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Differential equations, hysteresis, and time delay

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Abstract. We consider an ordinary differential equation with hysteresis and time delay in the hysteresis term and also a partial differential equation with hysteresis in the boundary conditions. Both are candidates for the description of a type of thermostat. The considered hysteresis relation is relay hysteresis in both models. The existence of a periodic solution to both equations is shown and the relationship between the two models is discussed.

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1. Introduction

We consider two differential equations with a hysteresis term of relay type, both equations arise from the same physical system. The derivation of both models is given in Section 2. The two differential equations are quite different in nature and complexity.

The first equation is an ordinary differential equation with hysteresis and delay:

$$\frac{du}{dt} = -\frac{1}{L}[\mathcal{F}(u)](t-\tau). \tag{1}$$

Here $\mathcal{F}$ represents a hysteresis operator of relay type, defined in Section 3. Hysteresis operators and their connection with differential equations have recently received considerable attention (see, e.g., [1], [2], [3] and the references given therein). The combination of hysteresis and delay is a relatively new topic. Recently, two papers have been written about this problem, see [4] and [5]. They consider O.D.E.s with hysteresis and delay and are mainly interested in the oscillatory behaviour of those equations. The hysteresis model they dicuss is a generalized play operator. In our paper we will show that the equation (1) coupled with the delayed relay operator has a periodic solution and that all solutions of the equation (1) will eventually coincide with this periodic solution. We will actually compute this periodic solution and its period using simple techniques from the O.D.E. theory and the definition and properties of the delayed relay operator.

The second equation we consider is a linear parabolic P.D.E. with hysteresis in the boundary conditions:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{if } 0 < x < L,\ t > 0, \tag{2}$$

$$\frac{\partial u}{\partial t}(0,t) = 0 \qquad \text{if } t > 0, \tag{3}$$

$$\frac{\partial u}{\partial x}(L,t) = -[\mathcal{F}(u(0,.))](t) \qquad \text{if } t > 0. \tag{4}$$

Here $\mathcal{F}$ represents again a hysteresis operator of relay type, applied to the function $u(0,t)$.

A parabolic P.D.E. with hysteresis in the boundary conditions of different types has been already studied before, see e.g. [6], [7], [8]. In [8], A. Friedman and L.-S. Jiang have studied equation (2) with boundary condition (3) and a different condition at $x = L$:

$$\frac{\partial u}{\partial x}(L,t) + u(L,t) = -[\mathcal{F}(u,(0,.))](t) \qquad \text{if } t > 0. \tag{5}$$

They proved the existence of a periodic solution of this equation. The constructed solution has two phases, namely

$$[\mathcal{F}(u(0,.))](t)\begin{cases} -1 \text{ if } 0 < t < T_1, \\ +1 \text{ if } T_1 < t < T. \end{cases} \tag{6}$$

for some $T_1 \in (0,T)$ and $u(x,t+t) = u(x,t)$. This two-phase solution is unique, if $-1 < \rho_1 < \rho_2 < 1$ and $1 - |\rho_1| < \delta$, $1 - \rho_2 < \delta$, where δ is sufficiently small and ρ_1, ρ_2 are threshold values of the relay operator. It is interesting to compare these results with those obtained here. The authors believe that the here considered model is more appropriate to be compared with the ordinary differential equation with time delay (1) since the boundary condition (4) is independent of the value of u at $x = L$.

In this paper we will compute a periodic solution of (2)–(4). We will show that at least one periodic solution exists for arbitrary ρ_1, ρ_2.

The period of periodic solutions is a convenient tool as well as technically important and is therefore used for the comparison of both models of the same system.

2. Physical motivation: Two models for one thermostat

A physical system described by the equations considered here is a solid body kept around a certain temperature range by a thermostat which heats the body with a constant heat rate a if it becomes colder than ρ_1 and which cools the body with a constant heat rate b if it becomes warmer than ρ_2. The thermostat reacts instantaneously as one would expect from an electronic device and is thus correctly described by a very simple hysteresis operator $\mathcal{F}$ defined in Section 3, the relay.

The essential complication examined here is the situation where the temperature of the body is taken at a place on the body different from the place where the heat is applied. To specify the system further, the body is taken to be a thin rod of length L and the thermostat is set up in such a way that the temperature to which it reacts is taken at one end of the rod (at $x = 0$) and heating and cooling is applied to the other end (at $x = L$).

