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Existence Results for Some Anisotropic Singular Problems via Sub-supersolutions

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Abstract. In this manuscript it is proved existence results for some singular problems involving an anisotropic operator. In the approach we combine sub-supersolutions, truncation arguments and the Schaefer's Fixed Point Theorem [23]. In this work it is not used approximation arguments as in [33, 37]

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1. Introduction

Partial differential equations involving anisotropic operators arise in several areas of science. For example, in physics, such operators are related with models that describes the dynamics of fluids with different conductivities in different directions. Another interesting example arises in Biology as a model that describes the spread of an epidemic disease in heterogeneous environments. Regarding the mentioned examples we point out the references [4, 6, 7]

Let Ω be a bounded domain with smooth boundary. In this paper we obtain existence results for the singular anisotropic problems

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) \ = \dfrac{1}{u^{\gamma}} + \beta f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{$(P_E)_\gamma$}$$

and

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) &= \frac{1}{v^{\gamma_1}} + \beta_1 f_1(x,v) \text{ in } \Omega, \\ -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial v}{\partial x_i}\Big|^{q_i-2}\frac{\partial v}{\partial x_i}\Big) &= \frac{1}{u^{\gamma_2}} + \beta_2 f_2(x,u) \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \qquad (P_S)_{\gamma_1,\gamma_2}$$

where $\gamma_i \in (0,1)$, $f_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous function and $\beta_i > 0$ are constants for $i = 0, 1, 2$ with $\gamma_0 := \gamma$, $f_0 := f$ and $\beta_0 := \beta$. Here $2 \le p_1 \le \cdots \le p_N < +\infty$ and $2 \le q_1 \le \cdots \le q_N < +\infty$ are real numbers.

Regarding the anisotropic operator note that if $p_i = 2, i = 1, \ldots, N$ we have the Laplacian operator. The problem $(P_E)_\gamma$ it was studied, in the Laplacian case, in several works in both bounded and unbounded domains, see for instance [3, 12, 13, 14, 15, 19, 21, 31, 32, 34, 39, 43, 44] and the references therein.

A related work with the p-Laplacian operator is the interesting paper [38] where the authors considered the problem

$$\begin{cases} -\Delta_p u = \frac{a(x)}{u^\gamma} + f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where a is a function satisfying certain hypotheses and $\gamma > 0$ is constant. Their arguments are mainly based in the well established regularity theory, the Vazquez's Strong Maximum Principle and on sub-supersolutions. We also quote the interesting papers [10, 11].

There is by now a large number of papers and an increasing interest about anisotropic problems. Some recent results can be found for example in the references [1, 2, 9, 26, 35, 36]. For example, in [2] the authors studied some anisotropic problems and in one of such problems they considered a sub-supersolution approach to study the equation

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = h(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathedory function.

There are few works regarding singular problems involving anisotropic operators. For example in [33] it was considered the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) \; = \frac{h}{u^{\gamma}} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \qquad (P)_{\gamma}$$

with $\gamma > 0$ and h is a function which belongs to a suitable Lebesgue space. By using perturbation arguments the author obtains existence of a positive solutions for (P_{γ}). A similar approach in considered in the related paper [37] where the author studied the equation

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) \; = \frac{h}{u^{\gamma(x)}} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where h, γ are functions that satisfy certain conditions.

Thus based on the previous commentaries we propose the study of the problems $(P_E)_{\gamma}$ and $(P_S)_{\gamma_1,\gamma_2}$. At least to our knowledge the problems studied in this manuscript were not considered in the literature. Below we describe the contributions of this work.

- It is considered a large class of operators which includes the Laplacian operator;
- Since it is not considered a perturbation of the problems proposed then our approach is different when compared to [33, 37];
- As far we know it is the first time that a sub-supersolution approach is considered for a singular anisotropic problem;
- Abstract results involving sub-supersolutions are proved. Such results are different from the one contained in [2];
- The lack of homogeneity of the anisotropic operator implies additional difficulties when one intends to consider a sub-supersolution approach;
- Due to the lack of regularity results for the anisotropic operator the approach of [38] is not applicable to our problems. In order to avoid such problem a refined estimate is needed, see Lemma 3.1. Besides that we combine the Hardy-Sobolev inequality with truncation arguments to estimate the singular term.

Unless otherwise stated it will be considered that $2 \le p_1 \le \cdots \le p_N < \overline{p}^{\star}$ and $2 \le q_1 \le \cdots \le q_N < \overline{q}^{\star}$ are real numbers with $\overline{p}, \overline{q} < N$, where $\overline{p} = \sum_{i=1}^{N} \frac{1}{p_i}$, $\overline{q} = \sum_{i=1}^{N} \frac{1}{q_i}$, $\overline{p}^{\star} := \frac{N\overline{p}}{N-\overline{p}}$ and $\overline{q}^{\star} := \dfrac{N\overline{q}}{N-\overline{q}}$.

In order to state the results of this paper some definitions are needed.

Let $\overrightarrow{p} := (p_1, \ldots, p_N)$. We say that $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a solution of $(P_E)_\gamma$ if for all $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega)$ the following equality holds:

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} = \int_\Omega \left(\frac{1}{u^\gamma} + \beta f(x,u)\right)\varphi.$$

We say that $(\underline{u}, \overline{u}) \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega) \times W^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ sub-supersolution for $(P_E)_\gamma$ if $\underline{u}(x) \le \overline{u}(x)$ a.e. in Ω, $\underline{u} = 0 \le \overline{u}$ on $\partial\Omega$ (that is $(\underline{u} - \overline{u})^+ \in W_0^{1,\overrightarrow{p}}(\Omega)$) and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \le \int_\Omega \left(\frac{1}{\underline{u}^\gamma} + \beta f(x,\underline{u})\right)\varphi \tag{1.1}$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \ge \int_\Omega \left(\frac{1}{\overline{u}^\gamma} + \beta f(x,\overline{u})\right)\varphi,$$

for all nonnegative functions $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Consider $\overrightarrow{q} := (q_1, \ldots, q_N)$. We say that $(u, v) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ is a solution for $(P_S)_{\gamma_1,\gamma_2}$ if

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} = \int_\Omega \left(\frac{1}{v^{\gamma_1}} + \beta_1 f_1(x,v)\right)\varphi$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega \left(\frac{1}{u^{\gamma_2}} + \beta_2 f_2(x,u)\right)\psi$$

for all $(\varphi, \psi) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$. We say that $(\underline{u}, \underline{v}) \in (W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,\overrightarrow{q}}(\Omega) \cap L^\infty(\Omega))$ and $(\overline{u}, \overline{v}) \in (W^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)) \times (W^{1,\overrightarrow{q}}(\Omega) \cap L^\infty(\Omega))$ is a sub-supersolution pair for $(P_S)_{\gamma_1,\gamma_2}$ if $\underline{u}(x) \le \overline{u}(x), \underline{v}(x) \le \overline{v}(x)$ a.e. in $\Omega, \underline{u} = 0 \le \overline{u}, \underline{v} = 0 \le \overline{v}$ on $\partial\Omega$ and the following inequalities hold:

$$\begin{cases} \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \le \int_\Omega \left(\frac{1}{\underline{v}^{\gamma_1}} + \beta_1 f_1(x,\underline{v})\right)\varphi, \\ \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{q_i-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} \le \int_\Omega \left(\frac{1}{\underline{u}^{\gamma_2}} + \beta_2 f_2(x,\underline{u})\right)\psi, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}, \ge \int_\Omega \left(\frac{1}{\underline{v}^{\gamma_1}} + \beta_1 f_1(x,\overline{v})\right)\varphi, \\ \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{v}}{\partial x_i}\right|^{q_i-2} \frac{\partial \overline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} \ge \int_\Omega \left(\frac{1}{\overline{u}^{\gamma_2}} + \beta_2 f_2(x,\overline{u})\right)\psi, \end{cases}$$

for all $(\varphi, \psi) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ with φ, ψ nonnegative functions.

