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Strong relaxation of the isothermal Euler system to the heat equation

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Abstract. We consider the system of isothermal Euler Equations with a strong damping. For large BV solutions, we show that the density converges to the solution to the heat equation when the friction coefficient ε^{-1} tends to infinity. Our estimates are already valid for small time, including in the initial layer. They are global in space (and even in time when the limits of the density are the same at $\pm\infty$) and they provide rates of convergence when $\varepsilon \to 0$.

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1. Statement of the results

In this paper, we consider the flow of a compressible fluid trough a porous medium, namely the Euler system with damping, written here in the one-dimensional case

$$\frac{\partial \rho^\varepsilon}{\partial t} + \frac{\partial}{\partial x}\left(\rho^\varepsilon u^\varepsilon\right) = 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}(\rho^\varepsilon u^\varepsilon) + \frac{\partial}{\partial x}\left(\rho^\varepsilon (u^\varepsilon)^2 + p(\rho^\varepsilon)\right) = -\frac{\rho^\varepsilon u^\varepsilon}{\varepsilon}. \tag{1.2}$$

In this model, ρ^ε is the density, u^ε is the velocity, p is the pressure and ε^{-1} the friction coefficient. In the isothermal case, the pressure is given by:

$$p(\rho) = \rho. \tag{1.3}$$

We consider the problem (1.1), (1.2), (1.3), with the initial data:

$$\rho^\varepsilon(0,x) = \rho_0(x), \qquad u^\varepsilon(0,x) = u_0(x). \tag{1.4}$$

We assume that the initial data are BV functions, i.e. functions with bounded variation, and satisfy:

$$\inf_{x\in\mathbb{R}} \rho_0(x) \geq \rho_{min} > 0, \quad \rho_{\pm\infty} = \lim_{x\to\pm\infty} \rho_0(x), \quad u_{0,\pm\infty} = \lim_{x\to\pm\infty} u_0(x). \tag{1.5}$$

In this paper we consider L^1 perturbations of Riemann data :

$$(\rho_0 - \overline{\rho}_0, u_0 - \overline{u}_0) \in L^1(\mathbb{R}), \quad \text{where } (\overline{\rho}_0, \overline{u}_0)(x) := (\rho_{\pm\infty}, u_{0,\pm\infty}),\ \pm x > 0. \tag{1.6}$$

The existence of an entropy solution of (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), is given for instance by the splitting scheme used in [23].
When $\varepsilon \to 0$, we are going to prove that the density converges to the solution r of the following heat equation, where $s = \varepsilon t$ is a "slow" time.

$$\frac{\partial r}{\partial s} - \frac{\partial^2 r}{\partial x^2} = 0, \qquad r(0,x) = \rho_0(x). \tag{1.7}$$

Our main result is:

Theorem 1.1. *Assume that (1.5) and (1.6) are satisfied. For any* $\varepsilon \in (0,1]$*, let* $(\rho^\varepsilon, u^\varepsilon)$ *be an entropy solution to (1.1), (1.2), (1.3), (1.4), constructed by the splitting scheme used in [23]. Then*
(i) There exists C*, such that for all* $T > 0$*:*

$$\int_0^T \int_{\mathbb{R}} |\rho^\varepsilon(t,x) - r(\varepsilon t, x)|^2 dxdt \leq C\varepsilon \left(1 + \sqrt{\varepsilon T}\right). \tag{1.8}$$

(ii) Furthermore, if $\varrho_{-\infty} = \varrho_{+\infty}$*, we have:*

$$\int_0^{+\infty} \int_{\mathbb{R}} |\rho^\varepsilon(t,x) - r(\varepsilon t, x)|^2 dxdt \leq C\varepsilon. \tag{1.9}$$

These estimates can be used to study two limits, first, when $\varepsilon \to 0$, and next when $t \to +\infty$ and $\varepsilon > 0$ is fixed. The first limit is clearly described in formulas (1.7) to (1.9). In order to describe the second limit, let us first recall that the solution of the heat equation converges for large time to the unique self-similar solution with the same limits at $\pm\infty$:

$$\frac{\partial \overline{r}}{\partial t} - \frac{\partial^2 \overline{r}}{\partial x^2} = 0, \quad \overline{r}(t,x) = \overline{r}(z) \text{ where } z = \frac{x}{\sqrt{t}} \quad \text{and } \overline{r}(\pm\infty) = \rho_{\pm\infty}, \tag{1.10}$$

see [8, 9]. Trivially, if $\rho_{+\infty} = \rho_{-\infty}$, $\overline{r}$ is a constant: $\overline{r} \equiv \rho_\infty$. In this case, we give a few results of convergence. In general, $\overline{r}$ has the following classical expression: $\overline{r}(z) = \rho_{-\infty} + (\rho_{+\infty} - \rho_{-\infty})/\sqrt{4\pi} \int_{-\infty}^z \exp(-w^2/4) dw.$
Combining this with Theorem 1.1, we give in Corollary 1.1 straightforward consequences of Theorem 1.1. Then, using a Hardy type of Lemma as in [9], we obtain better results of convergence in Theorem 1.2.
Since ε is fixed, we fix $\varepsilon = 1$ and we drop the superscript ε.

Corollary 1.1. *Under the same assumption as in Theorem 1.1, there exists* $C > 0$ *, such that any entropy solution constructed by this scheme satisfies:*

$$\frac{1}{T}\int_0^T \int_{\mathbb{R}} |\rho(t,x) - \overline{r}(x/\sqrt{t})|^2 dxdt \leq \frac{C}{\sqrt{T}}. \tag{1.11}$$

Furthermore, if $\rho_{-\infty} = \rho_{+\infty} = \rho_\infty$ *, we have:*

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |\rho(t,x) - \rho_\infty|^2 dx = 0. \tag{1.12}$$

In general, there is no better *rate* of convergence of a solution to the heat equation to the self-similar solution.
Combining now Theorem 1.1 and Corollary 1.1 with results of [9], and using the same variables, we improve the above rate of convergence of the density towards the self-similar solution $\overline{r}$.

Theorem 1.2. *Under the same assumption as in Theorem 1.1, for any fixed* $L \in]0,\infty]$ *, there exists* $C_L > 0$ *, such that for any entropy solution constructed by the same scheme satisfies:*

$$\int_{\mathbb{R}} |\rho(t,z\sqrt{t}) - \overline{r}(z)|^2 dz \leq \frac{C_\infty}{t^{1/4}}, \tag{1.13}$$

$$\int_{|z|\leq L} |\rho(t,z\sqrt{t}) - \overline{r}(z)|^2 dz \leq \frac{C_L}{\sqrt{t}}. \tag{1.14}$$

Furthermore, if $\rho_{-\infty} = \rho_{+\infty} = \rho_\infty$ *, we have:*

$$\int_{\mathbb{R}} |\rho(t,z\sqrt{t}) - \rho_\infty|^2 dz \leq \frac{C_\infty}{\sqrt{t}} \tag{1.15}$$

$$\int_{|z|\leq L} |\rho(t,z\sqrt{t}) - \rho_\infty|^2 dz \leq \frac{C_L}{t^{3/4}}. \tag{1.16}$$

Let us now comment these results. In Theorem 1.1, we consider the limit when $\varepsilon \to 0$ of *large weak* entropy solutions to (1.1), (1.2), (1.3), (1.4), and we prove the convergence towards the solution of the heat equation with the *same* initial datum $\overline{\rho}_0$. The convergence is established in the space $L^2_{t,x}$, *globally* in space, and *globally* in time if the limits $\rho_{\pm\infty}$ of the density when $x \to \pm\infty$ are the same. When $\rho_{+\infty} \neq \rho_{-\infty}$, the L^2 convergence is established in any strip $\left\{(t,x); 0 \leq t \leq \frac{T}{\varepsilon^{3-\delta}}, x \in \mathbb{R}\right\}$, for any $T > 0$ and $\delta > 0$. We note that e.g. results obtained by the compensated compactness [17, 18, 21] are in general *local.*
We also refer to [14], who studied the zero relaxation limit of a slightly more general system. They show the convergence, in the original variables (t,x), to the solution to $\partial_t \rho = 0,\ \rho(0,x) = \rho_0(x)$. Our result can be viewed as a *refinement* of

the above result, and $r - \rho_0$ as a corrector term.

In Corollary 1.1 and Theorem 1.1 we have combined the above results with the classical decay estimates for the heat equation, to study the limit $t \to +\infty, \varepsilon > 0$ fixed.

In the case of a general p-system, and for *small smooth* solutions, H. Ling and T.P. Liu have studied in [8], the limit $t \to +\infty$, $\varepsilon > 0$ fixed, and obtained precise rates of convergence when $t \to +\infty$ towards the self-similar solution to the corresponding porous media equation, using an appropriate shift of coordinates, for which some moment of the initial data vanishes.

On the other hand, the same limit $t \to +\infty$, $\varepsilon > 0$ fixed, has been studied in [9] for a p-system with a change of convexity, where the Authors show, for any $L > 0$, the L^2 convergence of *large weak* entropy solutions at time t, in a parabolic domain $\{x;\ x^2 \leq Lt\}$, but do not give any rate of convergence.

Let us emphasize that our results of convergence are not restricted to large times, but are already valid for *small* times, including in the initial layer. In this spirit observe that (1.9) would be optimal if the difference $(\rho^\varepsilon(t,.) - r(\varepsilon t,.))$ behaved like $\exp(-t/\varepsilon)$ when $\varepsilon \to 0_+$.

The outline of the proof of Theorem 1.1 is as follows. First, using the same scaling as in [19], see also [10, 17, 18], we introduce a slow time $s := \varepsilon t$, and define

$$\varrho^\varepsilon(s,x) = \rho^\varepsilon\left(\frac{s}{\varepsilon}, x\right); \qquad v^\varepsilon(s,x) = \frac{1}{\varepsilon} u^\varepsilon\left(\frac{s}{\varepsilon}, x\right), \tag{1.17}$$

(note the difference of notations for the density). In the new variables the system becomes:

$$\frac{\partial \varrho^\varepsilon}{\partial s} + \frac{\partial}{\partial x}\left(\varrho^\varepsilon v^\varepsilon\right) = 0, \tag{1.18}$$

$$\varepsilon^2 \frac{\partial}{\partial s}(\varrho^\varepsilon v^\varepsilon) + \varepsilon^2 \frac{\partial}{\partial x}\left(\varrho^\varepsilon (v^\varepsilon)^2\right) + \frac{\partial \varrho^\varepsilon}{\partial x} = -\varrho^\varepsilon v^\varepsilon, \tag{1.19}$$

with the initial data

$$\varrho^\varepsilon(0,x) = \rho_0(x), \qquad v^\varepsilon(0,x) = \frac{u_0(x)}{\varepsilon}. \tag{1.20}$$

Assuming that we can pass to the limit in (1.19), and combining with (1.18), we formally obtain (1.7).

Our method of proof uses a stream function wich plays the role of the Lagrangian mass coordinate, and a suitable integration by parts which provides *global* estimates, contrarily to the div-curl lemma, wich only implies *local* strong convergences. Also, we estimate the L^2 norm *in space and time* of the difference between the density and its limit.

