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Positive Solutions for Slightly Subcritical Elliptic Problems Via Orlicz Spaces

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Abstract. This paper concerns semilinear elliptic equations involving sign-changing weight function and a nonlinearity of subcritical nature understood in a generalized sense. Using an Orlicz–Sobolev space setting, we consider superlinear nonlinearities which do not have a polynomial growth, and state sufficient conditions guaranteeing the Palais–Smale condition. We study the existence of a bifurcated branch of classical positive solutions, containing a turning point, and providing multiplicity of solutions.

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1. Introduction

In this paper we study the classical positive solutions to the Dirichlet problem for a class of semilinear elliptic equations whose nonlinear term is of subcritical nature in a generalized sense and involves indefinite nonlinearities. More precisely, given $\Omega \subset \mathbb{R}^N$, $N > 2$, a bounded, connected open subset, with C^2 boundary $\partial\Omega$, we look for positive solutions to:

$$-\Delta u = \lambda u + a(x) f(u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a real parameter, $a \in C^1(\bar{\Omega})$ changes sign in Ω,

$$f(s) := g(s) + h(s), \qquad \text{with} \quad h(s) := \frac{|s|^{2^*-2} s}{[\ln(e + |s|)]^{\alpha}}, \tag{1.2}$$

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$2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\alpha > 0$ is a fixed exponent, and $f, g \in C^1(\mathbb{R})$ satisfy

$$(\mathrm{H}) \begin{cases} (\mathrm{H})_0 & \lim_{s\to 0} \frac{f(s)}{|s|^{p-2}s} = L_1, \quad \text{for some } L_1 > 0, \text{ and } p \in \left(2, \frac{2N}{N-2}\right] \\ (\mathrm{H})_\infty & \lim_{s\to \infty} \frac{g(s)}{|s|^{q-2}s} = L_2, \quad \text{for some } L_2 \geq 0, \text{ and } q \in \left(2, \frac{2N}{N-2}\right) \\ (\mathrm{H})_{g'} & |g'(s)| \leq C(1+|s|^{q-2}), \quad \text{for } s \in \mathbb{R}. \end{cases}$$

We will say that f satisfies hypothesis (H) whenever $(\mathrm{H})_0$, $(\mathrm{H})_\infty$, and $(\mathrm{H})_{g'}$ are satisfied. Since we are interested in positive solutions, we

$$\text{redefine } f \text{ to be zero on } (-\infty, 0], \tag{1.3}$$

note that, since $(\mathrm{H})_0$, $f(0) = 0$ and that

$$\lim_{s\to 0^+} \left(\frac{f(s)}{s} - L_1|s|^{p-2} \right) = 0. \tag{1.4}$$

When $\lambda = 0$, $a(x) \equiv 1$ and $g(s) \equiv 0$, this kind of nonlinearity has been studied in [5–7,16], and in [11] for the case of the $p-$laplacian operator, with $\alpha > \frac{p}{N-p}$. It is known the existence of uniform L^∞ a-priori bounds for any positive classical solution, and as a consequence, the existence of positive solutions. When $\alpha \to 0$, there is a positive solution blowing up at a non-degenerate point of the Robin function as $\alpha \to 0$, see [9] for details.

Let (λ_1, φ_1) stands for the first eigen-pair of the Dirichlet eigenvalue problem $-\Delta\varphi = \lambda\varphi$ in Ω, $\varphi = 0$ on $\partial\Omega$. From [10] it is known that $(\lambda_1, 0)$ is a bifurcation point of positive solutions (λ, u_λ) to the equation (1.1). If f behaves like $|u|^{p-2}u$ at zero with $2 \leq p \leq 2^*$, the influence of the negative part of the weight a is displayed under the sign of $\int_\Omega a(x)\varphi_1(x)^p\,dx$, where φ_1 is the first positive eigenfunction for $-\Delta$ in $H_0^1(\Omega)$. Specifically, whenever

$$\int_\Omega a(x)\varphi_1(x)^p\,dx < 0 \tag{1.5}$$

the bifurcation of positive solutions from the trivial solution set is 'on the right' of the first eigenvalue, in other words, for values of $\lambda > \lambda_1$. When

$$\int_\Omega a(x)\varphi_1(x)^p\,dx > 0$$

the bifurcation from the trivial solution set is 'on the left' of the first eigenvalue, in other words, for values of $\lambda < \lambda_1$.

Inspired by the work of Alama and Tarantello in [1], we will focus our attention to the case of $a(x)$ changing sign and (1.5) is being satisfied, and, among other things, we will prove the existence of a turning point for a value of the parameter $\Lambda > \lambda_1$, and in particular the existence of solutions when $\lambda = \lambda_1$. We will use local bifurcation and variational techniques.

All throughout the paper, for $v : \Omega \to \mathbb{R}$, $v = v^+ - v^-$ where

$$v^+(x) := \max\{v(x), 0\} \qquad \text{and} \qquad v^-(x) := \max\{-v(x), 0\}.$$

Let us also define

$$\Omega^{\pm} := \{x \in \Omega : \ \pm a(x) > 0\}, \qquad \Omega^{0} := \{x \in \Omega : \ a(x) = 0\},$$

and assume that both Ω^+, Ω^- are non empty sets.

For this nonlinearity the Palais–Smale condition of the energy functional becomes a delicate issue, needing Orlicz spaces and a Orlicz–Sobolev embedding theorem.

In order to prove (PS) condition, Alama and Tarantello ([1]) assume that the zero set Ω^0 has a non empty interior. This is also a common hypothesis for other authors when dealing with changing sign superlinear nonlinearities [8, 20, 23]. But this is a technical hypothesis. (PS)-condition will be proved in Proposition 3.1 without assuming that hypothesis. We neither use Ambrosetti-Rabinowitz condition.

Let us now denote

$$C_0 = \inf\{C \geq 0 : \ f'(s) + C \geq 0 \text{ for all } s \geq 0\}, \tag{1.6}$$

and remark that hypothesis (H) implies that $C_0 < +\infty$. Observe also that

$$f(s) + C_0 s \geq 0, \text{ for all } s \geq 0; \quad f(s)s + C_0 s^2 \geq 0, \text{ for all } s \in \mathbb{R}. \tag{1.7}$$

Let u be a weak solution to (1.1). By a regularity result, see Lemma 2.1, $u \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega})$. So by a *solution*, we mean a *classical solution.*

Assume that u is a non-negative nontrivial solution. It is easy to see that the solution is strictly positive. Indeed, adding $\pm C_0 a(x) u$ to the r.h.s. of the equation, splitting $a = a^+ - a^-$, taking into account (1.4) and (1.7), and letting in each side the nonnegative terms, we can write

$$\begin{aligned}\left(-\Delta + a^-(x)\left[\frac{f(u)}{u} + C_0\right] + C_0 a(x)^+\right) u \\ = \lambda u + a(x)^+ \left[f(u) + C_0 u\right] + C_0 a(x)^- u, \quad \text{in } \Omega.\end{aligned} \tag{1.8}$$

Now, the strong Maximum Principle implies that $u > 0$ in Ω, and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.

Our main result is the following theorem.

Theorem 1.1. *Assume that $g \in C^1(\mathbb{R})$ satisfies hypothesis* (H)*. Let $C_0 > 0$ be defined by* (1.6)*. If a changes sign in Ω, and* (1.5) *holds, then there exists a $\Lambda \in \mathbb{R}$,*

$$\lambda_1 < \Lambda < \min\left\{\lambda_1\big(\mathrm{int}\,(\Omega^0)\big), \quad \lambda_1\big(\mathrm{int}\,(\Omega^+ \cup \Omega^0)\big) + C_0 \sup a^+\right\}$$

and such that (1.1) *has a classical positive solution if and only if $\lambda \leq \Lambda$.*

Moreover, there exists a continuum (a closed and connected set) $\mathscr{C}$ of classical positive solutions to (1.1) *emanating from the trivial solution set at the bifurcation point $(\lambda, u) = (\lambda_1, 0)$ which is unbounded. Furthermore,*

(a) For every, $\lambda \in (\lambda_1, \Lambda)$, (1.1) *admits at least two classical ordered positive solutions.*

(b) For $\lambda = \Lambda$, problem (1.1) *admits at least one classical positive solution.*

(c) For every $\lambda \leq \lambda_1$, problem (1.1) *admits at least one classical positive solution.*

The paper is organized in the following way. Section 2 contains a regularity result and a non existence result. (PS)-condition and an existence of solutions result for $\lambda < \lambda_1$ based in the Mountain Pass Theorem will be proved in Sect. 3. A bifurcation result for $\lambda > \lambda_1$ is developed in Sect. 4. The main result is proved in Sect. 5. Appendix A contains some useful estimates. Orlicz spaces, and a Orlicz–Sobolev embeddings theorems, will be treated in Appendix B.

2. A Regularity Result and a Non Existence Result

Next, we recall a regularity Lemma stating that any weak solution is in fact a classical solution.

Lemma 2.1. *If $u \in H_0^1(\Omega)$ weakly solves (1.1) with a continuous function f with polynomial critical growth*

$$|f(x,s)| \leq C(1+|s|^{2^*-1}),$$

then, $u \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\mu}(\overline{\Omega})} \leq C\Big(1+\|u\|^{2^*-1}_{L^{(2^*-1)r}(\overline{\Omega})}\Big),$$

for any $r > N$ and $\mu = 1 - N/r$. Moreover, if $\partial\Omega \in C^{2,\mu}$, then $u \in C^{2,\mu}(\overline{\Omega})$.

Proof. Due to an estimate of Brézis-Kato [3], based on Moser's iteration technique [17], $u \in L^r(\Omega)$ for any $r > 1$; and by elliptic regularity $u \in W^{2,r}(\Omega)$, for any $r > 1$ (see [22, Lemma B.3] and comments below).

Moreover, by Sobolev embeddings for $r > N$ and interior elliptic regularity $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$. Furthermore, if $\partial\Omega \in C^{2,\alpha}$, then $u \in C^{2,\alpha}(\overline{\Omega})$. □

Proposition 2.2. *Let f satisfy hypothesis* (H) *and let C_0 be defined in (1.6). Assume that a changes sign in Ω.*

1. *Problem (1.1) does not admit a positive solution $u \in H_0^1(\Omega)$ for any*

$$\lambda \geq \lambda_1\big(\mathrm{int}\,(\Omega^+ \cup \Omega^0)\big) + C_0 \sup a^+.$$

2. *If $\mathrm{int}\,(\Omega^0) \neq \emptyset$, then $\lambda_1\big(\mathrm{int}\,(\Omega^0)\big) < +\infty$ and (1.1) does not admit a positive solution for any*

$$\lambda \geq \lambda_1\big(\mathrm{int}\,(\Omega^0)\big).$$

Proof. 1. Let $\lambda \geq \lambda_1\big(\mathrm{int}\,(\Omega^+ \cup \Omega^0)\big) + C_0 \sup a^+$, and assume by contradiction that there exists a non-negative non-trivial solution $u \in H_0^1(\Omega)$ to (1.1) for the parameter λ. Since the Maximum Principle $u > 0$ in Ω, see (1.8).

