Z. angew. Math. Phys. 54 (2003) 484–502
0044-2275/03/030484-19
DOI 10.1007/s00033-003-1068-1

Zeitschrift für angewandte Mathematik und Physik ZAMP

Rotation number for non-autonomous linear Hamiltonian systems I: Basic properties

Roberta Fabbri, Russell Johnson and Carmen Núñez

Abstract. This paper concerns the rotation number for a random family of linear non-autonomous Hamiltonian systems. Several definitions corresponding to its analytic and geometrical approaches are presented, and some of its main properties are studied, as its continuity and its relation with the presence of exponential dichotomy.

Mathematics Subject Classification (2000). Primary: 34A30, 37B55; Secondary: 34C10, 37J05.

Keywords. Hamiltonian system, rotation number, symplectic geometry.

1. Introduction

In basic papers published around 1960, Yakubovich [31, 32, 33] used argument functions defined on the real symplectic group to study the oscillatory properties of the solutions of the linear Hamiltonian system

$$\mathbf{z}' = JA(t)\,\mathbf{z}\,, \quad \mathbf{z} \in \mathbb{R}^{2n}\,, \quad n \geq 1\,. \tag{1.1}$$

As usual we have written J for the standard skew-symmetric matrix, $J = \left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$, where I_n and 0_n are the $n \times n$ identity and zero matrices. The function $A(\cdot)$ is bounded and measurable, with values in the set $S_{\mathbb{R}}(2n)$ of real symmetric $2n \times 2n$ matrices. In developing his theory, Yakubovich used a basic fact, pointed out earlier by Gel'fand and Lidskiĭ [11], concerning the topological structure of the symplectic group.

Somewhat later, V. Arnold [2] introduced his argument function on the manifold of Lagrange planes on $\mathbb{R}^{2n}$ and used it to study the Maslov index. This argument function can also be used to study oscillation problems for (1.1), as pointed out by Arnold himself in [3].

The first and second authors were partially supported by C.I.C.Y.T. under project PB98-0359, by G.N.A.M.P.A. and by M.U.R.S.T

The third author was partially supported by C.I.C.Y.T. under project PB98-0359 and by Junta de Castilla y Leon and European Community under project VA19/00B

In the course of the last fifteen years, it has developed that the argument functions of Yakubovich and Arnold can be put to use in a random context. Thus, one considers a family Ω of equations of the form (1.1) such that Ω is closed under translations of the argument t and compact in, say, the usual weak-$*$ topology on $L^\infty(\mathbb{R}, S_\mathbb{R}(2n))$. Such a family Ω always supports an ergodic measure μ [22]. In [15], a μ-dependent rotation number $\alpha = \alpha(\mu)$ for the family Ω of equations (1.1) was defined in terms of the time-average of the Arnold argument. See also Ruelle [28] for a related construction. This rotation number may be said to measure the average oscillation of the solutions of (1.1). It has remarkable properties, one of which is the following. Consider a random "Atkinson problem"

$$\mathbf{z}' = J(A(t) + \lambda B(t))\,\mathbf{z}\,, \tag{1.2}$$

where A and B are bounded measurable functions with values in $S_\mathbb{R}(2n)$, B is positive semi-definite, and Atkinson's non-degeneracy condition holds: namely, if $U(t)$ is the principal matrix solution of (1.1), then

$$\int_{-\infty}^{\infty} U^T(t)B^T(t)B(t)U(t)\,dt \geq \delta I_{2n}$$

for some $\delta > 0$. Then the rotation number becomes a function $\alpha = \alpha(\lambda)$ of the real parameter λ. It turns out that, if α is constant on an open interval $I \subset \mathbb{R}$, the equation (1.2) admits an exponential dichotomy over $\mathbb{R}$, for all $\lambda \in I$ ([17])

The rotation number and its basic properties can also be discussed beginning with the Yakubovich argument functions. The relevant analysis was carried out in [23]. In this paper the polar coordinates on the symplectic group of Barret [4] and Reid [26, 27] were used to good effect. The authors introduced a flow on a bundle of Lagrange planes; it turned out that the polar coordinates were the appropriate tool to study this flow and then to derive assertions concerning the ergodic limit which defines the rotation number.

The rotation number has been used to good effect in the context of the random linear regulator problem and the random feedback control problem of control theory. One considers these problems on the semi-infinite interval $[0,\infty)$. On a finite interval $[0,T]$ they can be solved in a standard way using a Riccati equation [10]. On the semi-infinite interval it is convenient to derive the connection between the local controllability of the control system

$$\mathbf{x}' = A(t)\,\mathbf{x} + B(t)\,\mathbf{u}\,, \tag{1.3}$$

and the existence of a minimizer of the regulator problem resp. the existence of a linear stabilizing feedback control, by using the relation between the rotation number and exponential dichotomy concept discussed above. This approach permits one to discuss in an efficient and complete way such properties of solutions as conservation of recurrence, robustness, and smoothness with respect to parameters by making use of results from the theory of exponential dichotomies (see [29] and [36]). See [18, 19] for the use of the rotation number in solving these basic control-theoretic problems.

It also turns out that the concepts of rotation number and exponential dichotomy permit a direct generalization of the Yakubovich Frequency Theorem [33, 34] from periodic control systems to general non-autonomous systems (1.3) with bounded measurable coefficients. The details will be presented in [9].

The present paper is the first of two in which we study the rotation number and its properties. After taking up some preliminary considerations in §2, we proceed in §3 to unify the approaches to the study of the rotation number illustrated in [15] and [23]. The unification will be carried out by working with the flow induced by the family Ω of systems (1.1) on the bundle of complex Lagrange planes with base Ω. We study the solutions of a complex Riccati equation in appropriate coordinates. (Incidentally, in the control-theoretic problem mentioned above, the role of the rotation number is effectively that of shifting a Riccati equation from the real to the complex domain -where its study is much more natural and informative.) In §4 we will state and prove a strong continuity result of the rotation number with respect to variation of the coefficient matrix in (1.1). Finally, in §5, we use the Schwarzmann homomorphism [30] defined by the flow on Ω to prove that the values of α are "quantized" on the set of coefficient matrices for which (1.1) admits an exponential dichotomy. More precisely, there is a countable subgroup G of the additive reals, which depends only on the flow on Ω and on the ergodic measure μ, such that if equations (1.1) have an exponential dichotomy, then $\alpha(\mu) \in G$.

In the sequel to this paper [8], we take up some other basic themes related to the rotation number. We will discuss the Floquet exponent w for equations (1.1); this is a complex quantity of which the rotation number is the imaginary part. Using the Floquet exponent, we will discuss the Kotani theory for the Atkinson problem (1.2), thus generalizing the original results of Kotani [20] for the random Schrödinger operator together with those of later authors regarding other spectral problems. We will also use w to reprove a basic trace formula for the rotation number $\alpha = \alpha(\lambda)$ of the Atkinson problem (1.2) [15]. Finally, we will discuss the gap labelling phenomenon for (1.2).

2. Preliminaries and basic results

This section is devoted to the statement of the problem we work with: we introduce a random non-autonomous linear Hamiltonian system and explain the way on which it induces a flow on the corresponding real Lagrange bundle, as well as the use of the polar coordinates to determine the evolution of this flow.

We begin by recalling some basic notions of topological dynamics which will be consistently used in what follows. Let Ω be a compact metric space. A *real continuous flow* on Ω is a continuous map $\sigma : \mathbb{R} \times \Omega \to \Omega$ such that $\sigma_0 = \mathrm{Id}_\Omega$ and $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for any $s, t \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t, \omega)$. The *orbit* of a point $\omega \in \Omega$ is given by the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. A Borel subset $A \subset \Omega$ is σ*-invariant* if $\sigma_t(A) = A$ for any $t \in \mathbb{R}$. A measure μ on Ω is σ*-invariant* if $\mu(\sigma_t(A)) = \mu(A)$

for any Borel subset $A \subset \Omega$ and any $t \in \mathbb{R}$. Finally, a normalized σ-invariant measure is *ergodic* if any invariant set has measure 0 or 1.