The outline of this paper is as follows. In section 2, we sketch the proof of Theorem 1.1 in the simpler case of compactly supported initial data, with a general pressure law, where there is no technical difficulty. In section 3, we give a BV estimate, uniform both in time and in ε, and we obtain a positive lower bound for ϱ^ε. In section 4, we give an entropy inequality and deduce various a priori estimates. In section 5, we introduce the Lagrangian mass coordinate ψ . In section 6, we use a modification of the div-curl lemma to conclude the proof of Theorem 1.1. In section 7, we study the large time behavior. Finally, in section 8, 9, and 10, we have collected the (tedious) proofs of some technical results.

2. The case of compactly supported initial data, with a general pressure law

In this Section, we give the main ideas to prove Theorem 1.1 in the simple case of compactly supported initial data. The key-tool is the stream-function ψ associated to the equation of mass conservation. Indeed, we multiply the momentum equation by ψ , and after some calculations, we are going to justify (1.9) in this simple case.
More precisely, let ρ_∞ , $u_\infty \in \mathbb{R}$, assume that the pressure law $p(\rho)$, satisfies $p \in C^0([0,+\infty)) \cap C^2((0,+\infty))$, and assume that: There exists a globally defined solution $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$, uniformly bounded in L^∞ , in ε , such that,

$$\forall T > 0, (\rho^\varepsilon(t,x) - \rho_\infty) \text{ and } (\rho^\varepsilon(t,x) u^\varepsilon(t,x) - \rho_\infty u_\infty) \tag{2.1}$$

are compactly supported, on $(0,T) \times \mathbb{R}$

$$\rho_{min} := \inf_{x\in\mathbb{R}} \rho_0(x) > 0 \tag{2.2}$$

$$p' > 0 \text{ and } p'' \geq 0 \text{ on }]0,+\infty[\tag{2.3}$$

For example, these asumptions are satisfied for $p(\rho) = \rho^\gamma$, $\gamma > 1$, see [21], see [15] for the case without source term. For $\gamma = 1$ see [22, 23] and see Section 3.

Theorem 2.1. *Assume that (2.1), (2.2), (2.3) are satisfied. Let r be the solution to*

$$\frac{\partial r}{\partial s} - \frac{\partial^2 p(r)}{\partial x^2} = 0, \qquad r(0,x) = \rho_0(x). \tag{2.4}$$

Then the sequence $(\rho^\varepsilon(t,x) - r(\varepsilon t, x))$ converges strongly in $L^2(\mathbb{R}^+ \times \mathbb{R})$ to 0 . More precisely, there exist a constant C , independent of ε , such that:

$$\int_0^{+\infty} \int_{\mathbb{R}} |\rho^\varepsilon(t,x) - r(\varepsilon t, x)|^2 dx dt \leq C\varepsilon. \tag{2.5}$$

Remark 2.1. In any case, even if the initial data do not avoid the vacuum, we obtain:

$$\int_0^{+\infty} \int_{\mathbb{R}} \left[p(\rho^\varepsilon(t,x)) - p(r(\varepsilon t,x))\right] \left[\rho^\varepsilon(t,x) - r(\varepsilon t,x)\right] dx dt \leq C\varepsilon. \tag{2.6}$$

Proof. Setting $\varrho^\varepsilon(s,x) := \rho^\varepsilon(s/\varepsilon, x)$, $v^\varepsilon(s,x) := u^\varepsilon(s/\varepsilon, x)/\varepsilon$, $p := p(r)$, $\pi^\varepsilon := p(\varrho^\varepsilon)$, the system rewrites:

$$\partial_s(\varrho^\varepsilon - r) + \partial_x(\varrho^\varepsilon v^\varepsilon + \partial_x p) = 0, \tag{2.7}$$

$$\varepsilon^2\partial_s(\varrho^\varepsilon v^\varepsilon) + \varepsilon^2\partial_x(\varrho^\varepsilon (v^\varepsilon)^2) + \partial_x(\pi^\varepsilon - p) = -(\varrho^\varepsilon v^\varepsilon + p). \tag{2.8}$$

Define the stream function z^ε from the first equation by:

$$\partial_x z^\varepsilon := \varrho^\varepsilon - r, \quad \partial_s z^\varepsilon := -(\varrho^\varepsilon v^\varepsilon + \partial_x p), \quad z^\varepsilon(0,x) := 0.$$

Then multiply (2.8) by z^ε and integrate by parts over the strip $[0,S]\times\mathbb{R}$, to obtain:

$$\varepsilon^2 \int_0^S \int_{\mathbb{R}} \varrho^\varepsilon v^\varepsilon(\varrho^\varepsilon v^\varepsilon + \partial_x p)dxds$$

$$+\varepsilon^2 \int_{\mathbb{R}} \varrho^\varepsilon v^\varepsilon z^\varepsilon(S,x)dx - \varepsilon^2 \int_0^S \int_{\mathbb{R}} \varrho^\varepsilon (v^\varepsilon)^2(\varrho^\varepsilon - r)dxds \tag{2.9}$$

$$:= A + B + C = \int_0^S \int_{\mathbb{R}} (\pi^\varepsilon - p)(\varrho^\varepsilon - r)dxds$$

$$+1/2\int_{\mathbb{R}} (z^\varepsilon)^2(S,x)dx := C + D. \tag{2.10}$$

Now, the classical entropy inequality:

$$\partial_t(\rho^\varepsilon (u^\varepsilon)^2/2 + P(\rho^\varepsilon)) + \partial_x\left[\rho^\varepsilon(u^\varepsilon)^3/2 + P'(\rho^\varepsilon)\rho^\varepsilon u^\varepsilon\right] + \rho^\varepsilon(u^\varepsilon)^2/\varepsilon \le 0,$$

where $P \ge 0$ and $P''(\rho) := p'(\rho)/\rho$, can be rewritten in (s,x) variables:

$$\partial_s(\varepsilon^2\varrho^\varepsilon(\widetilde{u}^\varepsilon)^2/2 + P(\varrho^\varepsilon)) + \partial_x\left[\varrho^\varepsilon(\widetilde{u}^\varepsilon)^3/2 + P'(\varrho^\varepsilon)\rho^\varepsilon \widetilde{u}^\varepsilon\right] + \varrho^\varepsilon(\widetilde{u}^\varepsilon)^2 \le 0. \tag{2.11}$$

Now, (2.11) implies

$$\varepsilon^2\int_{\mathbb{R}} \varrho^\varepsilon(v^\varepsilon)^2(S,x)dx + \int_0^S\int_{\mathbb{R}} \varrho^\varepsilon (v^\varepsilon)^2 dxds = O(1). \tag{2.12}$$

Now we can study the terms of (2.9),(2.10):

$$\begin{aligned} A &:= \varepsilon^2\int_0^S\int_{\mathbb{R}} \varrho^\varepsilon v^\varepsilon(\varrho^\varepsilon v^\varepsilon + \partial_x p)dxds \\ &= A_1 + A_2 = \varepsilon^2\int_0^S\int_{\mathbb{R}} \varrho^\varepsilon[\varrho^\varepsilon(v^\varepsilon)^2]dxds + \varepsilon^2\int_0^S\int_{\mathbb{R}} \varrho^\varepsilon v^\varepsilon \partial_x p dxds = \mathrm{O}\left(\varepsilon^2\right). \end{aligned}$$

Then, A_1 is bounded by (2.1) and (2.12). On the other hand, since r is the solution to (2.4), $\partial_x p(r)$ is classicaly bounded in L^2. Therefore, by Cauchy-Schwarz inequality, we obtain $A_2 = \mathrm{O}\left(\varepsilon^2\right)$.
Now, by (2.12) and Young inequality, we have:

$$\begin{aligned} B &:= \varepsilon^2\int_{\mathbb{R}} \varrho^\varepsilon v^\varepsilon z^\varepsilon(S,x)dx \le 2\varepsilon^2\int_{\mathbb{R}}(\varrho^\varepsilon v^\varepsilon)^2(S,x)dx + \varepsilon^2/8\int_{\mathbb{R}}(z^\varepsilon)^2(S,x)dx \\ &\le \mathrm{O}(1) + \varepsilon^2/8\int_{\mathbb{R}}(z^\varepsilon)^2(S,x)dx. \end{aligned}$$

Finally, $C := \varepsilon^2 \int_0^S \int_{\mathbb{R}} \varrho^\varepsilon (v^\varepsilon)^2 (\varrho^\varepsilon - r) dx ds = \mathrm{O}\left(\varepsilon^2\right)$ by (2.1) and by (2.12).
Combining these estimates for $0 < \varepsilon \leq 1$:

$$\int_0^S \int_{\mathbb{R}} (\pi^\varepsilon - p)(\varrho^\varepsilon - r) dx ds + 1/4 \int_{\mathbb{R}} (z^\varepsilon)^2 (S, x) dx = O(\varepsilon^2). \tag{2.13}$$

Since r (but perhaps not ϱ^ε) avoids the vacuum and since the pressure satisfies (2.3),

$$(p(x) - p(y))(x - y) \geq \alpha (x - y)^2, \qquad \forall x \geq \rho_{min}, \text{ and } y \geq 0,$$

with $\alpha := (p(\rho_{min}) - p(0))/\rho_{min} > 0$, which concludes the proof. □

3. A uniform BV estimate

For any fixed $\varepsilon > 0$, it is proven in [23] that a more general system - the Isothermal Euler-Poisson system - admits a globally defined weak entropy solution

$$(\rho^\varepsilon, u^\varepsilon) \in C^0((0, +\infty), L^1_{loc}(\mathbb{R})) \cap L^\infty_{loc}((0, +\infty), BV(\mathbb{R})).$$

We recall that BV is the space of functions of x with bounded variation, i.e. whose derivatives live in the space $M_1(\mathbb{R})$ of bounded measures on $\mathbb{R}$.
For the Isothermal Euler-Poisson system, see [10], it is not clear whether this BV estimate is uniform with respect to ε , and even with respect to time when $t \to +\infty$. Here, in this simpler case, this is indeed the case.

Proposition 3.1 (L^∞ and BV bounds). *There exists $0 < \varrho^*_{min} \leq \varrho^*_{max}$, and K such that $\forall \varepsilon > 0$, $\forall (s, x) \in (0, +\infty) \times \mathbb{R}$:*

$$0 < \varrho^*_{min} \leq \varrho^\varepsilon(s, x) \leq \varrho^*_{max}, \qquad TV \varrho^\varepsilon(s, .) \leq K, \tag{3.1}$$

$$|v^\varepsilon(s, x)| \leq \frac{K}{\varepsilon}, \qquad TV v^\varepsilon(s, .) \leq \frac{K}{\varepsilon}. \tag{3.2}$$

The proof of Proposition 3.1 It uses classical ideas, (see [13, 22, 23], ... and more recently [14])). For convenience, we do not give the classical proof.

