Milan J. Math. Vol. 84 (2016) 231–242
DOI 10.1007/s00032-016-0258-1
Published online October 17, 2016

Milan Journal of Mathematics

Diameter Two Properties, Convexity and Smoothness

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Abstract. We study smoothness and strict convexity of (the bidual) of Banach spaces in the presence of diameter 2 properties. We prove that the strong diameter 2 property prevents the bidual from being strictly convex and being smooth, and we initiate the investigation whether the same is true for the (local) diameter 2 property. We also give characterizations of the following property for a Banach space X: "For every slice S of B_X and every norm-one element x in S, there is a point $y \in S$ in distance as close to 2 as we want." Spaces with this property are shown to have non-smooth bidual.

Mathematics Subject Classification. 46B04, 46B20.

Keywords. Diameter 2 property; strictly convex norm; smooth norm; Daugavet property.

1. Introduction

Let X be a (real) Banach space and denote, as usual, by B_X and S_X its unit ball and unit sphere, respectively, and denote the topological dual of X by X^*.

Recall that (the norm of) a Banach space X is *strictly convex* if $\|\frac{x+y}{2}\| < 1$ when x and y are different points of S_X, and that (the norm of) X is *smooth* if for every $x \in S_X$ there is exactly one $x^* \in S_{X^*}$ such that $x^*(x) = 1$. It is well-known that X is smooth if X^* is strictly convex, and that X is strictly convex if X^* is smooth.

It is a classical result from 1948 of J. Dixmier [10, Théorème 20'] that X^{****} is never strictly convex unless X is reflexive. Several authors have independently strengthened Dixmier's result by showing that X^{***} is not smooth for X non-reflexive. (A partial list of authors can be found in [21]. Milman credits the result

The main results of this paper were presented in a talk by the fourth named author at the Seminario Matematico e Fisico di Milano in March 2016. The fourth named author was partially supported by MTM2014-54182-P and the Bulgarian National Scientific Fund under Grant DFNI-I02/10.

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to M. I. Kadets [18, Theorem 2.3].) Note that this result is sharp in the sense that James' space J has a renorming such that the third dual is strictly convex [21].

The purpose of this paper is to study the implications of the big-slice phenomena on smoothness and convexity. By a slice of B_X of X we mean a set of the form

$$S(x^*,\varepsilon) := \{x \in B_X : x^*(x) > 1-\varepsilon, x^* \in S_{X^*}, \varepsilon > 0\}.$$

Recall the following successively stronger "big-slice concepts", defined in [4]:

Definition 1.1. A Banach space X has the

- (i) *local diameter* 2 *property* (LD2P) if every slice of B_X has diameter 2.
- (ii) *diameter* 2 *property* (D2P) if every non-empty relatively weakly open subset of B_X has diameter 2.
- (iii) *strong diameter* 2 *property* (SD2P) if every finite convex combination of slices of B_X has diameter 2.

In Section 2 we prove that X^{**} can be neither strictly convex nor smooth if X has the SD2P. In fact, we prove that when X has the SD2P, then X^{**} contains an isometric copy of $L_1[0,1]$. We next ask whether it is possible that X can have (L)D2P while X^{**} is still strictly convex, and we give a partial answer; namely we prove that if X has a bimonotone basis and the D2P, then the unit sphere of X^{**} contains a line segment of length as close to 1 as we want.

Recall the following successively stronger "rotundity concepts":

Definition 1.2. A Banach space X is

- (i) *strictly convex* (or rotund) if every $x \in S_X$ is an extreme point in B_X, i.e., for every $y \in X$ we have that $y = 0$ whenever $\|x \pm y\| = 1$.
- (ii) *weakly midpoint locally uniformly rotund* (weakly MLUR) if every $x \in S_X$ is a weakly strongly extreme point of B_X, i.e., for every sequence (x_n) in X, we have that $x_n \to 0$ weakly whenever $\|x \pm x_n\| \to 1$.
- (iii) *midpoint locally uniformly rotund* (MLUR) if every $x \in S_X$ is a strongly extreme point of B_X, i.e., for every sequence (x_n) in X, we have that $x_n \to 0$ in norm whenever $\|x \pm x_n\| \to 1$.

