepsilon^2\mu\varphi + |\varphi|^{p-2}\varphi.$$

Setting $u(x) := \varepsilon^{-2/(p-2)}\varphi(x)$ and $\lambda = \varepsilon^{-2}$, this equation is transformed into

$$-\Delta u + \lambda a(x)u = \mu u + |u|^{p-2}u.$$

where $\lambda = \hbar^{-2}$.

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Equations of this type with subcritical nonlinearities (that is, with $p < 2^* = \frac{2N}{N-2}$ for $N \geq 3$) have been investigated extensively, see for example [1], [3], [4], [5], [11], [12], [13], [14], [16], [17], [20] , [21], [23], [26].

Here we investigate the existence and multiplicity of solutions of nonlinear Schrödinger equations with critical nonlinearity. More precisely, we consider the problem

$$\begin{cases} -\Delta u + \lambda a(x) u = \mu u + u^{2^*-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N)\,, \end{cases} \qquad (NS_{\lambda,\mu})$$

where $N \geq 4$, $2^* = \frac{2N}{N-2}$, $\lambda > 0$, $\mu \in \mathbb{R}$ and $a(x)$ satisfies the following assumptions:

$(A1)$ $a \in C(\mathbb{R}^N, \mathbb{R})$, $a \geq 0$, and $\Omega := \text{int } a^{-1}(0)$ is a nonempty bounded set with smooth boundary, and $\overline{\Omega} = a^{-1}(0)$.

$(A2)$ There exists $M_0 > 0$ such that

$$\mathcal{L}\{x \in \mathbb{R}^N : a(x) \leq M_0\} < \infty$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Recently Bartsch and Wang [4] considered the similar problem with subcritical nonlinearity

$$-\Delta u + (\lambda a(x) + 1) u = u^{p-1}, \quad u \in H^1(\mathbb{R}^N),\ N \geq 3,\ 2 < p < 2^*, \qquad (NS_{\lambda,0,p})$$

where $a(x)$ satisfies $(A1)$ and $(A2)$. They showed that, for λ large, this problem has a positive least energy solution, and that there exist $p_0 \in (2, 2^*)$ and a function $\Lambda : (p_0, 2^*) \to \mathbb{R}$ such that it has at least $\text{cat}\,(\Omega)$ positive solutions for any $\lambda \geq \Lambda(p)$, $p \geq p_0$. Here $\text{cat}\,(\Omega)$ stands for the Lusternik-Schnirelmann category of Ω. They also showed that a certain concentration behaviour of the solutions occurs as $\lambda \to \infty$.

A problem arises naturally: Are there similar results for the Schrödinger equation with critical nonlinearity u^{2^*-1}?

The leitmotiv of Bartsch and Wang's approach was that, for large λ, the Dirichlet problem

$$-\Delta u + u = u^{p-1},\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \qquad (D_{0,p})$$

is some kind of limit problem for $(NS_{\lambda,0,p})$. Benci and Cerami had previously shown [6] that problem $(D_{0,p})$ has at least $\text{cat}\,(\Omega)$ solutions if $p < 2^*$ but close enough to 2^*.

There is a great deal of work on elliptic equations with critical nonlinearity on bounded domains, see for example [2], [25], [27] and the references therein. We focus our attention on the the following results for the Dirichlet problem

$$\begin{cases} -\Delta u = \mu u + u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (D_\mu)$$

Brzis and Nirenberg [8] showed there is at least one solution of (D_μ) if $N \geq 4$ and $0 < \mu < \mu_1(\Omega)$, where $\mu_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$. Multiplicity results similar to those of [6] are also known for this problem. It was shown by Rey [24] for $N \geq 5$ and by Lazzo [19] for $N \geq 4$ that there is a $0 < \mu^{\#} < \mu_1(\Omega)$ such that (D_μ) has at least $\operatorname{cat}(\Omega)$ solutions for all $0 < \mu < \mu^{\#}$.

Motivated by these results we will show that, for μ small enough, problem (D_μ) is some kind of limit problem for $(NS_{\lambda,\mu})$ as $\lambda \to \infty$ and use the knowledge about (D_μ) to establish existence and multiplicity of solutions of $(NS_{\lambda,\mu})$. Moreover, as in the subcritical case [4], there is also a concentration behavior of the solutions as $\lambda \to \infty$. Before stating our results we give some definitions.

A solution u_λ of $(NS_{\lambda,\mu})$ is said to be a least energy solution if the energy integral

$$I_{\lambda,\mu}(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2} \left(|\nabla u|^2 + (\lambda a(x) - \mu) u^2 \right) - \frac{1}{2^*} |u|^{2^*} \right) dx$$

achieves its minimum at u_λ over all nontrivial solutions of $(NS_{\lambda,\mu})$.

A sequence of solutions (u_n) of $(NS_{\lambda_n,\mu})$ will be said to concentrate at a solution u of $(D)_\mu$ if a subsequence converges strongly to u in $H^1(\mathbb{R}^N)$ as $\lambda_n \to \infty$.