Two models are offered here:

1. **Detailed model.** The thermal behavior of the thin rod is described by the heat equation (2). At the end $x = 0$, no heat is applied as expressed in equation (3), while the heat applied at the other end $x = L$ is given by the state of the thermostat as given in equation (4). The result is a *partial differential equation with a hysteresis operator in the boundary condition.* An initial condition has to be applied, of course.
2. **Effective model.** On a heuristic level, the essential complication of measuring and heating in different places is expected to have the following effect: While the thermostat reacts instantaneously to the temperature at $x = 0$, it takes some time till the effect of heating at $x = L$ becomes significant in $x = 0$. How long does it take for the heat applied on one end of the rod to get to its other end? The answer can be given on dimensional grounds: The needed time is $\tau = L^2$. With this time delay, the heating prescribed by the thermostat will influence the temperature at $x = 0$. Of course, this is to be taken with a grain of salt but it allows to propose equation (1) as a description of the situation. All the particularities of the temperature field along the rod are eliminated and one is left with an *ordinary differential equation with a hysteresis and with time delay*, a significant simplification.

Remark 1. The time delay of the effective model suggested above is L^2. But on dimensional grounds one can only conclude that it has to be proportional to L^2. The factor of proportionality k in the dimensional argument is expected to be a number of the order 1. The time delay is then

$$\tau = kL^2. \tag{7}$$

To determine the constant k properly, one has to resort to the detailed model. If this is not possible and the value $\tau = L^2$ is used then it is to be expected that any results will be correct only to first order in L. This is of course useful only if L is small, $L \ll 1$.

The question is now whether there is a price to be paid for using the effective model or whether it gives in fact a good description comparable to what the detailed model provides. To answer this question, the solutions of these rather different equations have to be understood and compared. In particular, one may wish to compare the periods of periodic solutions calculated in Section 4.

Remark 2. To avoid further complexity, it is assumed throughout the paper that no heat is gained from or lost to the environment. Also, only the special case of the heat rates a, b of heating and cooling being equal, $a = b = 1$ is discussed in detail while the results for general a and b are only stated.

3. Delayed relay operator

In this section we will recall the basic definitions of hysteresis operators and define a special and probably the simplest kind of a hysteresis operator, the so called delayed relay operator. We will use this operator in the next sections in connection with the differential equations (1) and (2).

Mathematically, a hysteresis relationship between two functions u and w that are defined on some time interval $[0, t]$ and attain their values in some sets U and W, respectively, can be expressed as an operator equation with an operator $\mathcal{F}$:

$$w = \mathcal{F}[u]. \tag{8}$$

Hysteresis operators are characterized by two main properties:

(i) Memory: at any instant t, $w(t)$ depends on the previous evolution of u. We also assume that

$$\text{if } u_1 = u_2 \text{ in } [0, t], \text{ then } [\mathcal{F}(u_1)](t) = [\mathcal{F}(u_2)](t) \text{ (causality).} \tag{9}$$

(ii) Rate independence: the output w is invariant with respect to changes of the time scale, formally

$$\mathcal{F}[u] \circ \phi = [u \circ \phi] \tag{10}$$

for all inputs u and all increasing functions ϕ mapping the considered time interval onto itself.

At any instant t, the output $w(t)$ usually depends not only on $u|_{[0,t]}$, but also on the initial state of the system. Hence, the initial value $w_0 = w(0)$, or some equivalent information, must be prescribed. If necessary, we therefore write $\mathcal{F}(u, w_0)$ to make the dependence on w_0 explicit.