4. An entropy inequality

As in Section 2, it is easy to establish *all* the estimates in this Section where u_0 is compactly supported and if ρ_0 has the same limits as $x \to \pm\infty$. In the general case, see also [8], [9], ... the proof is (unfortunately!) much longer, and the estimates are no longer uniform in time. For clarity, we divide the proof in several steps.
In this section we establish the entropy inequality in order to control the kinetic

energy. Observe that the initial data are *not* in L^1, since they do not vanish at infinity. Therefore, we substract simple functions wich have the same limits as the solution at $\pm\infty$.
Let $m(x)$ a nonnegative smooth function with "unit mass", see [8].
Let $H(x) := \int_{-\infty}^{x} m(y)dy$. Therefore H is a regularization of the Heaviside function. We introduce $\forall (s,x) \in (0,+\infty)\times\mathbb{R}$:

$$V(s,x) \quad := \quad \frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)(u_{0,-\infty} + (u_{0,+\infty} - u_{0,-\infty})H(x)), \tag{4.1}$$

and we note that

$$V(s,\pm\infty) = v^\varepsilon(s,\pm\infty), \qquad \frac{\partial V}{\partial s} \quad = \quad -\frac{1}{\varepsilon^2}V. \tag{4.2}$$

The aim of this section is to deduce the following estimates from the entropy inequality.

Proposition 4.1 (Bounds on the kinetic energy). *There exists a constant* C, *independent of* ε, *such that for all* $S > 0$

$$\varepsilon^2 \int_{\mathbb{R}} \rho^\varepsilon (v^\varepsilon - V)^2(S,x)dx \quad \leq \quad C\left(1 + \Delta\varrho\sqrt{S}\right), \tag{4.3}$$

$$\int_0^S \int_{\mathbb{R}} \rho^\varepsilon (v^\varepsilon - V)^2(s,x)dxds \quad \leq \quad C\left(1 + \Delta\varrho\sqrt{S}\right), \tag{4.4}$$

$$\Delta\varrho \quad := \quad |\rho_{+\infty} - \rho_{-\infty}|. \tag{4.5}$$

<u>**Step 1**</u> : First, we need to recall a few bounds for the solution to the heat equation. The behavior is different if $\rho_{-\infty} = \rho_{+\infty}$ or $\rho_{-\infty} \neq \rho_{+\infty}$. The following bounds are optimal when $\Delta\varrho \neq 0$.

Lemma 4.1 (Bounds for the heat equation). r *is solution of (1.7) with BV initial data. We have the following bounds for all* $(s,S,x) \in (0,+\infty)^2 \times \mathbb{R}$, *where* $\Delta\varrho$ *is defined by (4.5)*

$$0 < \varrho^*_{min} \leq r(s,x) \leq \varrho^*_{max}, \tag{4.6}$$

$$\int_{\mathbb{R}} \left|\frac{\partial r}{\partial x}\right|(s,x)dx = O(1), \quad \int_0^S \int_{\mathbb{R}} \left(\frac{\partial r}{\partial x}\right)^2 (s,x)dx = O\left(1 + \Delta\varrho\sqrt{S}\right), \tag{4.7}$$

$$\int_{\mathbb{R}} \left|\frac{\partial^2 r}{\partial x^2}\right|(s,x)dxds = O\left(\frac{1}{\sqrt{s}}\right), \quad \int_0^S \int_{\mathbb{R}} \left|\frac{\partial^2 r}{\partial x^2}\right|(s,x)dxds = O\left(\sqrt{S}\right). \tag{4.8}$$

Proof. The results (4.6) are obvious. In order to prove first inequality (4.7), for instance, we differentiate (1.7) with respect to x, next we multiply by the sign of

$\partial_x r$, integrate over $\mathbb{R}$, and integrate by parts the second term: $\partial_s \int_{\mathbb{R}} |\partial_x r|(s,x)dx + \int_{\mathbb{R}} sign'(\partial_x r)(\partial_x^2 r)^2(s,x)dx = 0$. Therefore, $\partial_s \int_{\mathbb{R}} |\partial_x r|(s,x)dx \leq 0$, which gives us first the inequality (4.7).
If $\Delta\varrho = 0$, second inequality (4.7) is the classical energy estimate for heat equation. If $\Delta\varrho > 0$, we write $r(s,x) = (E(s,.) * \rho_0(.))(x)$ with $E(s,x) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{x^2}{4s}\right)$. Since $\int_{\mathbb{R}} E^2(s,x)dx = \mathrm{O}\left(\frac{1}{\sqrt{s}}\right)$ and $\partial_x r = E * \partial_x \rho_0$, we can estimate $\|\partial_x r(s,.)\|^2_{L^2(\mathbb{R})}$. Integrating in s , over $(0,S)$ we obtain the second inequality (4.7).
We prove (4.8) in a similar way. For instance, we see that inequalities are optimal, if $\Delta\varrho \neq 0$, by computing the exact solution with a Heaviside initial data.

□

Step 2: **Entropy inequality.** We need to substract suitable functions, in order to deal with integrable functions. In (s,x) variables, we have

$$\frac{\partial}{\partial s}\left(\varepsilon^2 \varrho^\varepsilon \frac{(v^\varepsilon)^2}{2} + \varrho^\varepsilon \ln(\varrho^\varepsilon)\right) + \frac{\partial}{\partial x}\left(\varepsilon^2 \varrho^\varepsilon \frac{(v^\varepsilon)^3}{2} + \varrho^\varepsilon v^\varepsilon \ln(\varrho^\varepsilon) + \varrho^\varepsilon v^\varepsilon\right) + \varrho^\varepsilon (v^\varepsilon)^2 \leq 0 \quad (4.9)$$

Let us define

$$\begin{aligned} \varphi(\rho) &:= \varphi(t,x,\rho) = \psi(\rho) - (\psi(r) + \psi'(r)(\rho - r)) \\ &= \rho \ln\left(\frac{\rho}{r}\right) - (\rho - r), \end{aligned} \quad (4.10)$$

where $\psi(\rho) := \rho\ln(\rho)$. By convexity $\varphi \geq 0$, and $\varphi(\varrho^\varepsilon)(s,\pm\infty) = 0$. Using (1.18), we obtain

$$\frac{\partial}{\partial s}\left(\varrho^\varepsilon \ln(\varrho^\varepsilon)\right) = \partial_s\left(\varphi(\varrho^\varepsilon)\right) - \partial_x((1+\ln r)\varrho^\varepsilon v^\varepsilon) + \varrho^\varepsilon v^\varepsilon \partial_x \ln r + \varrho^\varepsilon \partial_s \ln r - \partial_s r$$

Similarly, using (1.18), (1.19) and (4.2), we obtain (after some tedious calculations ...)

$$\begin{aligned} \varepsilon^2 \partial_s\left(\varrho^\varepsilon \frac{(v^\varepsilon)^2}{2}\right) &= \varepsilon^2 \partial_s\left(\varrho^\varepsilon \frac{(v^\varepsilon - V)^2}{2}\right) + \partial_x\left(\varrho^\varepsilon V\left(\varepsilon^2 v^\varepsilon\left(\frac{V}{2} - v^\varepsilon\right) - 1\right)\right) \\ &\quad + \varrho^\varepsilon (v^\varepsilon)^2\left(\varepsilon^2 \partial_x V\right) + \varrho^\varepsilon v^\varepsilon\left(-V(2+\varepsilon^2\partial_x V)\right) + \varrho^\varepsilon\left(V^2 + \partial_x V\right) \end{aligned}$$

Adding up the previous results we obtain the entropy inequality:

Lemma 4.2 (Entropy inequality).

$$\varepsilon^2\frac{\partial}{\partial s}\left(\varrho^\varepsilon\frac{(v^\varepsilon-V)^2}{2}\right)+\frac{\partial}{\partial s}\left(\varphi(\varrho^\varepsilon)\right)+\frac{\partial}{\partial x}Q_2+R_2\leq 0$$

$$\begin{aligned} Q_2 &= \varepsilon^2\varrho^\varepsilon\frac{(v^\varepsilon)^3}{2}+\varrho^\varepsilon v^\varepsilon\ln\left(\frac{\varrho^\varepsilon}{r}\right)+\varrho^\varepsilon V\left(\varepsilon^2 v^\varepsilon\left(\frac{V}{2}-v^\varepsilon\right)-1\right)\\ R_2 &= \varrho^\varepsilon(v^\varepsilon)^2(1+\varepsilon^2\partial_x V)+\varrho^\varepsilon v^\varepsilon\left(\partial_x\ln r-V(2+\varepsilon^2\partial_x V)\right)\\ &\quad+\varrho^\varepsilon\left(\partial_s\ln r+V^2+\partial_x V\right)-\partial_s r \end{aligned}$$

Step 3: In the next Lemma, we estimate $\partial_x Q_2$ and R_2 .

Lemma 4.3. $\displaystyle\int_0^S\int_{\mathbb{R}}\frac{\partial Q_2}{\partial x}dxds = O(\varepsilon)$; $R_2=\varrho^\varepsilon(v^\varepsilon-V)^2+R_3$;
$\displaystyle\int_0^S\int_{\mathbb{R}}|R_3|dxds\leq\frac{1}{2}\int_0^S\int_{\mathbb{R}}\varrho^\varepsilon(v^\varepsilon-V)^2dxds+O\left(1+\Delta\varrho\sqrt{S}\right)$.

Proof. For the first estimate, we obtain easily

$$\int_0^{+\infty}(Q_2(s,+\infty)-Q_2(s,-\infty))ds \quad=\quad \varepsilon(\rho_{-\infty}u_{0,-\infty}-\rho_{+\infty}u_{0,+\infty})$$

Now, we estimate R_2 . Replacing again $\varrho^\varepsilon(v^\varepsilon)^2$ by $\varrho^\varepsilon(v^\varepsilon-V)^2+2\varrho^\varepsilon v^\varepsilon V-\varrho^\varepsilon V^2$ we obtain:

$$\begin{aligned} R_3 &= \varrho^\varepsilon(v^\varepsilon-V)^2\varepsilon^2\partial_x V+\varrho^\varepsilon v^\varepsilon(\partial_x\ln r+\varepsilon^2 V\partial_x V)\\ &\quad+\varrho^\varepsilon(\partial_s\ln r+(1-\varepsilon^2V^2)\partial_x V)-\partial_s r\\ &= \left[\partial_x V\left(\varrho^\varepsilon(v^\varepsilon-V)^2\varepsilon^2+\varrho^\varepsilon v^\varepsilon\varepsilon^2 V+1-\varepsilon^2V^2\right)\right]\\ &\quad+\left[\varrho^\varepsilon v^\varepsilon\partial_x\ln r\right]+\left[\varrho^\varepsilon\partial_s\ln r-\partial_s r\right]\\ &:= R_3^1+R_3^2+R_3^3. \end{aligned}$$

Since $\varrho^\varepsilon,\varepsilon v^\varepsilon,\varepsilon V$ are bounded in L^∞ and $\partial_x V$ is bounded in L^1 , R_3^1 is bounded in L^1 , uniformly in ε and in s .
Now, since $\partial_s r=\partial_x^2 r$ we obtain $\|R_3^3\|_{L^1((0,S)\times\mathbb{R})}=\mathrm{O}\left(\sqrt{S}\right)$.
Finally we can estimate $|R_3^2|$:

$$\begin{aligned} |\varrho^\varepsilon v^\varepsilon\partial_x\ln r| &\leq |\sqrt{\varrho^\varepsilon}(v^\varepsilon-V)\sqrt{\varrho^\varepsilon}\partial_x\ln r|+|\varrho^\varepsilon V\partial_x\ln r|\\ &\leq \frac{1}{2}\varrho^\varepsilon(v^\varepsilon-V)^2+\frac{\varrho^\varepsilon}{2(r)^2}(\partial_x r)^2+\frac{\varrho^\varepsilon}{r}|V||\partial_x r| := a+b+c. \end{aligned}$$

By (4.6) and (4.7), $\|b\|_{L^1((0,S)\times\mathbb{R})}=\mathrm{O}\left(1+\Delta\varrho\sqrt{S}\right)$. Similarly, due to (4.7), and to the exponential decay of V in time, $\|c\|_{L^1((0,+\infty)\times\mathbb{R})}=\mathrm{O}(\varepsilon)$.