Now we are in position to state our first results.

Theorem 1.1. *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that there exists a sub-supersolution $(\underline{u}, \overline{u})$ for $(P_E)_\gamma$. Assume that $\underline{u}(x) \geq Cd(x)$ a.e. in Ω, where $C > 0$ is a constant with $d(x) := dist(x, \partial\Omega), x \in \Omega$. Then problem $(P_E)_\gamma$ has a solution u a.e. in Ω.*

Theorem 1.2. *Consider $f_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$ continuous functions. Suppose that there exists a sub-supersolution pair $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v}) \in$ for $(P_S)_{\gamma_1,\gamma_2}$ such that $\underline{u}(x), \underline{v}(x) \geq Cd(x)$ a.e. in Ω, where $C > 0$ is a constant and $d(x) := dist(x, \partial\Omega), x \in \Omega$. Then problem $(P_S)_{\gamma_1,\gamma_2}$ has a solution (u, v) with $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq v(x) \leq \overline{v}(x)$ a.e. in Ω.*

Consider the following hypothesis.

(H_1) There exists $0 < \delta < 1$ and a constant $c_1 > 0$ such that

$$-c_1 \leq f_i(x, t), \text{ for every } 0 \leq t \leq \delta, \text{ a.e. in } \Omega \text{ for } i = 0, 1, 2.$$

(H_2) There exists $r_i > 1$ and a constant $C \geq 0$ such that

$$f_i(x, t) \leq C(t^{r_i - 1} + 1), \text{ for every } t \geq 0 \text{ and } i = 0, 1, 2,$$

where $r_0 := r$ and $f_0 := f$.

As an application of Theorem 1.1 and 1.2 we obtain the following existence results under the conditions (H_1)–(H_2).

Theorem 1.3. *The following assertions are true.*

(i) *If (H_1) holds, then $(P_E)_\gamma$ has a solution for $\beta > 0$ small enough.*
(ii) *If (H_1)–(H_2) holds and $r_1 < p_1$, then $(P_E)_\gamma$ has a solution for all $\beta > 0$;*

Theorem 1.4. *The following assertions are true.*

(i) *If (H_1) holds, then $(P_S)_{\gamma_1,\gamma_2}$ has a solution for $\beta_i > 0$ small enough $i = 1, 2$.*
(ii) *If (H_1)–(H_2) holds and $r_1 < q_1$ and $r_2 < p_1$, then $(P_S)_{\gamma_1,\gamma_2}$ has a solution for all $\beta_i > 0, i = 1, 2$.*

The paper is organized as follows:

· Sections 2 and 3 is devoted to the needed properties of Anisotropic spaces and some auxiliary estimates;
· Section 4 contains the proofs of Theorems 1.1 and 1.2;
· In Section 5 it is proved the Theorems 1.3 and 1.4.

2. Preliminaires

In this section we present some basic facts regarding anisotropic spaces and results that will be used in this work. For more informations on anisotropic spaces we quote [2, 16, 17, 18, 24, 25, 29, 42].

In what follows we denote by Ω a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary. Let $1 < p_1 \leq p_2 \leq \cdots \leq p_N$ be real numbers and denote by $\overrightarrow{p}$ the vector $\overrightarrow{p} := (p_1, \ldots, p_N) \in \mathbb{R}^N$. We denote by $W^{1,\overrightarrow{p}}(\Omega)$ the space defined by

$$W^{1,\overrightarrow{p}}(\Omega) := \left\{ u \in W^{1,1}(\Omega); \quad \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1, \ldots, N \right\},$$

which is a Banach space when equipped with the norm

$$\|u\|_{1,\overrightarrow{p}} := \|u\|_{L^1} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}, \tag{2.1}$$

where $\|\cdot\|_{L^{p_i}}$ denotes the usual norm of $L^{p_i}(\Omega)$. It will be denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$ the Banach space defined by the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(\Omega)$ with respect to the norm $\|\cdot\|_{1,\overrightarrow{p}}$.

Consider $\overline{p}$ the harmonic mean of $p_i, i = 1, \ldots, N$, given by

$$\overline{p} := \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$$

and define $\overline{p}^\star := \dfrac{N\overline{p}}{N - \overline{p}}$ for $\overline{p} < N$. From [24] we have that there exists an embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$ which is continuous for $q \in [1, \overline{p}^\star]$ and compact in the case $q \in [1, \overline{p}^\star)$. Thus the norm

$$\|u\| := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}, \quad u \in W_0^{1,\overrightarrow{p}}(\Omega)$$

is equivalent to the norm given in (2.1).

Let Ψ_1 be the first eigenfunction of the p-Laplacian operator $(-\Delta_p, W_0^{1,p}(\Omega))$. There are constants $0 < l < L$ such that $ld(x) \leq \Psi_1(x) \leq Ld(x)$ a.e. in Ω, where $d(x) = dist(x, \partial\Omega), x \in \Omega$, see for instance [27, Page 121]. Thus it is possible to rewrite the Hardy-Sobolev inequality of [5, 30] as follows.

Lemma 2.1 (Hardy-Sobolev inequality). *If $u \in W_0^{1,p}(\Omega)$ with $1 < p \leq N$, then $\frac{u}{d^\tau} \in L^r(\Omega)$, for $\frac{1}{r} = \frac{1}{p} - \frac{1-\tau}{N}$, $0 \leq \tau \leq 1$, and*

$$\left\| \frac{u}{d^\tau} \right\|_{L^r(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

where $C > 0$ is a constant.

Repeating the same arguments of [20, Lemma 2.1] (or [40, Lema 2.1]) it is possible to obtain the following improvement of the mentioned result which we present a proof for convenience.