Now we explain the setting of our problem. Let $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma(t, \omega)$ be a continuous flow on the compact metric space Ω. ¿From now on we represent $\omega \cdot t = \sigma(t, \omega)$ and fix a normalized σ-ergodic measure m_0 on Ω. Given a measurable matrix-valued function $H : \Omega \to \mathfrak{sp}(n, \mathbb{R})$, we consider the family of $2n$-dimensional linear Hamiltonian systems

$$\mathbf{z}' = H(\omega \cdot t)\, \mathbf{z}, \qquad \omega \in \Omega\,. \tag{2.1}$$

Recall that $\mathfrak{sp}(n, \mathbb{R}) = \{H \in M_{\mathbb{R}}(2n) \mid H^T J + JH = 0_{2n}\}$, being J the standard symplectic matrix $\left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$. We assume some conditions on H, namely

$$\sup_{\omega \in \Omega} \|H(\omega \cdot s)\|_\infty < \infty\,,$$

where $\|\cdot\|_\infty$ represents essential supremum (i.e., the smallest real number m with $\|H(\omega \cdot t)\| \le m$ for Lebesgue-a.e. $t \in \mathbb{R}$; here $\|\cdot\|$ represents any fixed matrix norm), and, in addition, the map

$$\Omega \to \mathbb{R}\,, \quad \omega \mapsto \int_{\mathbb{R}} H(\omega \cdot t)\, \mathbf{z}(t)\, dt$$

is continuous for any vector function $\mathbf{z}$ on $L^1(\mathbb{R})$. In particular, the first condition assures that the matrix-valued function H belongs to $L^1(\Omega, m_0)$. Note that the n-dimensional random Schrödinger equation

$$-\mathbf{x}'' + G(\omega \cdot t)\, \mathbf{x} = 0\,, \qquad \omega \in \Omega\,, \tag{2.2}$$

determined by a symmetric $n \times n$ matrix-valued function G on Ω, is included in the general formulation (2.1) by taking $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix}\right]$ and $H = \left[\begin{smallmatrix} 0_n & I_n \\ G & 0_n \end{smallmatrix}\right]$, repeatedly used in what follows.

As explained in [19], the above set up includes non-autonomous systems with a very wide class of coefficient functions: the space Ω can be taken as the hull in the weak-$*$ topology of a measurable function $H_*(\cdot) \in L^\infty(\mathbb{R})$ taking values on the algebra of symplectic matrices. Slightly more generally we could take $H_*(\cdot) \in L^p_{\mathrm{loc}}(\mathbb{R})$ with $\sup_t \int_0^1 (\|H_*(t+s)\|)^p\, ds < \infty$ ($p \ge 1$; if $p = 1$ we impose the supplementary condition $\lim_{\varepsilon \to 0} \sup_t \int_0^\varepsilon \|H_*(t+s)\|\, ds = 0$).

One of these families of systems (2.1) induces in a natural way a linear skew-product flow on $\Omega \times \mathbb{C}^{2n}$: the orbit of the element $(\omega, \mathbf{z}_0)$ is $\{(\omega{\cdot}t, U(t,\omega)\, \mathbf{z}_0) \mid t \in \mathbb{R}\}$, where $U(t, \omega)$ represents the fundamental matrix solution of Eq. (2.1) for $\omega \in \Omega$ with $U(0, \omega) = I_{2n}$. Since the coefficient matrix of the system $H(\omega \cdot t)$ belongs to the Lie algebra of infinitesimally symplectic matrices, $U(t, \omega)$ lies in the symplectic group $Sp(n, \mathbb{R}) = \{G \in M_{\mathbb{R}}(2n) \mid G^T J G = J\}$. This property allows us to define a new skew-product flow on the real and complex Lagrange bundles, as we explain in what follows.

Recall that a *complex Lagrange plane* (resp. *real*) is an n-dimensional vector subspace $l \subset \mathbb{C}^{2n}$ (resp. $l \subset \mathbb{R}^{2n}$) such that $\mathbf{x}^T J \mathbf{y} = 0$ for all $\mathbf{x}$, $\mathbf{y} \in l$.

The spaces $\mathcal{L}_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{R}}$ of all complex and real Lagrange planes are compact manifolds of dimension $n(n+1)/2$. The symplectic character of $U(t,\omega)$ assures that $U(t,\omega)\,l_0$ lies in $\mathcal{L}_{\mathbb{C}}$ whenever $l_0 \in \mathcal{L}_{\mathbb{C}}$, and consequently the map $\tau : \mathbb{R}\times\Omega\times\mathcal{L}_{\mathbb{C}} \to \Omega\times\mathcal{L}_{\mathbb{C}},\ (t,\omega,l_0)\mapsto(\omega{\cdot}t, U(t,\omega)\,l_0)$ defines a linear skew-product flow on $\mathcal{K}_{\mathbb{C}} = \Omega\times\mathcal{L}_{\mathbb{C}}$, which can be obviously restricted to $\mathcal{K}_{\mathbb{R}} = \Omega\times\mathcal{L}_{\mathbb{R}}$. The conditions assumed on H assure the continuity of all these flows.

An element l of $\mathcal{L}_{\mathbb{C}}$ can be represented by a $2n\times n$ matrix $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ of range n with $L_1^TL_2 = L_2^TL_1$. The column vectors form the basis of the Lagrange subspace; so two matrices $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right]$ represent the same complex Lagrange plane if and only if there is a non-singular $n\times n$ complex matrix P such that $L_1 = M_1P$ and $L_2 = M_2P$. The set $S_{\mathbb{C}}(n)$ of symmetric complex $n\times n$ matrices parametrizes an open dense subset of $\mathcal{L}_{\mathbb{C}}$, $\mathcal{O} = \left\{\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \mid M\in S_{\mathbb{C}}(n)\right\}$. Taking these coordinates in (2.1), we obtain the Riccati equations

$$M' = -MH_3(\omega{\cdot}t)M - MH_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)M + H_2(\omega{\cdot}t)\,, \qquad \omega\in\Omega\,, \tag{2.3}$$

where $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right]$. The flow on $\Omega\times\mathcal{O}$ is then $(\omega, M_0){\cdot}t = (\omega{\cdot}t, M(t,\omega,M_0))$, where $M(t,\omega,M_0)$ satisfies Eq. (2.3) with initial data $M(0,\omega,M_0) = M_0$.

On the other hand, the space $\mathcal{L}_{\mathbb{R}}$ can be identified with the homogeneous space of left cosets $\mathcal{G}/\mathcal{H}$, where

$$\begin{aligned}
\mathcal{G} &= \left\{\begin{bmatrix} \Phi_1 & -\Phi_2 \\ \Phi_2 & \Phi_1 \end{bmatrix} \in M_{\mathbb{R}}(2n) \mid (\Phi_1 + i\Phi_2)^*(\Phi_1 + i\Phi_2) = I_n\right\} \simeq U(n,\mathbb{C})\,,\\
\mathcal{H} &= \left\{\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \in M_{\mathbb{R}}(2n) \mid R^TR = I_n\right\} \simeq O(n,\mathbb{R})\,.
\end{aligned}$$

As usual, the symbol * represents the transpose conjugate. The above identification allows to express the flow induced by (2.1) on the real Lagrange bundle in terms of the *generalized polar coordinates*, as explained in the next theorem, whose proof can be found in Reid [27]. The application of the polar transformation to the study of oscillation and comparison theorems for matrix differential equations was first presented by Barret [4] and subsequently extended by Reid [26].

Theorem 2.1. *Let* $\left[\begin{smallmatrix} L_1^0 \\ L_2^0 \end{smallmatrix}\right]$ *be a real Lagrange plane and* Φ_1^0, Φ_2^0 *and* R^0 $n\times n$ *real matrices such that* $\left[\begin{smallmatrix} L_1^0 \\ L_2^0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Phi_1^0R^0 \\ \Phi_2^0R^0 \end{smallmatrix}\right]$, *with* $\left[\begin{smallmatrix} \Phi_1^0 & -\Phi_2^0 \\ \Phi_2^0 & \Phi_1^0 \end{smallmatrix}\right] \in \mathcal{G}$ *and* R^0 *non-singular. Then the* $2n\times n$ *solution of* (2.1) *corresponding to the initial data* $\left[\begin{smallmatrix} L_1^0 \\ L_2^0 \end{smallmatrix}\right]$ *is*

$$\begin{bmatrix} L_1(t,\omega,L_1^0,L_2^0) \\ L_2(t,\omega,L_1^0,L_2^0) \end{bmatrix} = \begin{bmatrix} \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,R^0) \\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)\,R(t,\omega,\Phi_1^0,\Phi_2^0,R^0) \end{bmatrix},$$

where $\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)$, $\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)$ *and* $R(t,\omega,\Phi_1^0,\Phi_2^0,R^0)$ *are the solutions* *of*

$$\begin{aligned} \Phi_1' &= \Phi_2\, Q(\omega{\cdot}t, \Phi_1, \Phi_2)\,, \\ \Phi_2' &= -\Phi_1\, Q(\omega{\cdot}t, \Phi_1, \Phi_2)\,, \end{aligned} \tag{2.4}$$

$$R' = S(\omega{\cdot}t, \Phi_1, \Phi_2)\, R \tag{2.5}$$

given by the initial data Φ_1^0, Φ_2^0 *and* R^0 *respectively, with*

$$Q(\omega, \Phi_1, \Phi_2) = \begin{bmatrix}\Phi_1^T\ \Phi_2^T\end{bmatrix} JH(\omega)\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix} \quad \textit{and} \quad S(\omega, \Phi_1, \Phi_2) = \begin{bmatrix}\Phi_1^T\ \Phi_2^T\end{bmatrix} H(\omega)\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}.$$

Besides $R^T(t,\omega,\Phi_1^0,\Phi_2^0,R^0)\, R(t,\omega,\Phi_1^0,\Phi_2^0,R^0) = L_1^T(t,\omega,L_1^0,L_2^0)\, L_1(t,\omega,L_1^0,L_2^0) + L_2^T(t,\omega,L_1^0,L_2^0)\, L_2(t,\omega,L_1^0,L_2^0)$ *and* $\begin{bmatrix} \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0) & -\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0) \\ \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0) & \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)\end{bmatrix} \in \mathcal{G}$ *for all* $t \in \mathbb{R}$.