Many hysteresis operators also satisfy other typical properties:

1) Piecewise monotonicity:

$$\begin{cases} \forall (u, w_0) \in \mathrm{Dom}(\mathcal{F}), \forall [t_1, t_2] \subset [0, T], \\ \text{if } u \text{ is nondecreasing (resp. nonincreasing) in } [t_1, t_2], \\ \text{then so is } \mathcal{F}(u, w_0). \end{cases} \tag{11}$$

2) Order preservation:

$$\begin{cases} \forall (u_1, w_{10}), (u_2, w_{20}) \in \mathrm{Dom}(\mathcal{F}), \forall t \in (0,T], \\ \text{if } u_1 \le u_2 \text{ in } [0,t], \text{ and } w_{10} \le w_{20}, \text{ then} \\ \mathcal{F}(u_1, w_{10})(t) \le \mathcal{F}(u_2, w_{20})(t). \end{cases} \tag{12}$$

To define the delayed operator:
For any couple $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ with $\rho_1 < \rho_2$, we introduce the delayed relay operator

$$h_\rho : C^0([0,T]) \times \{-1, 1\} \to BV(0,T) \cup C_r^0([0,T)), \tag{13}$$

where $C_r^0([0,T))$ denotes the space of functions right-continuous in $[0,T)$. For any $u \in C^0([0,T])$ and any $\xi = -1$ or 1, $h_\rho(u,\xi) = w : [0,T] \to \{-1,1\}$ is defined as follows:

$$w(0) = \begin{cases} -1 & \text{if } u(0) \le \rho_1 \\ \xi & \text{if } \rho_1 < u(0) < \rho_2 \\ 1 & \text{if } u(0) \ge \rho_2 \end{cases} \tag{14}$$

for any $t \in (0,T]$, setting $X_t = \{\tau \in (0,t],\, u(\tau) = \rho_1 \text{ or } \rho_2\}$

$$w(t) = \begin{cases} w(0) & \text{if } X_t = \emptyset, \\ -1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = \rho_1, \\ 1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = \rho_2. \end{cases} \tag{15}$$

Then w is uniquely defined in $[0,T]$. For instance, let $u(0) < \rho_1$; then $w(0) = -1$, and $w(t) = -1$ as long as $u(t) < \rho_2$; if at some instant u reaches ρ_2, then w jumps up to 1, where it remains as long as $u(t) > \rho_1$; if later u reaches ρ_1, then w jumps down to -1; and so on, cf. Figure 1.

The relay operator is characterized by two threshold values $\rho_1 < \rho_2$ and two output values which we assume here to be equal to $+1$ and -1, respectively.

The delayed relay operator is a rate independent, piecewise monotone, order preserving and discontinuous hysteresis operator (in any sense).

For more details as well as for definitions of different kinds of hysteresis operators, see [1].

In the next sections we use the notation $[\mathcal{F}(u)](t) = w(t)$ with $w(t)$ defined above.

4. Main results

In this section we present statements and proofs of our main results. The discussion and comparison of the models is postponed to Section 5.

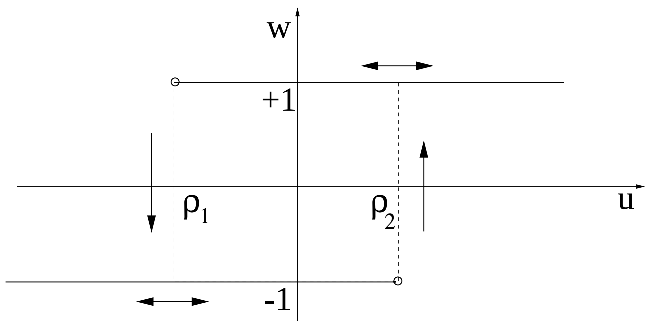

Figure 1. The relay operator. In our models, u has the meaning of the measured temperature while w describes the heat flow of cooling resp. heating.

4.1. The ordinary differential equation of the effective model

Theorem 1. *There exists a periodic continuous solution of equation (1) with period equal to* $2L(\rho_2 - \rho_1) + 4L^2$.

Proof. Computational. Suppose that $[\mathcal{F}(u)](t-\tau) = -1$ on $[0,\tau]$ and $u(0) = \rho_1$. Then

$$\frac{du}{dt} = \frac{1}{L} > 0 \text{ on } [0,\tau]. \tag{16}$$

The equation (16) has a solution $u_1 = \frac{1}{L} + \rho_1$ on $[0,\tau]$ and since this solution is increasing, the solution u of the equation (1) is equal to u_1 until $t = t_1 + \tau$, where t_1 is the time where $u_1(t_1) = \rho_2$. A simple computation gives us

$$\frac{1}{L}t_1 + \rho_1 = \rho_2,$$

which implies

$$t_1 = (\rho_2 - \rho_1)L.$$

At time $t = t_1 + \tau$, there is a jump at the hysteresis output, i.e.