It is clear that if X is weakly MLUR then X is strictly convex. Smith [22] observed using the Principle of Local Reflexivity that X is weakly MLUR if and only if every $x \in S_X$ is an extreme point of $B_{X^{**}}$. In particular, if X^{**} is strictly convex, then X is weakly MLUR. The converse is not true.

It was observed in [2, Proposition 1.3] that if X is weakly MLUR, then the LD2P implies the D2P by Choquet's lemma [11, Lemma 3.69]. In particular, the LD2P implies the D2P when X^{**} is strictly convex.

The main result of [2] is that there exists an equivalent norm $|\cdot|$ on $C[0,1]$ such that $X = (C[0,1], |\cdot|)$ is MLUR and has the (L)D2P. In fact X has the LD2P in the following stronger sense:

Definition 1.3. A Banach space X has the *local diameter* 2 *property+ (LD2P+)* if for every $\varepsilon > 0$, every slice S of B_X, and every $x \in S \cap S_X$ there exists $y \in S$ such that $\|x - y\| > 2 - \varepsilon$.

Let X be a Banach space and I the identity operator on X. Recall that X has the *Daugavet property* if the equation

$$\|I+T\| = 1 + \|T\|$$

holds for every rank 1 operator T on X. The Daugavet property can be characterized as follows (see [24] or [20]):

Theorem 1.4. *Let X be a Banach space. Then the following statements are equivalent.*

(i) *X has the Daugavet property.*
(ii) *The equation $\|I+T\| = 1+\|T\|$ holds for every weakly compact operator T on X.*
(iii) *For every $\varepsilon > 0$, every $x \in S_X$, and every $x^* \in S_{X^*}$, there exists $y \in S(x^*, \varepsilon)$ such that $\|x+y\| \geq 2-\varepsilon$.*
(iv) *For every $\varepsilon > 0$, every $x^* \in S_{X^*}$, and every $x \in S_X$, there exists $y^* \in S(x, \varepsilon)$ such that $\|x^* + y^*\| \geq 2-\varepsilon$.*
(v) *For every $\varepsilon > 0$ and every $x \in S_X$ we have $B_X = \overline{conv}(\Delta_\varepsilon(x))$, where $\Delta_\varepsilon(x) = \{y \in B_X : \|y-x\| \geq 2-\varepsilon\}$.*

In Section 3 we prove a similar characterization of the LD2P+, see Theorems 3.2 and 3.5. It is known that the dual of a Banach space with the Daugavet property is neither strictly convex nor smooth [17, Corollary 2.13]. In Corollary 3.7 we show that if X has the LD2P+, then X^{**} is not smooth. We also prove that just like the diameter two properties above the LD2P+ is inherited by ai-ideals (we postpone the definition of this concept till we need it).

The notation and conventions we use are standard and follow [16]. When considered necessary, notation and concepts are explained as the text proceeds.

2. Strict convexity and smoothness of X^{**}

A result of Day [8, Theorem 9] says that neither $\ell_1(\Gamma)$, Γ uncountable, nor ℓ_∞ have equivalent smooth renormings. So for example no equivalent norm on $C[0,1]$ has a bidual that is smooth or strictly convex.

Our first aim in this section is to prove that if a Banach space X has the SD2P, then the bidual is neither smooth nor strictly convex. Banach spaces which are M-ideals in their biduals are called *M-embedded*. It is known that non-reflexive M-embedded spaces have the SD2P [4, Theorem 4.10]. From [7, p. 109] (see also [14, Proposition I.1.7]) it is clear that the bidual of a non-reflexive M-embedded space is neither smooth nor strictly convex.

Let us note that in general we cannot say anything about the absence of smoothness or convexity in X and X^* in the presence of the SD2P. Indeed, there exists a smooth M-embedded renorming X of c_0 with strictly convex dual [14, Corollary III.2.12]. There also exists a strictly convex M-embedded space X with smooth dual [14, Remark IV.1.17].

We will need the following concept:

Definition 2.1. A sequence (x_n) in X is said to be *asymptotically isometric* ℓ_1, if there exists a sequence (δ_n) in $(0,1)$ decreasing to 0 and such that

$$\sum_{n=1}^{m}(1-\delta_n)|a_n| \leq \|\sum_{n=1}^{m} a_n x_n\| \leq \sum_{n=1}^{m}|a_n|$$

for each finite sequence $(a_n)_{n=1}^m$ in $\mathbb{R}$.

