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Exact solution of the Navier-Stokes equations for the oscillating flow in a duct of a cross-section of right-angled isosceles triangle

S. Tsangaris and N. W. Vlachakis

In memory of Klaus Oswatitsch as a teacher and scientist

Abstract. The Navier-Stokes equations have been solved in order to obtain an analytical solution of the fully developed laminar flow in a duct having a cross section of a right-angled, isosceles triangle. We obtained a solution for the case of oscillating pressure gradient flow. The pulsating flow is obtained by the superposition of the steady and oscillating pressure gradient solutions.

Keywords. Navier-Stokes, oscillatory flow.

Introduction

The analytical expressions for the developed, laminar, steady flow velocity distribution in straight channels (parallel walls), pipe of circular cross section (Hagen-Poisseuille law) and annular cross section refer back to G.G. Stokes (1851 and 1898), E. Hagenbach (1860) and H. Lamb (1879) respectively, [1, 2]. The velocity distribution for the laminar flow in a duct with rectangular cross section has been given by J. V. Boussinesq (1914), as referred to in Part II (Chapter II) in the excellent reviews by H.L. Dryden et al [3] and R. Berker [4]. Laminar steady flow for triangular cross section exists in the literature for the cases of the equilateral triangle and for the case of a right angled isosceles triangle B.G. Galerkin (1919), C.Kolossoff (1924), M. Paschoud (1924), [3, 5].

Laminar, developed, oscillating flow in straight ducts with a constant cross section and impermeable wall exists in the literature between parallel plates [2], for a circular [6-9], for an annular [10] and for a rectangular cross section [11-13]. For a triangular cross section an analytical solution for oscillating flow does not exist.

Reviews for the exact solutions of the Navier-Stokes equations can be found in the review articles of C.Y. Wang [14] and [15].

An analytical solution of the steady and oscillating flow in the straight duct with triangular cross is sought in the present paper and is successfully given in a series form for the case of isosceles, right angled triangle.

Methodology and solution

As governing equations, the Navier-Stokes equations for unsteady flow, of an incompressible fluid of constant viscosity, are used as a system with the continuity equation, both written in cartesian coordinates. We seek the analytical solution for both steady and oscillating flow conditions for the case of a straight duct with triangular cross section, confined by $x = 0$, $y = 0$ and $x + y = a$. By assuming fully developed flow ($u = 0$, $v = 0$), the continuity equation is satisfied when developed flow conditions across the z-axis of the straight duct are valid ($w = w(x, y, t)$). For the w component of the velocity, using the corresponding Navier-Stokes equation, the following partial differential equation should be satisfied, because we are interested here in the cases of the flow due to an oscillating pressure gradient:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + v\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \qquad -\frac{\partial p}{\partial z} = \frac{P}{\ell}\cos(\omega t). \tag{1}$$

P/ℓ is the amplitude of the imposed pressure gradient, while ω is the cyclic frequency of the oscillating pressure gradient. Taking the value $\omega = 0$ leads to the case of steady flow solution.

The boundary conditions that have to be fulfilled are the non-slip condition at the walls $x = 0$, $y = 0$, $x + y = a$ of the triangular cross section for the w component of the velocity ($w = 0$).

In order to define periodic and steady solutions we assume that w is periodic so that:

$$w = w_s \sin(\omega t) + w_c \cos(\omega t) \tag{2}$$

By introducing non-dimensional variables:

$$\tilde{x} = \frac{x}{a},\ \tilde{y} = \frac{y}{a},\ \tilde{w} = \frac{w}{Pa^2}\mu\ell,\ \alpha = a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \tag{3}$$

the equation (1) together with equation (2) are reduced to a system of non-homogeneous Helmholtz equations:

$$\begin{aligned} -\alpha^2\tilde{w}_c &= \frac{\partial\tilde{w}_s}{\partial\tilde{x}^2} + \frac{\partial\tilde{w}_s}{\partial\tilde{y}^2} \\ \alpha^2\tilde{w}_s &= 1 + \frac{\partial\tilde{w}_c}{\partial\tilde{x}^2} + \frac{\partial\tilde{w}_c}{\partial\tilde{y}^2} \end{aligned} \tag{4}$$

where α is the reduced frequency. The boundary conditions for $\tilde{w}_s$, $\tilde{w}_c$ result from using the boundary conditions for the velocity and the equation for oscillating flow assumption (2):

$$\begin{array}{ll}\tilde{w}_s(0,\tilde{y})=0 & \tilde{w}_s(\tilde{x},0)=0\ \tilde{w}_s(\tilde{x},1-\tilde{x})=0 \\ \tilde{w}_c(0,\tilde{y})=0, & \tilde{w}_c(\tilde{x},0)=0,\ \tilde{w}_c(\tilde{x},1-\tilde{x})=0\end{array}, \tag{5}$$