$$[\mathcal{F}(u)](t-\tau) = [\mathcal{F}(u)](t_1) = 1.$$

On the interval $[t_1+\tau, t_1+2\tau]$

$$\frac{du}{dt} = -\frac{1}{L} < 0, \tag{17}$$

which has a solution

$$u_2 = -\frac{1}{L}t + K.$$

The constant K can be computed from the continuity condition

$$u_2(t_1+\tau) = u_1(t_1+\tau),$$

which implies

$$K = 2\rho_2 - \rho_1 + \frac{2\tau}{L}.$$

Therefore the solution of the equation (1) on $[t_1 + \tau, t_1 + 2\tau]$ is

$$u_2(t) = -\frac{1}{L}t + 2\rho_2 - \rho_1 + \frac{2\tau}{L}.$$

This solution is decreasing in time, so the solution of the equation (1) is equal to u_2 until time $t_2 + \tau$, where t_2 is such that

$$u_2(t_2) = \rho_1 = -\frac{1}{L}t_2 + 2\rho_2 - \rho_1 + \frac{2\tau}{L}.$$

It follows immediately from the last equation that

$$t_2 = 2L(\rho_2 - \rho_1) + 2\tau.$$

At time t_2 there is again a jump, this time downwards in the hysteresis output, i.e.

$$[\mathcal{F}(u)](t_2) = -1,$$

so again by the same arguments as before, the solution u of the equation (1) on the interval $[t_2 + \tau, t_2 + 2\tau]$ is equal to

$$u_3(t) = \frac{1}{L}t + K^*.$$

The corresponding continuity condition gives us the value of the constant K^*:

$$u_2(t_2 + \tau) = u_3(t_2 + \tau),$$

which implies

$$K^* = -2\rho_2 + 3\rho_1 - \frac{4\tau}{L}.$$

Therefore the solution of the equation (1) on the interval $[t_2 + \tau, t_2 + 2\tau]$ is equal to

$$u_3(t) = \frac{1}{L}t - 2\rho_2 + 3\rho_1 - \frac{4\tau}{L}.$$

The constructed solution will be periodic if

$$u_3(t_3) = \rho_1 = \frac{1}{L}t_3 - 2\rho_2 + 3\rho_1 + \frac{4\tau}{L}.$$

This implies

$$t_3 = 2L(\rho_2 - \rho_1) + 4\tau.$$

This is the time it takes the constructed solution to come to its starting position, therefore t_3 is the period and the statement of the Theorem is proved. □

Remark 3. A similar computation can be provided for a little bit different, but perhaps more interesting model, for which the heat transfer rate is different for heating and cooling. This corresponds to a model with relay hysteresis operator with output values equal to $-a$ and $+b$ respectively, see also Figure 2, instead of -1 and $+1$ as in the model above. This means again that if $u(0) < \rho_1$; then $w(0) = -a$ and $w(t) = -a$ as long as $u(t) < \rho_2$; if at some instant u reaches ρ_2, then w jumps up to b, where it remains as long as $u(t) > \rho_1$; if later u reaches ρ_1, then w jumps down to $-a$; and so on, cf. Figure 2.

Let us denote the corresponding hysteresis operator as $\mathcal{F}_{a,b}$. We assume $a > 0$, $b > 0$. We claim that there exists a periodic solution of equation (1) with $\mathcal{F}$ replaced by $\mathcal{F}_{a,b}$ which period is equal to $T = L[\frac{1}{a} + \frac{1}{b}](\rho_2 - \rho_1) + \frac{L^2(a+b)^2}{ab}$. The idea of the proof of this statement is the same as the one of Theorem 1. Notice that for $a = b = 1$ the corresponding statements coincide.

