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Existence and continuous dependence of large solutions for the magnetohydrodynamic equations

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Abstract. Global solutions of the nonlinear magnetohydrodynamic (MHD) equations with general large initial data are investigated. First the existence and uniqueness of global solutions are established with large initial data in H^1. It is shown that neither shock waves nor vacuum and concentration are developed in a finite time, although there is a complex interaction between the hydrodynamic and magnetodynamic effects. Then the continuous dependence of solutions upon the initial data is proved. The equivalence between the well-posedness problems of the system in Euler and Lagrangian coordinates is also showed.

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1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids (cf. gases) in an electromagnetic field with a very broad range of applications. The dynamic motion of the fluids and the magnetic field strongly interact each other, and thus the hydrodynamic and electrodynamic effects are coupled. Plane magnetohydrodynamic flows are governed by the following equations (see [5, 7, 10, 15, 17]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \quad x \in \mathbf{R}, \quad t > 0, \\ (\rho u)_t + \left(\rho u^2 + p + \frac{1}{2}|\mathbf{b}|^2\right)_x = (\lambda u_x)_x, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x = (\mu \mathbf{w}_x)_x, \\ \mathbf{b}_t + (u\mathbf{b} - \mathbf{w})_x = (\nu \mathbf{b}_x)_x, \\ \mathcal{E}_t + \left(u\left(\mathcal{E} + p + \frac{1}{2}|\mathbf{b}|^2\right) - \mathbf{w}\cdot\mathbf{b}\right)_x = (\lambda u u_x + \mu \mathbf{w}\cdot\mathbf{w}_x + \nu \mathbf{b}\cdot\mathbf{b}_x + \kappa\theta_x)_x, \end{cases} \tag{1.1}$$

where ρ denotes the density of the flow, $u \in \mathbf{R}$ the longitudinal velocity, $\mathbf{w} \in \mathbf{R}^2$ the transverse velocity, $\mathbf{b} \in \mathbf{R}^2$ the transverse magnetic field, and θ the temperature; the longitudinal magnetic field is a constant which is taken to be one in (1.1); the total energy of the plane magnetohydrodynamic flow is

$$\mathcal{E} = \rho\left(e + \frac{1}{2}(u^2 + |\mathbf{w}|^2)\right) + \frac{1}{2}|\mathbf{b}|^2,$$

with the internal energy e; both the pressure p and the internal energy e are related with the density and temperature of the flow according to the equations of state: $p = p(\rho, \theta), e = e(\rho, \theta)$; $\lambda = \lambda(\rho, \theta)$ and $\mu = \mu(\rho, \theta)$ are the viscosity coefficients of the flow, $\nu = \nu(\rho, \theta)$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, $\kappa = \kappa(\rho, \theta)$ is the heat conductivity.

The equations in (1.1) describe the macroscopic behavior of the magnetohydrodynamic flow with dissipative mechanisms. This is a three-dimensional magnetohydrodynamic flow which is uniform in the transverse directions. Many efforts have been made because of its physical importance, complexity, and rich phenomena; see [1, 2, 3, 5, 7, 10, 11, 15, 17, 18, 19, 22, 23] and the references cited therein. In this paper we focus on an initial-boundary value problem for the magnetohydrodynamic flow of a perfect gas with the following equations of state:

$$p = R\rho\theta, \qquad e = c_v\theta,$$

where R is the gas constant, $c_v = R/(\gamma - 1)$ is the heat capacity of the gas at constant volume, and γ is the adiabatic exponent. We are interested in the well-posedness and continuous dependence of global solutions of the initial-boundary value problem. For small smooth initial data, the existence of global solution was proved in [10], and the large-time behavior was studied in [19]. For large initial data, these problems have additional difficulties because of the presence of the magnetic field and its interaction with the hydrodynamic motion of the flow of large oscillation.

In this paper we first establish the existence and uniqueness of global solutions to the initial-boundary value problem of (1.1) with initial data in H^1 and show that neither shock waves nor vacuum and concentration are developed in a finite time for such initial data. Different from the early results on the Navier-Stokes equations, such as [13] which considered the equations of a viscous heat-conductive perfect gas that does not allow heat to be generated by the magnetic field, we permit the generation of heat by the magnetic field as well as its interaction with the fluid motion. Mathematically, the argument of [13] can not be directly followed to solve our MHD problem because of the presence of magnetic field. For example, the lower and upper bounds of the density can not be obtained by the same argument. The main reason here is that the full pressure $p + |\mathbf{b}|^2/2$ in MHD does not have the simple special structure as the pressure p in the Navier-Stokes equations. We develop some new estimates and techniques to overcome these difficulties. We establish a new expression of the density for this magnetohydrodynamic system and successfully obtain its lower and upper bounds. Then we make other a-priori estimates and prove the existence of global solutions. The existence of global solutions is proved by extending the local solutions globally in time based on the global a-priori estimates of solutions with a physical growth condition of the heat conductivity. By introducing a Lagrangian variable, we transform the initial-boundary value problem of (1.1) in Euler coordinates into the corresponding initial-boundary value problem in Lagrangian coordinates. These two problems are equivalent for the solutions under consideration (see Section 2). We first obtain an entropy-type energy estimate involving the dissipative effects of viscosity, magnetic diffusion, and heat diffusion. After proving the lower and upper bounds of the density as stated earlier, all the required a-priori estimates are obtained subsequently with our careful analysis, and the existence of global solutions in H^1 (as well as in a Hölder space) is established in Section 3. After the proof of the existence, we show the continuous dependence of solutions on the initial data. For this purpose, we first prove the Lipschitz continuity of the density, and then we prove that the solutions depend continuously on the initial data with several careful estimates on a functional in Section 4.

More precisely, we focus on the initial-boundary value problem of (1.1) in a bounded spatial domain $\Omega = [0, 1]$ (without loss of generality) with the following initial condition and impermeable, thermally insulated boundaries:

$$\begin{aligned} &(\rho, u, \mathbf{w}, \mathbf{b}, \theta)|_{t=0} = (\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)(x), \quad x \in \Omega, \\ &(u, \mathbf{w}, \mathbf{b}, \theta_x)|_{\partial\Omega} = 0, \end{aligned} \tag{1.2}$$

where the initial data satisfies these compatibility conditions:

$$\begin{aligned} &(u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_{0x})|_{\partial\Omega} = 0, \\ &\big(\rho_0 u_0^2 + R\rho_0\theta_0 - \lambda u_{0x} + |\mathbf{b}_0|^2/2\big)_x\big|_{\partial\Omega} = 0, \\ &(\rho_0 u_0 \mathbf{w}_0 - \mathbf{b}_0 - \mu \mathbf{w}_{0x}, u_0 \mathbf{b}_0 - \mathbf{w}_0 - \nu \mathbf{b}_{0x})_x|_{\partial\Omega} = 0. \end{aligned} \tag{1.3}$$

For noninsulated boundary conditions, similar results can be obtained with some small modification. Although we deal with the initial-boundary value problem in this paper, our existence results can directly be generalized to the Cauchy problem by using the localization argument in [9] or the known localization lemma [12].

We state our main results of this paper in Section 2, prove the existence of global solutions in Section 3, and show the continuous dependence in Section 4.

2. Main Theorems

Consider the initial-boundary value problem (1.1)-(1.2) with positive lower and upper bounds of the initial density and temperature: $C_0^{-1} \le \rho_0 \le C_0, C_0^{-1} \le \theta_0 \le M_0,$ for some constants C_0 and M_0. Without loss of generality, we take $\int_0^1 \rho_0(x)dx = 1$. We first assume that $\rho, \theta > 0$, and then we will prove their positive lower bounds later. Introduce the Lagrangian variable:

$$y = y(x,t) = \int_0^x \rho(\xi, t) d\xi. \tag{2.1}$$

We have $0 \leq y \leq 1$ since y is increasing in x and $\int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx = 1$. We translate problem (1.1)-(1.2) in Euler coordinates into the following initial-boundary value problem in Lagrangian coordinates $(y,t), y \in \Omega = [0,1]$, a moving coordinate along the particle path:

$$v_t - u_y = 0, \tag{2.2a}$$

$$u_t + \left(p + |\mathbf{b}|^2/2\right)_y = \left(\frac{\lambda u_y}{v}\right)_y, \tag{2.2b}$$

$$\mathbf{w}_t - \mathbf{b}_y = \left(\frac{\mu \mathbf{w}_y}{v}\right)_y, \tag{2.2c}$$

$$(v\mathbf{b})_t - \mathbf{w}_y = \left(\frac{\nu \mathbf{b}_y}{v}\right)_y, \tag{2.2d}$$

$$E_t + \left(u\left(p + |\mathbf{b}|^2/2\right) - \mathbf{w}\cdot\mathbf{b}\right)_y = \left(\frac{\lambda u u_y + \mu \mathbf{w}\cdot\mathbf{w}_y + \nu \mathbf{b}\cdot\mathbf{b}_y + \kappa\theta_y}{v}\right)_y, \tag{2.2e}$$

with the initial-boundary conditions:

$$\begin{aligned} &(v,u,\mathbf{w},\mathbf{b},\theta)|_{t=0} = (v_0,u_0,\mathbf{w}_0,\mathbf{b}_0,\theta_0)(y), \quad y \in \Omega, \\ &(u,\mathbf{w},\mathbf{b},\theta_y)|_{\partial\Omega} = 0, \end{aligned} \tag{2.3}$$

where $v = 1/\rho$ is the specific volume, $p = R\theta/v$, $e = c_v\theta$, and

$$E = e + \frac{1}{2}(u^2 + |\mathbf{w}|^2) + \frac{1}{2}v|\mathbf{b}|^2.$$

From (2.2), we have

$$(c_v\theta)_t + p u_y = \left(\frac{\kappa\theta_y}{v}\right)_y + \frac{\lambda u_y^2}{v} + \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{b}_y|^2}{v}. \tag{2.4}$$