Let $\hat{\varphi}$ be the positive eigenfunction of $\big(-\Delta, H_0^1(\mathrm{int}\,(\Omega^+ \cup \Omega^0))\big)$ of L^2-norm equal to 1. For simplicity, we will also denote by $\hat{\varphi}$ the extension by 0 of $\hat{\varphi}$ in all Ω. By Hopf's maximum principle, we have $\frac{\partial\hat{\varphi}}{\partial\nu} < 0$ on $\partial\big(\mathrm{int}\,(\Omega^+ \cup \Omega^0)\big)$, where ν is the outward normal.

Again, if we multiply the equation (1.1) by $\hat{\varphi}$ and integrate along $\mathrm{int}\,\big(\Omega^+ \cup \Omega^0\big)$ we find, after integrating by parts,

$$0 > \int_{\partial(\mathrm{int}\,(\Omega^+ \cup \Omega^0))} u \frac{\partial\hat{\varphi}}{\partial\nu} d\sigma$$

$$+\int_{\mathrm{int}(\Omega^+\cup\Omega^0)}\Big[\lambda_1\big(\mathrm{int}\,(\Omega^+\cup\Omega^0)\big)-\lambda+C_0a^+(x)\Big]u\hat{\varphi}\,dx$$
$$=\int_{\Omega^+}a^+(x)\big[f(u)+C_0u\big]\hat{\varphi}\,dx>0,$$

a contradiction.

2. Let $\lambda\geq\lambda_1\big(\mathrm{int}\,(\Omega^0)\big)$ and, by contradiction, assume the existence of a positive solution $u\in H_0^1(\Omega)$ of problem (1.1) for the parameter λ. Let $\tilde{\varphi}$ be a positive eigenfunction associated to $\lambda_1\big(\mathrm{int}\,(\Omega^0)\big)<+\infty$. For simplicity, we will also denote by $\tilde{\varphi}$ the extension by 0 in all Ω. If we multiply equation (1.1) by $\tilde{\varphi}$ and integrate along Ω^0 we find, after integrating by parts,

$$\int_{\mathrm{int}\,(\Omega^0)}\nabla u\cdot\nabla\tilde{\varphi}\,dx=\lambda\int_{\mathrm{int}\,(\Omega^0)}u\tilde{\varphi}\,dx.$$

On the other hand

$$\int_{\mathrm{int}\,(\Omega^0)}\nabla u\cdot\nabla\tilde{\varphi}\,dx=\lambda_1(\mathrm{int}\,(\Omega^0))\int_{\mathrm{int}\,(\Omega^0)}\tilde{\varphi}u\,dx+\int_{\partial(\mathrm{int}\,(\Omega^0))}u\frac{\partial\tilde{\varphi}}{\partial\nu}d\sigma.$$

Hence

$$0>\int_{\partial(\mathrm{int}\,(\Omega^0))}u\frac{\partial\tilde{\varphi}}{\partial\nu}d\sigma=\Big(\lambda-\lambda_1\big(\mathrm{int}\,(\Omega^0)\big)\Big)\int_{\mathrm{int}\,(\Omega^0)}u\tilde{\varphi}\,dx\geq0,$$

a contradiction. □

3. An Existence Result for $\lambda<\lambda_1$

In this section, we prove the existence of a nontrivial solution to equation (1.1) for $\lambda<\lambda_1$, through the Mountain Pass Theorem.

3.1. On Palais–Smale Sequences

In this subsection, we define the framework for the functional J_λ associated to the problem $(1.1)_\lambda$. Hereafter, we denote by $\|\cdot\|$ the usual norm of $H_0^1(\Omega)$:

$$\|u\|=\left(\int_\Omega|\nabla u|^2\,dx\right)^{1/2}.$$

Given $f(s)=h(s)+g(s)$ defined by (1.2), let us denote by $F(s):=\int_0^s f(t)\,dt$. Observe that (1.7) implies the following

$$F(s)+\frac{1}{2}C_0s^2\geq0,\ \text{for all } s\geq0. \tag{3.1}$$

Consider the functional $J_\lambda:H_0^1(\Omega)\to\mathbb{R}$ given by

$$J_\lambda[v]:=\frac{1}{2}\int_\Omega|\nabla v|^2\,dx-\frac{\lambda}{2}\int_\Omega(v^+)^2\,dx-\int_\Omega a(x)F(v^+)\,dx.$$

Take note that for all $v\in H_0^1(\Omega)$, $J_\lambda\big[v^+\big]\leq J_\lambda[v]$.

The functional J_λ is well defined and belongs to the class C^1 with

$$J'_\lambda[v]\,\psi\ =\int_\Omega\nabla v\nabla\psi\,dx-\lambda\int_\Omega v^+\psi\,dx-\int_\Omega a(x)f(v^+)\psi\,dx,$$

for all $\psi \in H_0^1(\Omega)$. As a result, non-negative critical points of the functional J_λ correspond to non-negative weak solutions to (1.1).

The next Proposition establishes that *Palais–Smale sequences* are bounded whenever $\lambda < \lambda_1(\operatorname{int}\Omega^0)$, where $\lambda_1(\operatorname{int}\Omega^0)$ may be infinite.

Proposition 3.1. *Assume that* $g \in C^1(\mathbb{R})$ *fulfills hypothesis* (H) *and that* $\lambda < \lambda_1(\operatorname{int}\Omega^0) \le +\infty$.

Then any (PS) *sequence, that is, a sequence satisfying the conditions*

(J_1) $J_\lambda[u_n] \le C$,

(J_2) $|J_\lambda'[u_n]\,\psi| \le \varepsilon_n \|\psi\|$, *where* $\varepsilon_n \to 0$ *as* $n \to +\infty$

is a bounded sequence.

Proof. 1. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS) sequence in $H_0^1(\Omega)$ and, in contradiction, assume that $\|u_n\| \to +\infty$. Let us first prove the following claim:

Claim. Let $v \in H_0^1(\Omega)$ be the weak limit of $v_n = \frac{u_n}{\|u_n\|}$ and assume that $v_n \to v$, strongly in $L^{2^*-1}(\Omega)$ and a.e. Then $v = 0$ a.e. in Ω.

Assume that $v \not\equiv 0$ and write $\gamma_n = \|u_n\|$. Let $\omega_n := \{x \in \Omega : v_n^+(x) > 1\}$, then for any $\psi \in C_0^1(\Omega)$,

$$\left|\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}} \frac{(u_n^+(x))^{2^*-1}}{[\ln(e+\gamma_n\, v_n^+(x))]^\alpha}|\psi|\right| \le |v_n^+(x)|^{2^*-1}\|\psi\|_\infty, \qquad \forall x \in \omega_n.$$

Let $x \in \Omega \setminus \omega_n$, based on the estimates (A.1),

$$\left|\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}} \frac{(u_n^+(x))^{2^*-1}}{[\ln(e+\gamma_n\, v_n^+(x))]^\alpha}|\psi|\right| \le \left(|v_n^+(x)|^{2^*-2}\right)\|\psi\|_\infty \le \|\psi\|_\infty$$

Besides, by the reverse of the Lebesgue dominated convergence theorem, see for instance [2, Theorem 4.9, p. 94] , there exists $h_i \in L^1(\Omega)$, $1 \le i \le 3$ such that, up to a subsequence,

$$|v_n^+|^{2^*-1} \le h_1,\ |v_n^+|^{p-1} \le h_2,\ |v_n^+|^{2^*-2} \le h_3,\ \ a.e.\ x \in \Omega,$$

for all $n \in \mathbb{N}$, and therefore

$$\left|\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}} f(u_n^+)\psi\right| \le C\,(h_1+h_2+h_3+1))\,\|\psi\|_\infty \in L^1(\Omega).$$

By Lebesgue's dominated convergent theorem, we have

$$\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}} a(\cdot) f(u_n^+)\psi \to a(\cdot)(v^+)^{2^*-1}\psi \qquad \text{strongly in } L^1(\Omega).$$

We have used here that if $v^+(x) \ne 0$, then

$$\lim_{n\to+\infty} \frac{\ln(e+\gamma_n)}{\ln(e+\gamma_n\, v_n^+(x))} = 1,$$

and if $v^+(x) = 0$, then

$$\lim_{n\to+\infty}\left(\frac{\ln(e+\gamma_n)}{\ln(e+\gamma_n\, v_n^+(x))}\right)^\alpha |v_n^+(x)|^{2^*-1} \le \lim_{n\to+\infty} |v_n^+(x)|^{2^*-2} = 0.$$

On the other hand

$$\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}}\int_\Omega \nabla u_n\cdot\nabla\psi\,dx \to 0.$$

Hence, using (J_2) for an arbitrary test function ψ, multiplying by $\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}}$ and passing to the limit we find

$$\int_\Omega a(x)(v^+)^{2^*-1}\psi\,dx = 0 \quad \forall\psi\in C_0^1(\Omega).$$

In particular $v^+=0$ a.e. in $\Omega\setminus\Omega^0$.

Assume that $\operatorname{int}\Omega^0\neq\emptyset$, and that $\lambda<\lambda_1(\operatorname{int}\Omega^0)$. Thus, for any $\psi\in C_0^1(\operatorname{int}\Omega^0)$ we have from (J_2)

$$\int_{\operatorname{int}\Omega^0}\nabla u_n\cdot\nabla\psi\,dx - \lambda\int_{\operatorname{int}\Omega^0} u_n^+\psi\,dx = o(1).$$

Dividing by $\|u_n\|$ and passing to the limit we have

$$\int_{\operatorname{int}\Omega^0}\nabla v\cdot\nabla\psi\,dx = \lambda\int_{\operatorname{int}\Omega^0} v^+\psi\,dx.$$

From the Maximum Principle, $v\geq 0$ in $\operatorname{int}\Omega^0$. Since $\lambda<\lambda_1(\operatorname{int}\Omega^0)$ then it must be $v^+\equiv 0$ in $\operatorname{int}\Omega^0$. Hence $v^+\equiv 0$ in Ω.

On the other hand, taking u_n^- as a test function in the condition (J_2),

$$\left|-\int_\Omega|\nabla u_n^-|^2dx - \int_\Omega a(x)f(u_n^+)u_n^-dx\right| = \int_\Omega|\nabla u_n^-|^2dx \leq \epsilon_n\|u_n^-\|$$

so $\|u_n^-\|\to 0$ and then $v^-\equiv 0$, and we conclude the proof of the claim.

