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The Role of Surface Diffusion in Dynamic Boundary Conditions: Where Do We Stand?

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Abstract. In this study, we investigate reaction-diffusion and elliptic-like equations with two classes of dynamic boundary conditions, of reactive and reactive-diffusive type. We provide sharp upper and lower bounds on the dimension of the global attractor in all these cases. In particular, we emphasize how surface diffusion can act as a damping force in reducing the degree of complexity in these systems. We obtain a new Weyl asymptotic law for eigenvalue sequences associated with a family of perturbed Wentzell operators which is of independent interest.

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1. Introduction

Reaction–diffusion equations and elliptic equations, subject to various *dynamic* boundary conditions, are known to have a finite-dimensional asymptotic (in time) behavior (see, e.g., [17, 18, 19, 20, 21, 22], and references therein). Moreover, under natural assumptions on the nonlinearities and in the absence of external forces, these systems enjoy the property of global asymptotic stability, in the sense that any given solution trajectory will converge asymptotically as time goes to infinity to some equilibrium of the system (see [22] for elliptic equations and [20, 21, 41, 46] for reaction–diffusion equations). These properties also show up through the fact that such systems possess finite-dimensional global attractors and have a gradient structure. As a compact invariant subset of the phase space, the global attractor attracts images of all bounded sets (as time tends to infinity) and contains all of the nontrivial limit dynamics of the system in question. When the dynamics on the global attractor is finite dimensional, the limit dynamics of the infinite-dimensional dynamical system becomes equivalent to an appropriate finite dynamical system defined on a compact subset of $\mathbb{R}^N$ (see [35]). Generally speaking, the *dimension* of the global attractor is used to indicate the number of degrees of freedom needed to simulate the given system and is associated with the temporal and spatial complexity of the long-time dynamics. Additionally, the dimension of the global attractor may be used to suggest the correct resolution needed for numerical computations by relating it to a fundamental length scale of the original problem. Dynamic boundary conditions have been proposed and used in many applications. An enormous amount of literature for the rigorous treatment of dynamic boundary conditions in various contexts (such as, diffusion phenomena in thermodynamics, phase-transition phenomena in material science, climate science, control theory and special flows in hydrodynamics) has steadily grown over the last decade and is still presently growing at a rapid rate. Without being too exhaustive, we refer the reader to the references [18, 22, 28], and references therein, where such descriptions have been undertaken in detail.

Consider now the parabolic partial differential equation

$$\partial_t u = \nu \Delta u - f(u) + \lambda u, \quad (t,x) \in (0,+\infty) \times \Omega, \tag{1.1}$$

where $u = u(t,x) \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with boundary Γ of class $\mathcal{C}^2$, ν, λ are positive constants and f plays the role of a source/sink like nonlinearity. The function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be $C^{1,1}_{\text{loc}}$, that is, continuous and with a (locally) Lipschitz continuous first derivative, which satisfies, among other natural growth conditions (see Section 3),

$$f'(y) \geq -c_f, \text{ for all } y \in \mathbb{R}, \text{ for some } c_f > 0. \tag{1.2}$$

We equip (1.1) with dynamic boundary conditions of *pure-reactive* ($\delta = 0$) and *reactive-diffusive* type ($\delta > 0$), of the form

$$\partial_t u - \delta \Delta_\Gamma u + \nu \partial_{\mathbf{n}} u = 0, \quad \text{on } (0,\infty) \times \Gamma. \tag{1.3}$$

Here, $\mathbf{n} \in \mathbb{R}^n$ denotes the outward normal at Γ, $\partial_{\mathbf{n}} u$ is the outward normal derivative of u, Δ_Γ is the Laplace-Beltrami operator on Γ and $\delta \geq 0$ plays the role of a surface diffusion coefficient. Naturally, the system (1.1)-(1.3) is also equipped with initial conditions for u in $\overline{\Omega}$ at time $t = 0$.