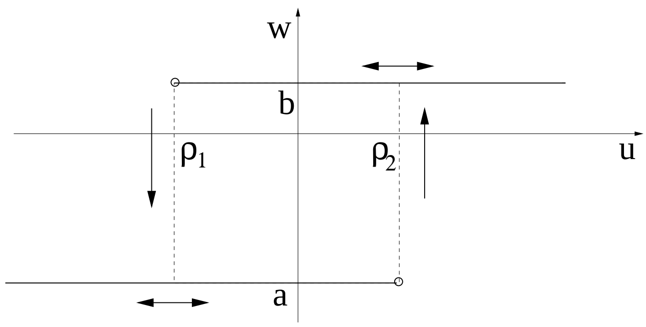

Figure 2. The general relay operator. The two output values of w, the heat flow of heating and cooling in response to the temperature $u(t)$ do not have the same absolute value.

4.2. The partial differential equation of the detailed model

Solutions of the partial differential equation (2) are supposed to be from a suitable space of functions, e.g., the space of continuous functions possessing the weak derivatives necessary to make sense of the equation.

Depending on the value of the right hand side of the boundary condition (4), a solution of equation (2) can be decomposed into heating and cooling phases. In a periodic solution, heating and cooling phases will alternate forever and this alternation has to be synchronized with the period of the solution. It is thinkable though that the period is reached only after many alterations.

Definition 1. A periodic solution is called two-phase if the period is equal to the length of two consecutive phases, a cooling and a heating phase.

Theorem 2. *There exists a periodic solution of the equation (2) with boundary conditions (3) and (4). This solution is a two-phase solution and it is a unique two-phase solution for $L < \rho_2 - \rho_1$ in the space of continuous functions possessing the weak derivatives necessary to make sense of the equation.*

Proof. Again by construction. Suppose that we have a solution of the equation (2), whose initial condition has a Fourier series $b_0 + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L}$.

For the given space of functions, this Fourier series always exists and it is easy to check from the results of the construction that the solutions does not leave this space.

Suppose also that

$$b_0 + \sum_{n=1}^{\infty} b_n = \rho_1. \tag{18}$$

This means we have a solution, which starts at time $t = 0$ at the jumping position, and

$$[\mathcal{F}(u(0,.)](0) = -1.$$

Then the solution of the equation (2) looks initially like

$$u_1(x,t) = \frac{t}{L} + \frac{x^2}{2L} + \left(b_0 - \frac{L}{6}\right) + \sum_{n=1}^{\infty} \left(b_n - \frac{2L(-1)^n}{n^2\pi^2}\right) \cos \frac{n\pi x}{L} \exp^{\frac{-n^2\pi^2 t}{L^2}}. \tag{19}$$

The solution was constructed by a Fourier series method with adjustment to nonhomogeneous boundary conditions. From the form of the solution it follows that this solution evaluated at $x = 0$ is eventually increasing in time and therefore there must exist a time t_1 for which there will be a jump in the hysteresis output. At this time the following equation must be satisfied:

$$u_1(0,t_1) = \rho_2 = \frac{t_1}{L} + \left(b_0 - \frac{L}{6}\right) + \sum_{n=1}^{\infty} \left(b_n - \frac{2L(-1)^n}{n^2\pi^2}\right) \exp^{\frac{-n^2\pi^2 t_1}{L^2}}. \tag{20}$$

At time t_1 there will be a change in the boundary condition because of the jump in the hysteresis output. The new boundary condition will be:

$$\frac{\partial u}{\partial x}(L,t) = -[\mathcal{F}(u(0,.))](t) = -1$$

and assuming continuity of our solution, we will get initial condition for an upper solution $u_2(x,t)$ in the form

$$u_2(x,0) = u_1(x,t_1) = \frac{t_1}{L} + \frac{x^2}{2L} + \left(b_0 - \frac{L}{6}\right) + \\ + \sum_{n=1}^{\infty} \left(b_n - \frac{2L(-1)^n}{n^2\pi^2}\right) \exp^{\frac{-n^2\pi^2 t_1}{L^2}} \cos \frac{n\pi x}{L}.$$

To find a solution of our equation by the Fourier series method, we need to have a Fourier series of the initial condition which in our case is

$$u_2(x,0) = \frac{t_1}{L} + b_0 + \\ + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2L(-1)^n}{n^2\pi^2} \right) \exp^{\frac{-n^2\pi^2 t_1}{L^2}} + \frac{2L(-1)^n}{n^2\pi^2} \right] \cos \frac{n\pi x}{L}.$$