Let

$$S = \inf_{u \in H^1 \setminus \{0\}} \frac{|\nabla u|_2^2}{|u|_{2^*}^2},$$

be the best Sobolev constant. We shall prove the following results:

Theorem 1. *Assume (A1) and (A2) hold and $N \geq 4$. Then, for every $0 < \mu < \mu_1(\Omega)$, there exists $\lambda(\mu) > 0$ such that $(NS_{\lambda,\mu})$ has a least energy solution u_λ for each $\lambda \geq \lambda(\mu)$.*

Theorem 2. *Assume (A1) and (A2) hold and that $N \geq 4$. Then there exist $0 < \mu^* < \mu_1(\Omega)$ and for each $0 < \mu \leq \mu^*$ two numbers $\Lambda(\mu) > 0$ and $0 < c(\mu) < \frac{1}{N} S^{\frac{N}{2}}$ such that, if $\lambda \geq \Lambda(\mu)$, then $(NS_{\lambda,\mu})$ has at least $\operatorname{cat}(\Omega)$ solutions with energy $I_{\lambda,\mu} \leq c(\mu)$.*

Theorem 3. *Every sequence of solutions (u_n) of $(NS_{\lambda_n,\mu})$ such that $\mu \in (0, \mu_1(\Omega))$, $\lambda_n \to \infty$ and $I_{\lambda_n,\mu}(u_n) \to c < \frac{1}{N} S^{\frac{N}{2}}$ as $n \to \infty$, concentrates at a solution of (D_μ).*

Thus, turning back to the nonlinear Schrödinger equation (S), our ansatz leads to solutions of the form $\psi(t,x) = \hbar^{2/(p-2)} e^{-i\mu\hbar t} u_\hbar(x)$ where $u_\hbar$ concentrates at a solution of the Dirichlet problem (D_μ) on the bottom Ω of the potential well as $\hbar \to 0$. This concentration behaviour is quite different from the one obtained by taking the usual ansatz $\phi(t,x) = e^{-i\nu\hbar^{-1}t}\varphi(x)$, $\varepsilon = \hbar$, and looking at the corresponding singularly perturbed elliptic equation

$$-\varepsilon^2 \Delta u + V(x)\varphi = |\varphi|^{p-2}\,\varphi.$$

It is well known that for $p < 2^*$ positive bound states of this equation concentrate at minima of the potential V and decay exponentially away from such minima, see for example [11], [13], [14], [16], [17], [20], [26]. For $p = 2^*$ solutions with this same kind of behaviour have recently been obtained by Chabrowsky and Yang [10] under special assumptions on the potential. Finally, we would like to mention that, in a recent paper [9], Chabrowski and Szulkin have considered the Schrödinger equation

$$-\Delta u + V(x)u = |u|^{2^*-2} u, \quad u \in H^1(\mathbb{R}^N),$$

with periodic potential V, $N \geq 4$ and critical nonlinearity. They showed the existence of one nontrivial solution provided that 0 lies in a spectral gap of the operator $-\Delta + V$ in $L^2(\mathbb{R}^N)$.

This paper is organized as follows. In section 2 we shall establish some compactness results for the variational problem related to $(NS_{\lambda,\mu})$. Section 3 is devoted to the proofs of Theorems 1 and 3. Theorem 2 will be proved in section 4.

2. Compactness conditions

Throughout this paper we always assume that $(A1)-(A2)$ hold and that $N \geq 4$. We denote by $\mu_1(\Omega)$ the first eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$, and write $|\cdot|_q$ for the L^q-norm for $q \in [1, \infty]$.

Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2 < \infty \right\}$$

be the Hilbert space endowed with the norm

$$\|u\| = \left(\|u\|_{H^1}^2 + \int_{\mathbb{R}^N} a(x)u^2 \right)^{1/2},$$

which is clearly equivalent to each of the norms

$$\|u\|_\lambda = \left(\|u\|_{H^1}^2 + \lambda \int_{\mathbb{R}^N} a(x)u^2 \right)^{1/2}$$

for $\lambda > 0$.

Lemma 4. *Let $\lambda_n \geq 1$ and $u_n \in E$ be such that $\lambda_n \to \infty$ and $\|u_n\|_{\lambda_n}^2 < K$. Then there is a $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^2(\mathbb{R}^N)$.*

Proof. Since $\|u_n\|^2 \leq \|u_n\|_{\lambda_n}^2 < K$ we may assume that $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$. Set $C_m = \{x : |x| \leq m, \ a(x) \geq 1/m\}$, $m \in \mathbb{N}$. Then

$$\int_{C_m} |u_n|^2 \leq m \int_{C_m} a u_n^2 \leq \frac{mK}{\lambda_n} \to 0 \text{ as } n \to \infty$$

for every m. This implies that $u(x) = 0$ for a.e. $x \in \mathbb{R}^N \backslash \Omega$. Hence, since $\partial\Omega$ is smooth, $u \in H_0^1(\Omega)$.