In the context of reaction–diffusion equations, dynamic boundary conditions have been rigorously derived in [28] based on first and second thermodynamical principles and their physical interpretation was also given in [27]. It is worth emphasizing that the derivation in [28] obtains the dynamic boundary condition (1.3) both as a sufficient and necessary condition for thermodynamic processes which incorporate thermodynamic sources located along the boundary Γ, and in which the second law plays a crucial role, while in [27] it has been introduced only as a sufficient condition. We shall denote the system (1.1), (1.3) as problem $(\text{RDE})_\delta$ and thus view the corresponding problem with $\delta > 0$ as a diffusive perturbation of $(\text{RDE})_0$ along the boundary Γ. Another problem (denoted here as $(\text{EE})_\delta$, $\delta \geq 0$) that we wish to consider in this contribution is the elliptic-parabolic system

$$\begin{cases} \nu\Delta u - \lambda u = 0, & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \nu\partial_{\mathbf{n}} u = \delta\Delta_\Gamma u - g(u), & \text{on } (0,\infty)\times\Gamma, \\ u|_{t=0} = \psi_0, & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where $\lambda \geq 0$, with $g \in C^{1,1}_{\text{loc}}(\mathbb{R})$ satisfying (1.2) among other natural assumptions. Recently, the system $(\text{EE})_\delta$ for $\delta \geq 0$ has been systematically investigated in [22] and one has now a rather complete picture of the well-posedness, blow-up phenomena, regularity and the asymptotic stability (in terms of finite dimensional global attractors and convergence to single equilibria) of classical solutions for this system. These issues are also complete for the parabolic system $(\text{RDE})_0$, see [17, 46]. For a more general system than (1.1), (1.3), we also refer the reader to [18, 33].

The main purpose of this study is to clarify the role of the additional term $-\delta\Delta_\Gamma u$ in either boundary conditions (1.3) or (1.4), and its both qualitative and quantitative effects on solutions and their long-time asymptotic behavior for the corresponding reaction–diffusion systems $(\text{RDE})_\delta$ and elliptic systems $(\text{EE})_\delta$, respectively. To set the scene, recall from [17] that, for a polynomial nonlinearity f satisfying (1.2), problem $(\text{RDE})_0$ generates a dynamical system on the phase-space $X_2 = L^2(\Omega)\times L^2(\Gamma)$, possessing a finite dimensional global attractor $\mathcal{G}_0$. Moreover, the Hausdorff and fractal dimensions of $\mathcal{G}_0$ satisfy the following upper and lower bounds:

$$c_0\left(\frac{\lambda}{\nu}\right)^{n-1}|\Gamma| \leq \dim_H(\mathcal{G}_0) \leq \dim_F(\mathcal{G}_0) \leq c_1\left(1+\frac{c_f+\lambda}{\nu}|\Gamma|^{1/(n-1)}\right)^{n-1}, \tag{1.5}$$

in dimension $n \geq 3$, with positive constants c_0, c_1 depending only on n and the shape of Ω but not its size, and are independent of ν and λ. We note that, for a fixed domain Ω, the estimates in (1.5) are sharp with respect to $\nu \to 0^+$ (for each fixed $\lambda > 0$), and for sufficiently large λ (if $\nu > 0$ is fixed). Analogous estimates in dimension $n = 2$ are also provided in [17] but these also depend on the "volume" of Ω while retaining the same exponent $n-1$ in (1.5).

Consider the reaction–diffusion system $(\text{RDE})_\delta$ in which the term $\delta\Delta_\Gamma u$, $\delta > 0$, provides, in addition to classical bulk diffusion, a diffusion mechanism present along the boundary Γ. A typical example in the theory of heat conduction (see [28]) arises when a given body Ω is in perfect thermal contact with a sufficiently thin metal sheet Γ, possibly of different material and completely insulating the internal body Ω from external contact with, say, a well-stirred hot or cold fluid. Now, a key question is to ask how much significance could such a "viscous" δ-regularization have on the system from both a quantitative and qualitative point of view. In considering the answer to this question, we prove in Section 3 that problem $(\text{RDE})_\delta$ for $\delta > 0$ generates yet another dynamical system on the energy space X_2 and that it possesses a finite dimensional global attractor $\mathcal{G}_\delta$. We then demonstrate that boundary diffusion has no essential qualitative impact on the energy estimates and regularity of individual solutions of $(\text{RDE})_\delta$, $\delta > 0$, other than the fact that the additional term generally enhances the boundary regularity of solutions by a fraction of 1/2. This, of course, is not that all surprising in light of a series of analytical results involving such systems and also proved recently (see, e.g., [17, 18, 19, 20, 21, 41, 33, 46], and the references therein). However what turns out to be remarkable is that the additional δ-regularization does have a significant qualitative impact from a global and *dynamical* point of view. For such systems, the Hausdorff and fractal dimensions of $\mathcal{G}_\delta$, $\delta > 0$, satisfy the following upper and lower bounds:

$$c_2 \left(\frac{\lambda}{\nu}\right)^{\frac{n}{2}} |\Omega| \leq \dim_H (\mathcal{G}_\delta) \leq \dim_F (\mathcal{G}_\delta) \leq c_3 \left(1 + \frac{c_f + \lambda}{\nu} |\Omega|^{2/n}\right)^{\frac{n}{2}}, \tag{1.6}$$

in dimension $n \geq 1$, where c_2, c_3 depend only on n and the shape of Ω but not its size, and are independent of $\nu, \delta > 0$ and $\lambda > 0$. In light of estimates (1.5)-(1.6), the degree of complexity of the "permanent regime" of $(\mathrm{RDE})_0$ changes significantly when surface diffusion is simply accounted for in the dynamical behavior of Γ. An heuristic explanation for this effect is simply given by the fact that the dynamic condition (1.3) when $\delta = 0$, is in fact a transport equation $\partial_t u + \mathbf{v} \cdot \nabla u = 0$ on Γ, in which the "flow" u is carried over from any point of Γ, in all directions normal to Γ, inside the bulk domain Ω with a constant velocity $\mathbf{v} = \nu n \in \mathbb{R}^n$. In this case, the mechanism for producing the observed dynamical behavior is determined solely by advective transport and the fact that in this case the boundary equation is purely hyperbolic. In the case $\delta > 0$, the dynamic condition (1.3) can be viewed as a combination of both advection and diffusive forces in which, of course, the additional δ-viscous regularization for $\delta > 0$ changes (1.3) into merely a parabolic equation on Γ.

On the other hand, for the elliptic-parabolic system $(\mathrm{EE})_\delta$, $\delta \geq 0$, we provide similar and comparable results. First, in Section 4 we devise a new approach to handle the well-posedness of the system $(\mathrm{EE})_\delta$, by viewing it as a *singular* perturbation (as $\varepsilon \to 0^+$) of a sequence of parabolic systems, of the form

$$\begin{cases} \varepsilon\partial_t u - \nu\Delta u + \lambda u = 0, & \text{in } (0,\infty) \times \Omega, \\ \partial_t u + \nu\partial_{\mathbf{n}} u = \delta\Delta_\Gamma u - g(u), & \text{on } (0,\infty) \times \Gamma, \\ u|_{t=0} = u_0 \text{ in } \Omega,\ u|_{t=0} = \psi_0 \text{ on } \Gamma, \end{cases} \tag{1.7}$$

where $\varepsilon \in (0, 1]$ is a given relaxation parameter. We then give optimal conditions on the nonlinearity g which allows to prove the global existence of strong solutions for the original problem $(\mathrm{EE})_\delta$, by first deducing sufficiently strong (uniform as $\varepsilon \to 0^+$) estimates for solutions of the system (1.7) and then by exploiting data reconstruction techniques, and employing compactness arguments (see Section 4.1) to pass to the limit. Furthermore, for a polynomial nonlinearity g which satisfies (1.2) we show that both problems $(\mathrm{EE})_0$ and $(\mathrm{EE})_\delta$, $\delta > 0$, generate a dynamical system on the phase-space $L^2(\Gamma)$, possessing a finite dimensional global attractor $\mathcal{E}_\delta$, $\delta \geq 0$. Going further to Section 4.2, arguments in the theory of infinite-dimensional dynamical systems imply the following sharp two-sided estimates on the Hausdorff and fractal dimensions of $\mathcal{E}_\delta$ in any space dimension $n \geq 2$, of the form

$$\begin{cases} c_4 \left(\frac{\zeta}{\nu}\right)^{n-1} |\Gamma| \leq \dim_H (\mathcal{E}_0) \leq \dim_F (\mathcal{E}_0) \leq c_5 \left(\frac{c_f}{\nu}\right)^{n-1} |\Gamma|, \\ c_4 \left(\frac{\zeta}{\delta}\right)^{\frac{n-1}{2}} |\Gamma| \leq \dim_H (\mathcal{E}_\delta) \leq \dim_F (\mathcal{E}_\delta) \leq c_5 \left(\frac{c_f}{\delta}\right)^{\frac{n-1}{2}} |\Gamma|, \end{cases} \tag{1.8}$$