The corresponding solution, computed again by Fourier series method will be

$$u_2(x,t) = -\frac{t}{L} - \frac{x^2}{2L} + \left(b_0 + \frac{t_1}{L} + \frac{L}{6} \right) + \\ + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2L(-1)^n}{n^2\pi^2} \right) \exp^{\frac{-n^2\pi^2 t_1}{L^2}} + \frac{4L(-1)^n}{n^2\pi^2} \right] \cos \frac{n\pi x}{L} \exp \frac{-n^2\pi^2 t}{L^2} \tag{21}$$

Again, from the form of the solution we can see that the solution will be eventually decreasing, this means, there must exist a time t_2 for which

$$u_2(0,t) = \rho_1 = -\frac{t_2}{L} + \left(b_0 + \frac{t_1}{L} + \frac{L}{6} \right) + \\ + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2L(-1)^n}{n^2\pi^2} \right) \exp^{\frac{-n^2\pi^2 t_1}{L^2}} + \frac{4L(-1)^n}{n^2\pi^2} \right] \exp^{-\frac{n^2\pi^2 t_2}{L^2}}.$$

At this time t_2 there will be a jump in the hysteresis output and in the same way as before our solution will change to u_3. The new boundary condition will be

$$\frac{\partial u}{\partial x}(L,t) = -[\mathcal{F}(u(0,.))](t) = 1$$

and the new initial condition

$$u_3(x,0) = u_2(x,t_2) = -\frac{t_2}{L} - \frac{x^2}{2L} + \left(b_0 + \frac{t_1}{L} + \frac{L}{6} \right) + \\ + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2L(-1)^n}{n^2\pi^2} \right) \exp^{\frac{-n^2\pi^2 t_1}{L^2}} + \frac{4L(-1)^n}{n^2\pi^2} \right] \cos \frac{n\pi x}{L} \exp^{-\frac{n^2\pi^2 t_2}{L^2}}.$$

Its Fourier series is

$$u_3(x,0) = -\frac{t_2}{L} + \left(b_0 + \frac{t_1}{L} \right) + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2L(-1)^n}{n^2\pi^2} \right) e^{\frac{-n^2\pi^2 (t_1+t_2)}{L^2}} + \\ + \frac{4L(-1)^n}{n^2\pi^2} e^{-\frac{n^2\pi^2 t_2}{L^2}} - \frac{2L(-1)^n}{n^2\pi^2} \right] \cos \frac{n\pi x}{L}.$$

Therefore the searched solution will now be, again computed by Fourier series method:

$$\begin{aligned} u_3(x,t) = & \frac{t}{L} + \frac{x^2}{2L} + \left(b_0 + \frac{t_1}{L} - \frac{t_2}{L} - \frac{L}{6}\right) + \\ & + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2L(-1)^n}{n^2\pi^2}\right) \exp^{\frac{-n^2\pi^2(t_1+t_2)}{L^2}} + \right. \\ & \left. + \frac{4L(-1)^n}{n^2\pi^2} \exp^{-\frac{n^2\pi^2 t_2}{L^2}} - \frac{4L(-1)^n}{n^2\pi^2} \right] \cos\frac{n\pi x}{L} \exp^{-\frac{n^2\pi^2 t}{L^2}}. \end{aligned} \tag{22}$$

To find a periodic solution, it will be sufficient to put

$$u_3(x,t) = u_1(x,t).$$

Comparing corresponding terms in the latest equation we get

$$t_1 = t_2 \tag{23}$$

as well as the condition for the coefficients b_n:

$$b_n = -\frac{2L(-1)^n}{n^2\pi^2} \frac{\left[1 - e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}{\left[1 + e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}. \tag{24}$$

This is the necessary condition for our solution to be periodic. We can easily see that this is also a sufficient condition.