2. In order to achieve a contradiction, we use a Hölder inequality, and properties on convergence into an Orlicz space, cf. Appendix B.

To this end, the analysis of Lemma A.2 gives us the existence of $\alpha^*>0$ such that the function $s\to\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$ is increasing along $[0,+\infty[$ if $\alpha\leq\alpha^*$. In this case, we will denote

$$m(s)=\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha} \tag{3.2}$$

If $\alpha>\alpha^*$ the function $s\to\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$ possesses a local maximum s_1 in $[0,+\infty[$. Let us denote by $\overline{s}_1$ the unique solution $s>s_1$ such that

$$\frac{s_1^{2^*-1}}{[\ln(e+s_1)]^\alpha}=\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$$

and define the non-decreasing function

$$m(s):=\begin{cases}\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha} & \text{if } s\notin[s_1,\overline{s}_1],\\ \frac{s_1^{2^*-1}}{[\ln(e+s_1)]^\alpha} & \text{if } s\in[s_1,\overline{s}_1].\end{cases} \tag{3.3}$$

It follows that

$$s\to M(s)=\int_0^s m(t)\,dt \qquad \text{is a} \quad N-\text{function in } [0,+\infty[. \tag{3.4}$$

By using

$$\lim_{s\to+\infty}\frac{\ln(e+s)}{\ln(e+2s)}=1 \qquad \text{and} \qquad \lim_{s\to 0}\frac{\ln(e+s)}{\ln(e+2s)}=1,$$

we get that

$$\lim_{s\to+\infty}\frac{m(2s)}{m(s)}<+\infty \qquad \text{and} \qquad \lim_{s\to 0^+}\frac{m(2s)}{m(s)}<+\infty,$$

which implies that there exists $K>0$ such that $m(2s)\le Km(s)$ for all $s\ge 0$ and consequently M satisfies the Δ_2-condition (B.1).

Since $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, it follows from (J_2) applied to $\psi = u_n$ that

$$\lim_{n\to\infty}\int_\Omega a(x)\frac{f(u_n^+)u_n}{\|u_n\|^2}\,dx=\lim_{n\to\infty}\int_\Omega a(x)\frac{f(u_n^+)}{\|u_n\|}\,v_n^+\,dx=1. \tag{3.5}$$

Since the Hölder inequality into Orlicz spaces, see Proposition B.11.(ii),

$$\int_\Omega\left|a(x)\frac{f(u_n^+)}{\|u_n\|}\,v_n^+\right|\,dx\le\frac{\|a\|_\infty}{\|u_n\|}\left\|f(u_n^+)\right\|_{M^*}\left\|v_n^+\right\|_M \tag{3.6}$$

By Theorem B.3 and Theorem B.12 we have

$$\|v_n-v\|_M\to 0. \tag{3.7}$$

Moreover, since there exists $C>0$ such that $m(s)\le Cs^{2^*-1}$, $M(s)\le Cs^{2^*}$ for all $s\ge 0$, and the sequence $\{u_n\}_{n\in\mathbb{N}}\subset H_0^1(\Omega)$, then, for each $n\in\mathbb{N}$, there exists a C_n such that

$$\int_\Omega|u_n^+|\,m\big(|u_n^+|\big)\le C_n,\quad \int_\Omega M\big(|u_n^+|\big)\le C_n.$$

By using definition B.8 of M^* and identities of Proposition B.9 we have

$$M^*\left(m\big(|u_n^+|\big)\right)=|u_n^+|\,m\big(|u_n^+|\big)-M\big(|u_n^+|\big)$$

then, for each $n\in\mathbb{N}$,

$$\int_\Omega M^*\left(m\big(|u_n^+|\big)\right)dx\le 2C_n.$$

Observe that $|f(s)|\le C(1+m(s))$, so then

$$\|f(u_n^+)\|_{M^*}\le C\left\|1+m\big(u_n^+\big)\right\|_{M^*}\le C\left[1+\int_\Omega M^*\Big(m\big(|u_n^+|\big)\Big)\right]\le C_n',$$

see Proposition B.11.(iii) and (i), concluding that the l.h.s. is bounded for each n.

Consequently, $a(x)\frac{f(u_n^+)}{\|u_n\|}\in L_{M^*}(\Omega)$, which is the dual of $L_M(\Omega)$ (see [15], Theorem 14.2).

On the other hand, from J_2, for all $\psi\in C_c^\infty(\Omega)$,

$$\left|\int_\Omega\nabla v_n\nabla\psi\,dx-\lambda_n\int_\Omega v_n\psi\,dx-\int_\Omega a(x)\frac{f(u_n^+)}{\|u_n\|}\psi\,dx\right|\le\frac{\varepsilon_n}{\|u_n\|}\|\psi\|. \tag{3.8}$$

Taking the limit, and since $C_c^\infty(\Omega)$ is dense in $L_M(\Omega)$ (see [13]),

$$\lim_{n\to\infty}\int_\Omega a(x)\frac{f(u_n^+)}{\|u_n\|}\,\psi\,dx=0, \qquad \text{for all} \quad \psi\in L_M(\Omega). \tag{3.9}$$

Moreover, since (3.7), $v_n \to v = 0$ in $L_M(\Omega)$, [2, Proposition 3.13 (iv)], and (3.9) imply

$$\lim_{n\to\infty} \int_\Omega a(x)\frac{f(u_n^+)}{\|u_n\|}\, v_n\, dx = 0,$$

which contradicts (3.5). This concludes the proof. □

Theorem 3.2. *Assume the hypothesis of Proposition 3.1 and let $\{u_n\}_{n\in\mathbb{N}}$ be a* (PS) *sequence in $H_0^1(\Omega)$.*

Then, there exists a subsequence, denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$u_n \to u \qquad in \quad H_0^1(\Omega).$$

Proof. From Proposition 3.1 we know that the sequence is bounded. Consequently, there exists a subsequence, denoted by $\{u_n\}_{n\in\mathbb{N}}$, and some $u \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \ \text{ weakly in } H_0^1(\Omega), \tag{3.10}$$

$$\int_\Omega a(x)g(u_n)|u_n - u|\, dx \to 0, \tag{3.11}$$

$$u_n \to u \ \text{ a.e.} \tag{3.12}$$

By testing (J_2) against $\psi = u_n - u$ and using (3.10), and (3.11) we get

$$\begin{aligned}\|u_n - u\|^2 &= \int_\Omega \nabla u_n \cdot \nabla(u_n - u)\, dx + o(1)\\ &\le \|a\|_\infty \int_\Omega \frac{|u_n|^{2^*-1}}{[\ln(e + |u_n|)]^\alpha}|u_n - u|\, dx + o(1).\end{aligned}$$

Claim.

$$\int_\Omega \frac{|u_n|^{2^*-1}}{[\ln(e + |u_n|)]^\alpha}|u_n - u| dx = o(1),$$

In order to prove this claim, we use, as in the above proposition, a Hölder inequality and a compact embedding into some Orlicz space, c.f. Appendix B.

By Theorem B.3 and Theorem B.12 we have

$$\|u_n - u\|_M \to 0, \tag{3.13}$$

where m, and M are defined by (3.2)–(3.4), as in the above proposition. On the other hand, because there exists $C > 0$ such that $m(s) \le Cs^{2^*-1}$ and $M(s) \le Cs^{2^*}$ for all $s \ge 0$, and the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, then

$$\|u_n\, m(|u_n|)\|_{L^1(\Omega)} \le C, \quad \|M(|u_n|)\|_{L^1(\Omega)} \le C \qquad \text{for all} \quad n \in \mathbb{N}$$

By using definition B.8 of M^* and identities of Proposition B.9 we have

$$M^*\left(m(|u_n|)\right) = |u_n|\, m(|u_n|) - M(|u_n|)$$

then

$$\int_\Omega M^*\left(m(|u_n|)\right) dx \le C$$

for all $n \in \mathbb{N}$. Finally, by inequality (B.5) of Proposition B.12 we get

$$\sup\left\{\|m(|u_n|)\|_{M^*},\ n \in \mathbb{N}\right\} \le C + 1.$$

Now, using Holder's inequality (B.6) and that $\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha} \le m(s)$ for all $s \ge 0$, we get

$$\int_\Omega \frac{|u_n|^{2^*-1}}{[\ln(e+|u_n|)]^\alpha}|u_n-u|dx \le \|u_n-u\|_M \,\|m(|u_n|)\|_{M^*} \le (C+1)\|u_n-u\|_M$$

and it follows from (3.13) that $\|u_n-u\| \to 0$. □

3.2. An Existence Result for $\lambda < \lambda_1$

The next theorem provides a solution to (1.1) for $\lambda < \lambda_1$ based on the Mountain Pass Theorem.

Theorem 3.3. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. Assume that the nonlinearity f defined by* (1.2) *satisfies* (H), *and that the weight $a \in C^1(\overline{\Omega})$. Then, the boundary value problem* $(1.1)_\lambda$ *has at least one classical positive solution for any $\lambda < \lambda_1$.*

Proof. We verify the hypothesis of the Mountain Pass Theorem, see [14, Theorem 2, Section 8.5]. Observe that the derivative of the functional $J'_\lambda : H_0^1(\Omega) \to H_0^1(\Omega)$ is Lipschitz continuous on bounded sets of $H_0^1(\Omega)$; also the (PS) condition is satisfied, see Proposition 3.1. Clearly $J_\lambda[0] = 0$.

1. Let now $u \in H_0^1(\Omega)$ with $\|u\| = r$, for $r > 0$ to be chosen below. Then,

$$J_\lambda[u] = \frac{r^2}{2} - \frac{\lambda}{2}\int_\Omega (u^+)^2\,dx - \int_\Omega a(x)F(u^+)\ dx. \tag{3.14}$$

From hypothesis (H) we have

$$\left|\int_\Omega a(x)G(u^+)\ dx\right| \le C\int_\Omega \left(|u|^p + |u|^q\right)dx \le C\left(r^p + r^q\right).$$

where $G(s) := \int_0^s g(t)\,dt$. Now, definition (1.2) implies that

$$\left|\int_\Omega a(x)F(u^+)\ dx\right| \le C\left(r^p + r^q + r^{2^*}\right).$$

In view of (3.14), and as a result of the Poincaré inequality, we get

$$J_\lambda[u] \ge \frac{1}{2}\left(1 - \frac{|\lambda|}{\lambda_1}\right) r^2 - C\left(r^p + r^q + r^{2^*}\right) \ge C_1 r^2,$$

taking $|\lambda| < \lambda_1$, $r > 0$ small enough, and using that $p,\ q,\ 2^* > 2$.