for sufficiently small δ, ν, where $0 < \zeta := g^{'}(z)$, $z \in \mathbb{R}$ is a fixed constant steady-state solution of (1.4) and c_4, c_5 depend only on n and the shape of Ω, Γ but not their "size", and are independent of ν, δ and $\lambda > 0$. What is also interesting to observe here is that, for a fixed domain Ω, λ and ζ, and as the bulk diffusion coefficient $\nu \to 0^+$, the permanent regime of the reaction–diffusion system $(\mathrm{RDE})_0$ and eliptic-parabolic system $(\mathrm{EE})_0$ bear the same degree of complexity as reflected in the dimension estimates (1.5) and the first of (1.8), respectively. By this token, perhaps we may argue that the parabolic equation in the bulk Ω is not much relevant to the global (and also, possibly local) dynamical behavior of problem $(\mathrm{RDE})_0$.

The theory of Dynamical Systems has always been driven by the need to understand concrete problems and hence it has incorporated a wide variety of mathematical tools from functional analysis and mathematical physics. An important link between the behavior of dynamical systems and spectral theory, which nowadays has itself grown in a large and separate field, is the study of the spectral properties of the underlying linear operators: when does a differential operator define a self-adjoint operator, when does it have a compact resolvent, and what asymptotic properties does its spectrum have? In particular, the asymptotic distribution of eigenvalues is one of the most important problems of the spectral theory of partial differential operators and since the pioneer work of H. Weyl in 1915, the validity of various asymptotic formulas for a diverse classes of differential operators in various situations have been established. Weyl asymptotic formulae for a given linear differential operator is intimately connected not only with the geometrical properties of the domain and the type of boundary conditions, but also to the dynamical properties of nonlinear partial differential equations associated with that linear operator. This body of work also collectively describes the spectral properties of a new class of (second-order) self-adjoint operators, referred here as the perturbed ($\delta > 0$) and unperturbed ($\delta = 0$) Wentzell Laplacians $A_W^{\nu,\delta}$, which are associated with the reaction–diffusion problems $(\mathrm{RDE})_\delta$ for $\delta \geq 0$. A interesting feature of the (un)perturbed Wentzell Laplacian is that it involves an eigenvalue problem for the Laplacian $-\nu\Delta$ which involves a boundary condition that depends on the eigenvalue explicitly. For the elliptic-parabolic system $(\mathrm{EE})_\delta$, a family of perturbed Steklov eigenvalue problems and the corresponding asymptotic eigenvalue distribution will play an essential role in establishing the sharp dimension estimates in (1.8). A large body of this work, in particular Section 2, is devoted to this new class of operators and complete proofs of their spectral properties. For instance, employing variational and perturbation methods we derive a new Weyl asymptotic law for the eigenvalue distribution of the perturbed Wentzell Laplacian; this will be also used to deduce (1.6). The importance of these laws will become more apparent in the subsequent Sections 3 and 4.

2. Weyl asymptotic laws for eigenvalues

To get started, we briefly elaborate in this section the relevant functional framework associated with our problems. In the first part we prove a basic fact about sesquilinear forms and the linear operators associated with them. We devote the final and second part of this section to characterizing all spectral properties of the so-called perturbed and unperturbed Wentzell Laplacian.

2.1. Perturbation of forms

Let A, B, C be three linear (possibly) unbounded self-adjoint operators such that $C = A + B$ with $D(C) = D(A)$ and $D(A) \subset D(B) \subset H$ (H is some Hilbert space). We assume that each of the operators A, B, C can be associated with the sesquilinear forms q_A, q_B and $q_C = q_A + q_B$, respectively, such that $D(q_C) = D(q_A) \subseteq D(q_B)$. More precisely, let us assume that

$$q_A : V_A \times V_A \to \mathbb{R},\ q_B : V_B \times V_B \to \mathbb{R},\ q_C : V_A \times V_A \to \mathbb{R}$$

are symmetric, closed and bounded from below on the corresponding spaces, see, e.g., [36]. By the second representation theorem for symmetric sesquilinear forms, the linear operator A associated with the form q_A is defined in the following way

$$\begin{aligned} D(A) &= \{u \in V_A : \exists f \in H \text{ such that } q_A(u,v) = (f,v)_H\,,\ \forall v \in V_A\}, \\ Au &= f. \end{aligned} \tag{2.1}$$