Finally, summing up these estimates, we obtain

$$\int_0^S \int_{\mathbb{R}} |R_3| dx ds \quad \leq \quad \frac{1}{2} \int_0^S \int_{\mathbb{R}} \varrho^\varepsilon (v^\varepsilon - V)^2 dx ds + \mathrm{O}\left(1 + \sqrt{S}\right),$$

which concludes the proof if $\Delta\varrho \neq 0$, i.e. if $\rho_{-\infty} \neq \rho_{+\infty}$. Note that $\mathrm{O}\left(\sqrt{S}\right)$ comes from the estimate on the derivative of r. So, if $\Delta\varrho = 0$ we can replace r by the constant $\rho_{+\infty}$, so that the above estimates are now uniform in S.

□

We now are able to prove Proposition 4.1:

Step 4 : **Proof of Proposition 4.1**: Integrating the entropy inequality (4.9) over $(0,S) \times (-L,+L)$, using the previous Lemma and passing to the limit when L goes to infinity, we obtain forall $S > 0$:

$$\varepsilon^2 \int_{\mathbb{R}} \varrho^\varepsilon \frac{(v^\varepsilon - V)^2}{2}(S,x) dx + \int_{\mathbb{R}} \varphi(\varrho^\varepsilon)(S,x) dx + \int_0^S \int_{\mathbb{R}} \varrho^\varepsilon \frac{(v^\varepsilon - V)^2}{2}(s,x) dx ds$$
$$= \mathrm{O}\left(1 + \Delta\varrho\sqrt{S}\right),$$

wich implies (4.3) and (4.4), since φ is nonnegative.

5. Euler system rewritten

Again, since the involved functions are not integrable on $\mathbb{R}$, we need to establish a few technical lemmas in this Section, before proving Theorem 1.1 in Section 6. These (tedious) calculations can be skipped at the first reading. We first rewrite Euler system (1.18), (1.19) in terms of $(\varrho^\varepsilon - r)$, and then we establish some useful bounds.

Lemma 5.1 (Euler system rewritten). *In* $(s,x) = (\varepsilon t, x)$, *functions* ϱ^ε *and* $v^\varepsilon(s,x) := u^\varepsilon(t,x)/\varepsilon$ *are solutions of*

$$\frac{\partial}{\partial s}(\varrho^\varepsilon - r) + \frac{\partial}{\partial x}\left(\varrho^\varepsilon v^\varepsilon + \frac{\partial r}{\partial x}\right) \quad = \quad 0, \tag{5.1}$$

$$\left.\begin{array}{l} \varepsilon^2 \dfrac{\partial}{\partial s}\left(\varrho^\varepsilon (v^\varepsilon - V)\right) + \varepsilon^2 \dfrac{\partial}{\partial x}\left(\varrho^\varepsilon (v^\varepsilon - V)^2\right) \\ + \dfrac{\partial}{\partial x}(\varrho^\varepsilon - r) \\ + \varepsilon^2 \dfrac{\partial}{\partial x}\left(\varrho^\varepsilon V (v^\varepsilon - V)\right) + \varepsilon^2 \varrho^\varepsilon v^\varepsilon \dfrac{\partial}{\partial x} V \end{array}\right\} \quad = \quad -\left(\varrho^\varepsilon (v^\varepsilon - V) + \frac{\partial r}{\partial x}\right) \tag{5.2}$$

Proof. The first equation is obvious. As to (5.2), observe that

$$\begin{aligned}
\varepsilon^2\partial_s(\varrho^\varepsilon v^\varepsilon) &= \varepsilon^2\partial_s(\varrho^\varepsilon(v^\varepsilon - V)) + \varepsilon^2\partial_s(\varrho^\varepsilon V)\\
z\varepsilon^2\partial_s(\varrho^\varepsilon V) &= \varepsilon^2\partial_s(\varrho^\varepsilon)V + \varepsilon^2\varrho^\varepsilon\partial_s V\\
&= -\varepsilon^2\partial_x(\varrho^\varepsilon v^\varepsilon)V - \varrho^\varepsilon V = -\varepsilon^2\partial_x(\varrho^\varepsilon v^\varepsilon V) + \varepsilon^2\varrho^\varepsilon v^\varepsilon\partial_x V - \varrho^\varepsilon V\\
\varepsilon^2\partial_x(\varrho^\varepsilon(v^\varepsilon)^2) &= \varepsilon^2\partial_x(\varrho^\varepsilon(v^\varepsilon - V)^2) + \varepsilon^2\partial_x(\varrho^\varepsilon 2v^\varepsilon V) - \varepsilon^2\partial_x(\varrho^\varepsilon V^2)
\end{aligned}$$

Adding these three equations, we obtain equation (5.2). □

We deduce from (5.1) and (5.2) the following crude bounds:

Lemma 5.2 (Bounds on $(\varrho^\varepsilon - r)$). $\forall s \in (0,+\infty)$: $\int_{\mathbb{R}} |\varrho^\varepsilon - r|(s,x)dx = O\Big(\dfrac{s}{\varepsilon} + s^{\frac{1}{2}}\Big)$, $\qquad \int_{\mathbb{R}} |\varrho^\varepsilon - r|^2(s,x)dx = O\Big(\dfrac{s}{\varepsilon} + s^{\frac{1}{2}}\Big)$.

Proof. Multiplying (5.1) by a regularisation of the sign of $(\varrho^\varepsilon - r)$, using the chain rule formula for BV functions and passing to the limit, we obtain the following inequality in the sense of measures

$$\partial_s|\varrho^\varepsilon - r| \le |\partial_x(\varrho^\varepsilon v^\varepsilon)| + |\partial_x^2 r|$$

Using (4.8) and the bounds of Proposition 3.1, we obtain:

$$\partial_s|\varrho^\varepsilon - r|(s,.)(\mathbb{R}) \le \mathrm{O}\left(\frac{1}{\varepsilon}\right) + \int_{\mathbb{R}} |\partial_x^2 r|(s,x)dx$$

Integrating over $(0,S)$ we obtain the first result of the Lemma, and $(\varrho^\varepsilon - r)$ is bounded in L^∞, which concludes the proof. □

Lemma 5.3 (Crude bounds). $\forall s,\ S \in (0,+\infty)$ *:*

$$\int_{\mathbb{R}} |v^\varepsilon - V|(s,x)dx = O\left(1 + \frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)\right), \tag{5.3}$$

$$\int_0^S \int_{\mathbb{R}} |v^\varepsilon - V|(s,x)dxds = O(S+\varepsilon), \tag{5.4}$$

$$\int_{\mathbb{R}} |\varrho^\varepsilon v^\varepsilon - \rho_0 V|(s,x)dx = O\left(1 + \frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)\right), \tag{5.5}$$

$$\int_0^S \int_{\mathbb{R}} |\varrho^\varepsilon v^\varepsilon - \rho_0 V|(s,x)dxds = O(S+\varepsilon). \tag{5.6}$$

Proof. Multiplying (5.2) again by a regularization of the sign of $\varrho^\varepsilon(v^\varepsilon - V)$ and proceding as in Lemma 5.2, we obtain, from Proposition 3.1, in the sense of $L^\infty((0,+\infty), M^1(\mathbb{R}))$:

$$\varepsilon^2\frac{dY}{ds} + Y = \mathrm{O}(1) \qquad \text{where } Y(s) = \int_{\mathbb{R}} |\varrho^\varepsilon(v^\varepsilon - V)|(s,x)dx.$$

Now, since $Y(0) = \mathrm{O}\left(\frac{1}{\varepsilon}\right)$, we obtain by Gronwall's Lemma:
$Y(s) = \mathrm{O}\left(1 + \frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)\right)$, wich implies (5.3) after integration.
Using now the momentum equation (1.19), we see that:
$\varepsilon^2\partial_s(\varrho^\varepsilon v^\varepsilon - \rho_0 V) + (\varrho^\varepsilon v^\varepsilon - \rho_0 V) = -\varepsilon^2\partial_x(\varrho^\varepsilon(v^\varepsilon)^2) - \partial_x\varrho^\varepsilon$.
Using again Proposition 3.1 and the above arguments, we obtain (5.5). □

6. Proof of Theorem 1.1

In this section, we establish Theorem 1.1 and its Corollary 1.1. The idea is to introduce the "stream function" ψ^ε, defined in equation (6.1) below, which plays the role of the Lagrangian mass coordinate, see [2], and then to multiply (5.2) by ψ^ε and then integrate by parts. ([8]) exploited a similar idea, but used it in a very different way. We also note that the spirit of this integration by parts is very similar to using the div-curl lemma, as in ([1, 9, 18, 19, 21]), ..., but gives *global* results of strong convergence, in contrast with most of the above-mentionned compensated compactness results which only yield, strong but *local* convergence results.
In view of equation (5.1) we define the following stream function ψ^ε by:

$$\frac{\partial\psi^\varepsilon}{\partial x} = \varrho^\varepsilon - r, \quad \frac{\partial\psi^\varepsilon}{\partial s} = -\left(\varrho^\varepsilon v^\varepsilon + \frac{\partial r}{\partial x}\right), \quad \psi^\varepsilon(0,-\infty) = 0. \tag{6.1}$$

Using (6.1) and Lemma 5.2, we see that

$$\begin{gathered}\psi^\varepsilon(0,x) = 0, \frac{\partial}{\partial s}\psi^\varepsilon(s,\pm\infty) = -\rho_0(\pm\infty)V(s,\pm\infty),\\ \psi^\varepsilon(s,x) = \mathrm{O}\left(\varepsilon + \frac{s}{\varepsilon} + s^{\frac{1}{2}}\right)\end{gathered} \tag{6.2}$$

Again, to control ψ^ε at infinity we introduce
$\Psi(s,x) := \varepsilon\exp\left(-\frac{s}{\varepsilon^2}\right)\rho_0(x)(u_{0,-\infty} + (u_{0,+\infty} - u_{0,\infty})H(x))$. We note that
$\Psi(s,x) = \mathrm{O}\left(\varepsilon\exp\left(-\frac{s}{\varepsilon^2}\right)\right)$, and that $\frac{\partial\Psi}{\partial s} = -\rho_0 V$.