2. Now, fix some element $0 \le u_0 \in H_0^1(\Omega)$, $u_0 > 0$ in Ω^+, $u_0 \equiv 0$ in Ω^-. Let $v = tu_0$ for a certain $t = t_0 > 0$ to be selected a posteriori. Since

$$f(tu_0) = |t|^{2^*-2}t\,f(u_0)\left(\frac{\ln(e+|u_0|)}{\ln(e+|tu_0|)}\right)^\alpha + g(tu_0), \tag{3.15}$$

then $f(tu_0)/t \to +\infty$ as $t \to +\infty$ in Ω^+.

From definition, and integrating by parts,

$$F(s) = \int_0^s \left(\frac{t^{2^*-1}}{\ln(e+t)^\alpha} + g(t)\right)dt$$

$$= \frac{1}{2^*}\, sh(s) + G(s) + \frac{\alpha}{2^*}\int_0^s \left(\frac{1}{\ln(e+t)}\right)^{\alpha+1} \frac{t^{2^*}}{e+t}\, dt.$$

It can be easily seen that $\lim_{s\to+\infty} \frac{G(s)}{sf(s)} = 0$.

Therefore, using l'Hôpital's rule we can write

$$\lim_{s\to+\infty} \frac{F(s)}{sf(s)} = \frac{1}{2^*} \in \left(0, \frac{1}{2}\right), \tag{3.16}$$

hence

$$\lim_{t\to+\infty} \frac{F(tu_0)}{tu_0 f(tu_0)} = \frac{1}{2^*} \in \left(0, \frac{1}{2}\right) \qquad \text{in} \quad \Omega^+. \tag{3.17}$$

Let $C_0 \geq 0$ be such that $F(s) + \frac{1}{2}C_0 s^2 \geq 0$ for all $s \geq 0$ (see (1.7)), and let

$$\widetilde{\Omega}_\delta^+ := \{x \in \Omega^+ \ : \ a(x) = a^+(x) > \delta\}. \tag{3.18}$$

By definition, $u_0 \equiv 0$ in Ω^-, so, introducing $\pm\frac{1}{2}C_0(tu_0)^2$, splitting the integral, and using (3.17)–(3.18) we obtain

$$\begin{aligned} -\int_\Omega a(x)F(tu_0)\, dx &= -\int_{\Omega^+} a^+(x)F(tu_0)\, dx \\ &\leq \frac{C_0 t^2}{2}\int_{\Omega^+} a^+(x)u_0^2\, dx - \int_{\widetilde{\Omega}_\delta^+} a^+(x)\left[\frac{1}{2}C_0(tu_0)^2 + F(tu_0)\right] dx \\ &\leq C + \frac{C_0 t^2}{2}\int_{\Omega^+} a^+(x)u_0^2\, dx - \frac{\delta t^2}{2}\int_{\widetilde{\Omega}_\delta^+} \left[C_0 u_0^2 + \frac{u_0 f(tu_0)}{2^* t}\right] dx. \end{aligned}$$

Hence, there exists a positive constant $C > 0$ such that

$$\begin{aligned} J_\lambda[tu_0] &= \frac{t^2}{2}\|u_0\|^2 - t^2\frac{\lambda}{2}\|u_0\|^2_{L^2(\Omega)} - \int_{\Omega^+} a^+(x)F(tu_0) \\ &\leq C(1+t^2) - \frac{\delta\, t^2}{2}\int_{\widetilde{\Omega}_\delta^+} \left[C_0(u_0)^2 + \frac{u_0 f(tu_0)}{2^* t}\right] dx < 0 \end{aligned}$$

for $t = t_0 > 0$ big enough.

Step 3. We have at last checked that all the hypothesis of the Mountain Pass Theorem are accomplished. Let

$$\Gamma := \{\mathbf{g} \in C\big([0,1]; H_0^1(\Omega)\big) : \mathbf{g}(0) = 0,\ \mathbf{g}(1) = t_0 u_0\},$$

then, there exists $c \geq C_1\, r^2 > 0$ such that

$$c := \inf_{\mathbf{g}\in\Gamma} \max_{0\leq t\leq 1} J_\lambda[\mathbf{g}(t)]$$

is a critical value of J_λ, that is, the set $\mathscr{K}_c := \{v \in H_0^1(\Omega) : J_\lambda[v] = c,\ J'_\lambda[v] = 0\} \neq \emptyset$. Thus there exists $u \in H_0^1(\Omega)$, $u \geq 0$, $u \neq 0$ such that for each $\psi \in H_0^1(\Omega)$, we have

$$\int_\Omega \nabla u \cdot \nabla\psi\, dx = \int_\Omega \big[\lambda u^+ + a(x)f(u^+)\big]\psi\, dx. \tag{3.19}$$

and thereby, u is a nontrivial weak solution to (3.19). By Lemma 2.1, u is a classical solution, and by (1.8), $u > 0$ in Ω. □

4. A Bifurcation Result for $\lambda > \lambda_1$

Next Proposition uses Crandall-Rabinowitz's local bifurcation theory, see [10], and Rabinowitz's global bifurcation theory, see [19].

Proposition 4.1. *Let us define*

$$\Lambda := \sup\{\lambda > 0 : \text{(1.1)}_\lambda \textit{ admits a positive solution}\}.$$

If (1.5) holds then,

$$\lambda_1 < \Lambda < \min\Big\{\lambda_1\big(\mathrm{int}\,(\Omega^0)\big)\,, \quad \lambda_1\big(\mathrm{int}\,(\Omega^+ \cup \Omega^0)\big) + C_0 \sup a^+\Big\}$$

where $C_0 > 0$ is such that $f(s) + C_0 s \geq 0$ for all $s \geq 0$, (see definition (1.6)).

Moreover, there exists an unbounded continuum (a closed and connected set) $\mathscr{C}$ of classical positive solutions to (1.1) emanating from the trivial solution set at the bifurcation point $(\lambda, u) = (\lambda_1, 0)$.

Proof. Proposition 2.2 establish the upper bounds for Λ. Next, we concentrate our attention in proving that $\Lambda > \lambda_1$. Choosing λ as the bifurcation parameter, we check that the conditions of Crandall - Rabinowitz's Theorem [10] are satisfied. For $r > N$, we define the set $W^{2,r}_+ := \{u \in W^{2,r}(\Omega) : u > 0 \text{ in } \Omega\}$, and consider $W^{2,r}_+(\Omega) \cap W^{1,r}_0(\Omega)$ endowed with the topology of $W^{2,r}(\Omega)$. If $r > N$, we have that $W^{2,r}_+(\Omega) \cap W^{1,r}_0(\Omega) \hookrightarrow C^{1,\mu}_0(\Omega)$ for $\mu = 1 - \frac{N}{r} \in (0,1)$. Moreover, from Hopf's lemma, we know that if $\tilde{u}$ is a positive solution to (1.1) then $\tilde{u}$ lies in the interior of $W^{2,r}_+(\Omega) \cap W^{1,r}_0(\Omega)$.

We consider the map $\mathscr{F} : \mathbb{R} \times W^{2,r}_+(\Omega) \cap W^{1,r}_0(\Omega) \to L^r(\Omega)$ for $r > N$,

$$\mathscr{F} : (\lambda, u) \to -\Delta u - \lambda u - a(x) f(u)$$

The map $\mathscr{F}$ is a continuously differentiable map. Since hypothesis (i), $g(0) = 0$, and so $a(x)F(0) = 0$, $\mathscr{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$, and since $F_u(x, 0) = 0$,

$$D_u\mathscr{F}(\lambda_1, 0)w := -\Delta w - \lambda_1 w,$$
$$D_{\lambda,u}\mathscr{F}(\lambda_1, 0)w := -w.$$

Observe that

$$N\big(D_u\mathscr{F}(\lambda_1, 0)\big) = span[\varphi_1], \qquad \mathrm{codim}\, R\big(D_u\mathscr{F}(\lambda_1, 0)\big) = 1,$$
$$D_{\lambda,u}\mathscr{F}(\lambda_1, 0)\varphi_1 = -\varphi_1 \notin R\big(D_u\mathscr{F}(\lambda_1, 0)\big),$$

where $N(\cdot)$ is the kernel, and $R(\cdot)$ denotes the range of a linear operator.

Hence, the hypotheses of Crandall-Rabinowitz's Theorem are satisfied and $(\lambda_1, 0)$ is a bifurcation point. Thus, decomposing

$$C^{1,\mu}_0(\overline{\Omega}) = span[\varphi_1] \oplus Z,$$

where $Z = span[\varphi_1]^\perp$, there exists a neighborhood $\mathscr{U}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times C^{1,\mu}_0(\overline{\Omega})$, and continuous functions $\lambda(s), \tilde{w}(s)$, $s \in (-\varepsilon, \varepsilon)$, $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $\tilde{w} : (-\varepsilon, \varepsilon) \to Z$ such that $\lambda(0) = \lambda_1$, $\tilde{w}(0) = 0$, with $\int_\Omega \tilde{w}\varphi_1\, dx = 0$, and the only nontrivial solutions to (1.1) in $\mathscr{U}$, are

$$\big\{\big(\lambda(s), s\varphi_1 + s\,\tilde{w}(s)\big) : \ s \in (-\varepsilon, \varepsilon)\big\}. \tag{4.1}$$

Set $u = u(s) = s\varphi_1 + s\,\tilde{w}(s)$. Note that by continuity $\tilde{w}(s) \to 0$ as $s \to 0$, which guarantees that $u(s) > 0$ in Ω for all $s \in (0, \varepsilon)$ small enough.

Next, we show that $\lambda(s) > \lambda_1$ for all s small enough. Since (3.15), and hypothesis $(\mathrm{H})_0$ on f, note that $\frac{a(x)f(su)}{s^{p-1}u^{p-1}} \to L_1 a(x)$ as $s \to 0$. In fact, as $\tilde{w}(s) \to 0$ uniformly as $s \to 0$, hypothesis $(\mathrm{H})_0$ yields

$$\frac{a(x)f\big(s\varphi_1 + s\,\tilde{w}(s)\big)}{s^{p-1}\big(\varphi_1 + \tilde{w}(s)\big)^{p-1}} \longrightarrow L_1 a(x) \text{ uniformly in } \Omega \qquad \text{as} \quad s \to 0.$$

Hence, multiplying and dividing by $\big(\varphi_1 + \tilde{w}(s)\big)^{p-1}$, we deduce

$$\frac{1}{s^{p-1}} \int_\Omega a(x) f\big(u(s)\big)\varphi_1 \underset{s\to 0}{\to} L_1 \int_\Omega a(x)\varphi_1^p.$$

Now we prove that $\lambda(s) > \lambda_1$ arguing by contradiction. Assume that there is a sequence $(\lambda_n, u_n) = \big(\lambda(s_n), u(s_n)\big)$ of bifurcated solutions to (1.1) in $\mathscr{U}$, with $\lambda(s_n) \le \lambda_1$. Multiplying $(1.1)_{\lambda_n}$ by φ_1 and integrating by parts

$$0 \le \frac{\big(\lambda_1 - \lambda(s_n)\big)}{s_n^{p-1}} \int_\Omega u(s_n)\varphi_1 = \frac{1}{s_n^{p-1}} \int_\Omega a(x) f\big(u(s_n)\big)\varphi_1 \to L_1 \int_\Omega a(x)\varphi_1^p < 0$$

which yields a contradiction, and consequently, $\Lambda > \lambda_1$.