Problem (1.1)-(1.2) and problem (2.2)-(2.3) are equivalent (see the argument below). Our main interest is to study the behavior of solutions of this problem with physical equations of state and various physical viscosity coefficients λ, μ, magnetic diffusivity ν, and heat conductivity κ, which generally depend on the density ρ and the temperature θ. For concreteness, in this paper we focus on the physical case for polytropic gases such that λ, μ, and ν are constants, and κ depends on the temperature θ with $C_1 \leq \kappa(\theta)/(1+\theta^r) \leq C_2$ for some positive constants C_1, C_2, and $r \geq 2$. The growth condition assumed on κ is motivated by the physical fact: $\kappa \propto \theta^{5/2}$ for important physical regimes (see [5, 22]). For problem (2.2)-(2.3), we will see that, if the initial data is in H^1 (or a Hölder space), then the solution will be at least in H^1 (or the Hölder space), and neither shock waves nor vacuum and concentration are developed in a finite time. We also study the continuous dependence of global solutions and prove that the solutions depend continuously on the initial data. Let $(v_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j), j = 1, 2,$ be two solutions to problem (2.2)-(2.3) with the corresponding initial data. Define a functional

$$D(t) = \|(v_1 - v_2, u_1 - u_2, \mathbf{w}_1 - \mathbf{w}_2, \mathbf{b}_1 - \mathbf{b}_2, \theta_1 - \theta_2, v_{1y} - v_{2y})(\cdot, t)\|^2_{L^2(\Omega)}$$
$$+ \|(u_1 - u_2, \mathbf{w}_1 - \mathbf{w}_2, \mathbf{b}_1 - \mathbf{b}_2, \theta_1 - \theta_2)_y\|^2_{L^2(\Omega\times(0,t))}.$$

Then we have

Theorem 2.1. *Suppose that there are some positive constants C_0 and M_0 such that*

$$C_0^{-1} \le v_0 \le C_0, \quad C_0^{-1} \le \theta_0 \le M_0,$$
$$\|(u_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^4} + \|\theta_0\|_{L^2} \le C_0, \quad \|(v_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)\|_{H^1} \le M_0.$$

Then, for some constants $C = C(C_0, T) > 0$, independent of M_0, and $M = M(C_0, M_0, T) > 0$,
(1). *Problem* (2.2)-(2.3) *has a global solution $(v, u, \mathbf{w}, \mathbf{b}, \theta) \in L^\infty(0, T; H^1(\Omega))$ for any fixed $T > 0$ such that, for each $(y, t) \in \Omega \times [0, T]$,*

$$C^{-1} \le v(y,t) \le C, \quad C^{-1} \le \theta(y,t) \le M, \quad |(u, \mathbf{w}, \mathbf{b})(y,t)| \le M,$$
$$\|(u, \mathbf{w}, \mathbf{b})\|_{L^2(0,T;L^4\cap H^1)} + \|\theta\|_{L^2(0,T;L^2\cap H^1)} \le C,$$
$$\|(v, u, \mathbf{w}, \mathbf{b}, \theta)_y(\cdot, t)\|^2_{L^2(\Omega)}$$
$$+ \|(v_y, u_y, \mathbf{w}_y, \mathbf{b}_y, \theta_y, v_t, u_t, \mathbf{w}_t, \mathbf{b}_t, v_{yt}, u_{yy}, \mathbf{w}_{yy}, \mathbf{b}_{yy}, \theta_{yy})\|^2_{L^2(\Omega\times(0,t))} \le M; \tag{2.5}$$

(2). *Furthermore, if $v_0 \in W^{1,\infty}(\Omega)$, $(v_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j), j = 1, 2,$ are two solutions to* (2.2)-(2.3) *in* (1), *then*

$$D(t) \le MD(0), \quad 0 \le t \le T. \tag{2.6}$$

That is, the solutions depend continuously on the initial data.

We also remark that, if $v_0 \in C^{1+\alpha}(\Omega)$ and $(u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in C^{2+\alpha}(\Omega)$ for some $\alpha \in (0,1)$, we can conclude, using the standard method (cf. [9, 16, 21]) and the Schauder estimates (cf. [14, 8]), that there exists a unique classical solution $v \in C^{1+\alpha, 1+\alpha/2}(\Omega \times [0,T])$ and $(u, \mathbf{w}, \mathbf{b}, \theta) \in C^{2+\alpha, 1+\alpha/2}(\Omega \times [0,T])$ for any fixed $T > 0$ to (2.2)-(2.3) satisfying (2.5).

The existence of local solutions is known from the standard method based on the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval (cf. [20]). The global existence of solutions will be proved by the method of extending the local solutions with respect to time based on a-priori global estimates. We will establish these a-priori estimates and prove the continuous dependence of the solutions on the initial data in Sections 3 and 4.

The results in Theorem 2.1 in Lagrangian coordinates can easily be converted to equivalent statements for the corresponding results in Euler coordinates. In particular, we have

Theorem 2.2. *Suppose that there exist some positive constants* C_0 *and* M_0 *such that*

$$C_0^{-1} \le \rho_0 \le C_0, \quad C_0^{-1} \le \theta_0 \le M_0,$$
$$\|(u_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^4} + \|\theta_0\|_{L^2} \le C_0, \quad \|(\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)\|_{H^1} \le M_0.$$

Then, for some constants $C = C(C_0, T) > 0$ *and* $M = M(C_0, M_0, T) > 0$ *,*
(1). *Problem* (1.1)-(1.2) *has a global solution* $(\rho, u, \mathbf{w}, \mathbf{b}, \theta) \in L^\infty(0, T; H^1(\Omega))$ *for any fixed* $T > 0$ *such that, for each* $(x,t) \in \Omega \times [0,T]$ *,*

$$\begin{aligned}
&C^{-1} \le \rho(x,t) \le C, \quad C^{-1} \le \theta(x,t) \le M, \quad |(u, \mathbf{w}, \mathbf{b})(x,t)| \le M,\\
&\|(u, \mathbf{w}, \mathbf{b})\|_{L^2(0,T;L^4 \cap H^1)} + \|\theta\|_{L^2(0,T;L^2 \cap H^1)} \le C,\\
&\|(\rho, u, \mathbf{w}, \mathbf{b}, \theta)_x(\cdot, t)\|^2_{L^2(\Omega)}\\
&+ \|(\rho_x, u_x, \mathbf{w}_x, \mathbf{b}_x, \theta_x, \rho_t, u_t, \mathbf{w}_t, \mathbf{b}_t, \rho_{xt}, u_{xx}, \mathbf{w}_{xx}, \mathbf{b}_{xx}, \theta_{xx})\|^2_{L^2(\Omega \times (0,t))} \le M;
\end{aligned} \tag{2.7}$$

(2). *Furthermore, if* $\rho_0 \in W^{1,\infty}(\Omega)$ *, then the solution of* (1.1)-(1.2) *is unique.*

If the initial data is in a Hölder space, then the corresponding solution is also in the Hölder space. The corresponding statement concerning the continuous dependence is more subtle, however, owing to the fact that the change of variables from Lagrangian to Euler coordinates is solution-dependent (see [4]). The uniqueness of solutions in Euler coordinates can be shown as follows.

Assume that $(\rho_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j)(x,t)$, $j = 1, 2,$ are two solutions of (1.1) with the same initial-boundary data (1.2)-(1.3). Then, through the Lagrangian variable

$$y_j = y_j(x,t) = \int_0^x \rho_j(\xi, t) d\xi, \tag{2.8}$$

which is a one-to-one correspondence between (x,t) and (y_j, t) , both solutions $(\rho_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j)(x(y_j t), t), j = 1, 2,$ satisfy (2.2)-(2.3) with $(y,t) = (y_j, t)$. Since the solution of the problem (2.2)-(2.3) is unique, we conclude that there exists $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)(y,t) \in L^2(0,T; H^1)$ such that

$$(\rho_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j)(x(y_j,t),t) = (\rho, u, \mathbf{w}, \mathbf{b}, \theta)(y_j, t). \tag{2.9}$$

Notice that $y_j(x,t), j = 1, 2,$ are Lipschitz and satisfy

$$y_{jt} + u_j(x(y_j,t),t) y_{jx} = 0, \quad y_1(x,0) = y_2(x,0) = \int_0^x \rho_0(\xi, 0) d\xi. \tag{2.10}$$

That is, by (2.9),

$$y_{jt} + u(y_j,t) y_{jx} = 0, \quad y_1(x,0) = y_2(x,0) = \int_0^x \rho_0(\xi, 0) d\xi, \tag{2.11}$$

which implies $y_1(x,t) = y_2(x,t)$ (see [4]). Therefore, from (2.8) and (2.9), we conclude

$$(\rho_1, u_1, \mathbf{w}_1, \mathbf{b}_1, \theta_1)(x,t) = (\rho_2, u_2, \mathbf{w}_2, \mathbf{b}_2, \theta_2)(x,t),$$

that is, the solution is unique.

