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A New Calabi–Bernstein Type Result in Spatially Closed Generalized Robertson–Walker Spacetimes

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Abstract. The aim of this article is to study the uniqueness of a complete spacelike hypersurface Σ^n immersed with constant mean curvature H in a spatially closed generalized Robertson–Walker spacetime $\overline{M}^{n+1} = -I \times_f M^n$, whose Riemannian fiber M^n has positive curvature. Supposing that the warping function f is such that $-\log f$ is convex and $Hf' \leq 0$ along Σ^n, we show that Σ^n must be isometric to a totally geodesic slice of $\overline{M}^{n+1}$. When $\overline{M}^{n+1}$ is a Lorentzian product space, we obtain a new Calabi–Bernstein type result concerning the CMC spacelike hypersurface equation.

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1. Introduction and statements of the results

In the last years, the study of spacelike hypersurfaces in Lorentzian spacetimes has been of substantial interest from both physical and mathematical points of view. For instance, it was pointed out by Marsdan and Tipler in [18] and Stumbles in [24] that spacelike hypersurfaces with constant mean curvature in a spacetime play an important role in General Relativity, since they can be used as initial hypersurfaces where the constraint equations can be split into a linear system and a nonlinear elliptic equation.

From the mathematical point of view, this interest is mostly due to the fact that such hypersurfaces exhibit nice Bernstein-type properties, and one can truly

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say that the first remarkable result in this direction were obtained in 1970, when Calabi [14] established the well-known Calabi–Bernstein theorem:

> *The only complete maximal surfaces in the* 3*-dimensional Lorentz–Minkowski spacetime* $\mathbb{L}^3$*, that is, spacelike surfaces with zero mean curvature, are the spacelike planes.*

The non-parametric version of this theorem asserts that the only entire maximal graphs in $\mathbb{L}^3$ are the affine functions. Cheng and Yau [15] extended this result to complete maximal hypersurfaces in $\mathbb{L}^{n+1}$. It is worth to recall that a spacelike hypersurface is called maximal when its mean curvature is identically zero.

A natural generalization of the thematic discussed above is to study the problem of uniqueness for complete constant mean curvature spacelike hypersurfaces immersed in a generalized Robertson–Walker (GRW) spacetime. This matter has attracted many authors and induced a vast and interesting literature (see, for instance, [3, 4, 5, 7, 8, 11, 12, 17, 21, 22, 23]). At this point, we recall that a GRW spacetime is a Lorentzian warped product $-I \times_f M^n$, with 1-dimensional negative base I, fiber a general Riemannian manifold and arbitrary warping function f. In particular case, when the fiber is assumed to be of constant sectional curvature and the dimension of the spacetime is 3, the GRW spacetime is a (classical) *Robertson–Walker* spacetime. Thus, GRW spacetimes widely extend *Robertson–Walker* spacetimes, and they include, for instance, the Einstein-de Sitter spacetime, Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime.

If the fiber of a GRW spacetime is compact, then it is called *spatially closed*. As it was observed by Aledo et al. [5], the subfamily of spatially closed GRW spacetimes has been very useful to get closed cosmological models. On the other hand, a number of observational and theoretical arguments on the total mass balance of the universe suggests the convenience of adopting open cosmological models. Even more, a spatially closed GRW spacetime violates the holographic principle whereas a GRW spacetime with non-compact fiber could be a suitable model compatible with that principle (cf. [9, 13, 16]). There again, nowadays is commonly accepted the theory of inflation. In this setting, it is natural to think that expansion must occur in the physical space at the same time and in the same manner. A notable fact in this theory is that distant regions in our universe cannot have any interaction. Notice that although the physical space in instants after the inflation may not be exactly a model manifold, in large scale the GRW spacetimes may be a good model to get an approach to this reality.

Our purpose in this paper is to study the uniqueness of a complete spacelike hypersurface immersed with constant mean curvature in a spatially closed generalized Robertson–Walker spacetime $\overline{M}^{n+1} = -I \times_f M^n$, whose fiber M^n has positive curvature. In what follows, we will assume that the orientation N of the spacelike hypersurface is *future-pointing*, which means that its angle function $\langle N, \partial_t \rangle < 0$ where ∂_t stands for the coordinate vector field induced by the universal time on $\overline{M}^{n+1}$. The mean curvature H taken with respect to such choice of orientation N is called the *future mean curvature* of Σ^n. Now, we are in position to state our main result.