We now show that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Let $F = \{x \in \mathbb{R}^N : a(x) \le M_0\}$ with M_0 as in (A_2), and let $F^c = \mathbb{R}^N \backslash F$. Then

$$\int_{F^c} u_n^2 \le \frac{1}{\lambda_n M_0} \int_{F^c} \lambda_n a u_n^2 \le \frac{K}{\lambda_n M_0} \to 0 \text{ as } n \to \infty..$$

Setting $B_R^c = \mathbb{R}^N \backslash B_R$, where $B_R = \{x \in \mathbb{R}^N : \ |x| \le R\}$, and choosing $r \in (1,\ N/(N-2))$, $r' = r/(r-1)$, we have

$$\int_{B_R^c \cap F} (u_n - u)^2 \le |u_n - u|_{2r}^2 \mathcal{L}(B_R^c \cap F)^{1/r'} \le c\|u_n - u\|^2 \mathcal{L}(B_R^c \cap F)^{1/r'} \to 0$$

as $R \to \infty$, due to $(A2)$. Finally, since $u_n \to u$ in L^2_{loc},

$$\int_{B_R} (u_n - u)^2 \to 0 \text{ as } n \to \infty.$$

□

Let $A_\lambda := -\Delta + \lambda a$ be the selfadjoint operator acting on $L^2(\mathbb{R}^N)$ with form domain E. We denote by $(\cdot, \cdot)$ the L^2-inner product and write

$$(A_\lambda u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda a u v)$$

for $u, v \in E$. Set $a_\lambda := \inf \sigma(A_\lambda)$, the infimum of the spectrum of A_λ. Observe that

$$0 \le a_\lambda = \inf \{(A_\lambda u,\ u) :\ u \in E, |u|_2 = 1\}$$

and that a_λ is nondecreasing in λ.

Lemma 5. *For each $\mu \in (0,\ \mu_1(\Omega))$, there is $\lambda(\mu) > 0$ such that $a_\lambda \ge (\mu + \mu_1(\Omega))/2$ for $\lambda \ge \lambda(\mu)$. Consequently,*

$$\alpha_\mu \|u\|_\lambda^2 \le ((A_\lambda - \mu)u,\ u)$$

for all $u \in E$, $\lambda \ge \lambda(\mu)$, where $\alpha_\mu := (\mu_1(\Omega) + \mu)/(\mu_1(\Omega) + 2 + 3\mu)$.

Proof. Assume, by contradiction, there exists a sequence $\lambda_n \to \infty$ such that $a_{\lambda_n} < (\mu + \mu_1(\Omega))/2$ for all n and $a_{\lambda_n} \to \tau \le (\mu + \mu_1(\Omega))/2$. Let $u_n \in E$ be such that $|u_n|_2 = 1$ and $((A_{\lambda_n} - a_{\lambda_n})u_n, u_n) \to 0$. Then

$$\begin{aligned} \|u_n\|_{\lambda_n}^2 &= \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + (1 + \lambda_n a)|u_n|^2\right) \\ &= ((A_{\lambda_n} - a_{\lambda_n})u_n,\ u_n) + (1 + a_{\lambda_n})|u_n|_2^2 \\ &\le 2(1 + \mu_1(\Omega)) \end{aligned}$$

for all n large. By Lemma 4 there is a $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^2(\mathbb{R}^N)$. Therefore $|u|_2 = 1$ and $\liminf_{n\to\infty} |\nabla u_n|_2^2 \ge |\nabla u|_2^2$.

It follows that

$$\begin{aligned}\int_\Omega (|\nabla u|^2 - \tau u^2) &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - a_{\lambda_n} u_n^2) \\ &\leq \liminf_{n\to\infty} ((A_{\lambda_n} - a_{\lambda_n})u_n,\ u_n) = 0.\end{aligned}$$

and, hence, that

$$\int_\Omega |\nabla u|^2 \leq \tau \leq (\mu + \mu_1(\Omega))/2 < \mu_1(\Omega).$$

This is a contradiction because, by definition, $\mu_1(\Omega) \leq |\nabla u|_2^2$ for all $u \in H_0^1(\Omega)$ with $|u|_2 = 1$. The result follows. □

Consider the functional

$$\begin{aligned}I_{\lambda,\mu}(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda a u^2 - \mu u^2\right) - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} \\ &= \frac{1}{2}((A_\lambda - \mu)u,\ u) - \frac{1}{2^*}|u|_{2^*}^{2^*}.\end{aligned}$$

Then $I_{\lambda,\mu} \in \mathcal{C}^1(E, \mathbb{R})$ and critical points of $I_{\lambda,\mu}$ are solutions of

$$-\Delta u + \lambda a(x) u = \mu u + |u|^{2^*-2} u, \qquad u \in H^1(\mathbb{R}^N).$$

Recall that a sequence $(u_n) \subset E$ is called a (PS) $_c$ sequence (for $I_{\lambda,\mu}$) if $I_{\lambda,\mu}(u_n) \to c$ and $I'_{\lambda,\mu}(u_n) \to 0$ as $n \to \infty$. $I_{\lambda,\mu}$ is said to satisfy the (PS) $_c$ condition if any (PS) $_c$ sequence contains a convergent subsequence.