We also need the following trivial result:

Lemma 6.1. $2\alpha \geq \beta \implies \int_0^{+\infty} \frac{s^\alpha}{\varepsilon^\beta}\exp\left(-\frac{s}{\varepsilon^2}\right)ds = O\left(\varepsilon^2\right).$

Now we use the stream function: multiply equation (5.2) by ψ^ε and integrate on

$(0,S)\times(-L,L)$ to obtain

$$\begin{aligned}
&\int_0^S\int_{-L}^{+L}\varepsilon^2\partial_s\left(\varrho^\varepsilon(v^\varepsilon-V)\right)\psi^\varepsilon dxds+\int_0^S\int_{-L}^{+L}\varepsilon^2\partial_x\left(\varrho^\varepsilon(v^\varepsilon-V)^2\right)\psi^\varepsilon dxds\\
&+\int_0^S\int_{-L}^{+L}\partial_x(\varrho^\varepsilon-r)\psi^\varepsilon dxds\\
&+\int_0^S\int_{-L}^{+L}\varepsilon^2\partial_x\left(\varrho^\varepsilon V(v^\varepsilon-V)\right)\psi^\varepsilon dxds+\int_0^S\int_{-L}^{+L}\varepsilon^2\varrho^\varepsilon v^\varepsilon\partial_x V\psi^\varepsilon dxds\\
&:=A^\varepsilon+B^\varepsilon+C^\varepsilon+D^\varepsilon+E^\varepsilon=F^\varepsilon:=\int_0^S\int_{-L}^{+L}-\left(\varrho^\varepsilon(v^\varepsilon-V)+\partial_x r\right)\psi^\varepsilon dxds,
\end{aligned}\tag{6.3}$$

We are going to estimate all the terms in (6.3), after having integrated by parts the terms A^ε to E^ε, and then will pass to the limit when $L\to\infty$. For clarity, we treat separately all these terms in the following Proposition, whose proof is postponed at the end of the paper, and then we add all these results, to obtain the key inequality and finally to prove Theorem 1.1.

Proposition 6.1.

$$|A^\varepsilon|=\left|\int_0^S\int_{-L}^{+L}\varepsilon^2\frac{\partial}{\partial s}\left(\varrho^\varepsilon(v^\varepsilon-V)\psi^\varepsilon\right)dxds\right|\le\frac{1}{4}\int_{-L}^{+L}(\psi^\varepsilon-\Psi)^2(S,x)dx + O\left(\varepsilon^2\left(1+\Delta\varrho\sqrt{S}\right)\right)\tag{6.4}$$

$$|B^\varepsilon| = \left|\int_0^S\int_{\mathbb{R}}\left(\varepsilon^2\frac{\partial}{\partial x}\left(\varrho^\varepsilon(v^\varepsilon-V)^2\right)\psi^\varepsilon\right)dxds\right| = O\left(\varepsilon^2\left(1+\Delta\varrho\sqrt{S}\right)\right)\tag{6.5}$$

$$C^\varepsilon = \int_0^S\int_{\mathbb{R}}\left(\frac{\partial}{\partial x}(\varrho^\varepsilon-r)\psi^\varepsilon\right)dxds=-\int_0^S\int_{\mathbb{R}}(\varrho^\varepsilon-r)^2dxds\tag{6.6}$$

$$|D^\varepsilon| = \left|\int_0^S\int_{\mathbb{R}}\left(\varepsilon^2\frac{\partial}{\partial x}\left(\varrho^\varepsilon V(v^\varepsilon-V)\right)\psi^\varepsilon\right)dxds\right| = O\left(\varepsilon^2\right)\tag{6.7}$$

$$|E^\varepsilon| = \left|\int_0^S\int_{\mathbb{R}}\left(\varepsilon^2\varrho^\varepsilon v^\varepsilon\frac{\partial V}{\partial x}\psi^\varepsilon\right)dxds\right| = O\left(\varepsilon^2\right)\tag{6.8}$$

$$F^\varepsilon=\int_0^S\int_{\mathbb{R}}\left(-\left(\varrho^\varepsilon(v^\varepsilon-V)+\frac{\partial r}{\partial x}\right)\psi^\varepsilon\right)dxds = \int_{\mathbb{R}}\frac{(\psi^\varepsilon-\Psi)^2}{2}(S,x)dx+O\left(\varepsilon^2\right)\tag{6.9}$$

Proof of Theorem 1.1: We first rewrite equation (6.3) under the form

$-C^\varepsilon + F^\varepsilon = A^\varepsilon + B^\varepsilon + D^\varepsilon + E^\varepsilon$, i.e:

$$-\int_0^S \int_{-L}^{+L} \partial_x(\varrho^\varepsilon - r)\psi^\varepsilon dxds + \int_0^S \int_{-L}^{+L} -\left(\varrho^\varepsilon(v^\varepsilon - V) + \partial_x r\right)\psi^\varepsilon dxds$$
$$= \int_0^S \int_{-L}^{+L} \varepsilon^2 \partial_s \left(\varrho^\varepsilon(v^\varepsilon - V)\right)\psi^\varepsilon dxds + \int_0^S \int_{-L}^{+L} \varepsilon^2 \partial_x \left(\varrho^\varepsilon(v^\varepsilon - V)^2\right)\psi^\varepsilon dxds$$
$$+ \int_0^S \int_{-L}^{+L} \varepsilon^2 \partial_x \left(\varrho^\varepsilon V(v^\varepsilon - V)\right)\psi^\varepsilon dxds + \int_0^S \int_{-L}^{+L} \varepsilon^2 \varrho^\varepsilon v^\varepsilon \partial_x V \psi^\varepsilon dxds \quad (6.10)$$

Adding the results of Proposition 6.1, (6.10) implies for all $S > 0$:

$$\int_0^S \int_{\mathbb{R}} (\varrho^\varepsilon - r)^2 dxds + \frac{1}{4}\int_{\mathbb{R}} (\psi^\varepsilon - \Psi)^2(S,x)dx \le \mathrm{O}\left(\varepsilon^2\left(1 + \Delta\varrho\sqrt{S}\right)\right), \quad (6.11)$$

which concludes the proof of Theorem 1.1. □

7. Asymptotic behavior for large time

Theorem 1.1 gives a rate of convergence of ϱ^ε to r when $\varepsilon \to 0$, in the space $L^2((0,S)\times\mathbb{R})$. The question is now, to deduce a *pointwise* estimate in time, for fixed ε, from the integral estimates (1.8) and (1.9).
Now ε is fixed, say $\varepsilon = 1$, and we write (ρ, u) instead of $(\rho^\varepsilon, u^\varepsilon)$. We begin by the easy proof of Corollary 1.1. Next, using some Hardy type of Lemmas, see [7], and the entropy inequality we prove Theorem 1.2.

7.1. Proof of Corollary 1.1

We rewrite (1.8) under the form: $\displaystyle\frac{1}{T}\int_0^T \int_{\mathbb{R}} |\rho(t,x) - r(t,x)|^2 dxdt = O\left(\frac{1}{\sqrt{T}}\right)$.
The classical decay in L^2 for the heat equation:
$\displaystyle\int_{\mathbb{R}} |r(t,x) - \overline{r}(x/\sqrt{t})|^2 dx = O(1/\sqrt{t})$,
wich is sharp for L^1 initial data, implies (1.11). Similarly, if $\rho_{+\infty} = \rho_{-\infty}$ inequality (1.9) rewrites:
$\displaystyle\int_0^{+\infty} h(t)dt < \infty$, with $\displaystyle h(t) := \int_{\mathbb{R}} |\rho(t,x) - r(t,x)|^2 dx$.
Using the chain rule formula for BV functions, see [4, 27], we see that
$\displaystyle\forall t \ge 1\,,\ \left|\frac{dh}{dt}\right| \le 2\|\rho - r\|_{L^\infty}\int_{\mathbb{R}} |\partial_t\rho - \partial_t r(t,x)|\, dx \le C$.
Therefore, h is Lipschitz continuous and integrable on $(0,+\infty)$. Consequently $h(t) \to 0$ as $t \to \infty$. And, finally, $\displaystyle\int_{\mathbb{R}} |r(t,x) - \rho_\infty|^2 dx = O(1/\sqrt{t})$, which concludes the proof of Corollary 1.1. □

7.2. Estimates à la Hardy

Here is a first useful result, see [9] (pp 365-366), and see [25] for related ideas.

Lemma 7.1 ([9]). *Let* g *be a nonnegative function defined on* $]0,+\infty[$ *, and* C *be a positive constant.*
Assume that $\lim_{T\to+\infty} \frac{1}{T}\int_0^T g(t)dt = 0$, *and* $\frac{dg}{dt}(t) \le \frac{C}{t}$, $0 < t$.
Then for any $D > \sqrt{2C}$ *, and for large* T *,* $g(T) \le D\sqrt{\frac{1}{T}\int_0^T g(t)dt}$.

For convenience we recall the proof given in [9].
Proof. Let $T > 0$ fixed and τ given by $C\ln(T/\tau) = g(T)$, i.e.
$\tau := T\exp(-g(T)/C) \le T$. Let $h(t) := g(t) - C\ln(t)$ then $h'(t) \le 0$. Since
$(T-\tau)h(T) = \int_\tau^T (t-\tau)h'(t)dt + \int_\tau^T h(t)dt,$ we have $(T-\tau)h(T) \le \int_\tau^T h(t)dt.$
Therefore, $(T-\tau)(g(T) - C\ln(T)) \le \int_0^T g(t)dt - C\int_\tau^T \ln(t)dt,$ and finally:

$$(T-\tau)g(T) - (T-\tau)C\ln(T) + C\int_\tau^T \ln(t)dt \le \int_0^T g(t)dt. \tag{7.1}$$

Introducing $\Lambda(s) := \exp(-s) + s - 1$ with $s := g(T)/C$, (7.1) rewrites, after a few calculations: $\Lambda\left(\frac{g(T)}{C}\right) \le \frac{1}{CT}\int_0^T g(t)dt$, which implies the result for any $D > \sqrt{2C}$, since $\Lambda^{-1}(s) \sim \sqrt{2\,s}$ when $s \to 0$. □
We are now able to deduce the following lemma for the weighted Cesaro mean value.

Lemma 7.2. *Let* $h \ge 0$ *defined on* $(0,+\infty)$ *,* c *a positive constant, and* α,β,γ *nonnegative constants. Assume that:*

$\frac{1}{T}\int_0^T t^\gamma h(t)dt \le \frac{c}{T^\beta}$, $T > 0$, *and* $\frac{dh}{dt}(t) \le \frac{c}{t^\alpha}$, $t > 0$, *and* $\alpha \le 1 < \alpha + \beta + \gamma$.

Then $h(T) \to 0$ *when* $T \to +\infty$ *. More precisely:*

$$\exists C > 0; \quad h(T) \le CT^{-\mu}, \quad \textit{where } \mu := \frac{\alpha+\beta+\gamma-1}{2}.$$

Proof. Let be $g(t) := t^{\alpha-1}h(t)$, then $dg/dt = O(1/t)$, since $h(t) \le h(0) + ct^{1-\alpha}/(1-\alpha)$ for $\alpha < 1$, and $1/T\int_0^T t^{\gamma+1-\alpha}g(t)dt = O(1/T^\beta)$. Define $s := t^\lambda$, $S := T^\lambda$, with $\lambda := \gamma + 2 - \alpha > 0$, $\nu := (\beta+\gamma+1-\alpha)/\lambda > 0$, and $f(s) := g(t)$.