Finally, Rabinowitz's global bifurcation Theorem [19] states that, in fact, the set $\mathscr{C}$ of positive solutions to (1.1) emanating from $(\lambda_1, 0)$ is a continuum (a closed and connected set) which is either unbounded, or contains another bifurcation point, or contains a pair of points (λ, u), $(\lambda, -u)$ with $u \ne 0$. Since (1.8), any non-negative non-trivial solution is strictly positive, and moreover $(\lambda_1, 0)$ is the only bifurcation point to positive solutions, so $\mathscr{C}$ can not reach another bifurcation point. Since (1.3), neither $\mathscr{C}$ contains a pair of points (λ, u), $(\lambda, -u)$ with $u \ne 0$, which states that $\mathscr{C}$ is unbounded, ending the proof. □

5. Proof of Theorem 1.1

First we prove an auxiliary result.

Proposition 5.1. *For each $\lambda \in (\lambda_1, \Lambda)$, the following holds:*

(i) Problem $(1.1)_\lambda$ admits a positive solution

$$u_\lambda = \inf\big\{u(x) : u > 0 \textit{ solving } (1.1)_\lambda\big\},$$

in other words u_λ is minimal.

(ii) Moreover, the map $\lambda \to u_\lambda$ is strictly monotone increasing, that is, if $\lambda < \mu < \Lambda$, then $u_\lambda(x) < u_\mu(x)$ for all $x \in \Omega$, and $\frac{\partial u_\lambda}{\partial \nu}(x) > \frac{\partial u_\mu}{\partial \nu}(x)$ for all $x \in \partial\Omega$.

(iii) Furthermore, u_λ is a local minimum of the functional J_λ.

Proof. (i.a) Step 1. Existence of positive solutions for any $\lambda \in (\lambda_1, \Lambda)$.

Let $\lambda \in (\lambda_1, \Lambda)$ be fixed. By definition of Λ, there exists a $\lambda_0 \in (\lambda, \Lambda)$ such that the problem $(1.1)_{\lambda_0}$ admits a positive solution u_0. It is easy to verify that $u_0 > 0$ is a supersolution to $(1.1)_\lambda$. Indeed, for any $\psi \in H_0^1(\Omega)$ with $\psi \geq 0$ in Ω

$$\int_\Omega \nabla u_0 \cdot \nabla \psi \, dx - \lambda \int_\Omega u_0 \psi \, dx - \int_\Omega a(x) f(u_0) \psi \, dx = (\lambda_0 - \lambda) \int_\Omega u_0 \psi \, dx \geq 0.$$

Moreover, for every $\delta > 0$ satisfying

$$0 < \delta < \left(\frac{\lambda - \lambda_1}{2L_1 \, \|a^-\|_\infty} \right)^{\frac{1}{p-2}} \frac{1}{\|\varphi_1\|_\infty} \tag{5.1}$$

the function $\underline{u} = \delta \varphi_1$ is a subsolution for $(1.1)_\lambda$ whenever $\lambda > \lambda_1$. Let $\delta > 0$ satisfying (5.1) and such that $g(s) \geq 0$ for any $s \in [0, \delta \|\varphi_1\|_{L^\infty(\Omega)}]$. For any $\psi \in H_0^1(\Omega)$, $\psi > 0$ with in Ω we deduce

$$\begin{aligned}
&\delta \int_\Omega \nabla \varphi_1 \cdot \nabla \psi \, dx - \lambda \delta \int_\Omega \varphi_1 \psi \, dx - \int_\Omega a(x) f(\delta \varphi_1) \psi \, dx \\
&\quad = -(\lambda - \lambda_1) \delta \int_\Omega \varphi_1 \psi \, dx - \int_\Omega a(x) f(\delta \varphi_1) \psi \, dx \\
&\quad = -(\lambda - \lambda_1) \delta \int_\Omega \varphi_1 \psi \, dx - \int_\Omega a(x) \left[\frac{(\delta \varphi_1)^{2^*-1}}{[\ln(e + \delta \varphi_1)]^\alpha} + g(\delta \varphi_1) \right] \psi \, dx \\
&\quad \leq -(\lambda - \lambda_1) \delta \int_\Omega \varphi_1 \psi \, dx + \|a^-\|_\infty \int_\Omega \big[h(\delta \varphi_1) + g(\delta \varphi_1) \big] \psi \, dx < 0.
\end{aligned}$$

This allows us to take $\underline{u} = \delta \varphi_1$ as a subsolution for $(1.1)_\lambda$ with $\underline{u} < u_0$. The sub- and supersolution method now guarantees a positive solution u to $(1.1)_\lambda$, with $\underline{u} \leq u \leq u_0$.

(i.b) Step 2. Existence of a minimal positive solution u_λ for any $\lambda \in (\lambda_1, \Lambda)$. To show that there is in fact a minimal solution, for each $x \in \Omega$ we define

$$\underline{u}_\lambda(x) := \inf \big\{ u(x) : \, u > 0 \text{ solving } (1.1)_\lambda \big\}.$$

Firstly, we claim that $\underline{u}_\lambda \geq 0$, $\underline{u}_\lambda \not\equiv 0$. Assume that $\underline{u}_\lambda \equiv 0$ by contradiction. This would yield a sequence u_n of positive solutions to $(1.1)_\lambda$ such that $\|u_n\|_{C(\overline{\Omega})} \to 0$ as $n \to \infty$, or in other words, $(\lambda, 0)$ is a bifurcation point from the trivial solution set to positive solutions. Set $v_n := \frac{u_n}{\|u_n\|_{C(\overline{\Omega})}}$. Observe that v_n is a weak solution to the problem

$$-\Delta v_n = \lambda v_n + a(x) f(u_n) / \|u_n\|_{C(\overline{\Omega})} \text{ in } \Omega; \qquad v_n = 0 \text{ on } \partial\Omega. \tag{5.2}$$

It follows from $(\mathrm{H})_0$ that $\frac{a(x) f(u_n)}{\|u_n\|_{C(\overline{\Omega})}} \to 0$ in $C(\overline{\Omega})$ as $n \to \infty$. Therefore, the right-hand side of (5.2) is bounded in $C(\overline{\Omega})$. Hence, by the elliptic regularity, $v_n \in W^{2,r}(\Omega)$ for any $r > 1$, in particular for $r > N$. Then, the Sobolev embedding theorem implies that $||v_n||_{C^{1,\alpha}(\overline{\Omega})}$ is bounded by a constant C that is independent of n. Then, the compact embedding of $C^{1,\mu}(\overline{\Omega})$ into $C^{1,\beta}(\overline{\Omega})$ for $0 < \beta < \mu$ yields, up to a subsequence, $v_n \to \Phi \geq 0$ in $C^{1,\beta}(\overline{\Omega})$. Since $\|v_n\|_{C(\overline{\Omega})} = 1$, we have that $\|\Phi\|_{C(\overline{\Omega})} = 1$. Hence, $\Phi \geq 0$, $\Phi \not\equiv 0$.

Using the weak formulation of equation (5.2), passing to the limit, and taking into account that λ is fixed and $v_n \to \Phi$, we obtain that $\Phi \geq 0$, $\Phi \not\equiv 0$, is a weak solution to the equation

$$-\Delta \Phi = \lambda \Phi \text{ in } \Omega, \qquad \Phi = 0 \text{ on } \partial\Omega.$$

Then, by the maximum principle, it follows that $\Phi = \varphi_1 > 0$, the first eigenfunction, and $\lambda = \lambda_1$ is its corresponding eigenvalue, which contradicts that $\lambda > \lambda_1$.

Secondly, we show that $\underline{u}_\lambda$ solves $(1.1)_\lambda$. We argue on the contrary. Observe that the minimum of any two positive solutions to $(1.1)_\lambda$ furnishes a supersolution to $(1.1)_\lambda$. Assume that there are a finite number of solutions to $(1.1)_\lambda$, then $\underline{u}_\lambda(x) := \min\big\{u(x) : u > 0 \text{ solves } (1.1)_\lambda\big\}$ and $\underline{u}_\lambda$ is a supersolution. Choosing ε_0 small enough so that $\varepsilon_0\varphi_1 < \underline{u}_\lambda$, the sub- supersolution method provides a solution $\varepsilon_0\varphi_1 \le v \le \underline{u}_\lambda$. Since v is a solution and $\underline{u}_\lambda$ is not, then $v \le \underline{u}_\lambda$, $v \ne u$, contradicting the definition of $\underline{u}_\lambda$, and achieving this part of the proof.

Assume now that there is a sequence u_n of positive solutions to $(1.1)_\lambda$ such that, for each $x \in \Omega$, $\inf u_n(x) = \underline{u}_\lambda(x) \ge 0$, $\underline{u}_\lambda \not\equiv 0$. Let $\underline{u}_1 := \min\{u_1, u_2\}$. Choosing ε_1 small enough so that $\varepsilon_1\varphi_1 < \underline{u}_1$, the sub- supersolution method provides a solution $\varepsilon_1\varphi_1 \le v_1 \le \underline{u}_1$. We reason by induction.

Let $\underline{u}_n := \min\{v_{n-1}, u_{n+1}\}$. Choosing ε_n small enough so that $\varepsilon_n\varphi_1 < \underline{u}_n$, the sub- supersolution method provides a solution $\varepsilon_n\varphi_1 \le v_n \le \underline{u}_n \le v_{n-1}$. With this induction procedure, we build a monotone sequence of solutions v_n, such that

$$0 < v_n \le \underline{u}_n \le v_{n-1} \le \underline{u}_{n-1} \le \cdots \le v_1. \tag{5.3}$$

Since monotonicity and Lemma 2.1, $\|v_n\|_{C(\overline{\Omega})} \le \|v_1\|_{C(\overline{\Omega})}$, by elliptic regularity, $\|v_n\|_{C^{1,\mu}(\overline{\Omega})} \le C$ for any $\mu < 1$, and by compact embedding $v_n \to v$ in $C^{1,\beta}(\overline{\Omega})$ for any $\beta < \alpha$. Using the weak formulation of equation $(1.1)_\lambda$, passing to the limit, and taking into account that λ is fixed, we obtain that v is a weak solution to the equation $(1.1)_\lambda$. Hence $v(x) \ge \underline{u}_\lambda > 0$. Moreover, since (5.3), $v_n(x) \downarrow v(x)$ pointwise for $x \in \Omega$, so $\inf v_n(x) = v(x)$. Also, and due to (5.3), $\underline{u}_n(x) \downarrow v(x)$ pointwise for $x \in \Omega$, and $\inf \underline{u}_n(x) = v(x)$.