Therefore, with these coordinates, the linear skew-product flow τ induced by Eqns. (2.1) on the compact metric space $\mathcal{K}_\mathbb{R} = \Omega \times \mathcal{L}_\mathbb{R}$ can be expressed in the following way: if $\begin{bmatrix}\Phi_1^0\\ \Phi_2^0\end{bmatrix}$ is a real Lagrange plane with $\Phi_1^0 + i\Phi_2^0$ unitary and $\Phi_1(t,\omega,\Phi_1^0,\Phi_2^0)$ and $\Phi_2(t,\omega,\Phi_1^0,\Phi_2^0)$ are the matrix solutions of Eqns. (2.4) with initial data Φ_1^0 and Φ_2^0, then

$$\tau(t,\omega,\Phi_1^0,\Phi_2^0) = (\omega{\cdot}t, \Phi_1(t,\omega,\Phi_1^0,\Phi_2^0), \Phi_2(t,\omega,\Phi_1^0,\Phi_2^0))$$

defines the equation of the flow on $\mathcal{K}_\mathbb{R}$. The relation $M = \Phi_2\Phi_1^{-1}$ provides the change between the different coordinates introduced on the dense subset of $\mathcal{K}_\mathbb{R}$ parametrized by the real symmetric matrices.

3. Rotation number for linear Hamiltonian systems

Now we summarize several different ways appearing in the literature to define the rotation number of the family of linear Hamiltonian systems (2.1), establishing at the same time the equivalence of all these approaches. An ergodic representation for the rotation number is also provided.

3.1. In terms of any argument on the real symplectic group

Our first definition of the rotation number is related to the evolution of the argument of a symplectic fundamental matrix solution of the Hamiltonian system.

The concept of argument of a symplectic matrix appears in the generalization of the Sturm theory for two-dimensional systems to linear periodic Hamiltonian systems of higher dimension. In order to analyze the oscillation properties of one of these systems, Yakubovich [31, 32] employs geometrical methods, in contrast to the analytical methods previously used by different authors. The starting point of his work is the fact that the real symplectic group $Sp(n,\mathbb{R})$ can be identified with a solid torus: it is homeomorphic to the topological product of a simply connected space and the unit circumference $\mathbb{S}^1$. This important property is proved by Gel'fand and Lidskiĭ [11], who use it to characterize stability regions of linear periodic Hamiltonians. The position of the projection of a symplectic matrix over $\mathbb{S}^1$ determines an angle, called by them the *argument* of the matrix.

Yakubovich identifies the *oscillatory* character of a periodic Hamiltonian with the property of unbounded increment of this argument along the curve determined by a symplectic fundamental matrix solution of the system. Following his idea, it is natural to define a rotation number as the mean increment of the argument, and that is exactly what is done in [23] in the general case.

As pointed out in [32], Gel'fand and Lidskiĭ's definition of the argument is difficult to manage (and it is not clear that the concept of oscillation agrees with the usual one for the two-dimensional case, defined in terms of the number of zeros of the solutions, as well as with the previously introduced ones for higher dimension). However, according to the results of [31], it is possible to define several different arguments for a symplectic matrix, which are equivalent in a sense that will be explained below, and which clarify these points.

The general definition of an argument function on the group $Sp(n,\mathbb{R})$ can be found in Yakubovich and Starzhinskii [35]: an *argument* of symplectic matrices is a real countable-valued relation Arg on $Sp(n,\mathbb{R})$ such that, if $(\operatorname{Arg} V)_0$ is any value of $\operatorname{Arg} V$, the other ones are

$$(\operatorname{Arg} V)_m = (\operatorname{Arg} V)_0 + 2m\pi\,, \quad m \in \mathbb{Z}\,,$$

each of the different branches is a locally continuous function, and there exists a closed symplectic curve $V(t)$ of index 1 with $\Delta \operatorname{Arg} V(t) = 2\pi$. Here the symbol Δ stands for the argument increment.

Let V be a real symplectic matrix, and represent by $\arg$ the usual argument of a complex number.

(i)If $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$, define

$$\begin{aligned} \operatorname{Arg}_1 V &= \arg\det(V_1 - iV_2)\,,\\ \operatorname{Arg}_2 V &= \arg\det(V_3 - iV_4)\,,\\ \operatorname{Arg}_3 V &= \arg\det(V_1 + iV_3)\,,\\ \operatorname{Arg}_4 V &= \arg\det(V_2 + iV_4)\,. \end{aligned}$$

(ii)If $T,\, S \in Sp(n,\mathbb{R})$ and $j = 1,\dots,4$, define

$$\operatorname{Arg}^j_{T,S} V = \operatorname{Arg}_j(TVS)\,.$$

(iii)Let $\mu_j,\, j = 1,\dots,n$ be the eigenvalues of the first type of the matrix V; i.e. those eigenvalues with modulus less than 1 or those for which there exist eigenvectors $\mathbf{v}_j$ satisfying $i\mathbf{v}_j^* J\mathbf{v}_j > 0$ (see [35]). Define

$$\operatorname{Arg}_* V = \sum_{j=1}^{n} \arg \mu_j\,.$$

As proved in [31], all these arguments are *equivalent* in the following sense (although non-equivalent arguments also exist): for any two of the previous argument functions Arg' and Arg'', there exists a uniform constant $c > 0$ such that for any continuous curve $V : [t_1, t_2] \to Sp(n, \mathbb{R})$ the inequality

$$\left| \Delta \operatorname{Arg}' V(t)\big|_{t_1}^{t_2} - \Delta \operatorname{Arg}'' V(t)\big|_{t_1}^{t_2} \right| < c$$

is satisfied when a continuous branch of each argument is taken along the curve. From now on, Arg will represent any argument equivalent to one of those listed above. The one appearing in [11] is included among these.

The function Arg allows us to define the rotation number for a random family of linear Hamiltonian systems. Let $V(t,\omega) = \left[\begin{smallmatrix} V_1(t,\omega) & V_3(t,\omega) \\ V_2(t,\omega) & V_4(t,\omega) \end{smallmatrix}\right]$ be a real symplectic fundamental matrix solution of (2.1), and consider the limit

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg} V(t,\omega)\,, \tag{3.1}$$

where a continuous branch of the argument is taken along the curve. Clearly, the equivalence of the arguments guarantees the independence of the limit with respect to the choices of Arg and the symplectic fundamental matrix solution. We call α the *rotation number of* (2.1) *with respect to* m_0. As shown below, α is well-defined, i.e. the limit exists and takes the same value for m_0-a.e. $\omega \in \Omega$, and in addition it admits an ergodic representation in terms of the generalized polar coordinates. Note that this definition and this representation extend the definition and the representation of rotation number for two-dimensional systems, introduced for the almost periodic Schrödinger case by Johnson and Moser in [16].

Proposition 3.1. *There is a* σ*-invariant subset* $\Omega_0 \subset \Omega$ *with* $m_0(\Omega_0) = 1$ *such that the limit* (3.1) *exists for every* $\omega \in \Omega_0$ *and takes the same constant value*

$$\alpha = \int_{\mathcal{K}_\mathbb{R}} \operatorname{tr} Q(\omega, \Phi_1, \Phi_2)\, d\nu \tag{3.2}$$

for every normalized τ*-invariant measure* ν *on* $\mathcal{K}_\mathbb{R}$ *projecting onto* m_0.