Lemma 2.2. *Let $a \in (W_0^{1,\overrightarrow{p}}(\Omega))'$. There exists an unique solution $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ of the problem*

$$\begin{cases} -\Big[\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big)\Big] = a \ \text{in } \Omega, \\ u = 0 \ \text{on } \partial\Omega. \end{cases}$$

Proof. Let $T : W_0^{1,\overrightarrow{p}}(\Omega) \to (W_0^{1,\overrightarrow{p}}(\Omega))'$ be the operator given by

$$\Big\langle Tu, \phi \Big\rangle = \sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}.$$

Since $p_i \geq 2$, $i = 1, \ldots, N$ the inequality

$$\begin{aligned} &\left(\Big|\frac{\partial u(x)}{\partial x_i}\Big|^{p_i-2}\frac{\partial u(x)}{\partial x_i} - \Big|\frac{\partial v(x)}{\partial x_i}\Big|^{p_i-2}\frac{\partial v(x)}{\partial x_i}\right)\left(\frac{\partial u(x)}{\partial x_i} - \frac{\partial v(x)}{\partial x_i}\right) \\ &\geq C_i\Big|\frac{\partial u(x)}{\partial x_i} - \frac{\partial v(x)}{\partial x_i}\Big|^{p_i}, \end{aligned} \tag{2.2}$$

holds for some $C_i > 0$, $i = 1, \ldots, N$. Therefore

$$\Big\langle Tu - Tv, u - v \Big\rangle > 0 \ \text{ for all } \ u, v \in W_0^{1,\overrightarrow{p}}(\Omega) \ \text{ with } \ u \neq v.$$

If $\|u\| \to +\infty$, we can assume that

$$\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} \geq 1, \text{for all } i = 1, 2, \ldots, N.$$

Hence, since $1 < p_1 \leq p_i$, for all $i = 1, 2, \ldots, N$, we have

$$\sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} \geq \sum_{i=1}^{N}\Big(\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i}\Big)^{\frac{p_1}{p_i}} \geq \frac{1}{N^{p_1-1}}\Big(\sum_{i=1}^{N}\Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i}}\Big)^{p_1},$$

which implies

$$\lim_{\|u\|\to\infty} \frac{\Big\langle Tu, u \Big\rangle}{\|u\|} = +\infty.$$

Then it follows by the Minty-Browder's Theorem [8, Theorem 5.16] that there exists an unique function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that $Tu = a$. □

Lemma 2.3. (See [41].) *Assume that $\phi : [0, \infty) \to [0, \infty)$ is a nonincreasing function such that if $h > k > k_0$, for some $\alpha > 0$, $\beta > 1$, $\phi(h) \leq C(\phi(k))^\beta/(h-k)^\alpha$. Then $\phi(k_0 + d) = 0$, where $d^\alpha = C2^{\frac{\alpha\beta}{\beta-1}}\phi(k_0)^{\beta-1}$ and C is a positive constant.*

The next two results can be found in [20, Lemma 2.4] (or [40, Lema 2.4]) and [20, Lema 2.2] (or [40, Lemma 2.2]), respectively. The proofs are presented for completeness.

Lemma 2.4. *Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a solution to problem*

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

such that $f \in L^r(\Omega)$ with $r > \overline{p}^*/(\overline{p}^* - p_1)$. Then $u \in L^\infty(\Omega)$. Moreover

$$\|u\|_\infty \leq \frac{C\|f\|_r^{\frac{1}{p_1-1}}|\Omega|^{\frac{\beta-1}{\alpha}}}{S^{\frac{1}{p_1-1}}}, \tag{2.3}$$

where β, α, S and C are constants that do not depend on u.

Proof. Define $v_k = sign(u)(|u| - k)^+$, $k \in \mathbb{R}$. It follows that $v_k \in W_0^{1,\overrightarrow{p}}(\Omega)$ and $\frac{\partial v}{\partial x_i} = \frac{\partial v_k}{\partial x_i}$ in the set $A(k) = \{x \in \Omega : |u(x)| > k\}$. Denote by $|A(k)|$ the Lebesgue measure of $A(k)$. By considering the test function v_k and using the Hölder inequality, we obtain that

$$\sum_{i=1}^{N}\int_{A(k)}\Big|\frac{\partial v_k}{\partial x_i}\Big|^{p_i} = \int_\Omega f v_k \leq \Big(\int_\Omega |v_k|^{p^*}\Big)^{\frac{1}{p^*}}\Big(\int_\Omega |f|^r\Big)^{\frac{1}{r}}|A(k)|^{1-\left(\frac{1}{p^*}+\frac{1}{r}\right)}.$$

From [22] we have

$$0 < S := \inf_{\substack{u\in D^{1,\overrightarrow{p}}(\mathbb{R}^N)\\ \|u\|_{p^*}=1}} \sum_{i=1}^{N}\frac{1}{p_i}\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_i}}^{p_i}.$$

Using the fact that $p_i \geq p_1 > 1$, we obtain that

$$S\Big(\int_\Omega |u|^{p^*}\Big)^{\frac{p_1}{p^*}} \leq \sum_{i=1}^{N}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i}, \text{ for all } u \in W_0^{1,\overrightarrow{p}}(\Omega).$$

Therefore

$$S\Big(\int_{A(k)} |v_k|^{p^*}\Big)^{\frac{p_1-1}{p^*}} \leq \Big(\int_\Omega |f|^r\Big)^{\frac{1}{r}}|A(k)|^{1-\left(\frac{1}{p^*}+\frac{1}{r}\right)}.$$

Note that if $0 < k < h$, $A(h) \subset A(k)$ and

$$|A(h)|^{\frac{1}{p^*}}(h-k) = \Big(\int_{A(h)}(h-k)^{p^*}\Big)^{\frac{1}{p^*}} \leq \Big(\int_{A(k)}|v_k|^{p^*}\Big)^{\frac{1}{p^*}},$$

then

$$|A(h)| \leq \frac{1}{(h-k)^{p^*}}\frac{1}{S^{\frac{p^*}{p_1-1}}}\|f\|_r^{\frac{p^*}{p_1-1}}|A(k)|^{\frac{p^*}{p_1-1}\left[1-\left(\frac{1}{p^*}+\frac{1}{r}\right)\right]}.$$

Since $r > \frac{p^*}{p^*-p_1}$, we have $\beta := \frac{p^*}{p_1-1}\big[1-\big(\frac{1}{p^*}+\frac{1}{r}\big)\big] > 1$. Therefore, if we define

$$\phi(h) = |A(h)|,\ \alpha = p^*,\ \beta = \frac{p^*}{p_1-1}\Big[1-\Big(\frac{1}{p^*}+\frac{1}{r}\Big)\Big],\ k_0 = 0,$$

we have that ϕ is a nonincreasing function and

$$\phi(h) \leq \frac{C}{(h-k)^\alpha}\phi(k)^\beta, \text{ for all } h > k > 0.$$

By Lemma 2.3, we have $\phi(d) = 0$ for $d = C\|f\|_r^{\frac{1}{p_1-1}}|\Omega|^{\frac{\beta-1}{\alpha}}/S^{\frac{1}{p_1-1}}$, then

$$\|u\|_\infty \leq \frac{C\|f\|_r^{\frac{1}{p_1-1}}|\Omega|^{\frac{\beta-1}{\alpha}}}{S^{\frac{1}{p_1-1}}}. \qquad \square$$

Lemma 2.5. *Let Ω be a bounded domain and consider $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying*

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) \leq -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial v}{\partial x_i}\Big|^{p_i-2}\frac{\partial v}{\partial x_i}\Big) & in \ \ \Omega, \\ u \leq v \ \ on \ \ \partial\Omega, \end{cases}$$

then $u \leq v$ a.e. in Ω.