On the other hand, by construction $\underline{u}_n \le u_{n+1}$, so, for each $x \in \Omega$, $v(x) = \inf \underline{u}_n(x) \le \inf u_n(x) = \underline{u}_\lambda(x)$. Therefore, and by definition of $\underline{u}_\lambda$, necessarily $v = \underline{u}_\lambda$, proving that $\underline{u}_\lambda$ solves $(1.1)_\lambda$, and achieving the proof of step 2.

(ii) The monotonicity of the minimal solutions is concluded from a sub- supersolution method. Reasoning as in step 1, u_μ is a strict supersolution to $(1.1)_\lambda$, so $w := u_\mu(x) - u_\lambda(x) \ge 0$, $w \not\equiv 0$. Moreover, $w = 0$ on $\partial\Omega$, and we can always choose $c_0 := C_0\|a\|_\infty > 0$ where C_0 is defined by (1.6), so that $a^-(x)f'(s) + c_0 \ge 0$ and $a^+(x)f'(s) + c_0 \ge 0$ for all $s \ge 0$, then

$$\Big(-\Delta + a^-(x)f'\big(\theta u_\mu + (1-\theta)u_\lambda\big) + c_0\Big)w = (\mu - \lambda)u_\mu + \lambda w \\ + \big[a^+(x)f'\big(\theta u_\mu + (1-\theta)u_\lambda\big) + c_0\big]w > 0 \text{ in } \Omega,$$

finally, the Maximum Principle implies that $w > 0$ in Ω, and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$, ending the proof of step 3.

(iii) Since [4, Theorem 2] if there exists an ordered pair of L^∞ bounded sub and supersolution $\underline{u} \le \overline{u}$ to $(1.1)_\lambda$, and neither $\underline{u}$ nor $\overline{u}$ is a solution to $(1.1)_\lambda$, then there exist a solution $\underline{u} < u < \overline{u}$ to $(1.1)_\lambda$ such that u is a local minimum of J_λ at $H_0^1(\Omega)$.

Reasoning as in (i), $\overline{u} := u_\mu$ with $\mu > \lambda$ is a strict supersolution to $(1.1)_\lambda$, and $\underline{u} := \delta\varphi_1$ is a strict sub-solution for $\delta > 0$ small enough, such that $\underline{u}(x) < \overline{u}(x)$ for each $x \in \Omega$. This achieves the proof. □

Proof of Theorem 1.1. Theorem 3.3 provides the existence of positive solutions for $\lambda < \lambda_1$, and Proposition 5.1 provide the existence of minimal positive solutions for $\lambda \in (\lambda_1, \Lambda)$.

(a) Step 1. Existence of a second positive solution for $\lambda \in (\lambda_1, \Lambda)$.
Fix an arbitrary $\lambda \in (\lambda_1, \Lambda)$, and let u_λ be the minimal solution to $(1.1)_\lambda$ given by Proposition 5.1, minimizing J_λ. A second solution follows seeking a solution through variational arguments [12, Theorem 5.10] and the Mountain Pass procedure shown below.

First, reasoning as in Proposition 5.1(iii), we get a local minimum $\tilde{u}_\lambda > 0$ of J_λ. If $\tilde{u}_\lambda \neq u_\lambda$, then $\tilde{u}_\lambda$ is the second positive solution, ending the proof. Assume that $\tilde{u}_\lambda = u_\lambda$.

Now we reason as in [12, Theorem 5.10] on the nature of local minima. Thus, either

- (i) there exists $\varepsilon_0 > 0$, such that $\inf \big\{ J_\lambda(u) : \|u - \tilde{u}_\lambda\| = \varepsilon_0 \big\} > J_\lambda(\tilde{u}_\lambda)$, in other words, $\tilde{u}_\lambda$ is a strict local minimum, or
- (ii) for each $\varepsilon > 0$, there exists $u_\varepsilon \in H_0^1(\Omega)$ such that J_λ has a local minimum at a point u_ε with $\|u_\varepsilon - \tilde{u}_\lambda\| = \varepsilon$ and $J_\lambda(u_\varepsilon) = J_\lambda(\tilde{u}_\lambda)$.

Let us assume that (i) holds, since otherwise case (ii) implies the existence of a second solution.

Consider now the functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ given by $I_\lambda[v] = J_\lambda[u_\lambda + v] - J_\lambda[u_\lambda]$, more specifically

$$I_\lambda[v] := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{\lambda}{2}\int_\Omega (v^+)^2\,dx - \int_\Omega \tilde{G}_\lambda(x, v^+)\,dx.$$

where

$$\begin{aligned}\tilde{G}_\lambda(x,s) &:= a(x)\big[F(u_\lambda(x)+s) - F(u_\lambda(x)) - f(u_\lambda(x))s\big]\\ &= a(x)\left[\frac{1}{2}f'(u_\lambda(x))s^2 + o(s^2)\right].\end{aligned}$$

Obviously $I_\lambda[v^+] \le I_\lambda[v]$, and observe that $I'_\lambda[v] = 0 \iff J'_\lambda[u_\lambda + v] = 0$.

Fix now some element $0 \le v_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v_0 > 0$ in Ω^+, $v_0 \equiv 0$ in Ω^-. Let $v = tv_0$ for a certain $t = t_0 > 0$ to be selected a posteriori, and evaluate

$$I_\lambda[tv_0] = \frac{1}{2}t^2\left(\|\nabla v_0\|^2_{L^2(\Omega)} - \lambda\|v_0\|^2_{L^2(\Omega)}\right) - \int_\Omega \tilde{G}_\lambda(x, tv_0)\,dx.$$

Reasoning as in the proof of Theorem 3.3 for large positive t, since $F(t)/t^2 \to \infty$ as $t \to \infty$, and using also (3.1) we obtain that

$$\begin{aligned}I_\lambda[tv_0] &\le C(1+t+t^2) - \int_{\Omega^+} a^+(x)\Big[F(u_\lambda + tv_0) + \frac{1}{2}C_0(u_\lambda + tv_0)^2\Big]\\ &\le C(1+t+t^2) - \delta\int_{\tilde{\Omega}^+_\delta}\Big[F(u_\lambda + tv_0) + \frac{1}{2}C_0(u_\lambda + tv_0)^2\Big]\,dx,\end{aligned}$$

so

$$I_\lambda[tv_0] < 0$$

for $t = t_0$ big enough, and where $\widetilde{\Omega}_\delta^+$ is defined by (3.18). Thus, the Mountain Pass Theorem implies that if

$$\Gamma := \{\mathbf{g} \in C\big([0,1]; H_0^1(\Omega)\big) : \mathbf{g}(0) = 0,\ I_\lambda[\mathbf{g}(1)] < 0\},$$

then, there exists $c > 0$ such that

$$c := \inf_{\mathbf{g}\in\Gamma} \max_{0\le t\le 1} I_\lambda[\mathbf{g}(t)]$$

is a critical value of I_λ, and thereby $\mathscr{K}_c := \{v \in H_0^1(\Omega) : I_\lambda[v] = c,\ I'_\lambda[v] = 0\}$ is non empty.

Since for any $\mathbf{g} \in \Gamma$ we have $I_\lambda[\mathbf{g}^+(t)] \le I_\lambda[\mathbf{g}(t)]$ for all $t \in [0,1]$, it follows that $\mathbf{g}^+ \in \Gamma$, and we derive the existence of a sequence v_n such that

$$I_\lambda[v_n] \to c, \qquad \|I'_\lambda[v_n]\| \to 0, \qquad v_n \ge 0.$$

On the other hand, $w_n := u_\lambda + v_n$ is a (PS) sequence for the original functional J_λ. Since Theorem 3.2, if $\lambda < \lambda_1(\operatorname{int}\Omega^0)$, $v_n \to v_\lambda$ en $H_0^1(\Omega)$, so $I'_\lambda[v] = 0$ and $I_\lambda[v] = c > 0$, hence $v_\lambda \ge 0$ is a nontrivial critical point of I_λ. Consequently, $w_\lambda := u_\lambda + v_\lambda$ is a positive critical point of J_λ, such that, for each $\psi \in H_0^1(\Omega)$, we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla\psi\, dx = \int_\Omega \Big(\lambda w_\lambda + a(x) f(w_\lambda)\Big)\psi\, dx,$$

and thereby $w_\lambda := u_\lambda + v_\lambda \ge u_\lambda$, $w_\lambda \ne u_\lambda$ is a second positive solution to $(1.1)_\lambda$.

(b) Step 2. Existence of a classical positive solution for $\lambda = \Lambda$.
We prove the existence of a solution for $\lambda = \Lambda$. For each $\lambda \in (\lambda_1, \Lambda)$, problem (1.1) admits a minimal positive weak solution u_λ and $\lambda \to u_\lambda$ is increasing, see Proposition 5.1. Taking the monotone pointwise limit, let us define

$$u_\Lambda(x) := \lim_{\lambda\uparrow\Lambda} u_\lambda(x).$$

We next see that $\|u_\Lambda\| < +\infty$, reasoning on the contrary. Assume that there exists a sequence of solutions $u_n := u_{\lambda_n}$ such that $\|u_{\lambda_n}\| \to +\infty$ as $\lambda_n \to \Lambda$. Set $v_n := u_n/\|u_n\|$, then there exists a subsequence, again denoted by v_n such that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, and $v_n \to v$ in $L^p(\Omega)$ for any $p < 2^*$ and a.e. Arguing as in the claim of Proposition 3.1, $v \equiv 0$. Moreover

$$\lim_{n\to\infty} \int_\Omega a(x)\frac{f(u_n)}{\|u_n\|}\, v_n\, dx = 1. \tag{5.4}$$

On the other hand, from the weak formulation, for all $\psi \in C_c^\infty(\Omega)$,

$$\int_\Omega \nabla v_n \cdot \nabla\psi\, dx = \lambda_n \int_\Omega v_n\psi\, dx + \int_\Omega a(x)\frac{f(u_n)}{\|u_n\|}\psi\, dx. \tag{5.5}$$

Taking the limit, and since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$

$$\lim_{n\to\infty} \int_\Omega a(x)\frac{f(u_n)}{\|u_n\|}\, \psi\, dx = 0, \qquad \text{for all} \quad \psi \in L^2(\Omega). \tag{5.6}$$

Since Lemma 2.1, $u \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega})$ and so $a(x)\frac{f(u_n)}{\|u_n\|} \in L^2(\Omega)$. Moreover $v_n \to v = 0$ in $L^2(\Omega)$. Hence [2, Proposition 3.13 (iv)], and (5.6) imply

$$\lim_{n\to\infty} \int_\Omega a(x)\frac{f(u_n)}{\|u_n\|}\, v_n\, dx = 0,$$

which contradicts (5.4) and yields $\|u_\Lambda\| < +\infty$.