Lemma 6. *If $\mu \in (0,\ \mu_1(\Omega))$ and $\lambda \geq \lambda(\mu)$, then every (PS)$_c$ sequence (u_n) for $I_{\lambda,\mu}$ is bounded in E, and satisfies*

$$\lim_{n\to\infty} ((A_\lambda - \mu)u_n,\ u_n) = \lim_{n\to\infty} |u_n|_{2^*}^{2^*} = Nc. \tag{2.1}$$

Proof. By definition,

$$I_{\lambda,\mu}(u_n) - \frac{1}{2^*} I'_{\lambda,\mu}(u_n)u_n = \frac{1}{N}((A_\lambda - \mu)u_n,\ u_n) \tag{2.2}$$

and

$$I_{\lambda,\mu}(u_n) - \frac{1}{2} I'_{\lambda,\mu}(u_n)u_n = \frac{1}{N}|u_n|_{2^*}^{2^*}. \tag{2.3}$$

Equation (2.2) and Lemma 5 imply that (u_n) is bounded in E, and taking limits in Equations (2.2) and (2.3) gives (2.1). □

Let

$$S = \inf_{u\in H^1\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_{2^*}^2},$$

be the best Sobolev constant. In the following, enlarging $\lambda(\mu)$ if necessary, we assume $\lambda(\mu) \geq \mu/M_0$, thus

$$\lambda M_0 - \mu \geq 0 \quad \text{for all } \lambda \geq \lambda(\mu). \tag{2.4}$$

Proposition 7. *If* $\mu \in (0,\ \mu_1(\Omega))$ *and* $\lambda \geq \lambda(\mu)$ *then* $I_{\lambda,\mu}$ *satisfies the* $(PS)_c$ *condition for all* $c < \frac{1}{N} S^{\frac{N}{2}}$ *, that is, any sequence* $(u_n) \subset E$ *with* $I_{\lambda,\mu}(u_n) \to c < \frac{1}{N} S^{\frac{N}{2}}$ *and* $I'_{\lambda,\mu}(u_n) \to 0$ *contains a convergent subsequence.*

Proof. By Lemma 6 (u_n) is bounded in E and we may assume without loss of generality that $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ in L^2_{loc} and $u_n(x) \to u(x)$ a.e. in $x \in \mathbb{R}^N$. A standard argument shows that u is a weak solution of

$$-\Delta u + \lambda a(x) u = \mu u + |u|^{2^*-2} u\,.$$

Let $w_n = u_n - u$. By the Brzis-Lieb lemma [7], [27],

$$|u_n|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + |w_n|_{2^*}^{2^*} + o(1) \tag{2.5}$$

and, since $I'_{\lambda,\mu}(u_n)u_n \to 0$, it is easy to check that

$$((A_\lambda - \mu)w_n,\ w_n) - |w_n|_{2^*}^{2^*} \to 0\,. \tag{2.6}$$

It follows from Equations (2.1), (2.5) and (2.6) that

$$((A_\lambda - \mu)w_n,\ w_n) \to b \quad \text{and} \quad |w_n|_{2^*}^{2^*} \to b \leq Nc < S^{\frac{N}{2}}\,.$$

As in the proof of Lemma 4 one shows that

$$\int_F w_n^2 \to 0 \quad \text{as } n \to \infty$$

where $F = \{x \in \mathbb{R}^N :\ a(x) \leq M_0\}$. Let $F^c = \mathbb{R}^N \backslash F$. Then, using Equation (2.4),

$$\begin{aligned} S|w_n|_{2^*}^2 &\leq |\nabla w_n|_2^2 \\ &\leq |\nabla w_n|_2^2 + \int_{F^c} (\lambda a - \mu) w_n^2 \\ &\leq ((A_\lambda - \mu)w_n,\ w_n) + \mu \int_F w_n^2 \\ &= ((A_\lambda - \mu)w_n,\ w_n) + o(1)\,. \end{aligned}$$

Passing to the limit yields $Sb^{2/2^*} \leq b$. Since $b < S^{\frac{N}{2}}$ it follows that $b = 0$. Hence $w_n \to 0$ in E. □

3. Proof of Theorems 1 and 3

The critical points of $I_{\lambda,\mu}$ lie on the Nehari manifold

$$\begin{aligned} \mathcal{M}_{\lambda,\mu} &= \{u \in E\backslash\{0\} :\ I'_{\lambda,\mu}(u)u = 0\} \\ &= \{u \in E\backslash\{0\} :\ ((A_\lambda - \mu)u, u) = |u|_{2^*}^{2^*}\}. \end{aligned}$$