So far, no condition on the parameter b_0 is imposed and also t_1 has not been determined yet. This can be done as follows: It follows from (18) and (25) that

$$\rho_1 = b_0 - \sum_{n=1}^{\infty} \frac{2L(-1)^n}{n^2\pi^2} \frac{\left[1 - e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}{\left[1 + e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}. \tag{25}$$

On the other hand it follows from (20) and (24) that

$$\rho_2 = \frac{t_1}{L} + b_0 + \sum_{n=1}^{\infty} \frac{2L(-1)^n}{n^2\pi^2} \frac{\left[1 - e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}{\left[1 + e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}, \tag{26}$$

where we used in the last equation the identity

$$\frac{L}{6} = -\sum_{n=1}^{\infty} \frac{2L(-1)^n}{n^2\pi^2}. \tag{27}$$

Combining (25) and (26), we get

$$t_1 = L(\rho_2 + \rho_1) - 2Lb_0, \tag{28}$$

or

$$t_1 = L(\rho_2 - \rho_1) - 2L\sum_{n=1}^{\infty} \frac{2L(-1)^n}{n^2\pi^2} \frac{\left[1 - e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}{\left[1 + e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}. \tag{29}$$

We will show that for given ρ_1, ρ_2 and L sufficiently small ($L < \rho_2 - \rho_1$), the equation (29) has a unique solution t_1. Then (28) determines b_0 uniquely and then this together with the previous will imply the statement of the theorem.

To show that (30) has a unique solution t_1, consider the function

$$f(t_1, L) = t_1 + 2l\sum_{n=1}^{\infty} \frac{2L(-1)^n}{n^2\pi^2} \frac{\left[1 - e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}{\left[1 + e^{-\frac{n^2\pi^2 t_1}{L^2}}\right]}. \tag{30}$$

t_1 is a solution if and only if

$$f(t_1, L) = L(\rho_2 - \rho_1) \tag{31}$$

or, alternatively

$$\frac{1}{L^2} f(t_1, L) = \frac{\rho_2 - \rho_1}{L} \tag{32}$$

The last expression has the advantage that the left hand side can be understood to be a function of a single variable only,

$$z := \frac{t_1}{L^2}, \tag{33}$$

and equation (32) can then be written in the form

$$h(z) := z + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} \frac{[1 - e^{-n^2\pi^2 z}]}{[1 + e^{-n^2\pi^2 z}]} = \frac{\rho_2 - \rho_1}{L} \tag{34}$$

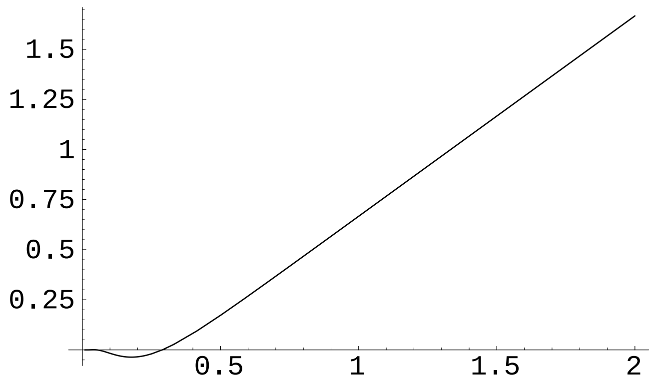

Figure 3. A graph of the function $h(z)$.

Simple estimates show that the function $h(z)$ takes any positive value only once for $h(z) > 1$ with corresponds to $L < \rho_2 - \rho_1$. It also follows that (34) has a solution t_1, so existence and uniqueness of the solution of Theorem 2 follows. □

Remark 4. The bound on L for uniqueness in Theorem 2 can be clearly improved. A numerical computation suggests that there may not be a bound at all since the function $h(z)$ appears to take any positive value exactly once (see Figure 3) but the authors did not find a suitable estimate for small values of z.

Remark 5. An analogous computation for the model from Remark 1, where the hysteresis operator $\mathcal{F}$ is replaced by $\mathcal{F}_{a,b}$ can be provided. In this case there also exists a periodic solution of the equation (2) with boundary conditions (3) and (4). The proof can be done in the same way as the proof of the Theorem 2, there will be some differences in the form of the corresponding solutions computed by the Fourier series method because of the different boundary conditions. For example the solution from (19) will be replaced by

$$u_1(x,t) = \frac{at}{L} + \frac{ax^2}{2L} + \left(b_0 - \frac{aL}{6}\right) + \sum_{n=1}^{\infty} \left(b_n - \frac{2aL(-1)^n}{n^2\pi^2}\right) \cos\frac{n\pi x}{L} e^{\frac{-n^2\pi^2 t}{L^2}}.$$