Proof. Using the test function $\phi = (u-v)^+ := max\{u-v, 0\} \in W_0^{1,\overrightarrow{p}}(\Omega)$ it follows that

$$\int_{\Omega \bigcap [u>v]} \sum_{i=1}^{N} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \leq 0.$$

From (2.2), we get $\|(u-v)^+\| \leq 0$, therefore $u \leq v$ a.e. in Ω. $\square$

3. An auxiliary estimate

The next estimate will play an important role in our arguments with respect to obtain apropriated sub-supersolutions.

Lemma 3.1. *Consider $p_1 \geq 2$ and $\mu > 0$. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ be the unique solution of the problem*

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = \mu \ in \ \Omega, \\ u = 0 \ \ on \ \partial\Omega. \end{cases} \tag{3.1}$$

Then there are constants $0 < \delta \leq 1$ and C, that do not depend on μ and u, such that

$$u(x) \geq C\min\left\{\mu^{\frac{1}{p_1-1}}, \mu^{\frac{1}{p_N-1}}\right\}\min\{\delta, d(x)\},$$

where $d(x) := dist(x, \partial\Omega)$, $x \in \Omega$.

Proof. There exists a constant $0 < \delta \leq 1$ small enough such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$, where $\Omega_\epsilon = \{x \in \Omega; d(x) < \epsilon\}$, $\epsilon > 0$, see [28, Lemma 14.16] and its proof. Consider

$$\theta > \max_{i=1,\ldots,N}\left\{\frac{1}{p_i-1}\right\} \tag{3.2}$$

and the function

$$v(x)=\begin{cases}\xi d(x), & \text{if} \quad d(x)<\delta,\\ \xi\delta+\displaystyle\int_{\delta}^{d(x)}\xi\left(\frac{2\delta-t}{\delta}\right)^{\theta}dt, & \text{if} \quad \delta\le d(x)<2\delta,\\ \xi\delta+\displaystyle\int_{\delta}^{2\delta}\xi\left(\frac{2\delta-t}{\delta}\right)^{\theta}dt, & \text{if} \quad 2\delta\le d(x),\end{cases}$$

where $\xi>0$ will be choosen before. Note that $v\in C_0^1(\overline{\Omega})$. Direct computations imply that if $x\in\Omega$ satisfies $d(x)<\delta$ with $\frac{\partial d(x)}{\partial x_i}\neq 0$, then

$$\begin{aligned}&-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\xi d)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\xi d)}{\partial x_i}\right)\\ &=-\sum_{i=1}^{N}\left(\left(sgn\left(\frac{\partial d}{\partial x_i}\right)\right)\xi^{p_i-1}(p_i-1)\left(\left(sgn\left(\frac{\partial d}{\partial x_i}\right)\right)\frac{\partial d}{\partial x_i}\right)^{p_i-2}sgn\left(\frac{\partial d}{\partial x_i}\right)\frac{\partial^2 d}{\partial x_i^2}\right)\\ &:=B(x),\end{aligned}\tag{3.3}$$

where $sgn(x)=1$ if $x\geq 0$ and $sgn(x)=-1$ if $x<0$. On other hand, we have in the case $\delta<d(x)<2\delta$ with $\frac{\partial d(x)}{\partial x_i}\neq 0$ that

$$\begin{aligned}&-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v_1}{\partial x_i}\right|^{p_i-2}\frac{\partial v_1}{\partial x_i}\right)\\ &=-\sum_{i=1}^{N}\xi^{p_i-1}\theta(p_i-1)\left(\frac{2\delta-d(x)}{\delta}\right)^{\theta(p_i-1)-1}\left(\frac{-1}{\delta}\right)\left(\left(sgn\left(\frac{\partial d}{\partial x_i}\right)\right)\frac{\partial d}{\partial x_i}\right)^{p_i-2}\left(\frac{\partial d}{\partial x_i}\right)^2\\ &\quad-\sum_{i=1}^{N}\left(\frac{2\delta-d(x)}{\delta}\right)^{\theta(p_i-1)}\left|\frac{\partial d}{\partial x_i}\right|^{p_i-2}\frac{\partial^2 d}{\partial x_i^2}\left(sgn\left(\frac{\partial d}{\partial x_i}\right)\right)\xi^{p_i-1}\\ &:=C(x).\end{aligned}\tag{3.4}$$

Thus

$$\int_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial\varphi}{x_i}=-\int_{\Omega}A\varphi,\forall\varphi\in W_0^{1,\overrightarrow{p}}(\Omega),$$

where $A(x)$ is equal to $B(x)$ and $C(x)$ when $d(x)<\delta$ with $\frac{\partial d(x)}{\partial x_i}\neq 0$ and $\delta<d(x)<2\delta$ with $\frac{\partial d(x)}{\partial x_i}\neq 0$, respectively, and zero when $\frac{\partial d(x)}{\partial x_i}=0$ or $d(x)>2\delta$.

From (3.2) and the fact that $p_i\geq 2$, $i=1,\ldots,N$, it follows, in the weak sense, that

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v_1}{\partial x_i}\right|^{p_i-2}\frac{\partial v_1}{\partial x_i}\right)\leq C\max\{\xi^{p_1-1},\ldots,\xi^{p_N-1}\},$$

where C is constant satisfying the statement of the result. Let $\mu>0$ and consider u to be the solution of (3.1).

Suppose that $\frac{\mu}{C}<1$, then consider $0<\xi<1$ such that $\max\{\xi^{p_1},\ldots,\xi^{p_N-1}\}=\xi^{p_1-1}=\frac{\mu}{C}$.

From Lemma 2.5 we have that

$$u(x) \geq v(x) \geq \xi \min\{\delta, d(x)\} = \left(\frac{\mu}{C}\right)^{\frac{1}{p_1-1}} \min\{\delta, d(x)\}.$$

Repeating the previous argument in the case $\frac{\mu}{C} \geq 1$ we obtain that

$$u(x) \geq v(x) \geq \xi \min\{\delta, d(x)\} = \left(\frac{\mu}{C}\right)^{\frac{1}{p_N-1}} \min\{\delta, d(x)\}.$$

The result is proved. □

4. Proof of Theorems 1 and 2

In this section we prove Theorems 1 and 2 by using Schaefer's Fixed Point Theorem combined with sub-supersolutions and a truncation argument.

Proof of Theorem 1.1. Consider $T : L^1(\Omega) \to L^1(\Omega)$ the operator given by

$$(Tu)(x) = \begin{cases} \underline{u}(x)^{-\gamma} + \beta f(x, \underline{u}(x)), & \text{if } u(x) \leq \underline{u}(x), \\ u(x)^{-\gamma} + \beta f(x, u(x)), & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \overline{u}(x)^{-\gamma} + \beta f(x, \overline{u}(x)), & \text{if } u(x) \geq \overline{u}(x). \end{cases}$$

Since $0 < \gamma < 1$, $\overline{u}(x) \geq \underline{u}(x) \geq Cd(x)$ a.e. in Ω and $d^{-\gamma} \in L^1(\Omega)$ (see [34, Page 726]) it follows that $\underline{u}^{-\gamma} \in L^1(\Omega)$. Note also that $\overline{u}^{-\gamma} \leq \underline{u}^{-\gamma}$. Thus, since $\underline{u}$ and $\overline{u} \in L^\infty(\Omega)$ we have that T is well defined.