From [12, Remark II.5.2] we have the following:

Definition 2.2. A Banach space X is said to be *octahedral* if for every finite dimensional subspace E of X and every $\varepsilon > 0$, there exists $y \in S_X$ such that for every $x \in E$ and every $\lambda \in \mathbb{R}$, we have

$$\|x + \lambda y\| \geq (1-\varepsilon)(\|x\| + |\lambda|).$$

Lemma 2.3. *If X is octahedral, then X contains an asymptotically isometric ℓ_1 sequence.*

The proof uses an idea of H. Pfitzner (see [19, Theorem 2]).

Proof. Let $(\delta_n) \subset (0,1)$ such that $\delta_n \to 0$. Let $\eta_1 = \frac{\delta_1}{2}$ and $\eta_{n+1} = \frac{1}{2}\min\{\eta_n, \delta_{n+1}\}$. We will find a sequence $(x_n) \subset S_X$ such that

$$\sum_{n=1}^{m}(1-\delta_n)|a_n| + \eta_m \sum_{n=1}^{m}|a_n| \leq \|\sum_{n=1}^{m} a_n x_n\| \tag{1}$$

by induction. The $m = 1$ step is trivial.

Assume (1) holds for a fixed $m \geq 1$. Choose $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{\eta_m - \eta_{m+1}}{1 - \delta_n + \eta_m}$$

for $n = 1, 2, \ldots, m$ and

$$\varepsilon \leq \delta_{m+1} - \eta_{m+1}.$$

Find, using the assumption and octahedrality, $x_{m+1} \in S_X$ such

$$(1-\varepsilon)\left(\sum_{n=1}^{m}(1-\delta_n)|a_n| + \eta_m \sum_{n=1}^{m}|a_n| + |a_{m+1}|\right) \leq \|\sum_{n=1}^{m+1} a_n x_n\|. \tag{2}$$

Then

$$\sum_{n=1}^{m+1}(1-\delta_n)|a_n| + \eta_{m+1} \sum_{n=1}^{m+1}|a_n| \leq \|\sum_{n=1}^{m+1} a_n x_n\| \tag{3}$$

because the left-hand side of (2) in this case will be greater than the left-hand side of (3). ∎

Remark 2.4. As noted above there is a smooth M-embedded Banach space X with strictly convex dual and a strictly convex M-embedded Banach space X with smooth dual. By [19, Theorem 2] X^* contains an asymptotically isometric ℓ_1 sequence whenever X is M-embedded. Hence the presence of an asymptotically isometric ℓ_1 sequence in a Banach space X does not prevent X from being strictly convex or smooth – even when X is a dual space.

Theorem 2.5. *If X has the* SD2P*, then X^{**} contains an isometric copy of $L_1[0,1]$.*

Proof. We know from [13, Theorem 2.4] that X has the SD2P if and only if X^* is octahedral. From Lemma 2.3 we know that an octahedral space contains an asymptotically isometric ℓ_1 sequence. From [9, Theorem 2] we then have that X^{**} contains an isometric copy of $L_1[0,1]$. □

Since $L_1[0,1]$ is neither smooth nor strictly convex the following corollary is immediate.

Corollary 2.6. *If X has the* SD2P*, then X^{**} is neither strictly convex nor smooth.*

From Corollary 2.6 a natural question arises: If X has the D2P, can X^{**} be strictly convex? (Recall from the Introduction that when X^{**} is strictly convex, LD2P and D2P for X must be the same thing.) We will give a negative answer to this question in the case X has a bimonotone basis in Proposition 2.10 below.

We start with an alternative description of the D2P.