3. Proof of Existence

In this section, we prove part (i) of Theorem 2.1. In order to establish the global existence, we need a-priori estimates of the solutions for $(y,t) \in \Omega \times [0,T]$ for any fixed $T>0$. We denote $C>0$ the generic constant depending only upon T and C_0, and denote M the generic constant depending upon M_0 besides upon T and C_0. Without loss of generality, we take $c_v = R = 1$. First, from equations (2.2) and the initial-boundary conditions (2.3), we have the following energy estimates.

Lemma 3.1.

$$\int_0^1 \left((v-1-\ln v)+(\theta-1-\ln\theta)+u^2+|\mathbf{w}|^2+v|\mathbf{b}|^2\right)dy \le C, \tag{3.1}$$

$$\int_0^t\int_0^1 \left(\frac{\kappa\theta_y^2}{v\theta^2}+\frac{\lambda u_y^2}{v\theta}+\frac{\mu|\mathbf{w}_y|^2}{v\theta}+\frac{\nu|\mathbf{b}_y|^2}{v\theta}\right)dyd\tau \le C. \tag{3.2}$$

Proof. Integrating the energy equation (2.2e) and using the boundary condition (2.3), we have

$$\begin{aligned}&\int_0^1 \left(\theta+\frac{1}{2}(u^2+|\mathbf{w}|^2+v|\mathbf{b}|^2)\right)dy\\ &=\int_0^1 \left(\theta_0+\frac{1}{2}(u_0^2+|\mathbf{w}_0|^2+v|\mathbf{b}_0|^2)\right)dy \le C.\end{aligned}$$

Set $\eta=(v-1-\ln v)+(\theta-1-\ln\theta)+\frac{1}{2}(u^2+|\mathbf{w}|^2+v|\mathbf{b}|^2)$. From (2.2), we obtain

$$\begin{aligned}\eta_t=&-\left(\frac{\kappa\theta_y^2}{v\theta^2}+\frac{\lambda u_y^2}{v\theta}+\frac{\mu|\mathbf{w}_y|^2}{v\theta}+\frac{\nu|\mathbf{b}_y|^2}{v\theta}\right)+u_y-\left(\frac{\kappa\theta_y}{\theta v}\right)_y\\ &+\left(\frac{\lambda u u_y}{v}+\frac{\mu\mathbf{w}\cdot\mathbf{w}_y}{v}+\frac{\nu\mathbf{b}\cdot\mathbf{b}_y}{v}+\frac{\kappa\theta_y}{v}-u\left(p+\frac{1}{2}|\mathbf{b}|^2\right)+\mathbf{w}\cdot\mathbf{b}\right)_y.\end{aligned}$$

Integrating the above equation over $[0,1]\times[0,t]$ yields Lemma 3.1. □

We now derive a representation of the specific volume v and prove the lower and upper bounds of the density, which is essential to establish the existence.

Lemma 3.2.

$$C^{-1} \le v(y,t) \le C. \tag{3.3}$$

Proof. The proof is divided in three steps.
1. Set

$$h(y,t)=\int_0^t \left(\frac{\lambda u_y}{v}-p-\frac{1}{2}|\mathbf{b}|^2\right)(y,\tau)d\tau+\int_0^y u_0(\xi)d\xi.$$

Then (2.2b) implies $u = h_y$. For a function $y(t) \in [0,1]$ of t, which will be determined later, we integrate in y from $y(t)$ to y on $h_y = u$ to have

$$h(y,t) = h(y(t),t) + \int_{y(t)}^{y} u(\xi,t)d\xi.$$

On the other hand, since $u_y = v_t$, then

$$h(y,t) = \lambda \ln v - \lambda \ln v_0 - \int_0^t \left(p + \frac{1}{2}|\mathbf{b}|^2\right)(y,\tau)d\tau + \int_0^y u_0(\xi)d\xi,$$

from the definition of h. Therefore,

$$v^{-1}\exp\left(\frac{1}{\lambda}\int_0^t p d\tau\right) = \exp\left(\frac{1}{\lambda}A(y,t) - \frac{1}{\lambda}\int_0^t \frac{1}{2}|\mathbf{b}|^2 d\tau\right),$$

where

$$A(y,t) = \int_0^y u_0(\xi)d\xi - h(y(t),t) - \int_{y(t)}^{y} u(\xi,t)d\xi - \lambda \ln v_0.$$

Multiplying by θ/λ and integrating over $[0,t]$, we have

$$\exp\left(\frac{1}{\lambda}\int_0^t p d\tau\right) = 1 + \int_0^t \frac{1}{\lambda}\theta(y,s)\exp\left(\frac{1}{\lambda}A(y,s) - \frac{1}{\lambda}\int_0^s \frac{1}{2}|\mathbf{b}|^2 d\tau\right)ds.$$

Hence, we conclude

$$\begin{aligned} v = &\exp\left(\frac{1}{\lambda}\int_0^t \frac{1}{2}|\mathbf{b}|^2 d\tau - \frac{1}{\lambda}A(y,t)\right) \times \\ &\left(1 + \int_0^t \frac{1}{\lambda}\theta(y,s)\exp\left(\frac{1}{\lambda}A(y,s) - \frac{1}{\lambda}\int_0^s \frac{1}{2}|\mathbf{b}|^2 d\tau\right)ds\right). \end{aligned}$$

2. We now determine $y(t)$. From the definition of h, we have

$$h_t = \frac{\lambda u_y}{v} - p - \frac{1}{2}|\mathbf{b}|^2.$$

Then, using $v_t = u_y$ and $h_y = u$, we obtain

$$(vh)_t - (\lambda u + uh)_y = -\left(u^2 + vp + \frac{1}{2}v|\mathbf{b}|^2\right).$$

Integrating the above equation over $[0,1] \times [0,t]$ yields

$$\begin{aligned} &\int_0^1 vh(y,t)dy \\ &= \int_0^1 v_0(y)\int_0^y u_0(\xi)d\xi dy - \int_0^t\int_0^1 \left(u^2 + vp + \frac{1}{2}v|\mathbf{b}|^2\right)(y,\tau)dyd\tau. \end{aligned}$$

Without loss of generality, we assume here $\int_0^1 v(y,t)dy = \int_0^1 v_0(y)dy = 1$. Then

$$\int_0^1 vh(y,t)dy = h(y(t),t)$$

for some $y(t) \in [0,1]$. Therefore,

$$\begin{aligned} v(y,t) =& P(y,t)\exp\left(\int_0^t Q(y,\tau)d\tau\right) \\ &+ \frac{1}{\lambda}\int_0^t \theta(y,s)P(y,t)P(y,s)^{-1}\exp\left(\int_s^t Q(y,\tau)d\tau\right)ds, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} P(y,t) &= \exp\left\{\frac{1}{\lambda}\int_0^1 v_0(y)\int_0^y u_0(\xi)d\xi dy \right. \\ &\qquad \left. +\frac{1}{\lambda}\int_{y(t)}^y u(\xi,t)d\xi - \frac{1}{\lambda}\int_0^y u_0(\xi)d\xi + \ln v_0\right\}, \\ Q(y,t) &= \frac{1}{2\lambda}|\mathbf{b}|^2 - \frac{1}{\lambda}\int_0^1\left(u^2 + vp + \frac{1}{2}v|\mathbf{b}|^2\right)dy. \end{aligned}$$

It is easy to see that, $C^{-1} \le P(y,t) \le C$, since $|\int_{y(t)}^y u(\xi,t)d\xi| \le \|u\|_{L^2(\Omega)} \le C$. Estimate (3.1) implies that $Q(y,t) \ge -C/\lambda$. Then the specific volume v has a positive lower bound since

$$v(y,t) \ge P(y,t)\exp\left(\int_0^t Q(y,\tau)d\tau\right) \ge C, \quad t \in [0,T].$$

3. From the convexity of $\theta - 1 - \ln\theta$ and Jensen's inequality, one has $c_1 \le \int_0^1 \theta(y,t)dy \le c_2$ for some positive constants c_1 and c_2, and

$$c_1 \le \theta(a(t),t) = \int_0^1 \theta(y,t)dy \le c_2,$$

for some $a(t) \in [0,1]$. Thus

$$c_1 \le \int_0^1 (u^2 + vp)dy \le C.$$

Integrating (2.4) over $[0,1]\times[0,t]$ and using Cauchy-Schwartz's inequality yield

$$\begin{aligned} &\int_0^t\int_0^1 \left(\frac{\lambda u_y^2}{v} + \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{b}_y|^2}{v}\right)dyd\tau \\ &= \int_0^1 \theta dy - \int_0^1 \theta_0(y)dy + \int_0^t\int_0^1 \frac{\theta u_y}{v}dyd\tau \\ &\le \int_0^1 \theta dy - \int_0^1 \theta_0(y)dy + \frac{1}{2}\int_0^t\int_0^1 \frac{\lambda u_y^2}{v}dyd\tau + C\int_0^t\int_0^1 \theta^2 dyd\tau. \end{aligned}$$