Proof. Choose $\operatorname{Arg} = \operatorname{Arg}_1$ and write $\left[\begin{smallmatrix} V_1(0,\omega) \\ V_2(0,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Phi_1^0 R^0 \\ \Phi_2^0 R^0 \end{smallmatrix}\right]$, with $\Phi_1^0 + i\Phi_2^0$ unitary and $\det R^0 > 0$. Then $\left[\begin{smallmatrix} V_1(t,\omega) \\ V_2(t,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Phi_1(t,\omega) R(t,\omega) \\ \Phi_2(t,\omega) R(t,\omega) \end{smallmatrix}\right]$, where $\Phi_1(t,\omega)$, $\Phi_2(t,\omega)$, $R(t,\omega)$ represent the solutions of (2.4) with respective initial values Φ_1^0, Φ_2^0, R^0. In particular, $\det(\Phi_1(t,\omega) - i\Phi_2(t,\omega))$ has modulus 1 and $\det R(t,\omega) > 0$ for every $t \in \mathbb{R}$. These facts and Liouville formula provide

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_1 V(t,\omega) &= \lim_{t\to\infty} \frac{1}{t} \arg\det(\Phi_1(t,\omega) - i\Phi_2(t,\omega)) \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} Q(\tau(s,\omega,\Phi_1^0,\Phi_2^0))\, ds\,. \end{aligned}$$

Therefore, Birkhoff ergodic theorem assures that, for any τ-invariant measure ν projecting onto m_0, this limit exists ν-a.e. on $\mathcal{K}_\mathbb{R}$ and defines a τ-invariant function. The independence of the limit of Φ_1^0 and Φ_2^0 and the ergodicity of m_0 allow us to assert that it is m_0-a.e. constant and hence it takes the value (3.2). $\square$

Remark 3.2. The rotation number for a Hamiltonian system $\mathbf{z}' = H\mathbf{z}$ with constant coefficients agrees with the sum of the imaginary parts of those eigenvalues of H of the first type which are purely imaginary, as can be immediately checked by using the argument Arg_* to obtain α. This is what one could reasonably expect (see Arnold and San Martin [1]). An analogous statement can be formulated in the periodic case, using now the characteristic exponents of the system.

3.2. Johnson's definition

The second definition of rotation number that we consider appears in the paper [15] of Johnson, in which the Floquet coefficient for a one-parameter family of random linear Hamiltonian systems is introduced, and its relation with the Weyl matrices and the spectral problem is studied. The analytic nature of the definition of the rotation number for real values of the parameter suggests a natural way to extend it to the complex plane. We consider this question in more detail in the second part of the present work [8].

In fact, the framework of the problem in [15] is quite more general, including the Hamiltonian systems (2.1) as a particular case: Johnson defines a rotation number for random linear systems whose coefficient matrices lie in the Lie algebra $\mathfrak{u}(p,q) = \{\widetilde{H} \in M_{\mathbb{C}}(p+q) \mid \widetilde{H}^* J_0 + J_0 \widetilde{H} = 0_{p+q}\}$, where $J_0 = \left[\begin{smallmatrix} -I_p & 0 \\ 0 & I_q \end{smallmatrix}\right]$ and $p \geq 1$, $q \geq 1$. The Iwasawa decomposition of this algebra allows the author to show the well-definedness of the rotation number, some of its properties, and its geometrical significance.

Coming back to our formulation, the symplectic algebra $\mathfrak{sp}(n,\mathbb{R})$ can be transformed in $\mathfrak{u}(n,n) \cap \mathfrak{sp}(n,\mathbb{C}) \subset \mathfrak{u}(n,n)$ via the map $H \mapsto K^{-1}HK$, where $K = \left[\begin{smallmatrix} iI_n & iI_n \\ -I_n & I_n \end{smallmatrix}\right]$, and hence one can define a rotation number for our system (2.1) by translating the coefficient matrix to the new algebra. This is the way followed in [15], which will be summarized below. But it is also possible and simpler to redefine the rotation number directly for the symplectic case, using exactly the same construction, and this is our next purpose.

Let us represent by $\mathcal{M}$ the set of the complex symmetric $n \times n$ matrices M with $\operatorname{Im} M > 0$. The Lie group $Sp(n,\mathbb{R})$ acts on $\mathcal{M}$ in the following way: a real symplectic matrix $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ induces a map $\widehat{V}$ of $\mathcal{M}$ into itself, given by

$$\widehat{V} \cdot M = (V_2 + V_4 M)(V_1 + V_3 M)^{-1};$$

i.e. $\left[\begin{smallmatrix} I_n \\ \widehat{V} \cdot M \end{smallmatrix}\right]$ coincides with the Lagrange plane $V \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$. Let $U(t,\omega) = \left[\begin{smallmatrix} U_1(t,\omega) & U_3(t,\omega) \\ U_2(t,\omega) & U_4(t,\omega) \end{smallmatrix}\right]$ be the fundamental matrix solution of (2.1) with $U(0,\omega) = I_{2n}$. For $t \in \mathbb{R}$ and $M_0 \in \mathcal{M}$, let $d_{M_0}\widehat{U}(t,\omega)$ be the Frèchet derivative at the point M_0 of the map

$\mathcal{M} \to \mathcal{M},\ M \mapsto \widehat{U}(t,\omega)\cdot M$. Then define

$$\alpha = -\lim_{t\to\infty} \frac{1}{2n}\,\frac{1}{t}\,\operatorname{Im}\ln\det d_{M_0}\widehat{U}(t,\omega)\,. \tag{3.3}$$

Now we will show the coincidence of the limits (3.3) and (3.1). In particular, it follows that this last limit exists m_0-a.e. on Ω and its value is independent of the choice of $M_0 \in \mathcal{M}$.

Proposition 3.3. *For every $M_0 \in \mathcal{M}$, the limit* (3.3) *agrees with the limit* (3.1).

Proof. A straightforward computation and the symplectic character of $U(t,\omega)$ show that

$$\begin{aligned} d_{M_0}\widehat{U}(t,\omega)\, M &= \big(U_4 - (U_2 + U_4M_0)(U_1+U_3M_0)^{-1}U_3\big)\, M\,(U_1+U_3M_0)^{-1} \\ &= (U_1^T + M_0U_3^T)^{-1}M(U_1+U_3M_0)^{-1}, \end{aligned}$$

where U_j represents $U_j(t,\omega)$. Consequently,

$$\det d_{M_0}\widehat{U}(t,\omega) = (\det(U_1(t,\omega)+U_3(t,\omega)M_0))^{-2n}$$

and hence

$$\begin{aligned} -\frac{1}{2n}\operatorname{Im}\ln\det d_{M_0}\widehat{U}(t,\omega) &= -\frac{1}{2n}\arg\det d_{M_0}\widehat{U}(t,\omega) \\ &= \arg\det(U_1(t,\omega)+U_3(t,\omega)M_0)\,. \end{aligned} \tag{3.4}$$

It is easy to check that the matrix $C_{M_0} = \begin{bmatrix} \operatorname{Im}^{-1/2}M_0 & 0 \\ \operatorname{Re}M_0\operatorname{Im}^{-1/2}M_0 & \operatorname{Im}^{1/2}M_0 \end{bmatrix}$ is symplectic. Therefore, Yakubovich's results above summarized assert the equivalence of Arg_3 and the new argument function defined by

$$\operatorname{Arg}^3_{I_{2n},C_{M_0}} V = \operatorname{Arg}_3(VC_{M_0})\,.$$

Since $\det\operatorname{Im}^{1/2}M_0 > 0$, we obtain

$$\operatorname{Arg}^3_{I_{2n},C_{M_0}} U(t,\omega) = \arg\det(U_1(t,\omega)+U_3(t,\omega)M_0)\,,$$

which together with (3.4) assures that the limits (3.3) and (3.1) agree. $\square$

As said before, the well-definedness of the rotation number is proved in [15] in a more general framework following a completely different argument, which also points out the geometrical significance of α . The main tool is the use of the Iwasawa decompositions (see [12]) for the Lie algebra $\mathfrak{u}(p,q)$ and the corresponding Lie group $U(p,q)$. For simplicity we restrict ourselves again to the symplectic case: we embed the Lie group $Sp(n,\mathbb{R})$ in $U(n,n)$ via $V \mapsto K^{-1}VK$ (recall that $K = \left[\begin{smallmatrix} iI_n & iI_n \\ -I_n & I_n \end{smallmatrix}\right]$) and note that its image is given by $U(n,n)\cap Sp(n,\mathbb{C}) = \{\widetilde{V} \in M_{\mathbb{C}}(2n) \mid \widetilde{V}^*J_0\widetilde{V} = J_0 \text{ and } \widetilde{V}^TJ\widetilde{V} = J\}$. Then $\widetilde{U}(t,\omega) = K^{-1}U(t,\omega)K$ is the fundamental matrix solution with value I_{2n} at $t=0$ of the systems $\widetilde{\mathbf{z}}' = \widetilde{H}(\omega t)\,\widetilde{\mathbf{z}}$, where $\widetilde{H}(\omega{\cdot}t) = K^{-1}H(\omega{\cdot}t)K$ belongs to the Lie algebra $\mathfrak{u}(n,n)\cap\mathfrak{sp}(n,\mathbb{C})$.