By Sobolev embedding and the Lebesgue dominated convergence theorem, $u_n \to u_\Lambda$ in $L^{2^*}(\Omega)$.

Now, by substituting $\psi = u_n$ in (5.5), using Hölder inequality and Sobolev embeddings we obtain

$$\Big[\|u_n\| \le \Lambda \|v_n\|_{L^2(\Omega)}\|u_n\| + C, \quad \text{with } \|v_n\|_{L^2(\Omega)} \to 0\Big] \Rightarrow \|u_n\| \le C.$$

By compactness, for a subsequence again denoted by u_n, $u_n \rightharpoonup u^*$ in $H_0^1(\Omega)$, $u_n \to u^*$ in $L^p(\Omega)$ for any $p < 2^*$ and a.e. By uniqueness of the limit, $u_\Lambda = u^*$. Finally, by taking limits in the weak formulation of u_n as $\lambda_n \to \Lambda$, we get

$$\int_\Omega \nabla u_\Lambda \cdot \nabla\psi = \Lambda \int_\Omega u_\Lambda \psi + \int_\Omega a(x) f(u_\Lambda)\psi\,.$$

Hence u_Λ is a positive weak solution to $(1.1)_\Lambda$. Lemma 2.1 yields that $u_\Lambda \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega})$ is a classical solution.

(c) Step 3. Existence of a classical positive solution for $\lambda \le \lambda_1$.
The existence of a classical positive solution for $\lambda < \lambda_1$ is done in Theorem 3.3. Let's look for a solution when $\lambda = \lambda_1$.

Since step 1, for any $\lambda \in (\lambda_1, \Lambda)$ there exists a second positive solution to $(1.1)_\lambda$. Let's denote it by $\tilde u_\lambda \neq u_\lambda$. Now, define the pointwise limit

$$\tilde u_{\lambda_1}(x) := \limsup_{\lambda\to\lambda_1}\ \tilde u_\lambda(x). \tag{5.7}$$

Reasoning as in step 2, $\|\tilde u_{\lambda_1}\| < +\infty$ and $\tilde u_{\lambda_1} \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega})$ is a classical solution to $(1.1)_{\lambda_1}$.

Moreover, $\tilde u_{\lambda_1} > 0$. Assume on the contrary that $\tilde u_{\lambda_1} = 0$. By the Crandall-Rabinowitz's Theorem [10], the only nontrivial solutions to (1.1) in a neighbourhood of the bifurcation point $(\lambda_1, 0)$ are given by (4.1)). Since Proposition 5.1, those are the minimal solutions u_λ, and due to $\tilde u_\lambda \neq u_\lambda$, $\tilde u_\lambda$ are not in a neighbourhood of $(\lambda_1, 0)$, contradicting the definition of $\tilde u_{\lambda_1}(x)$, (5.7)

Hence, $\tilde u_{\lambda_1} \ge 0$, and reasoning as in (1.8), the Maximum Principle implies that $\tilde u_{\lambda_1} > 0$. □

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A. Some Estimates

First, we prove an useful estimate of $\frac{\ln(e+s)}{\ln(e+as)}$.

Lemma A.1. *Let $0 < a \leq 1$ be fixed. Then for all $s \geq 0$,*

$$\frac{\ln(e+s)}{\ln(e+as)} \leq \ln\left(\frac{e}{a}\right) \leq \frac{1}{a}. \tag{A.1}$$

Proof. Denote $\ell(s) = \frac{\ln(e+s)}{\ln(e+as)}$ for all $s \geq 0$. Then $1 \leq \ell(s) \leq \ell(s_0)$ where $s_0 > 0$ is the unique value where $\ell'(s) = 0$. When computing s_0 we find

$$\ell'(s_0) = 0 \iff (e+as_0)\ln(e+as_0) - a(e+s_0)\ln(e+s_0) = 0$$

and therefore

$$\max \ell = \ell(s_0) = \frac{\ln(e+s_0)}{\ln(e+as_0)} = \frac{e+as_0}{a(e+s_0)}.$$

Notice that we have $\ell(s_0) \leq \frac{1}{a}$. In order to find a better upper bound of $\ln(\frac{e+as_0}{e+s_0})$ let us denote for all $s \geq 0$

$$\theta(s) = (e+as)\ln(e+as) - a(e+s)\ln(e+s).$$

Then, there exists $\chi \in (0, s_0)$ such that

$$0 - e(1-a) = \theta(s_0) - \theta(0) = \theta'(\chi)s_0 \Longrightarrow \frac{e(1-a)}{s_0} = -\theta'(\chi).$$

Then

$$-\theta'(s) = a\ln\left(\frac{e+s}{e+as}\right) \leq a\ln\left(\frac{1}{a}\right),$$

and

$$\frac{e(1-a)}{s_0} \leq a\ln\left(\frac{1}{a}\right) \Longrightarrow s_0 \geq \frac{e(1-a)}{a\ln\left(\frac{1}{a}\right)}.$$

Since $\frac{e+as}{a(e+s)}$ is decreasing,

$$\max_{s\geq 0}\ell(s)=\ell(s_0)=\frac{e+as_0}{a(e+s_0)}\leq\frac{e+\frac{e(1-a)}{\ln(\frac{1}{a})}}{ae+\frac{e(1-a)}{\ln(\frac{1}{a})}}$$
$$=\frac{\ln(1/a)+1-a}{a\ln(1/a)+1-a}\leq\ln(1/a)+1,$$

and the first inequality of (A.1) is achieved. The second one is obvious. □

Next lemma is about the variations of $h(s)=\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$ for $s\geq 0$.

Lemma A.2. *There exists* $\alpha^*>2(2^*-1)$ *such that* h *is an increasing function on* $]0,+\infty[$ *if and only if* $\alpha\leq\alpha^*$. *Moreover, if* $\alpha>\alpha^*$ *there exists* $s_1<s_2$ *such that* h *is increasing in* $[0,+\infty[\setminus]s_1,s_2[$.

Proof. We have

$$h'(s)=\frac{s^{2^*-2}}{[\ln(e+s)]^{\alpha+1}}\left((2^*-1)\ln(e+s)-\frac{\alpha s}{s+e}\right).$$

Let us define for $s\geq 0$,

$$\theta(s):=\ln(e+s)-\frac{\alpha}{2^*-1}\frac{s}{s+e},$$

so

$$h'(s)\geq 0\Longleftrightarrow\theta(s)\geq 0.$$

We have:

$$\begin{cases}\theta(0)=1,\\ \theta(s)\to+\infty\quad\text{as}\quad s\to+\infty,\\ \theta'(s)=\frac{s+e\left(1-\frac{\alpha}{2^*-1}\right)}{(e+s)^2}.\end{cases}$$

Hence:

(1) If $\frac{\alpha}{2^*-1}\leq 1$ then $\theta'(s)\geq 0$ for all $s\geq 0$ and in particular $\theta(s)\geq 0$ and therefore $h'(s)\geq 0$ for all $s\geq 0$;

(2) if $\frac{\alpha}{2^*-1}>1$ then

$$\theta'(s_0)=0\ \text{ for } s_0=e\left(\frac{\alpha}{2^*-1}-1\right).$$

Let us compute $\theta(s_0)$:

$$\theta(s_0)=\ln\left(\frac{\alpha}{2^*-1}\right)-\frac{\alpha}{2^*-1}+2,$$

and hence:

(i) if $\theta(s_0)\geq 0$ then $\theta(s)\geq 0$ for all $s\geq 0$ and therefore $h'(s)\geq 0$ for all $s\geq 0$;

(ii) if $\theta(s_0) < 0$ then there exists $s_1 < s_2$ such that

$$\theta(s) > 0 \quad \forall s \in [0,+\infty[\setminus]s_1,s_2[\implies h'(s) > 0 \quad \forall s \in [0,+\infty[\setminus]s_1,s_2[.$$

Notice that $t \to \ln t$ is greater that $t \to t-2$ somewhere between some $t_1 < 1$ and the value t^* =the unique solution > 2 of the equation

$$\ln t^* = t^* - 2.$$

Finally the statement of the lemma holds for $\alpha^* = t^*(2^* - 1)$. □

B. A Compact Embedding Using Orlicz Spaces

For references on Orlicz spaces see [15,21]. Throughout $\Omega \subset \mathbb{R}^N$ is an bounded open set. We will denote

$$\mathcal{L}(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi \text{ is Lebesgue measurable}\}.$$

Definition B.1. We will say that a function $M : [0,+\infty[\to [0,+\infty[$ is a N**-function** if and only if

(N1) M is convex, increasing and continuous,

(N2) $\lim_{s\to 0^+} \frac{M(s)}{s} = 0$,

(N3) $\lim_{s\to +\infty} \frac{M(s)}{s} = +\infty$.

The proof of the following property is trivial, we just quoted it for the sake of completeness.

Proposition B.2. *Any N-function M admits a representation of the form*

$$M(s) = \int_0^s m(t)dt$$

where $m : [0,+\infty[\to [0,+\infty[$ is a non-decreasing right-continuous function satisfying $m(0) = 0$ and

$$\lim_{s\to+\infty} m(s) = +\infty.$$

Thus, m is the right-derivative of M.

Our first aim is to prove the following result:

Theorem B.3. *Let $M : [0,+\infty[\to \mathbb{R}$ be a N-function such that*

$$\lim_{s\to+\infty} \frac{s^{2^*}}{M(s)} = +\infty.$$

Assume also that M satisfies the Δ_2**-condition**, *that is,*

$$\exists K > 0, \quad \forall s \in [0,+\infty[, \quad M(2s) \le KM(s). \tag{B.1}$$

Let $\{u_n\}_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ be a sequence satisfying

1. $\sup_{n\in\mathbb{N}} \|u_n\|_{2^*} < \infty$,
2. *there exists $u \in H_0^1(\Omega)$ such that $\lim_{n\to+\infty} u_n(x) = u(x)$ a.e.*

Then there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ *such that*

$$\lim_{k\to\infty}\int_\Omega M\Big(|u_{n_k}(x)-u(x)|\Big)dx=0. \tag{B.2}$$

In order to proof this theorem we need some definitions.