Then, by (3.2) and $\theta(y,t)=\theta(a(t),t)+\int_{a(t)}^{y}\theta_y d\xi$, we have

$$\begin{aligned}&\int_0^t\int_0^1\left(\frac{\lambda u_y^2}{v}+\frac{\mu|\mathbf{w}_y|^2}{v}+\frac{\nu|\mathbf{b}_y|^2}{v}\right)dyd\tau\\&\quad\leq C+C\int_0^t\int_0^1\frac{\kappa\theta_y^2}{v\theta^2}dy\int_0^1 vdyd\tau\leq C.\end{aligned}\tag{3.5}$$

From the above estimates, we derive that, for $s\leq t$,

$$\begin{aligned}\int_s^t Q(y,\tau)d\tau&=\frac{1}{\lambda}\int_s^t\left(\int_0^y\mathbf{b}\cdot\mathbf{b}_y d\xi-\int_0^1\left(u^2+vp+\frac{1}{2}v|\mathbf{b}|^2\right)dy\right)d\tau\\&\leq\frac{1}{\lambda}\int_s^t\int_0^1\left(\frac{|\mathbf{b}_y|^2}{2v}-(u^2+vp)\right)dyd\tau\leq C-\frac{c_1}{\lambda}(t-s).\end{aligned}\tag{3.6}$$

Since $\int_0^1\theta dy\leq C$, we have the following estimate:

$$\theta(y,t)^{1/2}=\theta(a(t),t)^{1/2}+\int_{a(t)}^y\frac{\theta_y}{2\theta^{1/2}}(\xi,t)d\xi\leq C+C\max_{y\in[0,1]}v\int_0^1\frac{\kappa\theta_y^2}{v\theta^2}dy.$$

From (3.4) and the above estimates,

$$\begin{aligned}\max_{y\in[0,1]}v&\leq C\,e^{-c_1t/\lambda}+C\int_0^t\theta\,e^{-c_1(t-s)/\lambda}ds\\&\leq C+C\int_0^t\max_{y\in[0,1]}v\int_0^1\frac{\kappa\theta_y^2}{v\theta^2}dy\,e^{-c_1(t-s)/\lambda}ds,\end{aligned}$$

and by Gronwall's inequality and (3.2), we conclude

$$\max_{y\in[0,1]}v\leq C\exp\left(C\int_0^t\int_0^1\frac{\kappa\theta_y^2}{v\theta^2}dy\,e^{-c_1(t-s)/\lambda}ds\right)\leq C.$$

That is, $v(y,t)\leq C$. This completes the proof of Lemma 3.2. □

Lemma 3.3.

$$\theta(y,t)\geq C^{-1}.\tag{3.7}$$

Proof. Set $\omega=1/\theta$. Multiplying (2.4) by $-\omega^2$, we obtain the following equation:

$$\omega_t=\left(\frac{\kappa\omega_y}{v}\right)_y+\frac{1}{4\lambda v}-\left(\frac{2\kappa\omega_y^2}{\omega v}+\frac{\omega^2}{v}(\mu|\mathbf{w}_y|^2+\nu|\mathbf{b}_y|^2)+\frac{\lambda\omega^2}{v}\left(u_y-\frac{1}{2\lambda\omega}\right)^2\right).$$

Multiplying the above equation by $2l\omega^{2l-1}$ with $l>0$, integrating in y over Ω , and using the boundary condition and Young's inequality, we have

$$\frac{d}{dt}\|\omega\|_{L^{2l}(\Omega)}^{2l}\leq\frac{l}{2\lambda}\int_0^1\frac{\omega^{2l-1}}{v}dy\leq\frac{l}{2\lambda}\|\omega\|_{L^{2l}(\Omega)}^{2l-1}\|v^{-1}\|_{L^{2l}(\Omega)},$$

and then $\frac{d}{dt}\|\omega\|_{L^{2l}(\Omega)} \le C$. Therefore, $\|\omega\|_{L^{2l}(\Omega)} \le C$. Taking $l \to \infty$, we obtain the lower bound of θ . □

Lemma 3.4.

$$\int_0^1 v_y^2(y,t)dy + \int_0^t\int_0^1 \left(\theta v_y^2 + u_y^2 + v_y^2\right)(y,s)dyds \le M. \tag{3.8}$$

Proof. In (2.2b), notice $(u_y/v)_y = (\ln v)_{ty} = (v_y/v)_t$. Then

$$\left(\frac{\lambda v_y}{v}\right)_t + \frac{\theta v_y}{v^2} = u_t + \frac{\theta_y}{v} + \mathbf{b}\cdot\mathbf{b}_y. \tag{3.9}$$

Multiplying the equation (2.2e) by K_1E and (2.2b) by K_2u^3 , taking the inner product of (2.2c) with $K_3|\mathbf{w}|^2\mathbf{w}$ and (2.2d) with $K_4|v\mathbf{b}|^2v\mathbf{b}$, respectively, for proper positive constants $K_j, 1 \le j \le 4$, integrating them over $[0,1]\times[0,t]$, adding all of them together, and using the Gronwall's inequality, we eventually obtain $\int_0^t\int_0^1 |\mathbf{b}\cdot\mathbf{b}_y|^2 dyds \le M$ from tedious calculations. Multiplying the equation (3.9) by v_y/v , integrating over $[0,1]\times[0,t]$, and using Cauchy-Schwartz's inequality and the following observation:

$$\frac{v_y}{v}u_t = (u(\ln v)_y)_t - u(\ln v)_{yt} = \left(u\frac{v_y}{v}\right)_t - \left(u\frac{v_t}{v}\right)_y + \frac{u_y^2}{v},$$

we obtain, by (3.7),

$$\begin{aligned}&\int_0^1 \frac{\lambda}{2}\left(\frac{v_y}{v}\right)^2 dy + \int_0^t\int_0^1 \frac{\theta}{v}\left(\frac{v_y}{v}\right)^2 dyds\\ &\le M + \frac{\lambda}{4}\int_0^1\left(\frac{v_y}{v}\right)^2 dy + \int_0^t\int_0^1 \frac{\lambda u_y^2}{v}dyds + \frac{1}{2}\int_0^t\int_0^1 \frac{\theta}{v}\left(\frac{v_y}{v}\right)^2 dyds.\end{aligned}$$

Therefore,

$$\int_0^1 v_y^2 dy + \int_0^t\int_0^1 \frac{\theta}{v}\left(\frac{v_y}{v}\right)^2 dyds \le M + \int_0^t\int_0^1 \frac{\lambda u_y^2}{v}dyds. \tag{3.10}$$

Although we can conclude the proof by using (3.5), we proceed the proof in the following way so that the bounds in the estimates do not have extra dependence on T . Multiplying (2.2b) by u , we get

$$\left(\frac{u^2}{2}\right)_t + \frac{\lambda u_y^2}{v} = \left(\frac{\lambda u u_y}{v}\right)_y - u\left(\frac{\theta_y}{v} - \frac{\theta v_y}{v^2} + \mathbf{b}\cdot\mathbf{b}_y\right).$$

Integrating it over $[0,1]\times[0,t]$, we obtain

$$\begin{aligned}
&\int_0^1 \frac{u^2}{2}dy + \int_0^t\int_0^1 \frac{\lambda u_y^2}{v}dyds\\
&\le \int_0^1 \frac{u_0^2}{2}dy + \int_0^t\int_0^1 \left(\frac{|u\theta_y|}{v} + \frac{|u(\theta - \int_0^1 \theta dy)v_y|}{v^2} + \frac{|uv_y|\int_0^1 \theta dy}{v^2}\right) dyds\\
&\qquad + \int_0^t\int_0^1 |u||\mathbf{b}\cdot\mathbf{b}_y|dyds\\
&\le M + \frac{1}{8}\int_0^t\int_0^1 \frac{\theta}{v}\left(\frac{v_y}{v}\right)^2 dyds + M\int_0^t \left(\int_0^1 \frac{u_y^2}{\theta}dy\right)\left(\int_0^1 v_y^2 dy\right) ds,
\end{aligned} \tag{3.11}$$

where we used the following estimates:

$$\int_0^t\int_0^1 u^2 dyds \le \int_0^t\int_0^1 \frac{u_y^2}{v\theta}dy\int_0^1 v\theta dy\, ds \le M\int_0^t\int_0^1 \frac{u_y^2}{v\theta}dyds \le M,$$

$$\int_0^t\int_0^1 \left|\theta - \int_0^1 \theta dy\right| dyds \le \int_0^t \left(\int_0^1 \frac{\theta_y^2}{v}dy\right)\left(\int_0^1 v dy\right) ds \le M.$$