Now we represent by $\mathcal{D}$ the open set of the complex symmetric $n\times n$ matrices N with $I_n - N^*N > 0$, which agrees with the unit open disk when $n=1$. The Lie group $U(n,n)$ acts on $\mathcal{D}$: $\widetilde{V} = \left[\begin{smallmatrix}\widetilde{V}_1 & \widetilde{V}_3\\ \widetilde{V}_2 & \widetilde{V}_4\end{smallmatrix}\right] \in U(n,n)$ induces a map $\mathcal{D}\to\mathcal{D}$, given by

$$\widehat{\widetilde{V}}\cdot N = (\widetilde{V}_2 + \widetilde{V}_4 N)(\widetilde{V}_1 + \widetilde{V}_3 N)^{-1}.$$

It is also known that this action can be extended to the closure $\overline{\mathcal{D}}$ (which is not possible for the action of $Sp(n,\mathbb{R})$ over $\mathcal{M}$): the map $\widehat{V}$ preserves the boundary of $\mathcal{D}$.

The rotation number is defined as

$$\alpha = -\lim_{t\to\infty}\frac{1}{2n}\,\frac{1}{t}\,\mathrm{Im}\ln\det d_{N_0}\widehat{\widetilde{U}}(t,\omega) \tag{3.5}$$

in [15], where in addition it is shown that the limit is independent of the choice of the element $N_0\in\overline{\mathcal{D}}$ and takes the same value m_0-a.e. The geometrical idea of this definition is that the rotation number must measure the average rotation due to the action of $\widetilde{U}(t,\omega)$ on the set $\mathcal{D}$ and its boundary.

In order to prove that the definition (3.5) is also correct we need some facts concerning the Iwasawa decompositions that now we recall. Any matrix $V\in Sp(n,\mathbb{R})$ can be written in a unique way as the product GS, where

$$G\in\mathcal{G}=\left\{\begin{bmatrix}\Psi_1 & -\Psi_2\\ \Psi_2 & \Psi_1\end{bmatrix}\in M_{\mathbb{R}}(2n)\;\middle|\;(\Psi_1+i\Psi_2)^*(\Psi_1+i\Psi_2)=I_n\right\},$$

$$S\in\mathcal{S}=\left\{\begin{bmatrix}A & B\\ 0_n & (A^T)^{-1}\end{bmatrix}\in M_{\mathbb{R}}(2n)\;\middle|\;\begin{array}{l}A \text{ is upper triangular with}\\ \quad\text{positive diagonal,}\\ A^{-1}B \text{ is symmetric}\end{array}\right\}$$

The decomposition for the symplectic fundamental matrix solution of (2.1)

$$U(t,\omega)=G(t,\omega)\,S(t,\omega)=\begin{bmatrix}\Psi_1(t,\omega) & -\Psi_2(t,\omega)\\ \Psi_2(t,\omega) & \Psi_1(t,\omega)\end{bmatrix}\begin{bmatrix}A(t,\omega) & B(t,\omega)\\ 0_n & (A^T)^{-1}(t,\omega)\end{bmatrix}$$

is continuous in t, and hence so is the corresponding decomposition for $\widetilde{U}(t,\omega)$,

$$\widetilde{U}(t,\omega)=\widetilde{G}(t,\omega)\,\widetilde{S}(t,\omega) \tag{3.6}$$

with $\widetilde{G}=K^{-1}GK=\left[\begin{smallmatrix}\Psi_1-i\Psi_2 & 0\\ 0 & \Psi_1+i\Psi_2\end{smallmatrix}\right]$ and $\widetilde{S}=K^{-1}SK$.

Proposition 3.4. *For every $N_0\in\overline{\mathcal{D}}$, the limit* (3.5) *agrees with the limit* (3.1).

Proof. The map induced on $\mathcal{D}$ by $\widetilde{G}(t,\omega)$, namely

$$\widehat{\widetilde{G}}(t,\omega)\cdot N=(\Psi_1(t,\omega)+i\Psi_2(t,\omega))N(\Psi_1(t,\omega)-i\Psi_2(t,\omega))^{-1},$$

is linear, and hence it agrees with its Frèchet derivative at any point. From this fact, the definition of the group $\mathcal{G}$ and (3.6) we obtain

$$\begin{aligned}\det d_{N_0}\widehat{\widetilde{U}}(t,\omega)&=\det\widehat{\widetilde{G}}(t,\omega)\det d_{N_0}\widehat{\widetilde{S}}(t,\omega)\\ &=\det{}^{2n}(\Psi_1(t,\omega)+i\Psi_2(t,\omega))\det d_{N_0}\widehat{\widetilde{S}}(t,\omega)\,.\end{aligned}$$

In addition, a technical result of [15] shows that $\lim_{t\to\infty}(1/t)\arg\det d_{N_0}\widehat{\widehat{S}}(t,\omega)=0$ for every $N_0\in\overline{D}$ and $\omega\in\Omega$. From here we conclude that the limit (3.5) is equal to

$$\alpha=\lim_{t\to\infty}\frac{1}{t}\arg\det(\Psi_1(t,\omega)-i\Psi_2(t,\omega)) \tag{3.7}$$

and consequently it is independent of the choice of $N_0\in\overline{\mathcal{D}}$. But this expression can be also obtained by choosing the argument Arg_1 in our first definition (3.1) and having in mind the equality $\left[\begin{smallmatrix}U_1(t,\omega)\\ U_2(t,\omega)\end{smallmatrix}\right]=\left[\begin{smallmatrix}\Psi_1(t,\omega)A(t,\omega)\\ \Psi_2(t,\omega)A(t,\omega)\end{smallmatrix}\right]$, since $\det A(t,\omega)>0$. This shows the good definition of (3.5) and its coincidence with the rotation number, and hence completes the proof. $\square$

Relation (3.7) shows that the limit (3.3) measures the index of rotation of the composition map of $\mathbb{R}\times\Omega\to U(n,\mathbb{C})$, $(t,\omega)\mapsto(\Psi_1-i\Psi_2)(t,\omega)$ and $U(n,\mathbb{C})\to\mathbb{S}^1$, $\Psi\mapsto\det\Psi$, where $U(n,\mathbb{C})$ is the group of the unitary $n\times n$ matrices (which can be identified with $\mathcal{G}$). This points out once more the geometrical significance of α. Compare (3.7) with the expression of the limit α in terms of the generalized polar coordinates appearing in the proof of Proposition 3.1.

3.3. In terms of the Arnold-Maslov index

The rotation number for the family (2.1) admits still another definition strongly based on Arnold's approach [2] to the theory of the Maslov index. This index theory, related to asymptotic methods in perturbation theory, is also a fundamental tool in the generalization of Sturm theory to linear Hamiltonian systems, as shown in the work of Arnold [3]: for the higher-dimensional Schrödinger equation $-\mathbf{x}''+G(t)\mathbf{x}=0$, instead of zeros of solutions one can consider moments at which a Lagrange plane evolving under the action of the corresponding system is *vertical*, i.e. it is represented by $\left[\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right]$ with $\det L_1=0$. Roughly speaking, the Maslov index measures the number of these moments.

In [2] Arnold characterizes the Maslov index for a closed curve in the space of real symplectic planes (whose previous definition is based on intersection index theory and hence difficult to manage) in terms of the rotation index of certain maps on $\mathbb{S}^1$ (see also Bott [5]). This is the idea which suggests the new approach to α (which also is used in [15]). To explain this definition and the connection with the preceding ones is our next purpose. To this end, we recall briefly the definition of the Maslov index for a closed curve in the set of real Lagrange planes $\mathcal{L}_\mathbb{R}$ and refer the reader to [2] for the details.