Definition B.4. Let $\mathcal{K}\subset\mathcal{L}(\Omega)$. We say that $\mathcal{K}$ has **equi-absolutely continuous integrals** if and only if $\forall\varepsilon>0$ there exists $h>0$ such that

$$\forall\varphi\in\mathcal{K},\forall A\subset\Omega \text{ mesurable },\ |A|<h\Longrightarrow\int_A|\varphi(x)|\,dx<\varepsilon.$$

Lemma B.5. *Let* $M:[0,+\infty[\to\mathbb{R}$ *be a* N*-function satisfying the* Δ_2 *condition* (B.1). *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable functions converging a.e. to some function* u *and such that the set*

$$\Big\{M\big(|u_n|\big):\ n\in\mathbb{N}\Big\}$$

has equi-absolutely continuous integrals. Then (B.2) *holds.*

Proof. Let fix $\varepsilon>0$ and let $\delta>0$ be such that

$$\forall n\in\mathbb{N},\ \forall A\subset\Omega \text{ mesurable },\ |A|<\delta\Longrightarrow\int_A M(|u_n|)dx\le\varepsilon.$$

Using Fatou's lemma we infer that also

$$\forall A\subset\Omega \text{ mesurable },\ |A|<\delta\Longrightarrow\int_A M(|u|)dx\le\varepsilon.$$

Let $\Omega_n=\{x\in\Omega:\ |u_n(x)-u(x)|>M^{-1}(\varepsilon)\}$. As a consequence of Egoroff's theorem, the sequence $(u_n)_{n\in\mathbb{N}}$ converge in measure to u so there exists $n_0\in\mathbb{N}$ such that

$$|\Omega_n|<\delta.$$

Then, using the convexity of M and (B.1) it comes

$$\begin{aligned}\int_\Omega M\big(|u_n-u|\big)\,dx&=\int_{\Omega_n}M\big(|u_n-u|\big)\,dx+\int_{\Omega\setminus\Omega_n}M\big(|u_n-u|\big)\,dx\\&\le\frac{1}{2}\left(\int_{\Omega_n}(M\big(2|u_n|\big)+M\big(2|u|\big)dx\right)+|\Omega|\,M\big(M^{-1}(\varepsilon)\big)\\&\le\frac{K}{2}\left(\int_{\Omega_n}\Big(M(|u_n|)+M(|u|)\Big)dx\right)+|\Omega|\varepsilon\le(K+|\Omega|)\varepsilon.\end{aligned}$$

□

In order to prove that, for the sequence of our theorem, the set

$$\Big\{M\big(|u_n|\big):\ n\in\mathbb{N}\Big\}$$

has equi-absolutely continuous integrals we are going to use the following lemma :

Lemma B.6. *Let $\mathcal{K} \subset \mathcal{L}(\Omega)$ and let $\Phi : [0, +\infty[\to [0, +\infty[$ be an increasing function satisfying*

$$\lim_{s\to+\infty} \frac{\Phi(s)}{s} = +\infty. \tag{B.3}$$

Suppose that there exists $D > 0$ such that

$$\sup_{u\in\mathcal{K}} \int_\Omega \Phi\big(|u|\big)dx \leq D. \tag{B.4}$$

Then all the functions $u \in \mathcal{K}$ are integrable and $\mathcal{K}$ has equi-absolutely continuous integrals (Valle Poussin's theorem).
Moreover, if $M : [0, +\infty[\to [0, +\infty[$ is a continuous increasing function satisfying

$$\lim_{s\to+\infty} \frac{M(s)}{s} = +\infty \text{ and } \lim_{s\to+\infty} \frac{\Phi(s)}{M(s)} = +\infty,$$

then the family $\mathcal{K}_1 = \{M\big(|u|\big) : u \in \mathcal{K}\}$ has equi-absolutely continuous integrals.

Proof. For the Valle Poussin's theorem see [18] page 159. To prove the second statement remark that the function $\tilde{\Phi} = \Phi \circ M^{-1}$ satisfies (B.3). Here M^{-1} stand for the right-hand inverse. □

Proof of theorem B.3. Let us take $\Phi(s) = |s|^{2^*}$. From hypothesis (1) of the theorem, the set $\mathcal{K} = \{u_n : n \in \mathbb{N}\}$ satisfies (B.4) for some $D > 0$. Then the conclusion follows from lemma B.5 and Lemma B.6 . □

Remark B.7. Whenever (B.2) is satisfied we say that the sequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ *converges in M-mean* to u.

One can formulate Theorem B.3 as a compact embedding of $H_0^1(\Omega)$ in some vector space endowed of the *Luxembourg norm associate to M* (see [15,21]). Instead, we are going to use the Orlicz-norm which is more suitable to our purposes. We will see later in Theorem B.12 that the convergence in M-mean implies the convergence with respect to the Orlicz-norm, provided that the Δ_2-condition is satisfied.

Definition B.8. Let M be a N-function. The **complementary of** M defined for all $s \geq 0$ is the function

$$M^*(s) := \max\big\{st - M(t) : t \geq 0\big\}.$$

As before, we give the following trivial result for the sake of completeness:

Proposition B.9. *If m is the right derivative of M then*

$$m^*(s) = \sup\{t : m(t) \leq s\}$$

is the right derivative of M^* and M^* is a N-function. Furthermore, for all $s \geq 0$ we have

$$sm(s) = M(s) + M^*(m(s)), \quad sm^*(s) = M(m^*(s)) + M^*(s).$$

Next, let us introduce the Orlicz norm associated to M :

Definition B.10. Let M be a N-function and let M^* be its complementary. Let us denote for any $v \in \mathcal{L}(\Omega)$

$$\rho(v, M^*) = \int_\Omega M^*\big(|v|\big)\, dx$$

and define the **Orlicz norm** of any $u \in \mathcal{L}(\Omega)$ by

$$\|u\|_M := \sup\left\{ \int_\Omega uv\, dx : \ v \in \mathcal{L}(\Omega),\ \rho(v, M^*) \le 1 \right\}.$$

$\|\cdot\|_M$ is a norm in the real vector space

$$L_M(\Omega) = \big\{u \in \mathcal{L}(\Omega) : \ \|u\|_M < +\infty\big\}.$$

(see [15] for the details). Let us prove the following less trivial properties:

Proposition B.11. (i) *For all* $u \in \mathcal{L}(\Omega)$*,*

$$\|u\|_M \le \int_\Omega M(|u|)\, dx + 1. \tag{B.5}$$

(ii) *For any* u *and* v *in* $\mathcal{L}(\Omega)$ *it holds*

$$\left|\int_\Omega uv\, dx\right| \le \|u\|_M\, \|v\|_{M^*} \text{ (Holder's inequality).} \tag{B.6}$$

(iii) *For any* u *and* v *in* $\mathcal{L}(\Omega)$ *we have* $\|u\|_M \le \|v\|_M$ *if* $|u| \le |v|$ *a.e.*

Proof. (i) This follows from the definition of $\|\cdot\|_M$ and the inequality $|uv| \le M(|u|) + M^*\big(|v|\big)$.

(ii) The divide the proof in 3 steps.
Step 1: For all $v \in \mathcal{L}(\Omega)$,

$$\left|\int_\Omega uv\, dx\right| \le \begin{cases} \|u\|_M & \text{if } \rho(v, M^*) \le 1 \\ \rho(v, M^*)\|u\|_M & \text{if } \rho(v, M^*) > 1 \end{cases}$$

Indeed, the first case follows directly from the definition. If $\rho(v, M^*) > 1$ then by convexity

$$M^*\left(\frac{|v|}{\rho(v, M^*)}\right) \le \frac{M^*\big(|v|\big)}{\rho(v, M^*)}$$

and therefore

$$\rho\left(\frac{|v|}{\rho(v, M^*)}, M^*\right) \le \frac{1}{\rho(v, M^*)} \int_\Omega M^*\big(|v|\big) dx = 1$$

and

$$\left|\int_\Omega u \frac{v}{\rho(v, M^*)}\, dx\right| \le \|u\|_M.$$

Step 2: If $\|u\|_M \le 1$ then $\rho\big(m(|u|), M^*\big) \le 1$.
Set $u_n = u\chi_{\{|u| \le n\}}$ for all $n \in \mathbb{N}$. Since u_n is bounded then $\rho(m(|u_n|), M^*) < +\infty$. Assume by contradiction that $\int_\Omega M^*\big(m(|u|)\big)\, dx > 1$ and let $n_0 \in \mathbb{N}$ be such that $\int_\Omega M^*\big(m(|u_{n_0}|)\big)\, dx > 1$. We have

$$M^*\big(m(|u_{n_0}|)\big) < M\big(|u_{n_0}|\big) + M^*\big(m(|u_{n_0}|)\big) = |u_{n_0}|\, m(|u_{n_0}|)$$

and therefore, by (i),

$$\rho\big(m(|u_{n_0}|), M^*\big) < \int_\Omega |u_{n_0}|\, m(|u_{n_0}|)\, dx \le \|u_{n_0}\|_M\, \rho\big(m(|u_{n_0}|), M^*\big)$$

which contradicts $\|u_{n_0}\|_M \le \|u\|_M \le 1$.

This is trivial from the definition of $\|u\|_M$, step 1 and the fact that $|u|m(|u|) = M(|u|) + M^*\big(m(|u|)\big)$.

Step 3: If $\|u\|_M \le 1$ then $\rho(u, M) \le \|u\|_M$.

Let us remark that for all $s \ge 0$

$$M^*(m(s)) + M(s) = sm(s).$$

Set $v_0 = m(|u|)$. From step 2, $\rho(v_0, M^*) \le 1$ and then

$$\rho(u, M) \le \rho(u, M) + \rho(v_0, M^*) = \int_\Omega uv_0\, dx \le \|u\|_M.$$

Now we prove Holder's inequality. From step 2 applied to M^* and $\frac{v}{\|v\|_{M^*}}$ we have $\rho\Big(\frac{v}{\|v\|_{M^*}}, M^*\Big) \le 1$, so then

$$\left|\int_\Omega u \frac{v}{\|v\|_{M^*}}\, dx\right| \le \|u\|_M$$

and Holder's inequality follows.

The proof of (iii) is trivial. □

Finally, we give the following compact embedding result:

Theorem B.12. *Let M be a N-function satisfying the Δ_2-condition (B.1) and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\Omega)$ such that*

$$\lim_{n\to\infty} \rho(u_n, M) = 0.$$

Then

$$\lim_{n\to\infty} \|u_n\|_M = 0.$$

Thus, the convergence in M-mean implies the converge with respect to the $\|\cdot\|_M$ norm.

Proof. Let $\varepsilon > 0$ and take $m \in \mathbb{N}$ such that $\frac{1}{2^{m-1}} < \varepsilon$. Using condition (B.1) we also have

$$\lim_{n\to\infty} \int_\Omega M(2^m |u_n|) dx = 0.$$

Let $n_0 \in \mathbb{N}$ be such that for all $n \ge n_0$ we have

$$\int_\Omega M(2^m |u_n|) dx < 1.$$

From step 1 of the proof in the previous proposition we have that for all $n \ge n_0$

$$\|2^m u_n\|_M \le \rho\big(2^m |u_n|, M\big) + 1 < 2,$$

which implies that

$$\|u_n\|_M < \frac{1}{2^{m-1}} < \varepsilon.$$ □

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