Let $v \in L^1(\Omega)$ be an arbitrary function. Note that by Lemmata 2.1 and 2.2 the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = Tv \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \qquad (P_1)$$

has an unique solution in $W_0^{1,\overrightarrow{p}}(\Omega)$. In fact, consider $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega)$, define

$$F(\varphi) := \int_\Omega (Tv)\varphi, \quad \varphi \in W_0^{1,\overrightarrow{p}}(\Omega).$$

Since $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$ it follows from the Lemma 2.1 that

$$\begin{aligned} \left|\int_\Omega (Tv)\varphi\right| &\leq C\left(\int_\Omega \left|\frac{\varphi}{d^\gamma}\right| + \int_\Omega |\varphi|\right) \\ &\leq C\left(\left\|\frac{\varphi}{d^\gamma}\right\|_{L^r(\Omega)} + \|\varphi\|\right) \\ &\leq C(\|\varphi\|_{W_0^{1,2}(\Omega)} + \|\varphi\|) \\ &\leq C\|\varphi\|, \end{aligned}$$

for some $r \geq 1$ and a constant $C > 0$. Hence $F \in (W_0^{1,\overrightarrow{p}}(\Omega))'$, then the Lemma 2.2 shows the claim. Therefore we can define an operator $\widetilde{S} : L^1(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)$, defined by $\widetilde{S}(v) = u$, where $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is the unique solution of (P_1). We claim that the operator $S : L^1(\Omega) \to L^1(\Omega)$ given by $S := i \circ \widetilde{S}$ is compact, where $i : W_0^{1,\overrightarrow{p}}(\Omega) \to L^1(\Omega)$ is the compact embedding given by [24]. In order to prove such claim consider (v_n) a sequence with $v_n \to v$ in $L^1(\Omega)$ and consider $u_n := S(v_n) \in W_0^{1,\overrightarrow{p}}(\Omega)$, $n \in \mathbb{N}$. From the definition of the operator S it follows that

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (Tv_n)\phi, \tag{4.1}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Using the test function $\phi = u_n, n \in \mathbb{N}$ in (4.1), the boundness in $L^1(\Omega)$ of (v_n), the boundness of f in $\overline{\Omega} \times [0, \|\overline{u}\|_{L^\infty(\Omega)}]$ and Lemma 2.1 we get

$$\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}}^{p_i} \le K\left(\int_\Omega \left|\frac{u_n}{d^\gamma}\right| + \int_\Omega |u_n|\right) \le K\left(\|u_n\| + 1\right), \tag{4.2}$$

The inequality $|t|^{p_1} \le 1 + |t|^{p_i}$, $i = 1, \dots, N$, for all $t \ge 0$ provides that

$$\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}^{p_1} \le K(\|u_n\| + 1),$$

for all $n \in \mathbb{N}$, where $K > 0$ is a constant that does not depend on $n \in \mathbb{N}$. Since $(a_1 + \cdots + a_N)^b \le C(a_1^b + \cdots + a_N^b)$ for $a_i \ge 0$, $i = 1, \dots, N$ and $b \ge 1$, where $C > 0$ depends only on N and b, it follows that

$$\left(\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}\right)^{p_1} \le K(\|u_n\| + 1), \tag{4.3}$$

for all $n \in \mathbb{N}$ with $K > 0$ being a constant that does not depend on n. Then it follows that the sequence (u_n) is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$. Using the compact embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^1(\Omega)$ we obtain, up to a subsequence, that $u_n \to u$ in $L^1(\Omega)$ for some $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, which implies that S is compact.

To verify the mentioned continuity let (v_n) be a sequence in $L^1(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$. Considering $u_n = S(v_n)$, $n \in \mathbb{N}$ and $u = S(v)$, we obtain from the definition of S that

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (Tv_n)\phi \tag{4.4}$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (Tv)\phi \tag{4.5}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Using $\phi = u_n$ as a test function and subtracting (4.5) from (4.4) we get

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N \left[\left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \right] \\ &= \int_\Omega (Tv_n) - (Tv)(u_n - u). \end{aligned} \tag{4.6}$$

Using Lemma 2.1 and the Lebesgue's Dominated Convergence Theorem it follows that the right-hand side of the equation (4.6) converges to zero. Then we have that $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$.

We claim that there is $R > 0$ such that if $u = \sigma S(u)$ with $\sigma \in [0,1]$ then $\|u\|_{L^1} < R$, where R is a constant that does not depend on u and σ. If $\sigma = 0$ then $u = 0$. Suppose that $\sigma \neq 0$. Then $S(u) = \frac{u}{\sigma}$, which implies the identitiy

$$\int_\Omega \sum_{i=1}^N \left| \frac{1}{\sigma} \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{1}{\sigma} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} = \int_\Omega (Tu)\phi, \tag{4.7}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Using $\phi = \frac{u}{\sigma}$ in (4.7), Lemma 2.1 and the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^1(\Omega)$ we obtain that

$$\begin{aligned} \frac{1}{\sigma^{p_1}} \int_\Omega \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \le \int_\Omega \sum_{i=1}^N \frac{1}{\sigma^{p_i}} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} &\le \frac{K}{\sigma} \left(\|u\| + \int_\Omega |u| \right) \\ &\le \frac{K}{\sigma} \|u\|, \end{aligned} \tag{4.8}$$

where $K > 0$ is a constant that does not depend on σ. Thus we obtain that

$$\begin{aligned} \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}^{p_i} &\le K \sigma^{p_1 - 1} \|u\| \\ &\le C \|u\|, \end{aligned} \tag{4.9}$$

where $C, K > 0$ are constants that do not depend on σ. Arguing as in (4.3) we get $\|u\|^{p_1-1} \le K$ for $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ with $u \neq 0$, where $K > 0$ is a constant that does not depend on σ and u. The claim is proved.

Thus by Schaefer's Fixed Point Theorem there exist $u \in L^1(\Omega)$ such that $u = S(u)$. Note also that by the definition of S we have $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ with

$$\int_\Omega \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} = \int_\Omega (Tu)\phi, \tag{4.10}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Note that $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ a.e. in Ω. In fact, by using $\phi = (\underline{u} - u)^+$ in (1) and (4.10) we get

$$\int_\Omega \sum_{i=1}^N \left[\left(\left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \underline{u}}{\partial x_i} - \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) \left(\frac{\partial (\underline{u} - u)^+}{\partial x_i} \right) \right] \leq 0.$$

Thus by inequality (2.2) we have $\underline{u}(x) \leq u(x)$ a.e. in Ω. The other inequality follows by using a similar reasoning with the test function $\phi = (u - \overline{u})^+$. □

Theorem 1.2 can be obtained by using the reasoning of the previous proof under some modifications. Below we present a proof.