$\mathcal{M}_{\lambda,\mu}$ is radially dipheomorphic to $\mathcal{V} = \{v \in E : |v|_{2^*} = 1\}$; the diffeomorphism is given by

$$\mathcal{V} \to \mathcal{M}_{\lambda,\mu}, \qquad v \mapsto ((A_\lambda - \mu)v, v)^{\frac{N-2}{4}} v.$$

For $u \in \mathcal{M}_{\lambda,\mu}$, the functional $I_{\lambda,\mu}$ is just

$$I_{\lambda,\mu}(u) = \frac{1}{N}((A_\lambda - \mu)u, u).$$

Hence,

$$c_{\lambda,\mu} := \inf_{u \in \mathcal{M}_{\lambda,\mu}} I_{\lambda,\mu}(u) = \frac{1}{N} \inf_{v \in \mathcal{V}} ((A_\lambda - \mu)v, v)^{\frac{N}{2}}.$$

Following Benci and Cerami [6] one can easily show that

Proposition 8. *If $u \in \mathcal{M}_{\lambda,\mu}$ is a critical point of $I_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u) < 2c_{\lambda,\mu}$ then u does not change sign. Hence, $|u|$ is a solution of $(NS_{\lambda,\mu})$.*

Proof. Since u is a critical point of $I_{\lambda,\mu}$, $((A_\lambda - \mu)u, w) = \int |u|^{2^*-2} uw$ for every $w \in E$. In particular for $w = u^\pm$ where $u^\pm = \pm \max\{\pm u, 0\}$. So, if both u^+ and u^- are nonzero, then $u^\pm \in \mathcal{M}_{\lambda,\mu}$ and $I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u^+) + I_{\lambda,\mu}(u^-) \geq 2c_{\lambda,\mu}$. This is a contradiction. $\square$

Similarly, for every domain $\mathcal{D} \subset \mathbb{R}^N$, we consider the functional

$$I_{\mu,\mathcal{D}}(u) = \frac{1}{2}\int_{\mathcal{D}} \left(|\nabla u|^2 - \mu u^2\right) - \frac{1}{2^*}\int_{\mathcal{D}} |u|^{2^*} = \frac{1}{2}((A_0 - \mu)u, u) - \frac{1}{2^*}|u|_{2^*}^{2^*}$$

on $H_0^1(\mathcal{D})$, associated to problem (D_μ). Its Nehari manifold

$$\mathcal{M}_{\mu,\mathcal{D}} = \{u \in H_0^1(\mathcal{D}) \backslash \{0\} : ((A_0 - \mu)u, u) = |u|_{2^*}^{2^*}\}$$

is radially diffeomorphic to $\mathcal{V}_{\mathcal{D}} = \left\{v \in H_0^1(\mathcal{D}) : |v|_{2^*} = 1\right\}$. Set

$$c(\mu, \mathcal{D}) := \inf_{u \in \mathcal{M}_{\mu,\mathcal{D}}} I_{\mu,\mathcal{D}}(u) = \frac{1}{N} \inf_{v \in \mathcal{V}_{\mathcal{D}}} ((A_0 - \mu)v, v)^{\frac{N}{2}}.$$

Lemma 9. *If $\mu \in (0,\ \mu_1(\Omega))$ and $\lambda \geq \lambda(\mu)$ then*

$$\frac{1}{N}(\alpha_\mu S)^{\frac{N}{2}} \leq c_{\lambda,\mu} < c(\mu, \Omega) < \frac{1}{N} S^{\frac{N}{2}}.$$

Proof. By Lemma 5, $\alpha_\mu \|v\|_{H^1}^2 \leq \alpha_\mu \|v\|_\lambda^2 \leq ((A_\lambda - \mu)v, v)$. Taking infima over $v \in \mathcal{V}$ gives the first inequality. Since $\mathcal{V}_\Omega \subset \mathcal{V}$ and $(A_\lambda v, v) = (A_0 v, v)$ for $v \in \mathcal{V}_\Omega$, it follows that $c_{\lambda,\mu} \leq c(\mu, \Omega)$. Now, Brzis and Nirenberg showed [8] that, for $\mu \in (0,\ \mu_1(\Omega))$, $c(\mu, \Omega) < \frac{1}{N} S^{\frac{N}{2}}$ and $c(\mu, \Omega)$ is achieved at some $\widetilde{u} > 0$. Therefore $c_{\lambda,\mu} < c(\mu, \Omega)$, because otherwise $c_{\lambda,\mu}$ would be also achieved at $\widetilde{u}$ which vanishes outside Ω, contradicting the maximum principle. $\square$

We are now ready to prove Theorems 1 and 3.