Proposition 2.7. *The following statements are equivalent:*

(i) *X has the* D2P*.*
(ii) *Whenever $\varepsilon > 0$, $x \in X$ with $\|x\| < 1$, and F is a finite dimensional subspace of X^*, there exist $y_1, y_2 \in F_\perp$ with $\|x+y_i\| < 1$, $i = 1,2$, such that $\|y_1 - y_2\| > 2-\varepsilon$.*
(iii) *Whenever $\varepsilon > 0$, $x \in X$ with $\|x\| < 1$, and E is a finite co-dimensional subspace of X, there exists $y_1, y_2 \in E$ with $\|x + y_i\| < 1$, $i = 1,2$, such that $\|y_1 - y_2\| > 2 - \varepsilon$.*

Proof. (i) ⇒ (iii). Let $\varepsilon > 0$, $x \in X$ with $\|x\| < 1$, and E a finite co-dimensional subspace of X. Assume without loss of generality that E does not contain x. Choose a finite dimensional subspace F of X which contains x and with the property that $X = E \oplus F$. Let P be a bounded linear projection onto F. For $\varepsilon/5 > \delta > 0$ put

$$W = \{w \in B_X : \|P(x - w)\| < \delta\}.$$

Note that W is a neighbourhood of x in the relative weak topology on B_X. Now, using (i), and that non-empty relatively weakly open subsets of B_X has diameter 2, we may pick w_1, w_2 in W, both of norm $< 1 - \delta$ and with $\|w_2 - w_1\| > 2 - 3\delta$.

Put $y_i = w_i - Pw_i$. Then y_1 and y_2 are both in E. Moreover, for $i = 1,2$, we have

$$\|x + y_i\| = \|Px + w_i - Pw_i\| \le \|P(x - w_i)\| + \|w_i\| < 1.$$

We also have

$$\begin{aligned}\|y_1 - y_2\| &= \|w_1 - w_2 - P(w_1 - w_2)\| \\ &\geq \|w_1 - w_2\| - 2\delta > 2 - 5\delta > 2 - \varepsilon\end{aligned}$$

since $\|P(w_1 - w_2)\| < 2\delta$.

(iii) $\Rightarrow$ (ii). This is obvious as any finite dimensional subspace of a dual space has a finite co-dimensional pre-annihilator.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$ and U a non-empty relatively weakly open subset of B_X. Let $x \in U$ with $\|x\| < 1$ and find a set of the form

$$V = \left(x + \bigcap_{k=1}^{n} (f_k)^{-1}(-\delta, \delta)\right) \bigcap B_X \subset U,$$

where $(f_k)_{k=1}^n \subset S_{X^*}$ and $\delta > 0$. Let

$$F = \operatorname{span}\{(f_k)_{k=1}^n\}.$$

As F is of finite dimension in X^*, there exist $y_1, y_2 \in F_\perp$ with $\|x + y_i\| < 1$, $i = 1, 2$, such that $\|y_2 - y_1\| > 2 - \varepsilon$. For $i = 1, 2$ we have $x + y_i \in V$ with $\|(x + y_1) - (x + y_2)\| > 2 - \varepsilon$. □

As a first application of Proposition 2.7 let us give a very simple proof of the following fact, known from [6].

Proposition 2.8. *If X has the* D2P *and Y is a subspace of X with finite co-dimension, then Y has the* D2P.

Proof. If $y \in Y$ with $\|y\| < 1$, $\varepsilon > 0$, and E is of finite co-dimension in Y, then E is also of finite co-dimension in X and the result follows from Proposition 2.7 (iii). □

Now we return to the problem whether X^{**} can be strictly convex if X has the D2P.

Definition 2.9. A Schauder basis $(e_k)_{k=1}^\infty$ for a Banach space X is *bimonotone* if the projections

$$P_{[n,m]}(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=n}^{m} a_k e_k.$$

satisfy $\|P_{[n,m]}\| = 1$ if $n \leq m$.

Proposition 2.10. *Suppose X has a bimonotone basis. Then, if X has the* D2P *and $\varepsilon > 0$, $S_{X^{**}}$ contains a line segment of length $> 1 - \varepsilon$.*

Proof. Let P_n be the natural projections associated to the basis (e_i) and put $Q_n = I - P_n$. Let $\varepsilon > 0$ and define $\varepsilon_n = \varepsilon / 2^{n+1}$.