Proof of Theorem 1.2. Consider $T, W : L^1(\Omega) \to L^1(\Omega)$ the operators given by

$$(Tz)(x) = \begin{cases} \underline{u}(x)^{-\gamma_1} + \beta_1 f_1(x, \underline{u}(x)), & \text{if } z(x) \leq \underline{u}(x), \\ z(x)^{-\gamma_1} + \beta_1 f_1(x, u(x)), & \text{if } \underline{u}(x) \leq z(x) \leq \overline{u}(x), \\ \overline{u}(x)^{-\gamma_1} + \beta_1 f_1(x, \overline{u}(x)), & \text{if } z(x) \geq \overline{u}(x), \end{cases}$$

and

$$(Wz)(x) = \begin{cases} \underline{v}(x)^{-\gamma_2} + \beta_2 f_2(x, \underline{v}(x)), & \text{if } z(x) \leq \underline{v}(x), \\ z(x)^{-\gamma_2} + \beta_2 f_2(x, z(x)), & \text{if } \underline{v}(x) \leq z(x) \leq \overline{v}(x), \\ \overline{v}(x)^{-\gamma_2} + \beta_2 f_2(x, \overline{v}(x)), & \text{if } z(x) \geq \overline{v}(x). \end{cases}$$

Let $h, \tilde{h} \in L^1(\Omega)$ be arbitrary functions. By Lemmata 2.1 and 2.2 it follows that the problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right) = Th \text{ in } \Omega, \\ -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i - 2} \frac{\partial v}{\partial x_i} \right) = W\tilde{h} \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{P_2}$$

has an unique solution in $W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$.

Consider in $L^1(\Omega) \times L^1(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ the maximum norm which will be denoted by $|\cdot|_{L^1 \times L^1}$ and $|\cdot|_{\overrightarrow{p},\overrightarrow{q}}$, respectively. Therefore we can define an operator $\overline{S} : L^1(\Omega) \times L^1(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$, defined by the equation $\overline{S}(\widetilde{h}, h) = (u, v)$, where $(u, v) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ is the unique solution of (P_2). Arguing as in the proof of Theorem 1.1 we have that the operator $S' : L^1(\Omega) \times L^1(\Omega) \to L^1(\Omega) \times L^1(\Omega)$ given by $S' := i' \circ \overline{S}$ is continuous and compact, where $i' : W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega) \to L^1(\Omega) \times L^1(\Omega)$ is a compact embedding which can be obtained by [24].

In what follows it will be proved that there is $R > 0$ such that, if

$$(u, v) = \theta S'(u, v) \text{ with } \theta \in [0, 1]$$

then we obtain that

$$|(u, v)|_{L^1 \times L^1} < R.$$

In fact, if $\theta = 0$ we get $(u,v) = (0,0)$. In the case $\theta \neq 0$, we have that

$$S^{'}(u,v) = \left(\frac{u}{\theta}, \frac{v}{\theta}\right).$$

From the definition of $S^{'}$ we have

$$\int_\Omega \sum_{i=1}^N \left|\frac{1}{\theta}\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{1}{\theta}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (Tv)\phi$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{1}{\theta}\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{1}{\theta}\frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega (Wu)\psi$$

for all $(\phi,\psi) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$. By considering $(\phi,\psi) = (u,v)$ we obtain that

$$\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}}^{p_i} \leq K\theta^{p_1-1}\|u\|_{1,\overrightarrow{p}}$$

and

$$\sum_{i=1}^N \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{q_i}}^{q_i} \leq K\theta^{q_1-1}\|v\|_{1,\overrightarrow{q}},$$

where $K > 0$ is a constant that does not depend on θ. Then it is possible to obtain $R > 0$ such that

$$|(u,v)|_{L^1\times L^1} < R.$$

Thus by Schaefer's Fixed Point Theorem, there exists $(u,v) \in L^1(\Omega) \times L^1(\Omega)$, such that

$$(u,v) = S^{'}(u,v) \text{ and } |(u,v)|_{L^1\times L^1} < R.$$

Therefore

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (Tv)\phi \tag{4.11}$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega (Wu)\psi, \tag{4.12}$$

for all $(\phi,\psi) \in W_0^{1,\overrightarrow{p}}(\Omega)\times W_0^{1,\overrightarrow{q}}(\Omega)$. By considering the pairs $(\phi,\psi) = ((\underline{u}-u)^+, (\underline{v}-v)^+)$ and $(\phi,\psi) = ((u-\overline{u})^+, (v-\overline{v})^+)$ in (4.11) and (4.12) we obtain that $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq v(x) \leq \overline{v}(x)$ a.e. in Ω, which concludes the proof. □

5. Proof of Theorem 1.3

We will start by constructing $\underline{u}$ for the all cases of the result. Let $\varepsilon > 0$ be chosen before. From Lemma 2.2 there exists $\underline{u} \in W_0^{1,\overrightarrow{p}}(\Omega)$, the unique solution of the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \underline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}\Big) = \varepsilon \text{ in } \Omega, \\ \underline{u} = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.1}$$

From Lemma 2.4 it follows that $\|\underline{u}\|_\infty \leq C\varepsilon^{\frac{1}{p_1-1}}$, where C is a constant that does not depend on ε. Therefore $\|\underline{u}\|_\infty \leq \delta$ for $\varepsilon > 0$ small enough, where δ is given in (H_1). Consider $\varepsilon_0 > 0$, which depends on α and β, such that

$$\frac{1}{\Big(C\varepsilon_0^{\frac{1}{p_1-1}}\Big)^\gamma} - \beta c_1 \geq \varepsilon_0. \tag{5.2}$$

Thus for all for all $0 < \varepsilon \leq \varepsilon_0$ we have from (5.2) and $(f)_1$ that

$$\begin{aligned} \frac{1}{\underline{u}^\gamma} + f(x,\underline{u}) &\geq \frac{1}{\Big(C\varepsilon^{\frac{1}{p_1-1}}\Big)^\gamma} - \beta c_1 \\ &\geq \frac{1}{\Big(C\varepsilon_0^{\frac{1}{p_1-1}}\Big)^\gamma} - \beta c_1 \\ &\geq \varepsilon \\ &= -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \underline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}\Big). \end{aligned} \tag{5.3}$$

Note that $\frac{\varphi}{\underline{u}^\gamma} \in L^1(\Omega)$ for all $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega)$.

In fact, since $p_1 \geq 2$ we have $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and by Lemma 3.1 we have $\underline{u}(x) \geq Kd(x)$ a.e. in Ω where K is a constant.

Let Ψ_1 be the first eigenfunction of the operator $(-\Delta, W_0^{1,2}(\Omega))$. Recall that there are constants $0 < l < L$ such that $ld(x) \leq \Psi_1(x) \leq Ld(x)$ a.e. in Ω. Thus by Lemma 2.1 it follows that $\frac{\varphi}{\underline{u}^\gamma} \in L^1(\Omega)$. From (6.5) and the integrability of $\frac{\varphi}{\underline{u}^\gamma}$ it follows that $\underline{u}$ satisfies (1.1).

Now consider the function $\overline{u}$ for the case i). Let $R > 0$ and denote by $B_R := B_R(0)$ an open ball centered at the origin with radius $R > 0$ and such that $\Omega \subset\subset B_R$.