Proof of Theorem 1. Let (u_n^λ) be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathcal{M}_{\lambda,\mu}$. By Ekeland's variational principle [15], [27], we may assume that it is a PS sequence. It follows from Proposition 7 and Lemma 9 that a subsequence converges to a least energy solution u_λ of $(NS_{\lambda,\mu})$. □

Proof of Theorem 3. Let (u_n) be a sequence of solutions of $(NS_{\lambda_n,\mu})$ such that $\mu \in (0,\ \mu_1(\Omega))$, $\lambda_n \to \infty$ and $NI_{\lambda_n,\mu}(u_n) = ((A_{\lambda_n} - \mu)u_n, u_n) \to Nc < S^{\frac{N}{2}}$. Then Lemmas 4 and 5 imply that there is a $u \in H_0^1(\Omega)$ such that a subsequence $u_n \rightharpoonup u$ weakly in E and $u_n \to u$ in $L^2(\mathbb{R}^N)$. Since u_n is a solution of $(NS_{\lambda_n,\mu})$,

$$\int_{\mathbb{R}^N} \nabla u_n \nabla v + \lambda_n a u_n v - \mu u_n v = \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v \quad \text{for all } v \in E.$$

If $v \in H_0^1(\Omega)$ then $\int \lambda_n a u_n v = 0$ for all n, so letting $n \to \infty$ we obtain

$$\int_{\mathbb{R}^N} \nabla u \nabla v - \mu u v = \int_{\mathbb{R}^N} |u|^{2^*-2} uv \quad \text{for all } v \in H_0^1(\Omega),$$

that is, u is a solution of (D_μ). Let $w_n = u_n - u$. Since $a(x) = 0$ for $x \in \Omega$, it is easy to see that

$$((A_{\lambda_n} - \mu)u_n, u_n) = ((A_0 - \mu)u, u) + ((A_{\lambda_n} - \mu)w_n, w_n) + o(1).$$

By the Brzis-Lieb lemma [7]

$$|u_n|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + |w_n|_{2^*}^{2^*} + o(1)$$

so, since $u_n \in \mathcal{M}_{\lambda_n,\mu}$ and $u \in \mathcal{M}_{\mu,\Omega}$,

$$((A_{\lambda_n} - \mu)w_n, w_n) - |w_n|_{2^*}^{2^*} = o(1)$$

We claim that $|w_n|_{2^*} \to 0$. Assume by contradiction that $|w_n|_{2^*}^{2^*} \to b > 0$. Then, since

$$\begin{aligned} S|w_n|_{2^*}^2 &\leq |\nabla w_n|_2^2 \\ &\leq ((A_{\lambda_n} - \mu)w_n, w_n) + o(1) \\ &= |w_n|_{2^*}^{2^*} + o(1), \end{aligned}$$

it follows that

$$S \leq |w_n|_{2^*}^{2^*-2} + o(1) \leq |u_n|_{2^*}^{2^*-2} + o(1)$$

and, therefore, that

$$S^{\frac{N}{2}} \leq \lim_{n\to\infty} |u_n|_{2^*}^{2^*} = c < S^{\frac{N}{2}}.$$

This is a contradiction. Consequently, $|w_n|_{2^*} \to 0$ and $((A_{\lambda_n} - \mu)w_n, w_n) \to 0$. Hence,

$$((A_0 - \mu)u, u) = \lim_{n\to\infty} ((A_{\lambda_n} - \mu)u_n, u_n) \tag{3.1}$$

Since $u_n = w_n$ in $\mathbb{R}^N \backslash \Omega$ and $a = 0$ in Ω,

$$\int a u_n^2 \leq \int \lambda_n a u_n^2 = \int \lambda_n a w_n^2 \leq ((A_{\lambda_n} - \mu) w_n, w_n) + o(1).$$

Therefore, $\int a u_n^2 \to 0$ and Equation (3.1) implies that $u_n \to u$ in E. □

As a consequence of Theorems 1 and 3 we have

Corollary 10. *For each* $\mu \in (0,\ \mu_1(\Omega))$, $\lim_{\lambda\to\infty} c_{\lambda,\mu} = c(\mu,\Omega)$.

Proof. By Lemma 9, $c_{\lambda,\mu} \to c \leq c(\mu,\Omega) < \frac{1}{N} S^{\frac{N}{2}}$ and, by Theorem 1, $c_{\lambda,\mu}$ is achieved for $\lambda \geq \lambda(\mu)$. So Theorem 3 implies that c is achieved by $I_{\mu,\Omega}$ on $\mathcal{M}_{\mu,\Omega}$. Hence, $c \geq c(\mu,\Omega)$. □

4. Proof of Theorem 2

To prove Theorem 2 we follow the method introduced by Benci and Cerami in [6].

Since Ω is a bounded smooth domain of $\mathbb{R}^N$, we may fix $r > 0$ small enough such that

$$\Omega_{2r}^{+} = \{x \in \mathbb{R}^N : \ \text{dist}\ (x,\Omega) < 2r\}$$

and

$$\Omega_r^{-} = \{x \in \Omega : \ \text{dist}\ (x,\partial\Omega) > r\}$$

are homotopically equivalent to Ω. Moreover, we may assume that $B_r = \{x \in \mathbb{R}^N : |x| < r\} \subset \Omega$. We define $c(\mu,r) := c(\mu,B_r)$. Then, arguing as in the proof of Lemma 9, we have that

$$c(\mu,\Omega) < c(\mu,r) < \frac{1}{N} S^{\frac{N}{2}}$$

for $0 < \mu < \mu_1(\Omega)$.