Theorem 1.1. *Let* $\overline{M}^{n+1} = -I \times_f M^n$ *be a spatially closed GRW spacetime, whose sectional curvature of the Riemannian fiber* M^n *is positive. Let* Σ^n *be a complete spacelike hypersurface immersed in* $\overline{M}^{n+1}$ *such that* $-\log f$ *is convex on* Σ^n*. If the future mean curvature* H *of* Σ^n *is constant and satisfies* $Hf' \leq 0$*, then* Σ^n *must be a totally geodesic slice of* $\overline{M}^{n+1}$*.*

The proof of Theorem 1.1 is given in Section 3. Now, let us consider that the ambient spacetime $\overline{M}^{n+1} = -I \times M^n$ is static and let $\Omega \subseteq M^n$ be a connected domain and let $u \in C^\infty(\Omega)$ be a smooth function such that $u(\Omega) \subseteq I$, then $\Sigma(u)$ will denote the graph over Ω determined by u, that is,

$$\Sigma(u) = \{(u(x), x) : x \in \Omega\} \subset -I \times M.$$

The graph is said to be entire if $\Omega = M^n$. The metric induced on Ω from the Lorentzian metric of the ambient space via $\Sigma(u)$ is

$$g_{\Sigma(u)} = -du^2 + g_M. \tag{1.1}$$

It can be easily seen that a graph $\Sigma(u)$ is a spacelike hypersurface if and only if $|Du|_M < 1$, Du being the gradient of u in M and $|Du|_M = \langle Du, Du\rangle_M^{1/2}$.

Furthermore, with a straightforward computation we verify that the vector field

$$N = \frac{1}{\sqrt{1 - |Du|_M^2}}(\partial_t + Du) \tag{1.2}$$

defines the future-pointing Gauss map of $\Sigma(u)$. Hence, from (1.2) we can verify that the shape operator A of $\Sigma(u)$ with respect to N is given by

$$AX = -\frac{1}{\sqrt{1 - |Du|_M^2}} D_X Du - \frac{\langle D_X Du, Du\rangle_M}{(1 - |Du|_M^2)^{3/2}} Du, \tag{1.3}$$

for every tangent vector fields $X \in \mathfrak{X}(\Sigma(u))$. Consequently, denoting by div the divergence operator on $\Sigma(u)$, from (1.3) we obtain that the future mean curvature function $H(u)$ associated to A is given by

$$H(u) = \operatorname{div}\left(\frac{Du}{n\sqrt{1 - |Du|_M^2}}\right).$$

The differential equation $H(u) = H$, with H constant, jointly with the constraint $|Du|_M < 1$ is called the *CMC spacelike hypersurface equation* in $\overline{M}$, and its solutions provide constant mean curvature spacelike graphs in $\overline{M}^{n+1}$.

Motivated by this previous digression, we will consider the following CMC spacelike hypersurface equation

$$(\mathbf{E}) \begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1 - |Du|_M^2}}\right) = H \\ \\ |Du|_M \leq c, \end{cases}$$

where H and $0 < c < 1$ are constant. We observe that (**E**) is uniformly elliptic. It is also interesting to observe that, in contrast to the case of graphs into a Riemannian product space, an entire spacelike graph $\Sigma(u)$ in a Lorentzian product space $-I \times M$ is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete on M. For instance, Albujer [2, Section 3] obtained explicit examples of non-complete entire maximal graphs in $-\mathbb{R} \times \mathbb{H}^2$. However, it follows from [4, Lemma 17] that when the fiber M is complete and $|Du|_M \leq c$, for a certain constant $0 < c < 1$, then $\Sigma(u)$ must be also complete. On an entire graph $\Sigma(u)$, the existence of such a constant c prevents that the tangent vector field to a divergent curve in $\Sigma(u)$ asymptotically approaches to a lightlike direction in the ambient space.

Taking into account this setting, it is not difficult to see that from Theorem 1.1 we get the following Calabi–Bernstein type result:

Theorem 1.2. *Let $\overline{M}^{n+1} = -I \times M^n$ be a Lorentzian product space, whose Riemannian fiber M^n is compact with positive sectional curvature. The only entire solutions of* (**E**) *are the constant functions $u = t_0$, with $t_0 \in I$.*

2. Preliminaries

Throughout this section we provide some basic notations and a couple of lemmas that will be useful in the proof of our main results. First of all, let M^n be a connected, n-dimensional, oriented Riemannian manifold, $I \subset \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ a positive smooth function. Also, in the product manifold $\overline{M}^{n+1} = I \times M^n$ let π_I and π_M denote the projection onto the factors I and M^n, respectively.