Consider $w \in W_0^{1,\overrightarrow{p}}(B_R)$, the solution of the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial w}{\partial x_i}\Big|^{p_i-2}\frac{\partial w}{\partial x_i}\Big) = 1 \text{ in } B_R, \\ w = 0 \text{ on } \partial B_R. \end{cases} \tag{5.4}$$

By Lemma 3.1 there exists a constant $C_0 > 0$ such that

$$w(x) \geq C_0 d_R(x) \text{ a.e. in } B_R, \tag{5.5}$$

where $d_R(x) := dist(x, \partial B_R)$, $x \in B_R$. Thus by (5.5) we obtain for $\xi > 0$ that

$$d_R(x)(\xi w)^{-\gamma} \leq C_0{}^{-\gamma} d_R(x)^{1-\gamma}\xi^{-\gamma} \leq C_0{}^{-\gamma}(2R)^{1-\gamma}\xi^{-\gamma} \text{ a.e. in } B_R, \tag{5.6}$$

which implies that $d_R(\xi w)^{-\gamma} \in L^\infty(\Omega)$. Thus there exists $\overline{u} \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$, the solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) = d_R(x)(\xi w(x))^{-\gamma} + M \text{ in } B_R, \\ \overline{u} = 0 \text{ on } \partial B_R, \end{cases} \tag{5.7}$$

where $M > 1$ is a fixed constant. Let $d_{\overline{\Omega}} := \min_{\overline{\Omega}} d_R(x) > 0$. Choose $\xi > 0$ small enough such that

$$(d_{\overline{\Omega}}\xi^{-\gamma} - 1)\|w\|^{-\gamma}_{L^\infty(B_R)} \geq 2M. \tag{5.8}$$

Since f is continuous it is possible to choose $\beta > 0$ small enough such that

$$M + \beta f(x, \overline{u}) \leq 2M \text{ a.e. in } \Omega. \tag{5.9}$$

Then by (5.8) and (5.9) we obtain that

$$(d_{\overline{\Omega}}\xi^{-\gamma} - 1)w^{-\gamma} \geq M + \beta f(x, \overline{u}) \text{ a.e. in } \Omega,$$

which implies the inequalities

$$\begin{aligned} d_R(x)(\xi w)^{-\gamma} + M &\geq 2M + \beta f(x, \overline{u}) + w^{-\gamma} \\ &\geq w^{-\gamma} + \beta f(x, \overline{u}) \quad \text{a.e. in } \Omega. \end{aligned} \tag{5.10}$$

Since $M > 1$ it follows from Lemma 2.5 that $\overline{u}(x) \geq w(x)$ a.e. in B_R. Then by (5.7) and (5.10) it follows that

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) \geq \overline{u}^{-\gamma} + \beta f(x, \overline{u}) \text{ a.e. in } \Omega.$$

Using again the fact that $M > 1$ we obtain from Lemma 2.5 and the equations (5.1) and (5.7) that $\overline{u}(x) \geq \underline{u}(x)$ a.e. in Ω. Thus the first part of the result is proved.

Regarding the second part of the result denote by $\overline{u} \in W_0^{1,\overrightarrow{p}}(\Omega)$, the unique solution of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) = \mu \text{ in } B_R, \\ \overline{u} = 0 \text{ on } \partial B_R. \end{cases} \tag{5.11}$$

Consider $\overline{\delta} > 0$ the distance of the sets ∂B_R and $\partial\Omega$. For all $n \in \mathbb{N}$ we have by Lemma 3.1 that

$$\begin{aligned}
&-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) - \overline{u}^{-\gamma} - \beta f(x,\overline{u})\\
&\geq \mu - \overline{u}^{-\gamma} - C\beta(\|\underline{u}\|_\infty^{r-1}+1)\\
&\geq \mu - \overline{u}^{-\gamma} - C\beta(\mu^{\frac{r-1}{p_1-1}}+1)\\
&\geq \mu - \frac{1}{\min\Big\{\big(\frac{\mu}{C}\big)^{\frac{\gamma}{p_1-1}}, \big(\frac{\mu}{C}\big)^{\frac{\gamma}{p_N-1}}\Big\}\min\{\delta,\overline{\delta}\}} - C\beta(\mu^{\frac{r-1}{p_1-1}}+1),
\end{aligned} \tag{5.12}$$

in Ω, where $\delta > 0$ is such that $d \in C^2(\overline{B_{3\delta}})$.

Since $r < p_1$ then it follows that the right-hand side of (5.12) is nonnegative for $\mu \geq 1$ large enough. For ε small we get $\varepsilon \leq 1 \leq \mu$. Thus by (5.1), (5.11) and Lemma 2.5 it follows that $\underline{u}(x) \leq \overline{u}(x)$ a.e. in Ω.

Therefore we have a sub-supersolution $(\underline{u},\overline{u})$ for $(P_E)_\gamma$ in all cases stated in Theorem 1.3. Thus by Lemma 1.1 we have the result. □

6. Proof of Theorem 1.4

We will start by constructing the function $\underline{u},\underline{v}$ for all cases of the result. Let $\varepsilon > 0$ be chosen before. From Lemma 2.2 there exists $(\underline{u},\underline{v}) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ such that

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \underline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}\Big) & = \varepsilon \text{ in } \Omega,\\ \underline{u} = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.1}$$

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \underline{v}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \underline{v}}{\partial x_i}\Big) & = \varepsilon \text{ in } \Omega,\\ \underline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.2}$$

From Lemma 2.4 it follows that $\|\underline{u}\|_\infty \leq C\varepsilon^{\frac{1}{p_1-1}}$ and $\|\underline{u}\|_\infty \leq C\varepsilon^{\frac{1}{q_1-1}}$ where C is a constant that does not depend on ε. Then it follows that $\|\underline{u}\|_\infty, \|\underline{v}\|_\infty \leq \delta$ for $0 < \varepsilon \leq \varepsilon_0$ with $\varepsilon_0 > 0$ small enough, where δ is given in (H_1). Consider also that

$$\frac{1}{\Big(C\varepsilon_0{}^{\frac{1}{p_1-1}}\Big)^{\gamma_1}} - \beta c_1 \geq \varepsilon_0 \tag{6.3}$$

and

$$\frac{1}{\Big(C\varepsilon_0{}^{\frac{1}{q_1-1}}\Big)^{\gamma_1}} - \beta c_1 \geq \varepsilon_0. \tag{6.4}$$

From (6.1), (6.2), (6.3), (6.4) and (H_1) it follows that

$$\begin{aligned}\frac{1}{\underline{v}^{\gamma_1}} + \beta f_1(x,\underline{v}) &\geq \frac{1}{\left(C\varepsilon^{\frac{1}{q_1-1}}\right)^{\gamma_1}} - \beta c_1 \\ &\geq \frac{\alpha}{\left(C\varepsilon_0{}^{\frac{1}{q_1-1}}\right)^{\gamma_1}} - \beta c_1 \\ &\geq \varepsilon \\ &= -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \underline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}\Big)\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}\frac{1}{\underline{u}^{\gamma_2}} + \beta f_2(x,\underline{u}) &\geq \varepsilon \\ &= -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \underline{v}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \underline{v}}{\partial x_i}\Big),\end{aligned} \tag{6.6}$$

which implies that $\underline{u}$ and $\underline{v}$ satisfy (1.2).