Let l_0 be the Lagrange plane generated by the n last coordinate vectors, that is, $l_0\equiv\left[\begin{smallmatrix}0_n\\ I_n\end{smallmatrix}\right]$, and define $\mathcal{C}=\{l\in\mathcal{L}_\mathbb{R}\mid\dim(l\cap l_0)\geq 1\}$. Obviously, $\mathcal{C}=\cup_{k=1}^n\mathcal{C}^k$, where $\mathcal{C}^k=\{l\in\mathcal{L}_\mathbb{R}\mid\dim(l\cap l_0)=k\}$. Each set $\mathcal{C}^k$ is an algebraic submanifold of $\mathcal{L}_\mathbb{R}$ of codimension $k(k+1)/2$. In particular, $\operatorname{codim}\mathcal{C}^1=1$. Moreover, $\mathcal{C}^1$ it is two-sidedly embedded in $\mathcal{L}_\mathbb{R}$; i.e. there exists a continuous vector field tangent to $\mathcal{L}_{\mathbb{R}}$ which is transversal to $\mathcal{C}^1$, and hence one can refer to the *positive* and *negative* sides of $\mathcal{C}^1$. The vector field is given at each point $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \in \mathcal{L}_{\mathbb{R}}$ by the velocity vector of the curve $t \mapsto e^{it} \cdot l \equiv \left[\begin{smallmatrix} \cos t\, L_1 - \sin t\, L_2 \\ \sin t\, L_1 + \cos t\, L_2 \end{smallmatrix}\right]$, and the positive side is chosen as the one towards which these velocity vectors are directed.

Let $\lambda : \mathbb{S}^1 \to \mathcal{L}_{\mathbb{R}}$ be a smooth closed curve, and assume that λ only intersects $\mathcal{C}$ transversally (and hence only in $\mathcal{C}^1$). Then the *Maslov index* of λ is given by

$$c(\lambda) = m_+ - m_- \,,$$

where m_+ (resp. m_-) is the number of intersection points for which λ passes from the negative side of $\mathcal{C}^1$ to the positive side (resp. from the positive to the negative). According to the results of [2] (see also Duistermaat [7]), the index map c is independent of the choice of l_0 and induces a group isomorphism $c : \pi_1(\mathcal{L}_{\mathbb{R}}) \to \mathbb{Z}$, where $\pi_1(\mathcal{L}_{\mathbb{R}})$ is the fundamental group of $\mathcal{L}_{\mathbb{R}}$. In particular the Maslov index is defined for any continuous loop on $\mathcal{L}_{\mathbb{R}}$.

Now we can give a new definition for the rotation number. Choose $l \in \mathcal{L}_{\mathbb{R}}$, and for each pair (t, ω) consider the curve $\lambda_{t,\omega,l} : [0, t] \to \mathcal{L}_{\mathbb{R}},\ s \mapsto U(s, \omega)\, l$. Deform $\lambda_{t,\omega,l}$ to a closed curve $\tilde{\lambda}_{t,\omega,l}$ by sliding the final point $U(t, \omega)\, l$ to l through $\mathcal{L}_{\mathbb{R}} - \mathcal{C}$, which is simply connected, and represent $m(t, \omega, l) = c(\tilde{\lambda}_{t,\omega,l})$. Then define

$$\alpha = -\lim_{t \to \infty} \frac{\pi}{t}\, m(t, \omega, l)\,. \tag{3.8}$$

The limit exists and is independent of the choices of l and ω (m_0-a.e.), as stated in the following proposition. Its proof is basically due to Arnold's results, but we include a brief sketch for reader's convenience.

Proposition 3.5. *For every $l \in \mathcal{L}_{\mathbb{R}}$, the limit* (3.8) *agrees with the limit* (3.1).

Proof. Each real Lagrange plane $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ can be represented as $l \equiv \left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right]$ with $\Phi_1 - i\Phi_2$ unitary: it suffices to take $\Phi_j = L_j P^{-1}$, where P is the unique definite positive square root of $L_1^T L_1 + L_2^T L_2$. Consequently, the map

$$\mathrm{Det}^2 : \mathcal{L}_{\mathbb{R}} \to \mathbb{S}^1, \quad \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \mapsto \det{}^2(\Phi_1 - i\Phi_2) = \frac{\det^2(L_1 - iL_2)}{\det(L_1^T L_1 + L_2^T L_2)} \tag{3.9}$$

is well-defined. In particular, the image of l does not depend on the representation chosen.

Let $\lambda : \mathbb{S}^1 \to \mathcal{L}_{\mathbb{R}}$ be a continuous loop. Define $\mathrm{Ind}(\lambda)$ as the rotation index of the composition $\mathrm{Det}^2 \lambda : \mathbb{S}^1 \to \mathbb{S}^1$; i.e. $1/(2\pi)$ times the increment along the circumference of a continuous determination of $\arg : \mathbb{S}^1 \to \mathbb{R}$. It is possible to extend Ind to an isomorphism $\mathrm{Ind} : \pi_1(\mathcal{L}_{\mathbb{R}}) \to \mathbb{Z}$. Arnold [2] shows that in fact $-\mathrm{Ind}$ and c are the same map (since they agree on a homotopy class), which provides a simple characterization for the Maslov index in the symplectic case.

Now return to the limit (3.8). The independence of the choice of l follows from the invariance of c under homotopies. Choose $l \equiv \left[\begin{smallmatrix} I_n \\ 0_n \end{smallmatrix}\right]$, and note that

$U(t,\omega)\, l \equiv \begin{bmatrix} U_1(t,\omega) \\ U_2(t,\omega) \end{bmatrix}$. This leads to

$$\begin{aligned} -\lim_{t\to\infty} \frac{\pi}{t}\, m(t,\omega,l) &= \lim_{t\to\infty} \frac{\pi}{t}\frac{1}{2\pi} \arg \frac{\det^2(U_1 - iU_2)(t,\omega)}{\det(U_1^T U_1 + U_2^T U_2)(t,\omega)} \\ &= \lim_{t\to\infty} \frac{1}{t} \arg\det(U_1(t,\omega) - iU_2(t,\omega)) \\ &= \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_1 U(t,\omega)\,, \end{aligned}$$

which proves the result. □

The arguments used in the proof of this result will be fundamental in Section 5, in which the relation between the properties rotation number and the occurrence of exponential dichotomy will be discussed.

Definition (3.8) shows that α/π measures the average number of oriented intersections with the Maslov cycle $\mathcal{C}$ of the curve determined in $\mathcal{L}_{\mathbb{R}}$ by the evolution of a real Lagrange plane under the flow determined by (2.1). Therefore, it extends to the random $2n$-dimensional case one of the usual ways to define α for the one-dimensional linear Schrödinger equation $-x'' + g(\omega{\cdot}t)\, x = 0$: $\alpha = \lim(\pi/t)m(t,\omega)$, where $m(t,\omega)$ is the number of zeros in $[0,t]$ of any solution of the equation (see [16]).

4. Continuous variation of the rotation number

The ergodic representation for the rotation number obtained in Paragraph 3.1 is the fundamental tool in the study of the continuity of the rotation number with respect to the $L^1(\Omega, m_0)$-topology on the set of potentials H that we are considering. According to Proposition 3.1,

$$\alpha(H) = \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr} Q_H(\omega, \Phi_1, \Phi_2)\, d\nu_H$$

for any τ_H-invariant measure ν_H projecting onto m_0. Here, τ_H represents the flow induced on $\mathcal{K}_{\mathbb{R}}$ by the family of Hamiltonian systems determined by H, $\alpha(H)$ is the corresponding rotation number, and $Q_H(\omega,\Phi_1,\Phi_2) = [\Phi_1^T, \Phi_2^T]\, JH(\omega) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$. The following technical lemma is also required.

Lemma 4.1. (i) *There exists a real function $T_H \in L^1(\Omega, m_0)$ such that*

$$|\operatorname{tr} Q_H(\omega,\Phi_1,\Phi_2)| \le T_H(\omega)$$

for all $\omega \in \Omega$ and $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \in \mathcal{L}_{\mathbb{R}}$.

(ii) *There exists a constant k such that*

$$|\operatorname{tr} Q_{H_1}(\omega,\Phi_1,\Phi_2) - \operatorname{tr} Q_{H_2}(\omega,\Phi_1,\Phi_2)| \le k\, \|H_1(\omega) - H_2(\omega)\|\,,$$

for all $\omega \in \Omega$ and $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \in \mathcal{L}_{\mathbb{R}}$, where $\|\cdot\|$ represents any norm on the set of matrices.

Proof. Let $|A|$ represent the only positive semidefinite square root of the matrix $A^T A$. It is known (see Reed and Simon [25]) that $\mathrm{tr}\,|\cdot|$ defines a norm on the set of matrices, with $|\,\mathrm{tr}\,A| \le \mathrm{tr}\,|A|$ and $\mathrm{tr}\,|AB| \le \mathrm{tr}\,|A|\,\mathrm{tr}\,|B|$. Write

$$\mathrm{tr}\,Q_H(\omega,\Phi_1,\Phi_2) = \frac{1}{2}\,\mathrm{tr}\left(\begin{bmatrix} \Phi_1^T & \Phi_2^T \\ \Phi_1^T & \Phi_2^T \end{bmatrix} JH(\omega) \begin{bmatrix} \Phi_1 & \Phi_1 \\ \Phi_2 & \Phi_2 \end{bmatrix}\right).$$

The results follow from this expression, the above properties, the compactness of $\mathcal{L}_{\mathbb{R}}$ and the equivalence of any pair of matrix norms. □

Remark 4.2. The function T_H is continuous if H is.