Define $s_1 = x_1 = \frac{(1-\varepsilon_1)e_1}{\|e_1\|}$. Assume that we have found a sequence $(x_i)_{i=1}^k$ each with finite support $\operatorname{supp}(x_i) = [l_i, r_i]$ such that if $s_k = \sum_{i=1}^k x_i$, then

- $\|s_k\| < 1$ and $\|s_k\| \geq \|s_{k-1}\|$
- $\|x_i\| > 1 - \varepsilon_i$ for $i = 1, 2, \ldots, k$.
- $r_i < l_{i+1}$ for $i = 1, 2, \ldots, k - 1$.

Let us show how to find x_{k+1}. Let $E = Q_{r_k}(X)$ and use Proposition 2.7 to find $y_1, y_2 \in E$ with $\|s_k + y_i\| < 1$ and $\|y_1 - y_2\| > 2 - 2\varepsilon_{k+1}$. Without loss of generality $\|y_1\| \geq \|y_2\|$ and y_1 has finite support. Let $x_{k+1} = y_1$. Then $\|s_{k+1}\| = \|s_k + x_{k+1}\| < 1$ and $\|s_k\| \leq \|s_{k+1}\|$ since the basis is monotone. We also have

$$2\|x_{k+1}\| = 2\|y_1\| \geq \|y_1 - y_2\| > 2 - 2\varepsilon_{k+1}$$

so $\|x_{k+1}\| > 1 - \varepsilon_{k+1}$.

Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. Then $z = w^* - \lim_{\mathcal{U}} s_m \in X^{**}$ exists with $\|z\| \leq 1$. For $\lambda \in [0,1]$, let

$$z_\lambda = w^* - \lim_{\mathcal{U}}(s_m - \lambda s_1) = z - \lambda s_1.$$

We have $z_\lambda = \lambda z_1 + (1-\lambda)z_0$. Note that $\|s_m - s_1\| \leq \|s_m\| < 1$ since the basis is bimonotone. Hence $\|z_1\| \leq 1$ and $\|z_0\| \leq 1$, so the line segment $[z_0, z_1]$ is contained in $B_{X^{**}}$. Let $R_n = P_{[l_n, r_n]}$ be the projection onto the support of x_n. We have $\|R_n\| = 1$ and

$$R_n^{**} z_\lambda = w^* - \lim_{\mathcal{U}} R_n(s_m - \lambda s_1) = x_n$$

and hence $\|z_\lambda\| \geq \|R_n^{**} z_\lambda\| = \|x_n\| > 1 - \varepsilon_n$ for all n which means that $\|z_\lambda\| = 1$. Thus $z_\lambda = \lambda z_1 + (1-\lambda)z_0$, $\lambda \in [0,1]$, is a line segment on the sphere. The segment has length $\|z_0 - z_1\| = \|s_1\| > 1 - \varepsilon$. □

3. The local diameter 2 property+

Let us recall from the Introduction the definition of the LD2P+.

Definition 3.1. We say that a Banach space X has the *local diameter* 2 *property+* (LD2P+) if for every $x^* \in S_{X^*}$, every $\varepsilon > 0$, every $\delta > 0$, and every $x \in S(x^*, \varepsilon) \cap S_X$ there exists $y \in S(x^*, \varepsilon)$ with $\|x - y\| > 2 - \delta$.

From [15, Theorem 1.4] and [24, Open problem (7) p. 95] the following is known.

Theorem 3.2. *Let X be a Banach space. Then the following statements are equivalent.*

(i) *The equation $\|I - P\| = 2$ holds for every norm 1 rank 1 projection P on X.*

(ii) *For every $\varepsilon > 0$, every $x^* \in S_{X^*}$ and every $x \in S(x^*, \varepsilon) \cap S_X$ there exists $y \in S(x^*, \varepsilon)$ with $\|x - y\| > 2 - \varepsilon$.*

(iii) *For every $x \in S_X$ and every $\varepsilon > 0$ we have $x \in \overline{conv}(\Delta_\varepsilon(x))$, where $\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| > 2 - \varepsilon\}$.*

From Lemma 3.3 below, which is due to Ivakhno and Kadets [15, Lemma 2.1], it is clear that the LD2P+ is equivalent to the statements in Theorem 3.2. Therefore every Daugavet space has the LD2P+. Note, however, that the converse is not true as the LD2P+ is stable by taking unconditional sums of Banach spaces which fails for spaces with the Daugavet property (see e.g. [15, Corollary 3.1]).