Then $\frac{1}{S}\int_0^S f(s)ds \leq \frac{c\lambda}{S^\nu}$. Since $tds = \lambda s dt$, we have $\frac{df}{ds} = \frac{dg}{dt}(t)\frac{dt}{ds} \leq \frac{c}{\lambda s}$. So, by Lemma (7.1), $f(S) \leq CS^{-\nu/2}$, wich concludes the proof, since $h(T) = T^{1-\alpha}f(T^\lambda)$. □

7.3. Proof of Theorem 1.2

Let us establish the estimates (1.13) to (1.16).

(i) Proof of inequality (1.13): By inequality (1.11), we obtain $\frac{1}{T}\int_0^T t^{1/2}\overline{h}(t)dt \leq \frac{C}{T^{1/2}}$, where $\overline{h}(t) := \int_{\mathbb{R}} |\rho(t, z\sqrt{t}) - \overline{r}(z)|^2 dz$. On the other hand,
$d\overline{h}/dt = 2\int_{\mathbb{R}} (\rho - \overline{r})(\partial_t \rho - 1/2/\sqrt{t}\partial_x \rho)(t, z\sqrt{t})dz = O(1/\sqrt{t}),$
using the BV estimates for the density, since $dz = dx/\sqrt{t}$, we obtain (1.13) by Lemma 7.2. □

(ii) Proof of inequality (1.15): Set $h(t) := \int_{\mathbb{R}} |\rho - r|^2(t, z\sqrt{t})dz$.
By inequality (1.9), $\int_0^{+\infty} t^{1/2}h(t)dt < +\infty$, i.e.: $\frac{1}{T}\int_0^T t^{1/2}h(t)dt \leq \frac{C}{T}$ and $dh/dt = O(1/\sqrt{t})$. Then, by Lemma 7.2, $h(t) = O(1/\sqrt{t})$. Furthermore, any solution to the heat equation converges towards a self-similar solution: $\int_{\mathbb{R}} |r(t, z\sqrt{t}) - \overline{r}(z)|^2 dz \leq 1/t$. We then can replace r by the self-similar solution $\overline{r}$ without losing any rate on the decay. □

(iii) Proof of inequalities (1.14), (1.16):
First of all, we need a "localised" entropy inequality to get a better rate of decay. Following [9], we use a test function to control the entropy on boundary of the parabolic domain $\{|x|^2 \leq Lt\}$, $L > 0$ be fixed and $\chi(x) := 1$ if $|x| \leq 1$, $\chi(x) := 0$ if $|x| \geq 2$. Defining:

$$H(t) := \frac{1}{\sqrt{t}}\int_{|x|\leq 2L\sqrt{t}} [\theta\varphi(\varrho)](t, x)dx, \text{ where } \theta(t, x) := \chi\left(\frac{x}{L\sqrt{t}}\right).$$

And following [9], we obtain:

Lemma 7.3. *There exists c such that* $\frac{dH}{dt}(t) \leq \frac{c}{t}$, $0 < t$.

On the other hand, by Corollary 1.1,

Lemma 7.4. *There exists* c *such that* $\forall T > 0,\ \frac{1}{T}\int_0^T \sqrt{t}H(t)dt \le c\left(\frac{\Delta\varrho}{\sqrt{T}} + \frac{1}{T}\right)$.

Proof. Using L^∞ bounds of ϱ and r and the quadratic behavior of φ at $\varrho = r$, we first have:

$$\delta(\varrho - r)^2 \le \varphi(\varrho) \le \delta^{-1}(\varrho - r)^2 \tag{7.2}$$

for some constant δ. Then,

$$\begin{aligned} \frac{1}{T}\int_0^T \sqrt{t}H(t)dt &= \frac{1}{T}\int_0^T \int_{\mathbb{R}} \theta\varphi(\varrho)dx \le \frac{1}{T}\int_0^T \int_{|x|\le 2L\sqrt{t}} \varphi(\varrho)dx \\ &\le \frac{1}{T}\int_0^T \int_{|x|\le 2L\sqrt{t}} \delta^{-1}(\varrho - r)^2 dx \le \frac{C}{\delta\sqrt{T}} \qquad \square \end{aligned}$$

Now we can establish inequalities (1.14), (1.16). Using the above Lemmas 7.2 to 7.4, we obtain $H(t) = O\left(1/\sqrt{t}\right)$. Using inequality (7.2), and replacing r by $\overline{r}$, we conclude the proof of Theorem 1.2. □

8. Appendix 1: Proof of Proposition 6.1

Proof of (6.4):

$$\begin{aligned} A^\varepsilon = \int_0^S \int_{-L}^{+L} \varepsilon^2 \frac{\partial}{\partial s}\left(\varrho^\varepsilon(v^\varepsilon - V)\right)\psi^\varepsilon dxds &:= a + b + c \\ &:= \int_0^S \int_{-L}^{+L} \varepsilon^2 \varrho^\varepsilon(v^\varepsilon - V)\left(\varrho^\varepsilon v^\varepsilon\right)dxds \\ + \int_0^S \int_{-L}^{+L} \varepsilon^2\varrho^\varepsilon(v^\varepsilon - V)\left(\partial_x r\right) + \int_{-L}^{+L} \varepsilon^2(\varrho^\varepsilon(v^\varepsilon - V)\psi^\varepsilon(S,x))dx. & \end{aligned} \tag{8.1}$$

We treat separately these three terms. To control (a), we use inequality (4.3), Proposition 3.1 , and Lemma 5.3.

$$\begin{aligned} a &:= \int_0^S \int_{-L}^{+L} \varepsilon^2\varrho^\varepsilon(v^\varepsilon - V)\left(\varrho^\varepsilon v^\varepsilon\right)dxds \\ &= \int_0^S \int_{-L}^{+L} \varepsilon^2(\varrho^\varepsilon)^2(v^\varepsilon - V)^2 dxds + \int_0^S \int_{-L}^{+L} \varepsilon^2(\varrho^\varepsilon)^2(v^\varepsilon - V)V dxds \\ &= \mathrm{O}\left(\varepsilon^2\left(1 + \Delta\varrho\sqrt{S}\right)\right) + \mathrm{O}(1)\int_0^S \varepsilon\mathrm{O}\left(1 + \frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)\right)\exp(-\frac{s}{\varepsilon^2})ds \\ &= \mathrm{O}\left(\varepsilon^2\left(1 + \Delta\varrho\sqrt{S}\right)\right). \end{aligned}$$

As to (b), by Proposition 4.1 and Lemma 4.1, we obtain

$$\begin{aligned} b &:= \int_0^S \int_{-L}^{+L} \varepsilon^2 \varrho^\varepsilon |v^\varepsilon - V| |\partial_x r| dx ds \\ &\leq \varepsilon^2 \left(\frac{1}{2} \int_0^S \int_{-L}^{+L} (\varrho^\varepsilon)^2 (v^\varepsilon - V)^2 dx ds + \frac{1}{2} \int_0^S \int_{-L}^{+L} (\partial_x r)^2 dx ds \right) \\ &= \mathrm{O}\left(\varepsilon^2 \left(1 + \Delta\varrho \sqrt{S} \right) \right). \end{aligned}$$

Now we use inequality (4.4). First, observe that:

$$c := \int_{-L}^{+L} \varepsilon^2 \varrho^\varepsilon (v^\varepsilon - V)(\psi^\varepsilon(S,x) - \Psi) dx \\ + \int_{-L}^{+L} \varepsilon^2 \varrho^\varepsilon (v^\varepsilon - V) \Psi(S,x) dx = c_1 + c_2.$$

We control c_1 by the classical inequality: $\nu > 0,\ \alpha\beta \leq \frac{\nu}{2}\alpha^2 + \frac{1}{2\nu}\beta^2$, with $\nu = 2$, $\alpha = \varepsilon^2 \varrho^\varepsilon (v^\varepsilon - V)$, $\beta = \psi^\varepsilon - \Psi$. Therefore, in view of (4.1)

$$\begin{aligned} |c_1| &\leq \varepsilon^2 \int_{-L}^{+L} \varepsilon^2 (\varrho^\varepsilon)^2 (v^\varepsilon - V)^2 dx + \frac{1}{4} \int_{-L}^{+L} (\psi^\varepsilon - \Psi)^2 (S,x) dx \\ &\leq \mathrm{O}\left(\varepsilon^2 \left(1 + \Delta\varrho\sqrt{S} \right) \right) + \frac{1}{4} \int_{-L}^{+L} (\psi^\varepsilon - \Psi)^2 (S,x) dx. \end{aligned}$$

On the other hand, by Lemma 5.1, the L^1 bound on $(v^\varepsilon - V)$ and the exponential decay of V imply:

$$|c_2| \leq \int_{-L}^{+L} \varepsilon^2 \mathrm{O}\left(1 + \frac{1}{\varepsilon} \exp\left(-\frac{S}{\varepsilon^2} \right) \right) \mathrm{O}\left(\varepsilon \exp\left(-\frac{S}{\varepsilon^2} \right) \right) dx = \mathrm{O}\left(\varepsilon^2 \left(1 + \Delta\varrho\sqrt{S} \right) \right).$$

Finally $c = c_1 + c_2$ is bounded by

$$|c| = \int_{-L}^{+L} \varepsilon^2 \varrho^\varepsilon \left| (v^\varepsilon - V)\psi^\varepsilon \right| (S,x) dx \\ \leq \mathrm{O}\left(\varepsilon^2 \left(1 + \Delta\varrho\sqrt{S} \right) \right) + \frac{1}{4} \int_{-L}^{+L} (\psi^\varepsilon - \Psi)^2 (S,x) dx,$$

and (6.4) follows.

Proof of (6.5): Since $\lim_{L \to +\infty} \int_0^S [\varepsilon^2 \varrho^\varepsilon (v^\varepsilon - V)^2 \psi^\varepsilon(s,.)]_{-L}^{+L} ds = 0$, we have by

(4.4).

$$\lim_{L\to+\infty}\int_0^S\int_{-L}^{+L}\left(\varepsilon^2\frac{\partial}{\partial x}\left(\varrho^\varepsilon(v^\varepsilon-V)^2\right)\psi^\varepsilon\right)dxds$$
$$=-\lim_{L\to+\infty}\int_0^S\int_{-L}^{+L}\varepsilon^2\varrho^\varepsilon(v^\varepsilon-V)^2(\varrho^\varepsilon-r)dxds=\mathrm{O}\left(\varepsilon^2\left(1+\Delta\varrho\sqrt{S}\right)\right).$$

Proof of (6.6): This Lemma is obvious. Note that at this stage we do not yet know if $(\varrho^\varepsilon-r)^2\in L^2$.
The key point to prove (6.7) and (6.8) is the exponential decay in time of V.