For $0 \neq u \in L^{2^*}(\Omega)$ we consider its center of mass

$$\beta(u) := \frac{\int_\Omega |u|^{2^*} x\, dx}{\int_\Omega |u|^{2^*} dx}.$$

Rephrasing Lazzo's results in [19] one has the following lemma.

Lemma 11. *There is a* $\mu^{\#} = \mu^{\#}(r) \in (0,\ \mu_1(\Omega))$ *such that, for* $0 < \mu \leq \mu^{\#}$,

i) $c(\mu,r) < 2c(\mu,\Omega)$, *and*

ii) $\beta(u) \in \Omega_r^{+}$ *for every* $u \in \mathcal{M}_{\mu,\Omega}$ *with* $I_{\mu,\Omega}(u) \leq c(\mu,r)$.

As in [4], we choose $R > 0$ with $\overline{\Omega} \subset B_R$ and set

$$\xi(t) = \begin{cases} 1 & 0 \leq t \leq R, \\ R/t & R \leq t\,. \end{cases}$$

Define

$$\beta_0(u) = \frac{\int_{\mathbb{R}^N} |u|^{2^*} \xi(|x|)x\,dx}{\int_{\mathbb{R}^N} |u|^{2^*}\,dx} \quad \text{for} \quad u \in L^{2^*}(\mathbb{R}^N)\backslash\{0\}.$$

Lemma 12. *There exist* $\mu^* = \mu^*(r) \in (0,\ \mu_1(\Omega))$ *and for each* $0 < \mu \le \mu^*$ *a number* $\Lambda(\mu) \ge \lambda(\mu)$ *with the following properties:*

i) $c(\mu, r) < 2c_{\lambda,\mu}$ *for all* $\lambda \ge \Lambda(\mu)$, *and*

ii) $\beta_0(u) \in \Omega_{2r}^+$ *for all* $\lambda \ge \Lambda(\mu)$ *and all* $u \in \mathcal{M}_{\lambda,\mu}$ *with* $I_{\lambda,\mu}(u) \le c(\mu, r)$.

Proof. Assertion *i)* follows immediately from Lemma 11 and Corollary 10. We now prove *ii)*. Assume, by contradiction, that for μ arbitrarily small there is a sequence (u_n) such that $u_n \in \mathcal{M}_{\lambda_n,\mu}$, $\lambda_n \to \infty$, $I_{\lambda_n,\mu}(u_n) \to c \le c(\mu, r)$ and $\beta_0(u_n) \notin \Omega_{2r}^+$. Then, by Lemma 4, there is $u_\mu \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_\mu$ weakly in E and $u_n \to u_\mu$ in $L^2(\mathbb{R}^N)$. We distinguish two cases:

Case 1: $|u_\mu|_{2^*}^{2^*} \le ((A_0 - \mu)u_\mu, u_\mu)$.

Let $w_n = u_n - u_\mu$. Since $a(x) = 0$ for $x \in \Omega$,

$$((A_{\lambda_n} - \mu)u_n, u_n) = ((A_0 - \mu)u_\mu, u_\mu) + ((A_{\lambda_n} - \mu)w_n, w_n) + o(1).$$

By the Brzis-Lieb lemma [7]

$$|u_n|_{2^*}^{2^*} = |u_\mu|_{2^*}^{2^*} + |w_n|_{2^*}^{2^*} + o(1)$$

so, since $u_n \in \mathcal{M}_{\lambda_n,\mu}$,

$$((A_{\lambda_n} - \mu)w_n, w_n) \le |w_n|_{2^*}^{2^*} + o(1)$$

We claim that $|w_n|_{2^*} \to 0$. Assume by contradiction that $|w_n|_{2^*}^{2^*} \to b > 0$. Then, since

$$\begin{aligned} S|w_n|_{2^*}^2 &\le |\nabla w_n|_2^2 \\ &\le ((A_{\lambda_n} - \mu)w_n, w_n) + o(1) \\ &\le |w_n|_{2^*}^{2^*} + o(1) \end{aligned}$$

we have that

$$S^{\frac{N}{2}} \le \lim_{n\to\infty} |u_n|_{2^*}^{2^*} = Nc < S^{\frac{N}{2}},$$

a contradiction. Consequently, $u_n \to u_\mu$ in $L^{2^*}(\mathbb{R}^N)$ and, therefore, $\beta_0(u_n) \to \beta(u_\mu)$. But, since $I_{\mu,\Omega}(u_\mu) \le \lim_{n\to\infty} I_{\lambda_n,\mu}(u_n) \le c(\mu, r)$, it follows from Lemma 11 that $\beta(u_\mu) \in \Omega_r^+$. This contradicts our assumption that $\beta_0(u_n) \notin \Omega_{2r}^+$.