The class of Lorentzian manifold which will be of our concern here is the one obtained by furnishing $\overline{M}^{n+1}$ with Lorentzian metric

$$\langle v, w\rangle_p = \langle(\pi_I)_* v, (\pi_I)_* w\rangle_{\pi_I(p)} + (f \circ \pi_I)(p)^2 \langle(\pi_M)_* v, (\pi_M)_* w\rangle_{\pi_M(p)},$$

for all $p \in \overline{M}^{n+1}$ and $v, w \in T_p\overline{M}$. In such case, we write

$$\overline{M}^{n+1} = -I \times_f M^n, \tag{2.1}$$

and say that $\overline{M}^{n+1}$ is the Lorentzian warped product with warping function f.

When M^n has constant sectional curvature, the warped product (2.1) is classically called a Robertson–Walker (RW) spacetime, an allusion to the fact that, for $n = 3$, it is an exact solution of Einstein's field equation (cf. Chapter 12 of [20]). After [7], the warped product (2.1) has usually been referred to as a Generalized Robertson–Walker (GRW) spacetime, and we shall stick to this usage along this paper. Furthermore, if warped function f is constant then it is called static GRW spacetime.

From now on, we deal with spacelike hypersurfaces immersed in a GRW spacetime. A smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ of an n-dimensional connected manifold Σ^n is said to be a spacelike hypersurface if the induced metric via ψ, is a Riemannian metric on Σ^n, which, as usual, is also denoted for $\langle .,. \rangle$. In that case, since

$$\partial_t = (\partial/\partial_t)_{(t,x)}, \quad (t,x) \in -I \times_f M^n,$$

is a unitary timelike vector field globally defined on the ambient spacetime, then there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n which is in the same time-orientation as ∂_t, moreover is easy to see that from Cauchy–Schwarz inequality, we get

$$\langle N, \partial_t \rangle \leq -1 < 0 \text{ on } \Sigma^n. \tag{2.2}$$

The mean curvature H taken with respect to such choice of orientation N is called the *future mean curvature* of Σ^n.

In this setting, let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\overline{M}^{n+1}$ and Σ^n, respectively. Then, the Gauss and Weingarten formulas of Σ^n are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N \tag{2.3}$$

and

$$AX = -\overline{\nabla}_X N, \tag{2.4}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here, $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the Weingarten operator of Σ^n, with respect to its orientation N.

A well-known fact is that the curvature tensor R of the spacelike hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of the ambient spacetime $\overline{M}^{n+1}$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top - \langle AX, Z \rangle AY + \langle AY, Z \rangle AX, \tag{2.5}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\ )^\top$ denote the tangencial component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n.

Now, let h denote the (vertical) height function naturally attached to Σ^n, namely, $h = (\pi_I)|_\Sigma$. Let $\overline{\nabla}$ and ∇ denote gradients with respect to the metrics of $-I \times_f M^n$ and Σ^n, respectively. A simple computation shows that the gradient of π_I on $-I \times_f M^n$ is given by

$$\overline{\nabla} \pi_I = -\langle \overline{\nabla} \pi_I, \partial_t \rangle = -\partial_t,$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla} \pi_I)^\top = -\partial_t^\top = -\partial_t - \langle N, \partial_t \rangle N. \tag{2.6}$$

In particular, we get

$$|\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1, \tag{2.7}$$

where $|\ \ |$ denotes the norm of a vector field on Σ^n. From Proposition 7.35 of [20] we have that

$$\overline{\nabla}_X \partial_t = \frac{f'}{f}(X + \langle X, \partial_t \rangle \partial_t), \tag{2.8}$$

for every $X \in \mathfrak{X}(\Sigma)$. In particular, for $t_0 \in \mathbb{R}$ fixed, it is not difficult to see that each slice $\{t_0\} \times M^n$ has constant mean curvature

$$H = \frac{f'(t_0)}{f(t_0)}$$

with respect to the unit normal vector field ∂t (see, for instance, [7]). Moreover, for our purpose, the Lemma 4.1 of [6] gives the following formula

$$\Delta h = (\log f)'(-n - |\nabla h|^2) - nH\langle N, \partial_t\rangle, \tag{2.9}$$

where $H = -\frac{1}{n}\mathrm{tr}(A)$ is the mean curvature of Σ^n with respect to N.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we will need to establish some key lemmas. The first one gives a suitable lower bound for the Ricci curvature of spacelike hypersurfaces immersed in a spatially closed GRW spacetime.