For the equations (4) the analytical solution, which satisfies the boundary conditions (5), can be determined by using a Fourier series analysis of $\tilde{w}_s$, $\tilde{w}_c$ for $\tilde{x},\tilde{y}$. The eigenfunction which satisfies the homogeneous Helmholtz equation as well as the boundary conditions (5) is given by P.M.Riz [16] and can be expressed as:

$$\tilde{w}_{mn}=\sin[\pi(m+n)\tilde{x}]\sin(\pi n\tilde{y})-(-1)^m\sin[\pi(m+n)\tilde{y}]\sin(\pi n\tilde{x}). \tag{6}$$

$\tilde{w}_s,\tilde{w}_c$ and 1 are expressed as Fourier expansions:

$$\tilde{w}_c=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left[[\sin[\pi(m+n)\tilde{x}]\sin(\pi n\tilde{y})-(-1)^m\sin(\pi n\tilde{x})\sin(\pi(m+n)\tilde{y})\right] \tag{7}$$

$$\tilde{w}_s=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\left[[\sin[\pi(m+n)\tilde{x}]\sin(\pi n\tilde{y})-(-1)^m\sin(\pi n\tilde{x})\sin(\pi(m+n)\tilde{y})\right] \tag{8}$$

$$1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\left[[\sin[\pi(m+n)\tilde{x}]\sin(\pi n\tilde{y})-(-1)^m\sin(\pi n\tilde{x})\sin(\pi(m+n)\tilde{y})\right] \tag{9}$$

where C_{mn} can be calculated by the relation:

$$C_{mn}=\frac{\int_D \tilde{w}_{mn}d\tilde{x}d\tilde{y}}{\int_D \tilde{w}_{mn}^2 d\tilde{x}d\tilde{y}} \tag{10}$$

D is the triangular domain. The calculation gives for C_{mn} the following relation:

$$C_{mn}=\frac{4}{\pi^2}[1-(-1)^m]\frac{1}{n(m+n)}\left[1-(-1)^m\frac{(m+n)^2+n^2}{(m+n)^2-n^2}\right] \tag{11}$$

Substituting the above double series expansions (7)-(9) in the system of differential equations (4) we get a system of algebraic equations, which have the following solution for the unknown coefficients A_{mn},B_{mn} :

$$\begin{aligned}A_{mn}&=C_{mn}\frac{\pi^2\left[(m+n)^2+n^2\right]}{\alpha^4+\pi^4\left[(m+n)^2+n^2\right]^2}\\ B_{mn}&=C_{mn}\frac{\alpha^4}{\alpha^2+\pi^4\left[(m+n)^2+n^2\right]^2}\end{aligned} \tag{12}$$

For the imposed oscillating pressure gradient the resulting periodic velocity can be written as:

$$\tilde{w} = \tilde{w}_a \cos(\omega t + \varphi) \tag{13}$$

where $\tilde{w}_a(\tilde{x}, \tilde{y})$ is the amplitude and $\varphi(\tilde{x}, \tilde{y})$ the phase angle resulting from the expression of $\tilde{w}(\tilde{x}, \tilde{y})$ as follows:

$$\tilde{w}_a = (\tilde{w}_s^2 + \tilde{w}_c^2)^{\frac{1}{2}}, \quad \varphi = \arctan\left(-\frac{\tilde{w}_s}{\tilde{w}_c}\right) \tag{14}$$

The defined analytic solution of the problem contains both the oscillatory as well as the steady solution of the problem. The steady solution can be obtained by substituting $\alpha = 0$ in the expressions (12).