Case 2: $|u_\mu|_{2^*}^{2^*} > ((A_0 - \mu)u_\mu, u_\mu)$.

In this case $tu_\mu \in \mathcal{M}_{\mu,\Omega}$ for some $t \in (0, 1)$ and, therefore,

$$c(\mu, \Omega) \le I_{\mu,\Omega}(tu_\mu) \le \frac{t^2}{N}((A_0 - \mu)u, u) < \lim_{n\to\infty} I_{\lambda_n,\mu}(u_n) \le c(\mu, r).$$

It follows that, for $n(\mu)$ large enough,

$$\left| |u_{n(\mu)}|_{2^*}^{2^*} - |tu_\mu|_{2^*}^{2^*} \right| \le N(c(\mu, r) - c(\mu, \Omega)).$$

Since $|c(\mu,r) - c(\mu,\Omega)| \to 0$ as $\mu \to 0$, this implies that $\left|\beta_0(u_{n(\mu)}) - \beta(tu_\mu)\right| < r$ for all μ sufficiently small. But, by Lemma 11, $\beta(tu_\mu) \in \Omega_r^+$, whereas $\beta_0(u_{n(\mu)}) \notin \Omega_{2r}^+$, a contradiction. □

As usual, for a given function $I : M \to \mathbb{R}$ we set $I^{\leq b} = \{z \in M : I(z) \leq b\}$. We shall need the following easy consequence of standard Lusternik-Schnirelmann theory.

Proposition 13. *Let $I : M \to \mathbb{R}$ be an even C^1-functional on a complete symmetric $C^{1,1}$-submanifold $M \subset V\backslash\{0\}$ of some Banach space V. Assume that I is bounded below and satisfies the Palais-Smale condition (PS)$_c$ for all $c \leq b$. Further, assume that there are maps*

$$X \xrightarrow{\iota} I^{\leq b} \xrightarrow{\beta} Y$$

whose composition $\beta \circ \iota$ is a homotopy equivalence, and that $\beta(z) = \beta(-z)$ for all $z \in M \cap I^{\leq b}$. Then I has at least cat(X) pairs $\{z, -z\}$ of critical points with $I(z) = I(-z) \leq b$.

Proof. Let Q be the quotient space of $I^{\leq b}$ obtained by identifying z with $-z$ and let $q : I^{\leq b} \to Q$ be the quotient map. Since I and β are even, they induce a functional $\widetilde{I} : Q \to \mathbb{R}$ and a map $\widetilde{\beta} : Q \to Y$ such that $\widetilde{\beta} \circ q \circ \iota : X \simeq Y$ is a homotopy equivalence. It follows easily that cat$(X) \leq$ cat(Q) [18]. Standard Lusternik-Schnirelmann theory [22], [25] now yields at least cat(X) critical points of the induced functional $\widetilde{I} : Q \to \mathbb{R}$, that is, at least cat$(X)$ pairs $\{z, -z\}$ of critical points of I with $I(z) = I(-z) \leq b$. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. For $0 < \mu \leq \mu^*$ and $\lambda \geq \Lambda(\mu)$, we define two maps

$$\Omega_r^- \xrightarrow{\iota} \mathcal{M}_{\lambda,\mu} \cap I_{\lambda,\mu}^{\leq c(\mu,r)} \xrightarrow{\beta_0} \Omega_{2r}^+$$

as follows: The map β_0 is the one defined above. Lemma 12 shows that it is well defined. Let $u_r \in H_0^1(B_r) \subset E$ be a minimizer of I_{μ,B_r} on $\mathcal{M}_{\mu,B_r}$ with $u_r > 0$ and set $\iota(x) = u_r(\cdot - x)$. Since $\iota(x) \equiv 0$ in $\mathbb{R}^N\backslash\Omega$ for every $x \in \Omega_r^-$, it follows that $\iota(x) \in \mathcal{M}_{\lambda,\mu}$ and that $I_{\lambda,\mu}(\iota(x)) = I_{\mu,B_r}(\iota(x)) = c(\mu,r)$. Since u_r is radially symmetric, $\beta_0(\iota(x)) = x$ for every $x \in \Omega_r^-$. Clearly, $I_{\lambda,\mu}(u) = I_{\lambda,\mu}(-u)$ and $\beta_0(u) = \beta_0(-u)$ for every $u \in E\backslash\{0\}$. On the other hand, $c(\mu,r) < \frac{1}{N}S^{\frac{N}{2}}$ [8] so, by Proposition 7, $I_{\lambda,\mu}$ satisfies (PS)$_c$ for all $c \leq c(\mu,r)$. It follows from Propositions 13 and 8 and Lemma 12 that $(NS_{\lambda,\mu})$ has at least cat(Ω) positive solutions. □

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