Adding (3.10) and $2\times$ (3.11), we get

$$\begin{aligned}
&\int_0^1 \left(u^2 + v_y^2\right) dy + \int_0^t\int_0^1 \left(u_y^2 + \theta v_y^2\right) dyds\\
&\quad \le M + M\int_0^t \left(\int_0^1 u_y^2 dy\right)\left(\int_0^1 v_y^2 dy\right) ds.
\end{aligned}$$

Then, from Gronwall's inequality and (3.2), we have

$$\int_0^1 \left(u^2 + v_y^2\right) dy + \int_0^t\int_0^1 \left(u_y^2 + \theta v_y^2\right) dyds \le M\exp\left(M\int_0^t\int_0^1 \frac{u_y^2}{\theta}dyds\right) \le M. \tag{3.12}$$

Since $c_1 \le \int_0^1 \theta(y,t)dy \le c_2$ and $\int_0^1 v_y^2 dy \le M$ from the above estimate,

$$\begin{aligned}
c_1\int_0^1 v_y^2 dy &\le \int_0^1 \theta(y,t)dy\int_0^1 v_y^2 dy\\
&\le \int_0^1 \left(\int_0^1 \theta(y,t)dy - \theta\right) v_y^2 dy + \int_0^1 \theta v_y^2 dy\\
&\le \frac{c_1}{2}\int_0^1 v_y^2 dy + M\int_0^1 \theta_y^2 dy\int_0^1 v_y^2 dy + \int_0^1 \theta v_y^2 dy.
\end{aligned}$$

Therefore, $\int_0^t\int_0^1 v_y^2 dyds \le M$ from (3.7) and (3.12). □

Lemma 3.5.

$$\int_0^1 u_y^2(y,t)dy + \int_0^t\int_0^1 u_{yy}^2(y,s)dyds \le M. \tag{3.13}$$

Proof. First we have the following interpolation inequalities: for $\delta > 0$,

$$u_y^2 \le \delta \int_0^1 u_{yy}^2 dy + (1+\delta^{-1})\int_0^1 u_y^2 dy, \quad \theta^2 \le 2\int_0^1 \theta_y^2 dy + 2\left(\int_0^1 \theta dy\right)^2.$$

Then, multiplying (2.2b) by u_{yy} and integrating it over $[0,1]\times[0,t]$, we have

$$\begin{aligned}
&\int_0^1 \frac{u_y^2}{2}dy + \int_0^t\int_0^1 \frac{\lambda u_{yy}^2}{v}dyds \\
&\quad \le \int_0^1 \frac{u_{0y}^2}{2}dy + \int_0^t\int_0^1 \left(\frac{\theta_y}{v} - \frac{\theta v_y}{v^2} + \mathbf{b}\cdot\mathbf{b}_y - \frac{\lambda u_y v_y}{v^2}\right) u_{yy}dyds \\
&\quad \le M + \frac{1}{2}\int_0^t\int_0^1 \frac{\lambda u_{yy}^2}{v}dyds + M\int_0^t\int_0^1 \left(u_y^2+\theta_y^2+v_y^2\right)dyds \\
&\quad \le M + \frac{1}{2}\int_0^t\int_0^1 \frac{\lambda u_{yy}^2}{v}dyds.
\end{aligned}$$

The lemma follows. □

Lemma 3.6.

$$|(u,\mathbf{w},\mathbf{b},\theta)(y,t)| + \int_0^1 |(\mathbf{w},\mathbf{b},\theta)_y|^2 dy + \int_0^t\int_0^1 |(\mathbf{w},\mathbf{b},\mathbf{w}_y,\mathbf{b}_y)_y|^2 dyds \le M. \tag{3.14}$$

Proof. Take the inner product of (2.2c) with $\mathbf{w}$, integrate it over $[0,1]\times[0,t]$, and use Cauchy-Schwartz's inequality to obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 |\mathbf{w}|^2 dy + \int_0^t\int_0^1 \frac{\mu|\mathbf{w}_y|^2}{v}dyds \\
&\quad \le \frac{1}{2}\int_0^1 |\mathbf{w}_0|^2 dy + \int_0^t\int_0^1 \left(\int_0^y |\mathbf{b}_y|d\xi\right)^2 dyds \\
&\quad \le M + \int_0^t\int_0^1 \frac{|\mathbf{b}_y|^2}{\theta}dy\int_0^1 \theta dyds \le M.
\end{aligned}$$

Similarly, we have by taking the inner product of (2.2d) with $v\mathbf{b}$ and integrating it over $[0,1]\times[0,t]$:

$$\begin{aligned}
&\frac{1}{2}\int_0^1 v^2|\mathbf{b}|^2dy+\int_0^t\int_0^1 \nu|\mathbf{b}_y|^2dyds\\
&\le \frac{1}{2}\int_0^1 v_0^2|\mathbf{b}_0|^2dy+\int_0^t\int_0^1\Big(v\mathbf{b}\cdot\mathbf{w}_y-\frac{\nu v_y}{v}\mathbf{b}\cdot\mathbf{b}_y\Big)\,dyds\\
&\le M+M\int_0^t\int_0^1\frac{|\mathbf{b}_y|^2}{\theta}dy\int_0^1\theta dyds+M\int_0^t\int_0^1\big(|\mathbf{w}_y|^2+v_y^2+|\mathbf{b}\cdot\mathbf{b}_y|^2\big)\,dyds\\
&\le M.
\end{aligned}$$

Multiplying (2.2c) by $\mathbf{w}_{yy}$, integrating it, and using the interpolation inequality:

$$|\mathbf{w}_y|^2\le(1+\delta^{-1})\int_0^1|\mathbf{w}_y|^2dy+\delta\int_0^1|\mathbf{w}_{yy}|^2dy,\quad\text{for any}\quad\delta>0,$$

one has

$$\begin{aligned}
&\frac{1}{2}\int_0^1|\mathbf{w}_y|^2dy\\
&\le M-M_1\int_0^t\int_0^1|\mathbf{w}_{yy}|^2dyds+M\int_0^t\int_0^1\left(|\mathbf{b}_y|+|v_y||\mathbf{w}_y|\right)|\mathbf{w}_{yy}|dyds\\
&\le M-\frac{3M_1}{4}\int_0^t\int_0^1|\mathbf{w}_{yy}|^2dyds+M\int_0^t\int_0^1|\mathbf{b}_y|^2dyds\\
&\qquad+M\int_0^t\max_{y\in\Omega}|\mathbf{w}_y|^2\int_0^1v_y^2dyds\\
&\le M-\frac{M_1}{2}\int_0^t\int_0^1|\mathbf{w}_{yy}|^2dyds,
\end{aligned}$$

and then

$$\int_0^1|\mathbf{w}_y|^2dy+\int_0^t\int_0^1|\mathbf{w}_{yy}|^2dyds\le M.$$

Using $v_t=u_y$, we rewrite (2.2d) as follows:

$$\mathbf{b}_t=-\frac{u_y}{v}\mathbf{b}+\frac{1}{v}\left(\mathbf{w}+\frac{\nu\mathbf{b}_y}{v}\right)_y.\tag{3.15}$$

Multiplying the above equation by $\mathbf{b}_{yy}$, integrating it, and using the similar interpolation inequalities, one has

$$\begin{aligned}\frac{1}{2}\int_0^1 |\mathbf{b}_y|^2dy \le M - M_2\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2dyds\\ + M\int_0^t\int_0^1 (|u_y||\mathbf{b}| + |\mathbf{w}_y| + |v_y||\mathbf{b}_y|)\,|\mathbf{b}_{yy}|dyds\\ \le M - \frac{3M_2}{4}\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2dyds + M\int_0^t\int_0^1 |\mathbf{w}_y|^2dyds\\ + M\max_{y,t}|\mathbf{b}|^2\int_0^t\int_0^1 u_y^2dyds + M\int_0^t \max_{y\in\Omega}|\mathbf{b}_y|^2ds\\ \le M - \frac{M_2}{2}\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2dyds + \frac{1}{4}\int_0^1 |\mathbf{b}_y|^2dy,\end{aligned}$$

and then

$$\int_0^1 |\mathbf{b}_y|^2dy + \int_0^t\int_0^1 |\mathbf{b}_{yy}|^2dyds \le M.$$

Multiplying (2.4) by $v^{-1}\int_0^\theta \kappa(\xi)d\xi$, one has the estimate $\int_0^1 \theta_y^2dy \le M$ by further similar arguments. Therefore, we have

$$\begin{aligned}\theta(y,t) = \theta(a(t),t) + \int_{a(t)}^y \theta_y d\xi \le M + \left(\int_0^1 \theta_y^2dy\right)^{1/2} \le M,\\ u^2 = \int_0^y 2uu_y d\xi \le 2\left(\int_0^1 u^2dy\right)^{1/2}\left(\int_0^1 u_y^2dy\right)^{1/2} \le M.\end{aligned}$$

Similarly, we conclude $|\mathbf{w}|^2 + |\mathbf{b}|^2 \le M$. This completes the proof of part (i) of Theorem 2.1. □

4. Continuous Dependence on the Initial Data

In this section we prove part (ii) of Theorem 2.1, the continuous dependence of global solutions on the initial data. First, with the global a-priori estimates in Section 3, we have