By integrating the expression of the velocity over the area of the triangle, the mean over the cross section velocity $\tilde{\tilde{w}}$ can be calculated:

$$\tilde{\tilde{w}} = \tilde{\tilde{w}}_a \cos(\omega t + \theta). \tag{15}$$

$\tilde{\tilde{w}}_a$ is the amplitude $\tilde{\tilde{w}}_a$ and θ is the phase angle, which are given by the following relations:

$$\tilde{\tilde{w}}_a = (\tilde{\tilde{w}}_s^2 + \tilde{\tilde{w}}_c^2)^{\frac{1}{2}} \qquad \theta = \arctan\left(-\frac{\tilde{\tilde{w}}_s}{\tilde{\tilde{w}}_c}\right) \tag{16}$$

$$\tilde{\tilde{w}}_s = \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^2}{\alpha^4 + \pi^4\left[(m+n)^2 + n^2\right]^2} C_{mn}^2 \tag{17}$$

$$\tilde{\tilde{w}}_c = \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\pi^2\left[(m+n)^2 + n^2\right]}{\alpha^4 + \pi^4\left[(m+n)^2 + n^2\right]^2} C_{mn}^2 \tag{18}$$

The double Fourier series converge, because the coefficients approach zero as m and n approach infinity. The truncation error for the Fourier series expression of the amplitude is of the order of 10^{-6} for $m = n = 11$ and of the order of 10^{-10} for $m = n = 71$ and is independent from the value of reduced frequency. For our calculations we used the values $m = n = 71$.

Results and discussion

For $\alpha = 0$, which is the case of steady flow, the amplitude shows a maximum maximorum value $\tilde{w}_{\max} = 0.0295$. This value is identical to the maximum velocity value, given in [3], while the phase angle is identical to zero.

The influence on flow of the oscillating pressure gradient is shown for two values of the characteristic reduced frequency of the flow field ($\alpha = 5$, $\alpha = 20$) in figures 1 and 2. Increasing the values of the reduced frequency causes a reduction of the velocity amplitude. For high values of the reduced frequency ($\alpha = 20$, figure 2) both the amplitude and phase angle ($\approx -\pi/2$) are nearly constant in

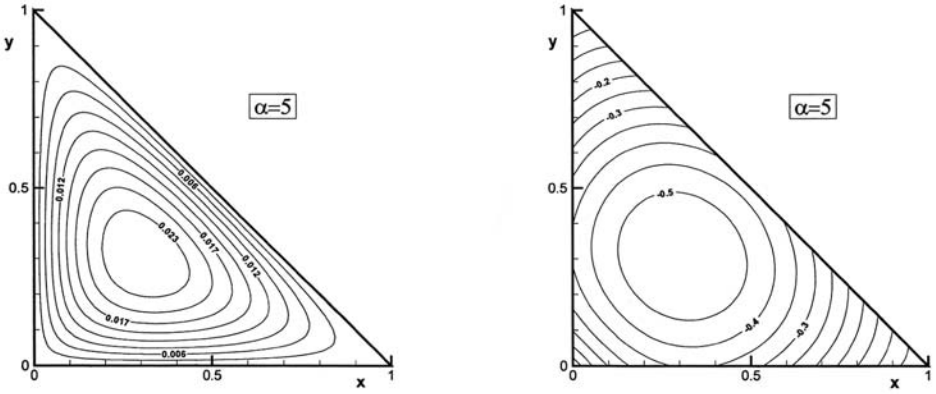

Figure 1. Iso-amplitude $\tilde{w}_a$ (left) and iso-phase angle φ (right) plot of the velocity in the cross-section of the right-angled isosceles duct, for oscillating flow with a reduced frequency $\alpha = 5$.