Theorem 4.3. *Assume that* $H = \lim_{n\to\infty} H_n$ *in the* $L^1(\Omega, m_0)$*-topology. Then*

$$\alpha(H) = \lim_{n\to\infty} \alpha(H_n)\,.$$

Proof. The argument used is standard in measure theory, the proof being simpler if the limit matrix H is supposed to be continuous. For each n, take a τ_{H_n}-invariant measure ν_{H_n} on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0. Assume that $(\nu_{H_n})_{n\in\mathbb{N}}$ converges weakly to a new measure ν_H on $\mathcal{K}_{\mathbb{R}}$ (each subsequence has a weakly convergent subsequence). Then ν_H is τ_H-invariant and projects onto m_0. In order to prove the result it suffices to check that

$$\alpha(H_n) - \alpha(H) = \int_{\mathcal{K}_{\mathbb{R}}} \mathrm{tr}\,Q_{H_n}\,d\nu_{H_n} - \int_{\mathcal{K}_{\mathbb{R}}} \mathrm{tr}\,Q_H\,d\nu_H \overset{n\to\infty}{\longrightarrow} 0\,. \tag{4.1}$$

Note first (ii) in Lemma 4.1 and the $L^1(\Omega, m_0)$-convergence imply that

$$\left|\int_{\mathcal{K}_{\mathbb{R}}} \mathrm{tr}\,Q_{H_n}\,d\nu_{H_n} - \int_{\mathcal{K}_{\mathbb{R}}} \mathrm{tr}\,Q_H\,d\nu_{H_n}\right| \le k\int_\Omega \|H_n(\omega) - H(\omega)\|\,dm_0 \overset{n\to\infty}{\longrightarrow} 0\,. \tag{4.2}$$

Take now $\varepsilon > 0$ and choose

- a constant $\delta > 0$ such that $\int_{\widetilde{\Omega}} T_H(\omega)\,dm_0 < \varepsilon$ if $\widetilde{\Omega} \subset \Omega$ and $m_0(\widetilde{\Omega}) < \delta$,
- a compact subset $K_\varepsilon \subset \Omega$ with $m_0(\Omega - K_\varepsilon) < \delta$ and a continuous function H^ε on Ω such that $H^\varepsilon|_{K_\varepsilon} = H|_{K_\varepsilon}$.

Consider $Q_{H^\varepsilon}(\omega,\Phi_1,\Phi_2) = [\Phi_1^T, \Phi_2^T]\,JH^\varepsilon(\omega)\left[\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right]$, continuous on $\mathcal{K}_{\mathbb{R}}$, and find

- an open subset $U_\varepsilon \subset \Omega$ with $K_\varepsilon \subset U_\varepsilon$ and

$$m_0(U_\varepsilon - K_\varepsilon) \sup_{\omega\in\Omega} |\,\mathrm{tr}\,Q_{H^\varepsilon}(\omega)| < \varepsilon\,,$$

- and a continuous function r on Ω with $\chi_{K_\varepsilon} \le r \le \chi_{U_\varepsilon}$.

Define now $\widetilde{Q}_H^\varepsilon(\omega,\Phi_1,\Phi_2) = r(\omega)\,Q_{H^\varepsilon}(\omega,\Phi_1,\Phi_2)$, continuous on $\mathcal{K}_{\mathbb{R}}$. Let ν be any measure on $\mathcal{K}_{\mathbb{R}}$ projecting onto m_0. Then

$$\int_{\mathcal{K}_{\mathbb{R}}} \mathrm{tr}\,Q_H\,d\nu = \int_{\mathcal{K}_{\mathbb{R}}} \mathrm{tr}\,\widetilde{Q}_H^\varepsilon\,d\nu - \int_{(U_\varepsilon - K_\varepsilon)\times\mathcal{L}_{\mathbb{R}}} \mathrm{tr}\,\widetilde{Q}_H^\varepsilon\,d\nu + \int_{(\Omega - K_\varepsilon)\times\mathcal{L}_{\mathbb{R}}} \mathrm{tr}\,Q_H\,d\nu\,.$$

Moreover, the definition of U_ε and (i) in Lemma 4.1 implies that

$$\left|\int_{(U_\varepsilon - K_\varepsilon)\times\mathcal{L}_{\mathbb{R}}} \operatorname{tr}\widetilde{Q}^{\varepsilon}_H\,d\nu\right| \le m_0(U_\varepsilon - K_\varepsilon)\sup_{\omega\in\Omega}|\operatorname{tr}\widetilde{Q}^{\varepsilon}_H(\omega)| < \varepsilon\,,$$

$$\left|\int_{(\Omega - K_\varepsilon)\times\mathcal{L}_{\mathbb{R}}} \operatorname{tr}Q_H\,d\nu\right| \le \int_{\Omega-K_\varepsilon} T_H(\omega)\,dm_0 < \varepsilon\,.$$

Consequently,

$$\left|\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr}Q_H\,d\nu_{H_n} - \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr}Q_H\,d\nu_H\right| < \left|\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr}\widetilde{Q}^{\varepsilon}_H\,d\nu_{H_n} - \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr}\widetilde{Q}^{\varepsilon}_H\,d\nu_H\right| + 4\varepsilon\,.$$

The weak convergence of the sequence of measures implies that

$$\left|\int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr}Q_H\,d\nu_{H_n} - \int_{\mathcal{K}_{\mathbb{R}}} \operatorname{tr}Q_H\,d\nu_H\right| \overset{n\to\infty}{\longrightarrow} 0\,. \tag{4.3}$$

Relations (4.2) and (4.3) assure the convergence of (4.1) and prove the result. □

5. Rotation number and the Schwarzmann homomorphism

This last section is devoted to establish a fact concerning the relation between the rotation number and the presence of an *exponential dichotomy* (or hyperbolic splitting) for the Hamiltonian system (2.1). More precisely, we will prove that if this family of systems admits an exponential dichotomy over Ω, then the corresponding rotation number α takes values on the image of the Schwarzmann homomorphism. Before stating this result, we recall the following:

Definition 5.1. The family of systems (2.1) has an *exponential dichotomy* (ED for short) over Ω if there exist two positive constants η, γ and a splitting $\Omega\times R^{2n} = L^+\oplus L^-$ of the real bundle into the Whitney sum of two τ-invariant closed subbundles, with the following properties:

(i) $\|U(t,\omega)\,\mathbf{z}_0\| \le \eta e^{-\gamma t}\|\mathbf{z}_0\|$ for every $t\ge 0$ and $(\omega,\mathbf{z}_0)\in L^+$,

(ii) $\|U(t,\omega)\,\mathbf{z}_0\| \le \eta e^{\gamma t}\|\mathbf{z}_0\|$ for every $t\le 0$ and $(\omega,\mathbf{z}_0)\in L^-$.

The concept of ED is a fundamental tool in several fields, such as the study of the invertibility of self-adjoint operators in different spaces (Massera and Schaefer [21]), bifurcation theory (Chenciner and Iooss [6]), study of invariant manifolds (Hirsch, Pugh and Shub [13]), analysis of homoclinic orbits (Palmer [24]), spectral theory for the Schrödinger operator (Johnson [14]) and control theory (Johnson and Nerurkar [17, 19]), among others.

Now we state and prove the result above mentioned.

Theorem 5.2. *Suppose that the family* (2.1) *has an ED over* Ω. *Then we have that* $2\alpha\in h\left(\check{H}^1(\Omega,\mathbb{Z})\right)$, *where* $h:\check{H}^1(\Omega,\mathbb{Z})\to\mathbb{R}$ *is the Schwarzmann homomorphism.*

Proof. We first define the Schwarzmann homomorphism. Let $H(\Omega,\mathbb{S}^1)$ be the set of homotopy classes of continuous maps $\phi:\Omega\to\mathbb{S}^1\subset\mathbb{C}$. The class $[\phi]$ contains a map ϕ such that the application

$$\omega\mapsto\frac{d}{dt}\phi(\omega{\cdot}t)\mid_{t=0}=\phi'(\omega)$$

is continuous. We can then define

$$h:H(\Omega,\mathbb{S}^1)\to\mathbb{R}\,,\quad [\phi]\mapsto \operatorname{Im}\int_\Omega\frac{\phi'(\omega)}{\phi(\omega)}\,dm_0\,.$$

It follows from the Birkhoff ergodic theorem that

$$h([\phi])=\lim_{t\to\infty}\frac{1}{t}\arg\phi(\omega{\cdot}t)\qquad m_0\text{-a.e.}\tag{5.1}$$

Schwarzmann [30] shows that the map h is well-defined and provides a homomorphism from the group $H(\Omega,\mathbb{S}^1)$ to the additive group of real numbers. Consider now the group of real Čech one-cocycles with integer values,

$$\check{H}^1(\Omega,\mathbb{Z})=\frac{H(\Omega,\mathbb{S}^1)}{C}\,,$$

where C is the subgroup of $H(\Omega,\mathbb{S}^1)$ given by the homotopy classes of the maps $\phi(\omega)=\exp 2ir(\omega)$, with $r:\Omega\to\mathbb{R}$ continuous. Equality (5.1) gives $h([\phi])=0$ for any $\phi\in C$, and consequently the map h also induces a homomorphism from $\check{H}^1(\Omega,\mathbb{Z})$ into $\mathbb{R}$. The map $h:\check{H}^1(\Omega,\mathbb{Z})\to\mathbb{R}$ is the Schwarzmann homomorphism.