Lemma 3.1. *Let Σ^n be a spacelike hypersurface immersed in a spatially closed GRW spacetime $-\mathbb{R} \times_f M^n$. Then, for all $X \in \mathfrak{X}(\Sigma)$, the Ricci curvature of Σ^n satisfies the inequality*

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) \geq\ & \alpha(n-1)|X|^2 + (\alpha - (\log f)'')(|X|^2|\nabla h|^2 + (n-2)\langle X, \nabla h\rangle^2) \\ & + \left(\frac{f'}{f}\right)^2 (n-1)|X|^2 + nH\langle AX, X\rangle + |AX|^2,\end{aligned}$$

where h and H denote, respectively, the height function and the future mean curvature of Σ^n and $\alpha = \min_M K_M$, with K_M being the sectional curvature of the Riemannian fiber M^n.

Proof. Let us consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss equation (2.5) that

$$\mathrm{Ric}_\Sigma(X,X) = \sum_i \langle \overline{R}(X,E_i)X, E_i\rangle + nH\langle AX, X\rangle + |AX|^2. \tag{3.1}$$

On the other hand, with a straightforward computation, we deduce

$$\begin{aligned}\overline{R}(X,E_i)X \ =\ & \overline{R}(X^*,E_i^*)X^* - \langle X,\partial_t\rangle \overline{R}(X^*,E_i^*)\partial_t - \langle E_i,\partial_t\rangle \overline{R}(X^*,\partial_t)X^* \\ & + \langle E_i,\partial_t\rangle\langle X,\partial_t\rangle \overline{R}(X^*,\partial_t)\partial_t - \langle X,\partial_t\rangle \overline{R}(\partial_t,E_i^*)X^* \\ & + \langle X,\partial_t\rangle^2 \overline{R}(\partial_t,E_i^*)\partial_t,\end{aligned}$$

where $X^* = X + \langle X,\partial_t\rangle\partial_t$ and $E_i^* = E_i + \langle E_i,\partial_t\rangle\partial_t$ are the projections on the tangent vector fields X and E_i onto the fibre M^n, respectively. Now, by repeated use of the formulas of Proposition 7.42 of [20] and using Eq. (2.6), we obtain

$$\begin{aligned}\langle \overline{R}(X,E_i)X,E_i\rangle &= \langle \overline{R}(X^*,E_i^*)X^*,E_i^*\rangle - \frac{f''}{f}(|X|^2+\langle X,\nabla h\rangle^2)\langle E_i,\nabla h\rangle^2 \\ &\quad +2\frac{f''}{f}(\langle X,E_i\rangle+\langle X,\nabla h\rangle\langle E_i,\nabla h\rangle)\langle E_i,\nabla h\rangle\langle X,\nabla h\rangle \\ &\quad -\frac{f''}{f}\langle X,\nabla h\rangle^2(1+\langle E_i,\nabla h\rangle^2).\end{aligned}$$

This data substituted in (3.3) yields

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) &= \sum_i \langle \overline{R}(X^*,E_i^*)X^*,E_i^*\rangle - \frac{f''}{f}(|X|^2+\langle X,\nabla h\rangle^2)|\nabla h|^2 \\ &\quad +2\frac{f''}{f}(1+|\nabla h|^2)\langle X,\nabla h\rangle^2 - \frac{f''}{f}(n+|\nabla h|^2)\langle X,\nabla h\rangle^2 \\ &\quad +nH\langle AX,X\rangle+|AX|^2.\end{aligned}$$

Hence, after some simple computations we obtain

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) &= \sum_i \langle \overline{R}(X^*,E_i^*)X^*,E_i^*\rangle - \frac{f''}{f}(|X|^2|\nabla h|^2 \\ &\quad +(n-2)\langle X,\nabla h\rangle^2)+nH\langle AX,X\rangle+|AX|^2. \end{aligned} \tag{3.2}$$

By using once more the Gauss equation we immediately have

$$\begin{aligned}\sum_i \langle \overline{R}(X^*,E_i^*)X^*,E_i^* &= \sum_i K_M(X^*,E_i^*)|X^*\wedge E_i^*|^2 \\ &\quad + \left(\frac{f'}{f}\right)^2 \sum_i |X^*\wedge E_i^*|^2.\end{aligned} \tag{3.3}$$