Lemma 3.3 (Ivakhno and Kadets). *Let $\varepsilon > 0$ and $x^* \in S_{X^*}$. Then for every $x \in S(x^*, \varepsilon) \cap S_X$ and every positive $\delta < \varepsilon$ there exist $y^* \in S_{X^*}$ such that $x \in S(y^*, \delta)$ and $S(y^*, \delta) \subset S(x^*, \varepsilon)$.*

In the proof of Theorem 3.5 below we will need the following weak*-version of Lemma 3.3. Its proof is more or less verbatim to that of Lemma 3.3 and will therefore be omitted.

Lemma 3.4. *Let $\varepsilon > 0$ and $x \in S_X$. Then for every $x^* \in S(x,\varepsilon) \cap S_{X^*}$ which attains its norm and every positive $\delta < \varepsilon$ there exist $y \in S_X$ such that $x^* \in S(y,\delta)$ and $S(y,\delta) \subset S(x,\varepsilon)$.*

We will now add to the list of statements in Theorem 3.2 statements similar to (ii) and (iv) in Theorem 1.4. As pointed out in [2, p. 232] the equivalence of (i) and (ii) in Theorem 3.5 below can be proved by a similar argument to the proof of [17, Lemma 1.5].

Theorem 3.5. *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X has the* LD2P+.
(ii) *For every $x \in S_X$, every $\varepsilon > 0$, every $\delta > 0$, and every $x^* \in S(x,\varepsilon) \cap S_{X^*}$ there exists $y^* \in S(x,\varepsilon)$ with $\|x^* - y^*\| > 2 - \delta$.*
(iii) *The equation $\|I - P\| = 1 + \|P\|$ holds for every weakly compact projection P on X.*

Proof. (i) $\Rightarrow$ (ii). By the Bishop-Phelps theorem we can assume without loss of generality that $x^* \in S(x,\varepsilon) \cap S_{X^*}$ attains its norm. Let $0 < \eta < \min\{\varepsilon, \delta/2\}$ and find by Lemma 3.4 $y \in S_X$ such that $x^* \in S(y,\eta)$ and $S(y,\eta) \subset S(x,\varepsilon)$. Note that $y \in S(x^*,\eta)$ and thus, since X has the LD2P+, we can find $z \in S(x^*,\eta)$ such that $\|y - z\| > 2 - \eta$. Hence there is $y^* \in S_{X^*}$ such that

$$y(y^*) - z(y^*) = (y - z)(y^*) > 2 - \eta.$$

From this we have $y(y^*) > 1 - \eta$ and $-z(y^*) > 1 - \eta$. It follows that $y^* \in S(x,\varepsilon)$ as $S(y,\eta) \subset S(x,\varepsilon)$. Moreover, using that $z \in S(x^*,\eta)$, we have

$$\begin{aligned}
\|x^* - y^*\| &\geq (x^* - y^*)(z) \\
&= x^*(z) - y^*(z) \\
&> 1 - \eta + 1 - \eta > 2 - \delta.
\end{aligned}$$

(ii) $\Rightarrow$ (i). The proof is identical to the proof of the converse except that one does not use the Bishop-Phelps theorem and that one uses Lemma 3.3 in place of Lemma 3.4.

(i) $\Rightarrow$ (iii). The proof is similar to that of [17, Theorem 2.3].

(iii) $\Rightarrow$ (i). This is clear as (iii) trivially implies (i) in Theorem 3.2. ☐

Note that from Theorem 3.5 we get

Corollary 3.6. *If B_{X^*} contains a weak*-denting point, in particular if X^* has the* RNP*, then X does not have the* LD2P+.

It is known that if X^{**} is smooth, then X^* has the RNP (see e.g. [23]), hence we have the following corollary.

Corollary 3.7. *If X has the* LD2P+*, then X^{**} is not smooth.*

It is known that all the diameter 2 properties in Definition 1.1 as well as the Daugavet property are inherited by certain subspaces called ai-ideals (see [5] and [1]). We will end this section by showing that this is true for the LD2P+ as well.