Lemma 4.1. *If* $v_0 \in W^{1,\infty}(\Omega)$, *then, for any solution* $(v, u, \mathbf{w}, \mathbf{b}, \theta)$ *to* (2.2)-(2.3) *in part (i) of Theorem 2.1,*

$$|v_y(y,t)| \le M. \tag{4.1}$$

Proof. For the convenience of differentiation, we rewrite (3.4) as follows:

$$v(y,t) = \exp\left(\ln P(y,t) + \int_0^t Q(y,\tau)d\tau\right) + \frac{1}{\lambda}\int_0^t \theta(y,s)\exp\left(\ln P(y,t) - \ln P(y,s) + \int_s^t Q(y,\tau)d\tau\right)ds.$$

Then, taking the derivative with respect to y,

$$\begin{aligned} &v_y(y,t) \\ &= P(y,t)\exp\left(\int_0^t Q(y,\tau)d\tau\right)\left((\ln P(y,t))_y + \int_0^t Q_y(y,\tau)d\tau\right) \\ &+ \frac{1}{\lambda}\int_0^t P(y,t)P(y,s)^{-1}\exp\left(\int_s^t Q(y,\tau)d\tau\right)\theta_y(y,s)ds \\ &+ \frac{1}{\lambda}\int_0^t P(y,t)P(y,s)^{-1}\exp\left(\int_s^t Q(y,\tau)d\tau\right)\times \\ &\qquad \theta(y,s)(\ln P(y,t) - \ln P(y,s))_y ds \\ &+ \frac{1}{\lambda}\int_0^t P(y,t)P(y,s)^{-1}\exp\left(\int_s^t Q(y,\tau)d\tau\right)\theta(y,s)\int_s^t Q_y(y,\tau)d\tau ds, \end{aligned}$$

where

$$(\ln P(y,t))_y = \frac{1}{\lambda}u(y,t) - \frac{1}{\lambda}u_0(y) + \frac{v_{0y}(y)}{v_0(y)}, \quad Q_y(y,t) = \frac{1}{\lambda}\mathbf{b}\cdot\mathbf{b}_y.$$

From the estimates in Section 3, we conclude

$$C^{-1} \le P(y,t),\ \theta(y,t) \le M, \quad |\partial_y \ln P(y,t)| \le M.$$

From (3.6), $\int_s^t Q(y,\tau)d\tau \le M$. Using the inequality

$$|\mathbf{b}_y|^2 \le 2\int_0^1 |\mathbf{b}_y|^2 dy + \int_0^1 |\mathbf{b}_{yy}|^2 dy,$$

we have

$$\left|\int_s^t Q_y(y,\tau)d\tau\right| \le \frac{3}{2\lambda}\int_0^t\int_0^1 |\mathbf{b}_y|^2 dy d\tau + \frac{1}{2\lambda}\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2 dy d\tau \le M.$$

Again, using a similar interpolation inequality, we have

$$\begin{aligned} &\int_0^t P(y,t)P(y,s)^{-1}\exp\left(\int_s^t Q(y,\tau)d\tau\right)|\theta_y(y,s)|ds \\ &\le \frac{1}{2}\int_0^t P(y,t)^2P(y,s)^{-2}\exp\left(2\int_s^t Q(y,\tau)d\tau\right)ds + \int_0^t\int_0^1 (\theta_y^2 + \frac{1}{2}\theta_{yy}^2)dyds \\ &\le M. \end{aligned}$$

Therefore, $|v_y| \le M$. This completes the proof of Lemma 4.1. □

We now start the proof of the continuous dependence of global solutions on the initial data by establishing the estimates in several lemmas. Set

$$G(t) = 2\sum_{j=1}^{2}\int_0^1 (|\partial_y(u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j)|^2 + |\partial_{yy}(u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j)|^2)dy.$$

Then, from (2.5), we have

$$\int_0^t G(s)ds \le M. \tag{4.2}$$

Using the interpolation inequality, we have

$$u_{jy}^2 \le 2\int_0^1 u_{jy}^2 dy + \int_0^1 u_{jyy}^2 dy \le G(t).$$

Similarly, for $j = 1, 2$, $\mathbf{w}_{jy}^2 + \mathbf{b}_{jy}^2 + \theta_{jy}^2 \le G(t)$. Applying (2.2) to both solutions $(v_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j)$, $j = 1, 2,$ we have the following lemmas.

Lemma 4.2.

$$\begin{aligned}
&\int_0^1 |(v_1 - v_2, u_1 - u_2, w_1 - w_2)|^2 dy - \int_0^1 |(v_{10} - v_{20}, u_{10} - u_{20}, w_{10} - w_{20})|^2 dy \\
&\le -\frac{3}{2}\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y} - u_{2y})^2 dyds - \frac{7}{4}\int_0^t\int_0^1 \frac{\mu}{v_1}|\mathbf{w}_{1y} - \mathbf{w}_{2y}|^2 dyds \\
&\quad + M\int_0^t (1 + G(s))\int_0^1 (v_1 - v_2)^2 dyds \\
&\quad + M\int_0^t\int_0^1 \left(|\mathbf{b}_1 - \mathbf{b}_2|^2 + (\theta_1 - \theta_2)^2\right) dyds.
\end{aligned} \tag{4.3}$$

Proof. Multiplying the following equation by $2(u_1 - u_2)$

$$(u_1 - u_2)_t = \left(\frac{\lambda u_{1y}}{v_1} - \frac{\lambda u_{2y}}{v_2} + \frac{\theta_2}{v_2} - \frac{\theta_1}{v_1} + \frac{1}{2}|\mathbf{b}_2|^2 - \frac{1}{2}|\mathbf{b}_1|^2\right)_y,$$

and integrating it over $[0,1] \times [0,t]$, we have

$$\begin{aligned}
&\int_0^1 (u_1 - u_2)^2 dy - \int_0^1 (u_{10} - u_{20})^2 dy \\
&\le -\frac{7}{4}\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y} - u_{2y})^2 dyds + M\int_0^t (1 + G(s))\int_0^1 (v_1 - v_2)^2 dyds \\
&\quad + M\int_0^t\int_0^1 \left(|\mathbf{b}_1 - \mathbf{b}_2|^2 + (\theta_1 - \theta_2)^2\right) dyds.
\end{aligned}$$

Multiplying the equation $(v_1 - v_2)_t = (u_1 - u_2)_y$ by $2(v_1 - v_2)$ and integrating over $[0,1]\times[0,t]$, for any constant $M'>0$, we have

$$\begin{aligned}&\int_0^1 (v_1-v_2)^2dy-\int_0^1 (v_{10}-v_{20})^2dy\\&\le \frac{1}{4M'}\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y}-u_{2y})^2dyds+M\int_0^t\int_0^1 (v_1-v_2)^2dyds.\end{aligned}$$

Taking $M'=1$, we obtain

$$\begin{aligned}&\int_0^1 (v_1-v_2)^2dy-\int_0^1 (v_{10}-v_{20})^2dy\\&\le \int_0^t\int_0^1 \left(\frac{1}{4}\frac{\lambda}{v_1}(u_{1y}-u_{2y})^2+M(v_1-v_2)^2\right)dyds.\end{aligned}$$

Taking $M'=2M$, we have

$$\begin{aligned}M\int_0^1 (v_1-v_2)^2dy\le M\int_0^1 (v_{10}-v_{20})^2dy+\frac{1}{8}\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y}-u_{2y})^2dyds\\+M\int_0^t\int_0^1 (v_1-v_2)^2dyds.\end{aligned}\tag{4.4}$$

Similarly, taking the inner product of the equation

$$(\mathbf{w}_1-\mathbf{w}_2)_t=\left(\frac{\mu\mathbf{w}_{1y}}{v_1}-\frac{\mu\mathbf{w}_{2y}}{v_2}+\mathbf{b}_1-\mathbf{b}_2\right)_y$$

with $2(\mathbf{w}_1-\mathbf{w}_2)$ and using Cauchy-Schwartz's inequality yield a similar estimate. Then adding these estimates together yields the result of Lemma 4.2. □

Lemma 4.3.

$$\begin{aligned}&\int_0^1 (\theta_1-\theta_2)^2dy\\&\le \int_0^1 (\theta_{10}-\theta_{20})^2dy-\frac{7}{4}\int_0^t\int_0^1 \frac{\kappa}{v_1}(\theta_{1y}-\theta_{2y})^2dyds\\&\quad+\frac{1}{4}\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y}-u_{2y})^2dyds+\frac{1}{4}\int_0^t\int_0^1 \frac{\mu}{v_1}|\mathbf{w}_{1y}-\mathbf{w}_{2y}|^2dyds\\&\quad+\frac{1}{4}\int_0^t\int_0^1 \nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2dyds\\&\quad+M\int_0^t (1+G(s))\int_0^1 \left((v_1-v_2)^2+(\theta_1-\theta_2)^2\right)dyds.\end{aligned}\tag{4.5}$$