On the other hand, it is not difficult to check that

$$\langle X^*,X^*\rangle_M\langle E_i^*,E_i^*\rangle_M = (1+\langle E_i,\nabla h\rangle^2)(|X|^2+\langle X,\nabla h\rangle^2)$$

and

$$\langle X^*,E_i^*\rangle_M^2 = \langle X,E_i\rangle^2+2\langle X,\nabla h\rangle\langle E_i,\nabla h\rangle\langle X,E_i\rangle+\langle X,\nabla h\rangle^2\langle E_i,\nabla h\rangle^2.$$

Hence, combining the above expressions we deduce

$$\sum_i |X^*\wedge E_i^*|^2 = (n-1)|X|^2+|X|^2|\nabla h|^2+(n-2)\langle X,\nabla h\rangle^2. \tag{3.4}$$

Now, substituting (3.3) in (3.2) and using Eq. (3.4) we infer

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) &= \sum_i K_M(X^*,E_i^*)|X^*\wedge E_i^*|^2 - (\log f)''(|X|^2|\nabla h|^2 \\ &\quad +(n-2)\langle X,\nabla h\rangle^2)+nH\langle AX,X\rangle+|AX|^2 \\ &\quad +(n-1)\left(\frac{f'}{f}\right)^2|X|^2.\end{aligned} \tag{3.5}$$

Therefore, the hypothesis on the sectional curvature K_M of Riemannian fiber jointly with (3.5) and (3.4) allow us to conclude that the Ricci curvature of Σ^n satisfies

$$\begin{aligned}\mathrm{Ric}_\Sigma(X,X) \geq\ & \alpha(n-1)|X|^2 + (\alpha - (\log f)'')(|X|^2|\nabla h|^2 + (n-2)\langle X, \nabla h\rangle^2) \\ & + \left(\frac{f'}{f}\right)^2 (n-1)|X|^2 + nH\langle AX, X\rangle + |AX|^2.\end{aligned}$$

This gives the requested result. □

The next key lemma is derived from Bochner's formula [10] and it follows the ideas of Proposition 3.1 in [17].

Lemma 3.2. *Let Σ^n be a spacelike hypersurface immersed with constant future mean curvature H in a spatially closed GRW spacetime $-\mathbb{R}\times_f M^n$. Then*

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \quad \geq \quad & \big(\alpha(n-1) - (\log f)''n\big)\,|\nabla h|^2(1+|\nabla h|^2) \\ & + \left|A(\nabla h) - \frac{f'}{f}\langle N, \partial t\rangle \nabla h\right|^2 + nH\frac{f'}{f}\langle N, \partial t\rangle |\nabla h|^2 \\ & + \left(\frac{f'}{f}\right)^2 (n + |\nabla h|^2)|\nabla h|^2 + |\mathrm{Hess}\, h|^2,\end{aligned}$$

where h denotes the height function on Σ^n and $\alpha = \min_M K_M$, with K_M being the sectional curvature of M^n.

Proof. Firstly, from Eq. (2.9) we obtain that

$$\nabla\Delta h \quad = \quad \nabla\left(\frac{f'}{f}\right)(-n - |\nabla h|^2) - \frac{f'}{f}\nabla|\nabla h|^2 - nH\nabla\langle N, \partial_t\rangle,$$

which can rewritten, using (2.7), as

$$\begin{aligned}\nabla\Delta h \quad = \quad & \left(\frac{f''f - (f')^2}{f^2}\right)(-n - |\nabla h|^2)\nabla h - 2\frac{f'}{f}\langle N, \partial_t\rangle \nabla\langle N, \partial_t\rangle \\ & -nH\nabla\langle N, \partial_t\rangle. \end{aligned} \tag{3.6}$$

Next, from (2.6) and (2.8) we have that

$$\begin{aligned}X\langle N, \partial_t\rangle \quad & = \quad \langle AX, \nabla h\rangle + \frac{f'}{f}\langle N, \langle X, \partial_t\rangle \partial_t\rangle \\ & = \quad \left\langle X, A(\nabla h) - \frac{f'}{f}\langle N, \partial_t\rangle \nabla h\right\rangle,\end{aligned}$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus,

$$\nabla\langle N, \partial_t\rangle = A(\nabla h) - \frac{f'}{f}\langle N, \partial_t\rangle \nabla h. \tag{3.7}$$