The operator A is selfadjoint on H and generates a (C_0)-semigroup $T_A = \{T_A(t) : t \geq 0\}$ satisfying $T_A(t) = T_A(t)^*$ and $\|T_A(t)\| \leq 1$ for all $t \geq 0$. Similar definitions are applied to the operators B, C. The eigenvalues $\{\lambda_{A,j}\}$, $\{\lambda_{B,j}\}$, $\{\lambda_{C,j}\}$, $j \in J$ (J is either $\mathbb{N}$ or $\mathbb{N}_0$) associated with the operators A, B and C, respectively, then obey the following min-max characterizations:

$$\begin{aligned} \lambda_{A,j} &= \min_{\substack{L_j \subset V_A,\\ \dim(L_j)=j}} \max_{0 \neq u \in L_j} \frac{q_A(u,u)}{\|u\|_H^2}, \\ \lambda_{B,j} &= \min_{\substack{L_j \subset V_B,\\ \dim(L_j)=j}} \max_{0 \neq u \in L_j} \frac{q_B(u,u)}{\|u\|_H^2}, \\ \lambda_{C,j} &= \min_{\substack{L_j \subset V_A,\\ \dim(L_j)=j}} \max_{0 \neq u \in L_j} \frac{q_C(u,u)}{\|u\|_H^2}, \quad q_C := q_A + q_B. \end{aligned} \tag{2.2}$$

We assume that each eigenvalue sequence for corresponding eigenvalue problems $Au = \lambda u$, $Bu = \lambda u$, satisfies the following Weyl asymptotic law:

$$\lambda_{A,j} = C_A j^p + o(j^p)\,,\ \lambda_{B,j} = C_B j^q + o(j^q)\,,\ \text{as } j \to \infty, \tag{2.3}$$

for some $C_A, C_B > 0$, and some $p, q \in \mathbb{R}_+$ with $p > q$.

We prove a simple but crucial result on the eigenvalue asymptotic formulae for $\lambda_{C,j}$ as j goes to infinity. Roughly speaking it states that $\lambda_{C,j}$ and $\lambda_{A,j}$ have the same asymptotic behavior at infinity when B is an "infinitesimal" small perturbation of A.

Lemma 2.1. *Suppose that* (2.3) *holds and that the form* q_B *is infinitesimally form bounded with respect to* q_A, *i.e., for every* $\varepsilon > 0$, *there is a positive constant* $C = C_\varepsilon > 0$, *independent of* u, *such that*

$$\|u\|_{V_B}^2 \leq \varepsilon \|u\|_{V_A}^2 + C_\varepsilon \|u\|_H^2, \text{ for every } u \in V_A. \tag{2.4}$$

Then the eigenvalue sequence $\{\lambda_{C,j}\}$, $j \in J$, *obeys the following Weyl law:*

$$\lambda_{C,j} = C_A j^p + o(j^p), \text{ as } j \to \infty. \tag{2.5}$$

Proof. By the description (2.2), it is clear that

$$\lambda_{C,j} \geq \lambda_{A,j}, \text{ for all } j \in J.$$

Thus, on account of (2.3), we immediately get

$$\liminf_{j\to\infty} \frac{\lambda_{C,j}}{j^p} \geq C_A. \tag{2.6}$$

By (2.4), we infer that

$$q_C(u,u) = q_A(u,u) + q_B(u,u) \leq (1+\varepsilon) q_A(u,u) + C_\varepsilon \|u\|_H^2.$$

Thus, from the description (2.2) we deduce

$$\lambda_{C,j} \leq (1+\varepsilon)\lambda_{A,j} + C_\varepsilon, \text{ for all } j \in J \tag{2.7}$$

and so we have

$$\limsup_{j\to\infty} \frac{\lambda_{C,j}}{j^p} \leq C_A(1+\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, together with (2.6) we immediately obtain the conclusion (2.5) as well. The proof is finished. □

2.2. The Wentzell Laplacian

In this subsection, we recall that a certain realization of $A = -\Delta$ with various Wentzell boundary conditions is self-adjoint and nonnegative on a proper Hilbert space. We shall refer to this realization as the Wentzell Laplacian. While these generation results are known to various experts in various forms and in a more general context (such as, more general elliptic second-order differential operators, with or without surface diffusion Δ_Γ in the boundary conditions), using different approaches based on energy methods, form methods and operator matrix methods, we choose to give proofs based on the form method for the sake of completeness. However, we refer the reader to [7, 15, 26] for an extensive survey of these results and the relevant literature (which lies outside the scope of this article). We point out that our main interest lies in a detailed study of the asymptotic behavior of the eigenvalues of the Wentzell Laplacian and not its generation properties. Henceforth, we shall derive a number of specific properties of the "Wentzell" eigenvalues associated with the perturbed and unperturbed Wentzell Laplacian, including a fairly precise description of their structure, of the regularity of the eigenfunctions, and also a number of variational and asymptotic results. To the best of our knowledge, these properties are fairly unknown to the scientific community.