Proof. Multiply the following equation

$$(\theta_1-\theta_2)_t = p_2u_{2y}-p_1u_{1y}+\left(\frac{\kappa\theta_{1y}}{v_1}-\frac{\kappa\theta_{2y}}{v_2}\right)_y + \frac{\lambda u_{1y}^2+\mu|\mathbf{w}_{1y}|^2+\nu|\mathbf{b}_{1y}|^2}{v_1}-\frac{\lambda u_{2y}^2+\mu|\mathbf{w}_{2y}|^2+\nu|\mathbf{b}_{2y}|^2}{v_2}$$

by $2(\theta_1-\theta_2)$ and integrate it to get

$$\begin{aligned}&\int_0^1(\theta_1-\theta_2)^2dy-\int_0^1(\theta_{10}-\theta_{20})^2dy\\&=2\int_0^t\int_0^1\left(\frac{\kappa\theta_{1y}}{v_1}-\frac{\kappa\theta_{2y}}{v_2}\right)(\theta_{2y}-\theta_{1y})dyds\\&\quad+2\int_0^t\int_0^1(p_2u_{2y}-p_1u_{1y})(\theta_1-\theta_2)dyds\\&\quad+2\int_0^t\int_0^1(\theta_1-\theta_2)\left(\frac{\lambda u_{1y}^2+\mu|\mathbf{w}_{1y}|^2+\nu|\mathbf{b}_{1y}|^2}{v_1}\right.\\&\qquad\left.-\frac{\lambda u_{2y}^2+\mu|\mathbf{w}_{2y}|^2+\nu|\mathbf{b}_{2y}|^2}{v_2}\right)dyds.\end{aligned}\tag{4.6}$$

Using Cauchy-Schwartz's inequality, we have

$$\begin{aligned}&2\int_0^t\int_0^1\left(\frac{\kappa\theta_{1y}}{v_1}-\frac{\kappa\theta_{2y}}{v_2}\right)(\theta_{2y}-\theta_{1y})dyds\\&\le-\frac{7}{4}\int_0^t\int_0^1\frac{\kappa}{v_1}(\theta_{1y}-\theta_{2y})^2dyds+M\int_0^t\max_{y\in[0,1]}\theta_{2y}^2\int_0^1(v_1-v_2)^2dyds,\end{aligned}$$

$$\begin{aligned}&2\int_0^t\int_0^1(p_2u_{2y}-p_1u_{1y})(\theta_1-\theta_2)dyds\\&=2\int_0^t\int_0^1\left(\left(\frac{\theta_2}{v_2}-\frac{\theta_1}{v_1}\right)u_{2y}+(u_{2y}-u_{1y})\frac{\theta_1}{v_1}\right)(\theta_1-\theta_2)dyds\\&\le\frac{1}{8}\int_0^t\int_0^1\frac{\lambda}{v_1}(u_{1y}-u_{2y})^2dyds+M\int_0^t\int_0^1(\theta_1-\theta_2)^2dyds\\&\quad+M\int_0^t\max_{y\in[0,1]}u_{2y}^2\int_0^1(\theta_1-\theta_2)^2dyds+M\int_0^t\max_{y\in[0,1]}u_{2y}^2\int_0^1(v_1-v_2)^2dyds,\end{aligned}\tag{4.7}$$

and

$$\int_0^t\int_0^1 \left(\frac{\lambda u_{1y}^2+\mu|\mathbf{w}_{1y}|^2+\nu|\mathbf{b}_{1y}|^2}{v_1}-\frac{\lambda u_{2y}^2+\mu|\mathbf{w}_{2y}|^2+\nu|\mathbf{b}_{2y}|^2}{v_2}\right)(\theta_1-\theta_2)dyds$$
$$\leq \frac{1}{8}\int_0^t\int_0^1 \left(\frac{1}{2}\frac{\lambda}{v_1}(u_{1y}-u_{2y})^2+\frac{\mu}{v_1}|\mathbf{w}_{1y}-\mathbf{w}_{2y}|^2+\nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2\right)dyds$$
$$+M\int_0^t \max_{y\in[0,1]}\sum_{j=1}^2(u_{jy}^2+|\mathbf{w}_{jy}|^2+|\mathbf{b}_{jy}|^2)\int_0^1 \left((v_1-v_2)^2+(\theta_1-\theta_2)^2\right)dyds.$$

Applying these estimates to (4.6) and from the definition of $G(t)$, we conclude (4.5). This completes the proof of Lemma 4.3. □

Lemma 4.4.

$$\begin{aligned}&\int_0^1|\mathbf{b}_1-\mathbf{b}_2|^2dy\\ &\leq M\int_0^1\left((v_{10}-v_{20})^2+|\mathbf{b}_{10}-\mathbf{b}_{20}|^2\right)dy-\frac{7}{2}\int_0^t\int_0^1\nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2dyds\\ &\quad+\frac{1}{2}\int_0^t\int_0^1\left(\frac{\lambda}{v_1}(u_{1y}-u_{2y})^2+\frac{\mu}{v_1}|\mathbf{w}_{1y}-\mathbf{w}_{2y}|^2\right)dyds\\ &\quad+M\int_0^t\int_0^1(v_{1y}-v_{2y})^2dyds\\ &\quad+M\int_0^t\int_0^1(1+G(s))\left((v_1-v_2)^2+|\mathbf{b}_1-\mathbf{b}_2|^2\right)dyds.\end{aligned}\tag{4.8}$$

Proof. Take the inner product of the following equation

$$(v_1\mathbf{b}_1-v_2\mathbf{b}_2)_t=\left(\frac{\nu\mathbf{b}_{1y}}{v_1}-\frac{\nu\mathbf{b}_{2y}}{v_2}+\mathbf{w}_1-\mathbf{w}_2\right)_y$$

with $2(v_1\mathbf{b}_1-v_2\mathbf{b}_2)$, and integrate it to get

$$\int_0^1|v_1\mathbf{b}_1-v_2\mathbf{b}_2|^2dy-\int_0^1|v_{10}\mathbf{b}_{10}-v_{20}\mathbf{b}_{20}|^2dy$$
$$=-2\int_0^t\int_0^1\left(\nu\frac{\mathbf{b}_{1y}}{v_1}-\frac{\mathbf{b}_{2y}}{v_2}\right)(v_1\mathbf{b}_1-v_2\mathbf{b}_2)_ydyds$$
$$+2\int_0^t\int_0^1(w_{1y}-w_{2y})(v_1\mathbf{b}_1-v_2\mathbf{b}_2)dyds.$$

Notice that

$$\int_0^1|v_1\mathbf{b}_1-v_2\mathbf{b}_2|^2dy\geq\frac{1}{2}\int_0^1v_1^2|\mathbf{b}_1-\mathbf{b}_2|^2dy-\int_0^1(v_1-v_2)^2|\mathbf{b}_2|^2dy.$$

Then, using (4.4), Lemma 4.1, and Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
&\int_0^1 |\mathbf{b}_1-\mathbf{b}_2|^2dy - M\int_0^1 \left((v_{10}-v_{20})^2+|\mathbf{b}_{10}-\mathbf{b}_{20}|^2\right)dy\\
&\le M\int_0^1 (v_1-v_2)^2dy - 4\int_0^t\int_0^1 \nu\left(\frac{\mathbf{b}_{1y}}{v_1}-\frac{\mathbf{b}_{2y}}{v_2}\right)(v_1\mathbf{b}_1-v_2\mathbf{b}_2)_y dyds\\
&\quad + 4\int_0^t\int_0^1 (w_{1y}-w_{2y})(v_1\mathbf{b}_1-v_2\mathbf{b}_2)dyds\\
&\le M\int_0^1 (v_{10}-v_{20})^2dy - \frac{7}{2}\int_0^t\int_0^1 \nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2dyds\\
&\quad + \frac{1}{2}\int_0^t\int_0^1 \left(\frac{\lambda}{v_1}(u_{1y}-u_{2y})^2+\frac{\mu}{v_1}|\mathbf{w}_{1y}-\mathbf{w}_{2y}|^2\right)dyds\\
&\quad + M\int_0^t\int_0^1 (1+G(s))\left((v_1-v_2)^2+|\mathbf{b}_1-\mathbf{b}_2|^2\right)dyds\\
&\quad + M\int_0^t\int_0^1 (v_{1y}-v_{2y})^2dyds.
\end{aligned}$$

This completes the proof of Lemma 4.4. □

Lemma 4.5.

$$\begin{aligned}
&\int_0^1 (v_{1y}-v_{2y})^2dy\\
&\le M\int_0^1 |(u_{10}-u_{20}, v_{10}-v_{20}, v_{10y}-v_{20y})|^2dy + M\int_0^t\int_0^1 (v_{1y}-v_{2y})^2dyds\\
&\quad + \left(\frac{1}{8}-\frac{M}{4}\right)\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y}-u_{2y})^2dyds\\
&\quad + \frac{1}{8}\int_0^t\int_0^1 \left(\frac{\kappa}{v_1}(\theta_{1y}-\theta_{2y})^2+\nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2\right)dyds\\
&\quad + M\int_0^t (1+G(s))\int_0^1 |(v_1-v_2, \theta_1-\theta_2, \mathbf{b}_1-\mathbf{b}_2)|^2dyds.
\end{aligned} \tag{4.9}$$