Hence we use this data in (3.6) to deduce

$$\begin{aligned}\nabla\Delta h &= (\log f)''(-n-|\nabla h|^2)\nabla h - 2\frac{f'}{f}\langle N,\partial_t\rangle A\nabla h \\ &\quad +2\left(\frac{f'}{f}\right)^2\langle N,\partial_t\rangle^2\nabla h - nHA(\nabla h) \\ &\quad +nH\frac{f'}{f}\langle N,\partial_t\rangle\nabla h. \end{aligned} \tag{3.8}$$

From Bochner's formula [10] jointly with (3.8) we have that

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 &= \langle\nabla\Delta h,\nabla h\rangle + \mathrm{Ric}_\Sigma(\nabla h,\nabla h) + |\mathrm{Hess}\, h|^2 \\ &= (\log f)''(-n-|\nabla h|^2)|\nabla h|^2 - 2\frac{f'}{f}\langle N,\partial_t\rangle\langle A\nabla h,\nabla h\rangle \\ &\quad +2\left(\frac{f'}{f}\right)^2\langle N,\partial_t\rangle^2|\nabla h|^2 - nH\langle A(\nabla h),\nabla h\rangle \\ &\quad +nH\frac{f'}{f}\langle N,\partial_t\rangle|\nabla h|^2 + \mathrm{Ric}_\Sigma(\nabla h,\nabla h) + |\mathrm{Hess}\, h|^2. \end{aligned} \tag{3.9}$$

On the other hand, we apply the lemma 3.1 for the vector field ∇h to arrive at

$$\begin{aligned}\mathrm{Ric}_\Sigma(\nabla h,\nabla h) &\geq \alpha(n-1)|\nabla h|^2 + (\alpha-(\log f)'')(n-1)|\nabla h|^2 \\ &\quad +\left(\frac{f'}{f}\right)^2(n-1)|\nabla h|^2 + nH\langle A(\nabla h),\nabla h\rangle + |A(\nabla h)|^2. \end{aligned}$$

Therefore, this last inequality combined with (3.9) yields

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 &= \big(-(\log f)''(n+|\nabla h|^2) + (\alpha-(\log f)''(n-1)|\nabla h|^2 + \alpha(n-1)\big)|\nabla h|^2 \\ &\quad +\left|A(\nabla h) - \frac{f'}{f}\langle N,\partial_t\rangle\nabla h\right|^2 + \left(\frac{f'}{f}\right)^2\langle N,\partial_t\rangle|\nabla h|^2 \\ &\quad +nH\frac{f'}{f}\langle N,\partial_t\rangle|\nabla h|^2 + \left(\frac{f'}{f}\right)^2(n-1)|\nabla h|^2 + |\mathrm{Hess}\, h|^2. \end{aligned}$$

Upon rearranging the terms above we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 &\geq \big(\alpha(n-1) - (\log f)''n\big)|\nabla h|^2(1+|\nabla h|^2) \\ &\quad +\left|A(\nabla h) - \frac{f'}{f}\langle N,\partial t\rangle\nabla h\right|^2 + nH\frac{f'}{f}\langle N,\partial t\rangle|\nabla h|^2 \\ &\quad +\left(\frac{f'}{f}\right)^2(n+|\nabla h|^2)|\nabla h|^2 + |\mathrm{Hess}\, h|^2. \end{aligned}$$

This that we wanted to prove. □

The following consequence of the generalized maximum principle of Omori-Yau [19, 25] is due to Akutagawa [1].

Lemma 3.3. *Let Σ^n denote an n-dimensional complete Riemannian manifold having Ricci curvature bounded from below. If $g \in \mathcal{C}^2(\Sigma)$ is nonnegative and satisfies $\Delta g \geq Cg^\beta$, for some real numbers $C > 0$ and $\beta > 1$, then g vanishes identically on Σ^n.*

Now, we are in position to prove our main uniqueness result.

Proof of Theorem 1.1. From Lemma 3.2 we have that

$$\Delta|\nabla h|^2 \geq 2\alpha(n-1)|\nabla h|^4,$$

where $\alpha = \min_M K_M > 0$. Thus, taking into account Lemma 3.1, we can apply Lemma 3.3 to obtain that $|\nabla h|^2$ vanishes identically on Σ^n and, consequently, Σ^n is a slice of $\overline{M}^{n+1}$. Moreover, taking into account that $Hf' \leq 0$, we see that the mean curvature H of Σ^n is, in fact, zero and Σ^n must be a totally geodesic slice of $\overline{M}^{n+1}$. □

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