A subspace X of a Banach space Y is called an *ideal* in Y if there exists a norm 1 projection P on Y^* with $\ker P = X^\perp$. X being an ideal in Y is in turn equivalent to X being locally 1-complemented in Y, i.e., for every $\varepsilon > 0$ and every finite dimensional subspace $E \subset Y$ there exists a linear $T : E \to X$ such that

(i) $Te = e$ for all $e \in X \cap E$.
(ii) $\|Te\| \le (1+\varepsilon)\|e\|$ for all $e \in E$.

Following [5] a subspace X of a Banach space Y is called an *almost isometric ideal (ai-ideal)* in Y if X is locally 1-complemented with almost isometric local projections, i.e., for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subset Y$ there exists $T : E \to X$ which satisfies (i) and

(ii′) $(1-\varepsilon)\|e\| \le \|Te\| \le (1+\varepsilon)\|e\|$ for all $e \in E$.

Note that an ideal X in Y is an ai-ideal if $P(Y^*)$ is a 1-norming subspace of Y^* [5, Proposition 2.1]. Ideals X in Y for which $P(Y^*)$ is a 1-norming subspace for Y are called *strict ideals*. An ai-ideal is, however, not necessarily strict (see [5, Example 1] and [3, Remark 3.2]).

Proposition 3.8. *Let Y have the* LD2P+ *and assume X is an ai-ideal in Y. Then X has the* LD2P+.

Proof. For $\delta > 0$, Z a subspace of Y, and $x \in S_Z$ put

$$\Delta_\delta^Z(x) = \{y \in B_Z : \|x - y\| > 2 - \delta\}.$$

Let $x \in S_X$, $\varepsilon > 0$, and $\alpha > 0$. We will show that there exists $z \in \mathrm{conv}\Delta_\varepsilon^X(x)$ with $\|x - z\| < \alpha$. The result will then follow from Theorem 3.2 (iii). First, since Y enjoys the LD2P+, we know that for any positive $\beta < \varepsilon$ and any positive $\gamma < \alpha$ we can find $y = \sum_{n=1}^N \lambda_n y_n$ with $(y_n)_{n=1}^N \subset \Delta_\beta^Y(x)$ such that $\|x - y\| < \gamma$. Now let $E = \mathrm{span}\{y_1, \ldots, y_N, x\}$ and pick a local projection $T : E \to X$ such that T is a $(1+\eta)$-isometry with $\eta > 0$ so small that $(1+\eta)\gamma + \eta < \alpha$, and $(1-\eta)(2-\beta) - \eta > 2 - \varepsilon$. Put $z_n = \frac{Ty_n}{\|Ty_n\|}$ and $z = \sum_{n=1}^N \lambda_n z_n$. As $Tx = x$ we get

$$\begin{aligned}
\|x - z\| &\le \|x - Ty\| + \|Ty - z\| \\
&\le \|T(x-y)\| + \sum_{n=1}^N \lambda_n \big|1 - \|Ty_n\|\big| \\
&< (1+\eta)\gamma + \max_{1 \le n \le N} \big|1 - \|Ty_n\|\big| \\
&\le (1+\eta)\gamma + \eta < \alpha.
\end{aligned}$$

Moreover, for every $1 \leq n \leq N$ we have,

$$\begin{aligned}\|x - z_n\| &= \|T(x - \frac{y_n}{\|Ty_n\|})\| \\ &\geq (1-\eta)\|x - \frac{y_n}{\|Ty_n\|}\| \\ &\geq (1-\eta)(\|x - y_n\| - \|y_n - \frac{y_n}{\|Ty_n\|}\|) \\ &\geq (1-\eta)(2 - \beta - \frac{\left|1 - \|Ty_n\|\right|}{\|Ty_n\|}\|y_n\|) \\ &\geq (1-\eta)(2 - \beta - \frac{\eta}{1-\eta}) > 2 - \varepsilon,\end{aligned}$$

Thus $(z_n)_{n=1}^N \subset \Delta_\varepsilon(x)$ and as $\alpha > 0$ is arbitrarily chosen, we are done. □

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Lecture given by S. Troyanski at the *Seminario Matematico e Fisico di Milano* on March 14, 2016

Received: May 22, 2016.

Accepted: September 23, 2016.