Proof of (6.7): Using the L^1 estimate of $(v^\varepsilon-V)$, the L^∞ bound of ψ^ε, the fast decay of V, we obtain after integration by parts:

$$D^\varepsilon=\int_0^S\int_{-L}^{+L}\left(\varepsilon^2\frac{\partial}{\partial x}\left(\varrho^\varepsilon V(v^\varepsilon-V)\right)\psi^\varepsilon\right)dxds$$
$$=-\int_0^S\int_{-L}^{+L}\varepsilon^2\varrho^\varepsilon V(v^\varepsilon-V)(\varrho^\varepsilon-r)dxds+\int_0^S[\varepsilon^2\varrho^\varepsilon V(v^\varepsilon-V)\psi^\varepsilon(s,.)]_{-L}^{+L}ds$$
$$\underset{L\to+\infty}{\longrightarrow}\quad \mathrm{O}(1)\int_0^S\varepsilon\mathrm{O}\left(1+\frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)\right)\exp(-\frac{s}{\varepsilon^2})ds+\mathrm{O}\left(\varepsilon^2\right)=\mathrm{O}\left(\varepsilon^2\right).$$

Proof of (6.8): Using the same estimate, we get similarly,

$$\begin{aligned}|E^\varepsilon| &= \left|\int_0^S\int_{\mathbb{R}}\left(\varepsilon^2\varrho^\varepsilon v^\varepsilon\frac{\partial V}{\partial x}\psi^\varepsilon\right)dxds\right|\\ &= \mathrm{O}(1)\int_0^S\varepsilon^2 1\frac{1}{\varepsilon}\exp\left(-\frac{s}{\varepsilon^2}\right)\left(\varepsilon+\frac{s}{\varepsilon}+s^{\frac12}\right)ds\\ &= \mathrm{O}(1)\int_0^S\left(\varepsilon+\frac{s}{\varepsilon}+s^{\frac12}\right)\exp\left(-\frac{s}{\varepsilon^2}\right)ds=\mathrm{O}\left(\varepsilon^2\right).\end{aligned}$$

Proof of (6.9): First observe that

$$-\left(\varrho^\varepsilon(v^\varepsilon-V)+\frac{\partial r}{\partial x}\right)\psi^\varepsilon=(\partial_s\psi^\varepsilon+\varrho^\varepsilon V)\psi^\varepsilon$$
$$=(\partial_s\psi^\varepsilon+\rho_0 V)\psi^\varepsilon+(\varrho^\varepsilon-\rho_0)V\psi^\varepsilon:=e+f.$$

By the same arguments (f) is $\mathrm{O}\left(\varepsilon^2\right)$ in $L^1((0,+\infty)\times\mathbb{R})$. As to (e), by (6.2),we have

$$(\partial_s\psi^\varepsilon+\rho_0 V)\psi^\varepsilon=(\partial_s\psi^\varepsilon-\partial_s\Psi)(\psi^\varepsilon-\Psi+\Psi)$$
$$=\partial_s\left(\frac{(\psi^\varepsilon-\Psi)^2}{2}\right)+\Psi(\partial_s\psi^\varepsilon+\rho_0 V):=g+h.$$

We now control $h = \Psi(\partial_s \psi^\varepsilon + \rho_0 V)$ as follows:

$$\Psi(\partial_s \psi^\varepsilon + \rho_0 V) \quad = \quad \Psi(-\varrho^\varepsilon v^\varepsilon - \partial_x r + \rho_0 V) = -\Psi(\varrho^\varepsilon v^\varepsilon - \rho_0 V) - \Psi \partial_x r,$$

we keep the term (g), which will give us a very nice information. The first term is obvious by Lemma (5.3) and the second one is controlled by (4.7). Finally, Proposition 6.1 is now completely proved. □

9. Appendix 2: Proof of Lemma 7.3

First, we prove a localised entropy inequality, following [9]. Using the notations of section 4 with $\varepsilon = 1$, in particular V is define by (4.1), we obtain, with similar calculations, the following result

Lemma 9.1. *Let $L > 0$, then, there exists c such that*

$$\begin{gathered}\partial_t \left(\varrho(u-V)^2/2 + \varphi(\varrho)\right) + \partial_x \left(\varrho(u-V)^3/2\right) \\ +\partial_x \left(\varrho(u-V)(\ln \varrho - \ln r)\right) + \varrho(u-V)^2 \le R, \\ \textit{with} \int_{|x|\le L\sqrt{T}} R(T,x)dx \le \frac{c}{\sqrt{T}} \qquad T > 0.\end{gathered}$$

Proof. We follow the proof of Lemma 4.2 with small modifications. We have:

$$\partial_t(\varrho \ln \varrho) = \partial_t \varphi(\varrho) - \psi'(r)\partial_x(\varrho u) + R_I, \quad \text{where} \int_{\mathbb{R}} |R_I|(t,x)dx = O\left(1/\sqrt{t}\right), \tag{9.1}$$

since $\psi(\varrho) = \varrho \ln \varrho = \varphi(\varrho) + \psi(r) + \psi'(r)(\varrho - r)$, $\partial_t \psi(\varrho) = \partial_t \varphi(\varrho) + \psi'(r)\partial_t(\varrho) + \psi''(r)(\varrho - r)\partial_t(r)$, and $\psi''(r)(\varrho - r)\partial_t(r) = O(\partial_t(r))$, $\int_{\mathbb{R}} |\partial_t(r)|(t,x)dx = O\left(1/\sqrt{t}\right)$.

$$\partial_t(\varrho u^2/2) = \partial_t \left(\varrho(u-V)^2/2\right) R_{II}, \quad \text{where} \int_{\mathbb{R}} |R_{II}|(t,x)dx = O\left(\exp(-t)\right) \tag{9.2}$$

since $R_{II} = \partial_t(\varrho(u-V)V) + \varrho(u-V)\partial_t V + \partial_t(\varrho V^2/2)$, and $|V| + |\partial_t V| = O(\exp(-t))$.
In the same way

$$\partial_t(\varrho u^3/2) = \partial_t \left(\varrho(u-V)^3/2\right) R_{III}, \quad \text{where} \int_{\mathbb{R}} |R_{III}|(t,x)dx = O\left(\exp(-t)\right), \tag{9.3}$$

since $\partial_x V = O(\exp(-t))$. Using entropy inequality (4.9) and adding up (9.1), (9.1), (9.3), we conclude the proof of this Lemma. □
Now, we can achieve the **proof of Lemma 7.3**:
Let be $\chi(x) := \exp(1/3)\exp\left(-(4-x^2)^{-1}\right)$ if $1 < |x| < 2$, then $\dfrac{(\chi')^2}{\chi}$ is bounded.

From the entropy inequality of Lemma 9.1, we get

$$\begin{aligned}&\left[\partial_t\left(\theta(\varrho(u-V)^2/2+\varphi(\varrho))\right)\right]\\&\quad+\left[\partial_x\left(\theta(\varrho(u-V)^3/2+\varrho(u-V)(\ln\varrho-\ln r))\right)\right]+\left[\theta\varrho(u-V)^2\right]\\&\qquad\leq\theta R+(\partial_t\theta)(\varrho(u-V)^2/2+\varphi(\varrho))+(\partial_x\theta)(\varrho(u-V)^3/2)\\&\qquad\qquad+(\partial_x\theta)(\varrho(u-V)(\ln\varrho-\ln r))\end{aligned}\tag{9.4}$$

i.e. $a+b+c\leq d+e+f+g$. Let $A=A(t):=\int_{|x|\leq L\sqrt{t}}a(t,x)dx$, ... $G=G(t):=\int_{|x|\leq L\sqrt{t}}g(t,x)dx$. We have clearly $D(t)=O\left(1/\sqrt{t}\right)$. Since $\partial_t\theta=0\,(1/t)$, we have also $E(t)=O\left(1/\sqrt{t}\right)$.

$(\partial_x\theta)(\varrho(u-V)^3/2)=(\varrho(u-V)^2/100)\times(\partial_x\theta)(u-V)$, then by Young inequality and the fact that $\partial_x\theta=O\left(1/\sqrt{t}\right)$, F is controlled by C .

$g=(\partial_x\theta)\left(\sqrt{\theta\varrho}(u-V)\right)\times((\partial_x\theta)(\ln\varrho-\ln r))\sqrt{\varrho})\leq\theta\varrho(u-V)^2/2+\dfrac{(\partial_x\theta)^2}{\theta}\varrho\ln(\varrho/r)$.

The first term of the right hand side is controlled by c and the last term is $O\,(1/t)$,
since $(\partial_x\theta)^2/\theta=O\,(1/t)$, then $G(t)=O\left(1/\sqrt{t}\right)$.
Adding all the previous results, the inequality (9.4) become
$\int_{|x|\leq L\sqrt{t}}\partial_t\left(\theta\varphi(\varrho)\right)dx+\int_{|x|\leq L\sqrt{t}}\theta\varrho(1-1/2-1/100)(u-V)^2dx\leq O\left(1/\sqrt{t}\right)$, which conclude the proof of this Lemma. □

10. Appendix 3: Proof of Proposition 3.1

For convenience, we briefly recall or adapt below the main steps of the proof.

Step 1: For each fixed $\varepsilon>0$, we rewrite system (1.18), (1.19) in the original (t,x) variables, and we use the same splitting as in [23]:

$$(i)\left\{\begin{array}{lcll}\dfrac{\partial\rho^\varepsilon}{\partial t}&+&\dfrac{\partial}{\partial x}(\rho^\varepsilon u^\varepsilon)&=&0,\\\dfrac{\partial}{\partial t}(\rho^\varepsilon u^\varepsilon)&+&\dfrac{\partial}{\partial x}(\rho^\varepsilon(u^\varepsilon)^2+\rho^\varepsilon)&=&0,\end{array}\right.;\ (ii)\left\{\begin{array}{lcl}\dfrac{\partial\rho^\varepsilon}{\partial t}&=&0,\\\dfrac{\partial u^\varepsilon}{\partial t}&=&-\dfrac{u^\varepsilon}{\varepsilon}.\end{array}\right.\tag{10.1}$$

System (10.1) (i) is the celebrated Nishida system, written in Eulerian coordinates. Next we briefly recall how these properties are preserved with the abovementioned splitting.

Step 2: **The simplified (Glimm) functional for the Nishida system**
We follow here the presentation of [23]. Consider the Riemann problem for the

system (10.1) (i) with data $U_- = (\rho_-, u_-)$ and $U_+ = (\rho_+, u_+)$

$$U(0,x) = (\rho(0,x), u(0,x)) = \begin{cases} U_-, & \text{if } x < 0, \\ U_+, & \text{if } x > 0, \end{cases}$$

The solution consists of two simple waves, separated by an intermediate constant state U_0. Let us introduce $\eta := \ln \rho$, and $f(\eta) := \begin{cases} 2\sinh\left(\frac{\eta}{2}\right) & \eta \geq 0 \\ \eta & \eta \leq 0 \end{cases}$. The intermediate constant state U_0 is uniquely determined by (10.2)

$$u_- - u_0 = f(\eta_0 - \eta_-), \qquad u_0 - u_+ = f(\eta_0 - \eta_+), \tag{10.2}$$

Let us now define the simplified functional: $S(U_-, U_+) := |\eta_0 - \eta_-| + |\eta_0 - \eta_+|$. Since η is monotone across simple waves, we have: $\forall t > 0,\ TV\eta(t,.) = S(U_-, U_+)$.

The following Lemma controls $S(U_-, U_+)$ in terms of the strength of the jumps in the initial data.