Regarding the functions $\overline{u}$ and $\overline{v}$ for the first case consider $w \in W_0^{1,\overrightarrow{p}}(\Omega)$ (5.4) and $\widetilde{w} \in W_0^{1,\overrightarrow{q}}(\Omega)$, the solution of the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \widetilde{w}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \widetilde{w}}{\partial x_i}\Big) = 1 \text{ in } B_R, \\ \widetilde{w} = 0 \text{ on } \partial B_R, \end{cases}$$

where B_R denotes an open ball centered at the origin with radius R such that $\Omega \subset\subset B_R$. By Lemma 3.1 there exists a constant $C_0 > 0$ such that

$$\min\{w(x), \widetilde{w}(x)\} \geq C_0 d_R(x) \text{ a.e. in } B_R.$$

Arguing as in (5.6) we obtain that

$$d_R(x)(\xi w)^{-\gamma_1} \leq C_0{}^{-\gamma_1}(2R)^{1-\gamma_1}\xi^{-\gamma_1} \text{ and } d_R(x)(\xi \widetilde{w})^{-\gamma_2} \leq C_0{}^{-\gamma_2}(2R)^{1-\gamma_2}\xi^{-\gamma_2},$$

a.e. in Ω.

Thus there exists $(\overline{u}, \overline{v}) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ such that

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) = d_R(x)(\xi\widetilde{w}(x))^{-\gamma_1} + M \text{ in } B_R, \\ \overline{u} = 0 \text{ on } \partial B_R, \end{cases}$$

and

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{v}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \overline{v}}{\partial x_i}\Big) = d_R(x)(\xi w(x))^{-\gamma_2} + M \text{ in } B_R, \\ \overline{u} = 0 \text{ on } \partial B_R, \end{cases}$$

where $M > 1$ is a fixed constant.

Let $d_{\overline{\Omega}} := \min_{\overline{\Omega}} d_R(x) > 0$. Choose $\xi > 0$ small enough such that

$$(d_{\overline{\Omega}}\xi^{-\gamma_1} - 1)\|\widetilde{w}\|^{-\gamma_1}_{L^\infty(B_R)} \geq 2M \text{ and } (d_{\overline{\Omega}}\xi^{-\gamma_2} - 1)\|w\|^{-\gamma_2}_{L^\infty(B_R)} \geq 2M.$$

Since f is continuous it is possible to choose $\beta_1, \beta_2 > 0$ small enough such that

$$M + \beta_1 f_1(x, \overline{v}) \leq 2M \text{ and } M + \beta_2 f_2(x, \overline{u}) \leq 2M \text{ a.e. in } B_R.$$

Then it follows that

$$(d_{\overline{\Omega}}\xi^{-\gamma_1} - 1)\widetilde{w}^{-\gamma_1} \geq M + \beta_1 f_1(x, \overline{v}) \text{ and } (d_{\overline{\Omega}}\xi^{-\gamma_2} - 1)w^{-\gamma_2} \geq M + \beta_2 f_2(x, \overline{u})$$

which implies the inequalities

$$\begin{aligned} d_R(x)(\xi\widetilde{w})^{-\gamma_1} + M &\geq 2M + \beta_1 f_1(x, \overline{v}) + \widetilde{w}^{-\gamma_1} \\ &\geq \widetilde{w}^{-\gamma_1} + \beta_1 f_1(x, \overline{v}) \end{aligned}$$

and

$$d_R(x)(\xi w)^{-\gamma_2} + M \geq w^{-\gamma_2} + \beta_2 f_2(x, \overline{u}) \text{ a.e. in } \Omega.$$

Since $M > 1$ it follows from Lemma 2.5 that $\overline{u}(x) \geq \widetilde{w}(x)$ and $\overline{v}(x) \geq w(x)$ a.e. in B_R. Then it follows that

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) \geq \overline{v}^{-\gamma_1} + \beta_1 f_1(x, \overline{v}) \text{ a.e. in } \Omega$$

and

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{v}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \overline{v}}{\partial x_i}\Big) \geq \overline{u}^{-\gamma_2} + \beta_2 f_2(x, \overline{u}) \text{ a.e. in } \Omega.$$

From Lemma 2.5 we have $\overline{u}(x) \geq \underline{u}(x)$ and $\overline{v}(x) \geq \underline{v}(x)$ a.e. in Ω. By Theorem 1.2 we have the first part of the result.

Regarding the last part of the result consider $(\overline{u}, \overline{v}) \in W_0^{1,\overrightarrow{p}}(B_R) \times W_0^{1,\overrightarrow{q}}(B_R)$ satisfying

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big) = \mu \text{ in } B_R, \\ \overline{u} = 0 \text{ on } \partial B_R, \end{cases}$$

and

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{v}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \overline{v}}{\partial x_i}\Big) = \mu \text{ in } B_R, \\ \overline{v} = 0 \text{ on } \partial B_R, \end{cases}$$

where $\mu > 0$ is a constant to be chosen before.

It follows by Lemma 2.4 that

$$\begin{aligned}
&-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{u}}{\partial x_i}\Big|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\Big)-\overline{v}^{-\gamma_1}-\beta_1 f_1(x,\overline{v})\\
&\geq \mu-\overline{v}^{-\gamma_1}-C\beta_1(\|\overline{v}\|_\infty^{r_1-1}+1)\\
&\geq \mu-\overline{v}^{-\gamma_1}-C\beta_1(\mu^{\frac{r_1-1}{q_1-1}}+1)\\
&\geq \mu-\frac{1}{\min\Big\{\big(\frac{\mu}{C}\big)^{\frac{\gamma_1}{q_1-1}},\big(\frac{\mu}{C}\big)^{\frac{\gamma_1}{q_N-1}}\Big\}\min\{\delta,\overline{\delta}\}}-C\beta_1(\mu^{\frac{r_1-1}{q_1-1}}+1),
\end{aligned}\tag{6.7}$$

and

$$\begin{aligned}
&-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial \overline{v}}{\partial x_i}\Big|^{q_i-2}\frac{\partial \overline{v}}{\partial x_i}\Big)-\overline{u}^{-\gamma_2}-\beta_2 f_2(x,\overline{u})\\
&\geq \mu-\frac{1}{\min\Big\{\big(\frac{\mu}{C}\big)^{\frac{\gamma_2}{p_1-1}},\big(\frac{\mu}{C}\big)^{\frac{\gamma_2}{p_N-1}}\Big\}\min\{\delta,\overline{\delta}\}}-C\beta_2(\mu^{\frac{r_2-1}{p_1-1}}+1)
\end{aligned}\tag{6.8}$$

in Ω, where $\overline{\delta}>0$ is the distance of the sets ∂B_R and $\partial\Omega$ and $\delta>0$ is small enough such that $d\in C^2(\overline{B_{3\delta}})$. Considering $\mu>0$ large enough in (6.7) and (6.8) we obtain the result because $r_1<q_1$ and $r_2<p_1$. □

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