Proof. From (3.9), we have the following equation:

$$\begin{aligned}
&\left(\frac{\lambda v_{1y}}{v_1}-\frac{\lambda v_{2y}}{v_2}\right)_t + \frac{\theta_1 v_{1y}}{v_1^2}-\frac{\theta_2 v_{2y}}{v_2^2}\\
&\quad = (u_1-u_2)_t + \frac{\theta_{1y}}{v_1}-\frac{\theta_{2y}}{v_2} + \mathbf{b}_1\cdot\mathbf{b}_{1y} - \mathbf{b}_2\cdot\mathbf{b}_{2y}.
\end{aligned}$$

Multiply the above equation by $\frac{\lambda v_{1y}}{v_1} - \frac{\lambda v_{2y}}{v_2}$ and integrate it to get

$$\begin{aligned}
&\int_0^1 \frac{\lambda^2}{2}\left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right)^2 dy - \int_0^1 \frac{\lambda^2}{2}\left(\frac{v_{10y}}{v_{10}} - \frac{v_{20y}}{v_{20}}\right)^2 dy \\
&= \int_0^t\int_0^1 \left(\frac{\theta_2 v_{2y}}{v_2^2} - \frac{\theta_1 v_{1y}}{v_1^2} + \frac{\theta_{1y}}{v_1} - \frac{\theta_{2y}}{v_2} + \mathbf{b}_1\cdot\mathbf{b}_{1y} - \mathbf{b}_2\cdot\mathbf{b}_{2y}\right)\times \\
&\qquad\qquad \left(\frac{\lambda v_{1y}}{v_1} - \frac{\lambda v_{2y}}{v_2}\right) dyds \\
&\quad + \int_0^t\int_0^1 (u_{1t} - u_{2t})\left(\frac{\lambda v_{1y}}{v_1} - \frac{\lambda v_{2y}}{v_2}\right) dyds.
\end{aligned}$$

Notice that

$$\begin{aligned}
&(u_1 - u_2)_t\left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right) \\
&\quad = ((u_1 - u_2)(\ln v_1 - \ln v_2)_y)_t - ((u_1 - u_2)(\ln v_1 - \ln v_2)_t)_y \\
&\qquad + (u_1 - u_2)_y\left(\frac{u_{1y}}{v_1} - \frac{u_{2y}}{v_2}\right).
\end{aligned}$$

Then we integrate it to get

$$\begin{aligned}
&\int_0^t\int_0^1 (u_{1t} - u_{2t})\left(\frac{\lambda v_{1y}}{v_1} - \frac{\lambda v_{2y}}{v_2}\right) dyds \\
&\leq M\int_0^1 |(u_{10} - u_{20}, v_{10} - v_{20}, v_{10y} - v_{20y})|^2 dy + 2\int_0^1 (u_1 - u_2)^2 dy \\
&\quad + \int_0^1 \frac{\lambda^2}{4}\left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right)^2 dy + \int_0^t\int_0^1 \lambda(u_{1y} - u_{2y})\left(\frac{u_{1y}}{v_1} - \frac{u_{2y}}{v_2}\right) dyds.
\end{aligned}$$

Since

$$\int_0^1 \frac{\lambda^2}{4}\left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right)^2 dy \geq \frac{1}{2}\int_0^1 \frac{\lambda^2(v_{1y} - v_{2y})^2}{4v_1^2}dy - \int_0^1 \frac{\lambda^2(v_1 - v_2)^2 v_{2y}^2}{4v_1^2 v_2^2}dy,$$

we also have

$$\int_0^1 (v_{1y} - v_{2y})^2 dy \leq M\int_0^1 (v_1 - v_2)^2 v_{2y}^2 dy + M\int_0^1 \frac{\lambda^2}{4}\left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right)^2 dy.$$

Thus, from (4.4) and (4.3),

$$\begin{aligned}
&\int_0^1 (v_{1y}-v_{2y})^2 dy - M\int_0^1 |(u_{10}-u_{20}, v_{10}-v_{20}, v_{10y}-v_{20y})|^2 dy \\
&\le M\int_0^t\int_0^1 (v_{1y}-v_{2y})^2 dyds + \frac{1}{8}\int_0^t\int_0^1 \left(\frac{\kappa}{v_1}(\theta_{1y}-\theta_{2y})^2 + \nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2\right) dyds \\
&\quad + \left(\frac{1}{8}-\frac{M}{4}\right)\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y}-u_{2y})^2 dyds \\
&\quad + M\int_0^t (1+G(s))\int_0^1 |(v_1-v_2, \theta_1-\theta_2, \mathbf{b}_1-\mathbf{b}_2)|^2 dyds.
\end{aligned}$$

This completes the proof of Lemma 4.5. □

We now add (4.3), (4.5), and (4.8)-(4.9) together to get

$$\begin{aligned}
&\int_0^1 \Bigg(|(v_1-v_2, u_1-u_2, \mathbf{w}_1-\mathbf{w}_2)|^2 + \frac{1}{2}|\mathbf{b}_1-\mathbf{b}_2|^2 \\
&\qquad\qquad +2(\theta_1-\theta_2)^2 + \frac{1}{2}(v_{1y}-v_{2y})^2 \Bigg) dy \\
&\quad + \left(\frac{3}{4}+\frac{M}{2}\right)\int_0^t\int_0^1 \frac{\lambda}{v_1}(u_{1y}-u_{2y})^2 dyds + \frac{5}{4}\int_0^t\int_0^1 \frac{\mu}{v_1}|\mathbf{w}_{1y}-\mathbf{w}_{2y}|^2 dyds \\
&\quad + \frac{5}{4}\int_0^t\int_0^1 \nu|\mathbf{b}_{1y}-\mathbf{b}_{2y}|^2 dyds + \frac{3}{2}\int_0^t\int_0^1 \frac{\kappa}{v_1}(\theta_{1y}-\theta_{2y})^2 dyds \\
&\le MD(0) + M\int_0^t\int_0^1 (v_{1y}-v_{2y})^2 dyds \\
&\quad + M\int_0^t (1+G(s))\int_0^1 |(v_1-v_2, \mathbf{b}_1-\mathbf{b}_2, \theta_1-\theta_2)|^2 dyds.
\end{aligned}$$

Using Gronwall's inequality and (4.2), we obtain

$$D(t) \le M\exp\left(M\int_0^t (1+G(s))ds\right) D(0) \le MD(0).$$

The proof of part (ii) of Theorem 2.1, is now complete.

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References

[1] R. Balescu, *Transport Processes in Plasmas I: Classical transport theory*, North-Holland, New York, 1988.

[2] M. Brio and C. C. Wu, An upwind differencing scheme for the equations of magnetohydrodynamics, *J. Comput. Phys.* 75 (1988). 400-422.

[3] H. Cabannes, *Theoretical Magnetofluiddynamics*, Academic Press, New York, 1970.

[4] G.-Q. Chen, The method of quasidecoupling for discontinuous solutions to conservation laws, *Arch. Rational Mech. Anal.* 121 (1992), 131-185.

[5] P. C. Clemmow and J. P. Dougherty, *Electrodynamics of Particles and Plasmas*, Addison-Wesley, New York, 1990.

[6] C. M. Dafermos and L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, *Nonlinear Anal.: Theory Methods Appl.* 6 (1982), 435-454.

[7] H. Freistühler and P. Szmolyan, Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves, *SIAM J. Math. Anal.* 26 (1995), 112-128.

[8] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1994.

[9] Y. I. Kanel, On a model system of equations of one-dimensional gas motion, *Differential Equations*, 4 (1968), 374-380.

[10] S. Kawashima and M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Proc. Japan Acad. Ser. A, Math. Sci.* 58 (1982), 384–387.

[11] S. Kawashima and Y. Shizuta, Magnetohydrodynamic approximation of the complete equations for an electromagnetic fluid, *Tsukuba J. Math.* 10 (1986), 131-149.

[12] V. Kazhikhov, Cauchy problem for viscous gas equations, *Siberian Math. J.* 23 (1982), 44-49.

[13] V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of initial-boundary-value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* 41 (1977), 273-282.

[14] S. Kruzkov, A priori estimates for the derivative of a solution to a parabolic equation and some of its applications, *Dokl. Akad. Nauk. SSSR*, 170 (1966), 501-504.

[15] A. G. Kulikovskiy and G. A. Lyubimov, *Magnetohydrodynamics*, Addison-Wesley, Reading, Massachusetts, 1965.

[16] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs 23, American Mathematical Society, Providence, Rhode Island, 1988.

[17] L. D. Laudau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed., Pergamon, New York, 1984.

[18] R. J. LeVeque, *Nonlinear Conservation Laws and Finite Volume Methods for Astrophysical Fluid Flow*, 27th Saas-Fee Advanced Course Lecture Notes, 1998.

[19] T.-P. Liu and Y. Zeng, *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws*, Memoirs Amer. Math. Soc. 599, 1997.

[20] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France* 90 (1962), 487-497.

[21] D. Wang, Global solution to the equations of viscous gas flows, *Proc. Royal Soc. Edinburgh*, 131A (2001), 437-449.

[22] L. C. Woods, *Principles of Magnetoplasma Dynamics*, Oxford University Press, New York, 1987.

[23] C. C. Wu, Formation, structure, and stability of MHD intermediate shocks, *J. Geophys. Res.* 95 (1990), 8149-8175.

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