The upper solution u_2 will be (compare with (21)):

$$\begin{aligned} u_2(x,t) = & -\frac{bt}{L} - \frac{bx^2}{2L} + \left(b_0 + \frac{at_1}{L} + \frac{bL}{6}\right) + \\ & + \sum_{n=1}^{\infty} \left[\left(b_n - \frac{2aL(-1)^n}{n^2\pi^2}\right) e^{\frac{-n^2\pi^2 t_1}{L^2}} + \frac{2L(a+b)(-1)^n}{n^2\pi^2}\right] \cos\frac{n\pi x}{L} \exp^{-\frac{n^2\pi^2 t}{L^2}}. \end{aligned}$$

The solution u_3 will be (again, compare with (23)):

$$u_3(x,t) = \frac{at}{L} + \frac{ax^2}{L} + \left(b_0 + \frac{at_1}{L} - \frac{bt_2}{L} - \frac{aL}{6}\right) + \\ + \sum_{n=1}^{\infty}\left[\left(b_n - \frac{2aL(-1)^n}{n^2\pi^2}\right)\exp^{\frac{-n^2\pi^2(t_1+t_2)}{L^2}} + \right. \\ \left. + \frac{2L(a+b)(-1)^n}{n^2\pi^2}\exp^{-\frac{n^2\pi^2 t_2}{L^2}} - \frac{2L(a+b)(-1)^n}{n^2\pi^2}\right]\cos\frac{n\pi x}{L}\exp^{-\frac{n^2\pi^2 t}{L^2}}.$$

In the same way as before, the assumption about periodicity of the solution implies the condition for the coefficients b_n, which in this case looks like:

$$b_n = -\frac{2L(a+b)(-1)^n}{n^2\pi^2}\frac{\left[1-e^{-\frac{n^2\pi^2 t_2}{L^2}}\right]}{\left[1-e^{-\frac{n^2\pi^2(t_1+t_2)}{L^2}}\right]} + \frac{2La(-1)^n}{n^2\pi^2},$$

and also a condition for t_1 and t_2 which takes the form

$$t_1 = \frac{b}{a}t_2. \tag{35}$$

A natural question now may be what the period of the periodic solution computed in the proof of Theorem 2 is and how this can be compared with the period of the periodic solution of the O.D.E. (1) with hysteresis and delay. Unfortunately, because of the complexity of the form of the computed periodic solution, this question can be answered only partially.

There is one problem with the last equation, namely that b_0 still depends on t_1 and in a nonlinear way. For $L \ll \rho_2 - \rho_1$ and thus $\frac{t_1}{L^2} \gg 1$, we can, using the indentity (27), compute approximately from (29) that

$$t_1 = L(\rho_2 - \rho_1) + \frac{L^2}{3}.$$

Therefore the approximated period for the periodic solution from the proof of the Theorem 2 is

$$T = 2L(\rho_2 - \rho_1) + \frac{2L^2}{3}. \tag{36}$$

5. Conclusion

The solutions of two equations, the heat equation with relay hysteresis on the boundary and an ordinary differential equation with hysteresis and time delay were considered. In both cases, periodic two-phase solutions were calculated and their periods determined (see Theorem 1 and Theorem 2):

$$T_{ODE} = 2L(\rho_2 - \rho_1) + 4kL^2 \qquad \text{for } L \ll 1 \tag{37}$$

$$T_{PDE} = 2L(\rho_2 - \rho_1) + \frac{2}{3}L^2 \qquad \text{for } L \ll \rho_2 - \rho_1 \tag{38}$$

The periods are found to agree to first order in L and can be made to agree to second order if L is small (i.e., $L \ll 1$ and $L \ll \rho_2 - \rho_1$) and k is chosen $\frac{1}{6}$.

Thus the ordinary differential equation of the effective model is a good description for a short rod and can be expected to work well for any spatially small system, though with a possibly different constant k.

If k is not determined, the obtained results still provide for the scaling of the precision of the model and are thus useful.

We would like to point out that other periodic solutions not considered here may exist. The question of the uniqueness of periodic solutions of (2)–(4) is an open problem and probably a very complex one. Periodic solutions which are not two-phase may exist. See [9, 10] for some ideas related to this problem.

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