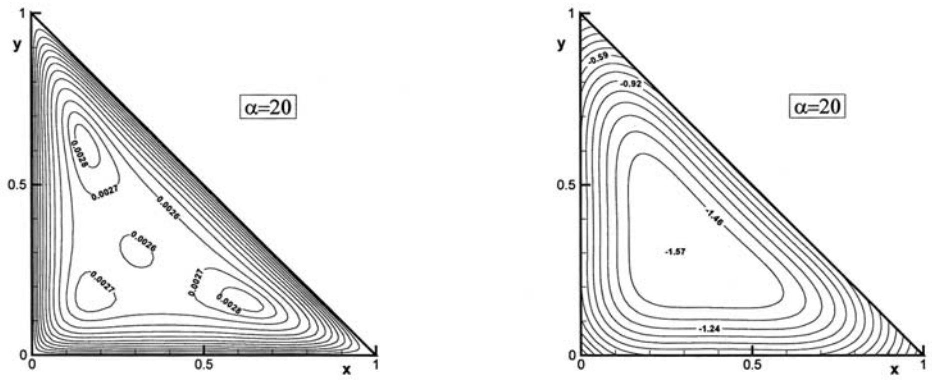

Figure 2. Iso-amplitude $\tilde{w}_a$ (left) and iso-phase angle φ (right) plot of the velocity in the cross-section of the right-angled isosceles duct, for oscillating flow with a reduced frequency $\alpha = 20$.

each cross section so that the velocity profiles become flat in the central region of the duct cross-section except of a region close to the walls of the duct, which shows a boundary layer behavior with a high velocity gradient close to the solid boundaries. In the region of the constant velocity the flow behaves inviscid showing amplitude $\tilde{w}_a = 1/\alpha^2$ and phase angle $\varphi = -\pi/2$ and this behavior is typical for the oscillating flows of high frequencies. The phase angle at the walls far from the apexes and for high values of α takes a value near to $-\pi/4$, which is also typical for such kind of flows (figure 2-right).

The velocity shows maximum values close to the solid walls, while in the centerline of the square duct the velocity has a local minimum value. There are three maximum maximorum values close to the three apexes of the triangular cross section, clearly shown in figure 2-right. Related flow phenomena are expected, and are discovered by E.G. Richardson and E. Tyler (1929) [7] and are known as "annular effect" [9].

Using the relations (15-18) the mean over the triangular cross section velocity amplitude $\tilde{\overline{w}}_a$ and the phase angle symbolized as θ can be calculated. In figure 3(left) the amplitude and in figure 3(right) the phase angles are plotted respectively as functions of the reduced frequency α.

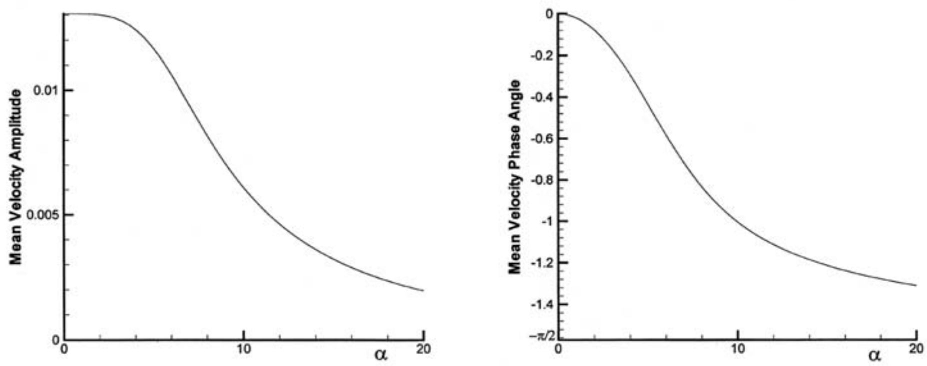

Figure 3. Mean over the right-angled isosceles cross section velocity amplitude $\tilde{\overline{w}}_a$ (left) and phase angle θ (right) as function of the reduced frequency α.

By increasing the reduced frequency α the amplitude of the mean velocity is reduced while the maximum of the mean velocity amplitude appears for the quasi steady flow ($\alpha = 0$), having the value $\tilde{\overline{w}}_a = 0.013$, which is identical to the value for steady flow given in literature [3]. The phase angle θ (fig. 3(left)) has a value close to zero for low values of α ($\theta \to 0$ as $\alpha \to 0$), totally damped by the viscous effects. For higher values of the reduced frequency ($\alpha = 20$) θ approaches the value $-\pi/2$ approaching the value for the inviscid flow case.

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S. Tsangaris
National Technical University of Athens
Department of Mechanical Engineering
Fluids Section
P.O.BOX 64070
15710 Zografou-Athens
Greece
e-mail: sgt@fluid. mech.ntua.gr

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