Lemma 10.1. $S(U_-, U_+) \leq \max(|u_- - u_+|, |\eta_- - \eta_+|)$.

Proof. We treat the four cases:

- (i) If $\eta_- \leq \eta_0 \leq \eta_+$ or $\eta_+ \leq \eta_0 \leq \eta_-$ we have: $S(U_-, U_+) = |\eta_0 - \eta_-| + |\eta_0 - \eta_+| = |\eta_- - \eta_+|$.
- (ii) if $\eta_0 \leq \eta_-$ and $\eta_0 \leq \eta_+$ we have: $|u_0 - u_-| = |\eta_0 - \eta_-|$, $|u_0 - u_+| = |\eta_0 - \eta_+|$, and $u_+ \leq u_0 \leq u_-$ by (10.2), since $\eta_0 - \eta_- \leq 0$ and $\eta_0 - \eta_+ \leq 0$. Then, we have: $S(U_-, U_+) = |\eta_0 - \eta_-| + |\eta_0 - \eta_+| = |u_0 - u_-| + |u_0 - u_+| = |u_- - u_+|$.
- (iii) If $\eta_0 \geq \eta_-$ and $\eta_0 \geq \eta_+$ we have: $|u_0 - u_-| = f(|\eta_0 - \eta_-|)$, $|u_0 - u_+| = f(|\eta_0 - \eta_+|)$, and $u_+ \geq u_0 \geq u_-$ by (10.2), since $\eta_0 - \eta_- \geq 0$ and $\eta_0 - \eta_+ \geq 0$.
  Then, using the fact that $|\eta| \leq f(|\eta|)$, we have:

$$\begin{aligned} S(U_-, U_+) &= |\eta_0 - \eta_-| + |\eta_0 - \eta_+| \leq f(|\eta_0 - \eta_-|) + f(|\eta_0 - \eta_+|) \\ &\leq |u_0 - u_-| + |u_0 - u_+| \leq |u_- - u_+| \qquad \square \end{aligned}$$

Step 3: **Extension to the full system**

Let $\varepsilon > 0$ be fixed. Again we follow [23], and the above-mentioned references. At each time step t_n, the splitting is the following:

(i) starting with piecewise constant data $U^n := (\rho^n, u^n) := \{(\rho_i^n, u_i^n),\ i \in \mathbb{Z}\}$ we first construct a new piecewise constant function $\overline{U}^{n+1} := (\overline{\rho}^n, \overline{u}^n) := \{(\overline{\rho}_i^n, \overline{u}_i^n),\ i \in \mathbb{Z}\}$ by the Glimm scheme.

(ii) starting now with this new initial data $\overline{U}^{n+1}$, we solve the stiff ordinary differential equation (10.1) from t_n to t_{n+1}, to construct the approximation U^{n+1} at time t_{n+1}. In other words,

$$\rho_i^{n+1} = \overline{\rho}_i^{n+1}, \qquad u_i^{n+1} = \exp(-k/\varepsilon)\ \overline{u}_i^{n+1}, \tag{10.3}$$

where $k = \Delta t$ is the time step.
Now define $S(U_-, U_+) := |\eta_0 - \eta_-| + |\eta_0 - \eta_+|$, and the simplified Glimm functional

$$N^n := \sum_{i \in \mathbb{Z}} S(\overline{U}_i^{n+1}, \overline{U}_{i+1}^{n+1}),$$

wich controls the total variation of the numerical solution. We recall that the total variation of a function $g : \mathbb{R} \to \mathbb{R}$, is:

$$TV(g) := \sup_{n \in \mathbb{N}, \quad x_0 < x_1 < \cdots < x_n} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|.$$

The following result is proven in [23].

Lemma 10.2 ([23]). *For any fixed* $\alpha \in [0,1]$ *, and for any* $\overline{U}_\pm := (\rho_\pm, u_\pm)$ *let us define*
$U_\pm := (\rho_\pm, \alpha u_\pm)$ *, then* $S(U_-, U_+) \leq S(\overline{U}_-, \overline{U}_+)$.

Using this result, we obtain finally the following crucial estimates

Lemma 10.3. $\forall n \geq 0, \qquad N^n \leq TV(u_0) + TV(\ln \rho_0)$.

Proof. Combining Lemma 10.2 with the well-known estimates on the Nishida system, we easily obtain for $n \geq 1$, $N^n = \sum_{i \in \mathbb{Z}} S(U_i^{n+1}, U_{i+1}^{n+1}) \leq \sum_{i \in \mathbb{Z}} S(\overline{U}_i^{n+1}, \overline{U}_{i+1}^{n+1}) \leq \sum_{i \in \mathbb{Z}} S(U_i^n, U_{i+1}^n) = N^{n-1}$. Moreover, we can control N^0 in term of the total variation of the initial data, uniformly with respect to ε and to the mesh size h . First, we note that for all sequences of nonnegative numbers a_j and b_j , we have $\sum_{l \in \mathbb{Z}} \max(a_j, b_j) \leq \sum_{l \in \mathbb{Z}} a_j + \sum_{l \in \mathbb{Z}} b_j$. Therefore, $N^0 \leq TV(u_0) + TV(\ln \rho_0)$ follows, by Lemma 10.1. □

We are now able to prove Proposition 3.1.

Step 4: **Proof of Proposition 3.1**
By Lemma 10.3 we know that (N^n) is bounded, uniformly in h , ε and n , and therefore in time. Therefore, the family of approximated solutions $(U_h^\varepsilon)_{h \to 0_+}$ constructed by the above scheme satisfies, uniformly in h and ε :

$$\sup_{t \geq 0} TV\left(\eta(U_h^\varepsilon)(t, .)\right) \leq N^0,$$

where we recall that $\eta(\rho, u) := \ln(\rho)$. Since the limits of $\eta(U(t,x))$ at $x = \pm\infty$ do not depend on t , we obtain a L^∞ bound for $\eta = \ln \rho$, uniform in ε and h . We also note that $f \in C^1(\mathbb{R})$ and $f(0) = 0$, so that $|u_\pm - u_0| \leq f'(|\eta_\pm - \eta_0|)|\eta_\pm - \eta_0|$. Using the previous inequality for the first step of the splitting and combining with

(10.1), we deduce that u is also bounded in BV, uniformly in h and ε. The velocity u is therefore uniformly bounded in L^∞, since its limit values at $x = \pm\infty$ are uniformly bounded by $|u_0(\pm\infty)|$, wich concludes the proof of Proposition 3.1 for approximate solutions.

Therefore we can pass to the limit as $h \to 0$, to get the same L^∞ and BV estimates for the weak entropy solution $U^\varepsilon := (\rho^\varepsilon, u^\varepsilon)$, uniformly in ε. Of course, these estimates also hold true with the new time s. □

We are now going to obtain other estimates by the entropy inequality.

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References

[1] Gui-Quiang Chen, Joseph W. Jerome, Bo Zhang, *Existence and the singular relaxation limit for the inviscid hydrodynamic energy model,* , J. Jerome (ed.), proceedings, Oxford University Press.

[2] Courant Friedrichs, *Supersonic Flow and Shock Waves* Applied Mathematical Sciences. Vol. 21. Springer-Verlag, New York - Heidelberg - Berlin 1977.

[3] S. Cordier and Y.J. Peng, Système d'Euler-Poisson non linéaire, existence globale de solutions faibles entropiques, *M.2.A.N.*, Vol. **32**(1) (1998), 1-23.

[4] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation.* Vol. 80. Birkhäuser, Boston-Basel-Stuttgart 1984.

[5] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Com.Pure.Appl.Math.,* **18** (1965), 698-715.

[6] T. Goudon, S. Junca and G. Toscani, *A Strengthened Version of Berry-Esseen's Theorem,* preprint, 1999.

[7] G. H. Hardy, Theorems relating to the summability and convergence of slowly oscillating series, *Proc. London. Math. Soc.* **8** (1910), 301-320.

[8] L. Hsiao and T. P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Com. Math. Phys.* **143** (1992), 599-605.

[9] L. Hsiao and D. Serre, Asymptotic behavior of large weak entropy solutions of the damped p-system, *J. Partial Diff. Eqs.* (1997), 355-368.

[10] S. Junca and M. Rascle, Relaxation of isothermal Euler-Poisson system to the drift-diffusion equations, *Quart. of Appl. Math.* **58** (2000), 511-521.

[11] A. Jungel, Y.-J. Peng, A hierarchy of hydrodynamic models for plasmas zero-relaxation-time limits, *C.P.D.E* **24** (1999), 1007-1033.

[12] A. Jungel, Y.-J. Peng, Zero-relaxation-time limits in the hydrodynamic equations for plasmas revisited, to appear in *ZAMP* (1999).

[13] T. P. Liu, Initial-boundary value problems for gas dynamics, *Arch. Rational Mech. Anal.* **32** (1979), 169-189.

[14] Tao Luo, Roberto Natalini and Tong Yang, *Global BV solutions to a p-system with relaxation,* Quaderno IAC 13/1998.

[15] P. L. Lions, B. Perthame and P. E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Comm. Pure and Appl. Math.* **49** (1996), 599-638.

[16] M. Luskin and J. B. Temple, The existence of a global weak solution for a model equation for a fluid flow in a pipe, *Com. Pure Appl. Math.* **35** (1982), 697-735.
[17] P. Marcati, A. Milani, The one-dimensional Darcy's law as limit of a compressible Euler flow, *J.D.E.* **84** (1990), 129-147.
[18] P. Marcati, A. Milani and P. Secchi, Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system, *Manuscripta Math.* **60** (1988), 49-69.
[19] P. Marcati and R. Natalini, Weak Solutions to a Hydrodynamic Model for Semiconductors and Relaxation to the Drift-Diffusion Equation, *Arch. Rat. Mech. Anal.* **129** (1995), 129-145.
[20] A. Matsumura and T. Nishida, Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.* **4** (1983), 445-464.
[21] P. Marcati and B. Rubino, Hyperbolic to parabolic relaxation theory for quasilinear first order systems, Proceedings of the IX International Conference on Waves and Stability in Continuous Media (Bari, 1997). *Rend. Circ. Mat. Palermo* (2) Suppl. 1998, **57**, 315–320.
[22] T. Nishida, Global solutions for an initial boundary value problem of a quasilinear hyperbolic system, *Japan Acad.* **44** (1968), 642-646.
[23] F. Poupaud, M. Rascle and J.P. Vila, Global Solutions to the Isothermal Euler-Poisson System with Arbitrarily Large Data, *J.D.E.* **123** (1995), 93-121.
[24] D. Serre, *Systèmes de lois de conservation,* Diderot, Paris 1996.
[25] W. Shen and S. Zheng, On the coupled Cahn-Hilliard equations, *C.P.D.E.* (1993), 701-727.
[26] J. Smoller, *Shock Waves and Reaction-Diffusion Equations,* Springer, New York 1983.
[27] A.I. Volpert, *The spaces BV and quasilinear equations, Math USSR Sb.* **73** (1967), 225-267.
[28] S. Zheng, *Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems,* Pitman series Monographs and Surveys in Pure and Applied Math. Vol. 76, Pitman 1995.

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