Now consider the decomposition $\Omega\times\mathbb{R}^{2n}=L^+\oplus L^-$ provided by the ED and note that, for each $\omega\in\Omega$, the fibers

$$l^\pm(\omega)=L^\pm\cap\big(\{\omega\}\times\mathbb{R}^{2n}\big)$$

are real Lagrange planes: the symplectic character of $U(t,\omega)$ assures that $\mathbf{x}^TJ\mathbf{y}=\mathbf{x}^TU^T(t,\omega)JU(t,\omega)\mathbf{y}$ for any $t\in\mathbb{R}$ and any pair of vectors $\mathbf{x},\,\mathbf{y}\in\mathbb{R}^{2n}$, and hence the behavior of the solutions on $+\infty$ (resp. $-\infty$) described in Definition 5.1 implies $\mathbf{x}^TJ\mathbf{y}=0$ for any pair of vectors $\mathbf{x},\,\mathbf{y}\in l^+(\omega)$ (resp. $\mathbf{x},\,\mathbf{y}\in l^-(\omega)$).

We can now define $\widetilde{\phi}$ as the composed map of $\Omega\to\mathcal{L}_\mathbb{R},\,\omega\mapsto l(\omega)$ and $\mathcal{L}_\mathbb{R}\to\mathbb{S}^1,\,l\mapsto\operatorname{Det}^2 l$, where this last application is defined by (3.9). The map $\widetilde{\phi}$ is well-defined and continuous, since the subbundles given by the ED are closed. In addition, according to equality (5.1) and the proof of Proposition 3.5, $2\alpha=h([\widetilde{\phi}])$, which completes the proof. □

This result is the starting point to obtain a gap labelling formula for the spectral problems corresponding to (2.1) and (2.2). We refer the reader to [8] for the details.

Acknowledgement

The authors thank Professors R. Obaya and S. Novo for their help and useful comments.

References

[1] L. Arnold, L. San Martin, A multiplicative ergodic theorem for rotation numbers, *J. Dynam. Differential Equations* **1**(1) (1989), 95–119.

[2] V.I. Arnold, On a characteristic class entering in a quantum condition, *Func. Anal. Appl.* **1** (1967), 1–14.

[3] V.I. Arnold, The Sturm theorems and symplectic geometry, *Func. Anal. Appl.* **19** (1985), 251–259.

[4] J.H. Barret, A Prüfer transformation for matrix differential equations, *Proc. Amer. Math. Soc.* **8** (1957), 510–518.

[5] R. Bott, On the iteration of closed geodesic and the Sturm intersection theory, *Commun. Pure and Applied Math.* **9** (2) (1956), 171–206.

[6] A. Chenciner, G. Iooss, Bifurcations de tores invariants, *Arch. Rational Mech. Anal.* **69** (1979), 109–198.

[7] J.J. Duistermaat, On the Morse index in variational calculus, *Adv. Math.* **21** (1976), 173–195.

[8] R. Fabbri, R. Johnson, C. Núñez, Rotation number for non-autonomous linear Hamiltonian systems II: The Floquet exponent, *Z. angew. Math. Phys.* **54** (2003).

[9] R. Fabbri, R. Johnson, C. Núñez, On the Yakubovich frequency theorem for non-autonomous control processes, *Diser. Cont. Dyr. Sys.* **9**(3) (2003), 677–704.

[10] W. Fleming, R. Rishel, *Deterministic and Optimal Control*, Springer-Verlag, Berlin, 1975.

[11] I.M. Gel'fand, V.B. Lidskiĭ, On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients, *Amer. Mat. Soc. Transl.* (2) **8** (1958), 143–181.

[12] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[13] W.M. Hirsch, C.C. Pugh, M. Shub, Invariant manifolds, *Lecture Notes in Math.* **583**, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[14] R. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients, *J. Differential Equations* **61** (1986), 54–78.

[15] R. Johnson, m-functions and Floquet exponents for linear differential systems, *Ann. Mat. Pura Appl.* **147** (1987), 211–248.

[16] R. Johnson, J. Moser, The rotation number for almost periodic differential equations, *Comm. Math. Phys.* **84** (1982), 403–438.

[17] R. Johnson, M. Nerurkar, Exponential dichotomy and rotation number for linear Hamiltonian systems, *J. Differential Equations* **108** (1994), 201–216.

[18] R. Johnson, M. Nerurkar, Stabilization and random linear regulator problem for random linear control processes, *J. Math. Anal. Appl.* **197** (1996), 608–629.

[19] R. Johnson, M. Nerurkar, Controllability, stabilization, and the regulator problem for random differential systems, *Mem. Amer. Math. Soc.* **646**, Amer. Math. Soc., Providence, 1998.

[20] S. Kotani, Lyapunov indices determine absolutely continuous spectrum of stationary random 1-dimensional Schrödinger equations, Proc. of Taniguchi Sympos., S.A.Katata (1982), 225–247.

[21] J. Massera, J. Schaeffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York, London, 1966.

[22] V. Nemytskii, V. Stepanoff, *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, NJ, 1960.

[23] S. Novo, C. Núñez, R. Obaya, Ergodic properties and rotation number for linear Hamiltonian systems, *J. Differential Equations* **148** (1998), 148–185.

[24] K.J. Palmer, Exponential dichotomies, the shadowing-lemma and transversal homoclinic points, *Dynam. Report.* **1** (1988), 265–306.

[25] M. Reed, B. Simon, *Method of Modern Mathematical Physics I: Funtional Analysis. Revised and Enlarged Edition*, Academic Press Inc., San Diego, 1980.

[26] W.T. Reid, A Prüfer transformation for differential systems, *Pac. J. Math.* **8** (1958), 575–584.
[27] W.T. Reid, *Sturmian Theory for Ordinary Differential Equations*, Applied Mathematical Sciences **31**, Springer-Verlag, New York, 1980.
[28] D. Ruelle, Rotation numbers for diffeomorphisms and flows, *Ann. Inst. Henri Poincaré, Phys. Théor.* **42** (1985), 109–115.
[29] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, *J. Differential Equations* **27** (1978), 320–358.
[30] S. Schwarzmann, Asymptotic cycles, *Ann. Math.* **66** (2) (1957), 270–284.
[31] V.A. Yakubovich, Arguments on the group of symplectic matrices, *Mat. Sb.* **55** (97) (1961), 255–280 (Russian).
[32] V.A. Yakubovich, Oscillatory properties of the solutions of canonical equations, *Amer. Math. Soc. Transl. Ser.* 2 **42** (1964), 247–288.
[33] V.A. Yakubovich, A linear-quadratic optimization problem and the frequency theorem for periodic systems I, *Siberian Math. J.* **27** (4) (1986), 614–630.
[34] V.A. Yakubovich, Linear-quadratic optimization problem and the frequency theorem for periodic systems II, *Siberian Math. J.* **31** (6) (1990), 1027–1039.
[35] V.A. Yakubovich, V.M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients*, John Wiley & Sons, Inc., New York, 1975.
[36] Y. Yi, A generalized integral manifold theorem, *J. Differential Equations* **102** (1993), 153–187.

Roberta Fabbri and Russell Johnson
Dipartimento di Sistemi e Informatica
Università degli Studi di Firenze
Via Santa Marta 3
50139 Firenze
Italy
e-mail: fabbri@dsi.unifi.it, johnson@ingfi1.ing.unifi.it

Carmen Núñez
Departamento de Matemática Aplicada a la Ingeniería
Universidad de Valladolid
Paseo del Cauce s/n
47011 Valladolid, Spain
e-mail: carnun@wmatem.eis.uva.es

(Received: May 2, 2001; revised: September 26, 2001)