To this end, let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with Lipschitz continuous boundary Γ. We recall that the natural space for our problems is $L^p\left(\overline{\Omega}, d\mu\right)$, where

$$d\mu = dx \oplus dS,$$

dx denotes the Lebesgue measure on Ω and dS denotes the natural surface measure dS on Γ. It is easy to see that $L^p\left(\overline{\Omega}, d\mu\right)$ may be identified

$$L^p\left(\overline{\Omega}, d\mu\right) = L^p\left(\Omega, dx\right) \times L^p\left(\Gamma, dS\right), \ 1 \leq p < \infty. \tag{2.8}$$

Since μ is also a Radon measure on $\mathbb{B}\left(\overline{\Omega}\right)$, $L^\infty\left(\overline{\Omega}, d\mu\right)$ can be identified with $L^\infty\left(\Omega, dx\right) \times L^\infty\left(\Gamma, dS\right)$ with norm

$$||u||_{X_\infty} := \max\left\{\|u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Gamma, dS)}\right\}.$$

Let now $U = (u, v)$, where $u : \Omega \to \mathbb{R}$ and $v : \Gamma \to \mathbb{R}$ are measurable functions such that

$$\int_\Omega |u(x)|^p\, dx + \int_\Gamma |v(x)|^p\, dS < \infty.$$

We define the norm $\|\cdot\|_p^*$ of U as follow

$$\|U\|_p^* = \left(\int_\Omega |u(x)|^p\, dx + \int_\Gamma |v(x)|^p\, dS\right)^{1/p},$$

for $1 \leq p < \infty$, and observe that the $L^p\left(\overline{\Omega}, d\mu\right)$ norm and the $\|\cdot\|_p^*$ norm are the same. From now on denote this norm by $||\cdot||_{X_p}$. Moreover, if we identify every $u \in C\left(\overline{\Omega}\right)$ with $U = (u, u_\Gamma) \in C(\Omega) \times C(\Gamma)$, where $u_\Gamma \overset{\text{def}}{=} \text{trace}(u) \in C(\Gamma)$ we define X_p to be the completion of $C\left(\overline{\Omega}\right)$ in the norm $||\cdot||_{X_p}$. But one can easily show that $X_p = L^p\left(\overline{\Omega}, d\mu\right)$ (see [14]). In general, any function $v \in X_p$ will be of the form $v = (v_1, v_2)$ with $v_1 \in L^p(\Omega, dx)$ and $v_2 \in L^p(\Gamma, dS)$, and there need not be any connection between v_1 and v_2. Finally, let $\mathcal{V}_\delta$, $\delta \geq 0$, be the completion of $C^1\left(\overline{\Omega}\right)$ in the norm

$$\|u\|_{\mathcal{V}_\delta}^2 := \int_\Omega \left(|u(x)|^2 + \nu |\nabla u(x)|^2\right) dx + \int_\Gamma \left(|u(x)|^2 + \delta |\nabla_\Gamma u(x)|^2\right) dS,$$

where ∇_Γ denotes the surface gradient on Γ. Note that for any $f \in \mathcal{V}_\delta$, we have $f \in H^1(\Omega)$ so that f_Γ makes sense in the trace sense. The space $\mathcal{V}_\delta$ is topologically isomorphic to $H^1(\Omega) \times H^1(\Gamma)$ if $\delta > 0$, and $\mathcal{V}_0 = H^1(\Omega)$.

Let us now recall that $X_2 = L^2(\Omega, dx) \times L^2(\Gamma, dS)$ is also Hilbert space when equipped with the canonical inner product

$$\langle U, V\rangle_{X_2} = \langle u_1, v_1\rangle_{L^2(\Omega)} + \langle u_2, v_2\rangle_{L^2(\Gamma, dS)},$$

for all $U = (u_1, u_2) \in X_2$, $V = (v_1, v_2) \in X_2$. For all $\delta \geq 0$, we also define the linear space

$$\mathcal{W}_\delta := \left\{(u_1, u_2) \in \mathcal{V}_\delta : u_2 = \text{trace}(u_1)\right\}.$$

We emphasize that $\mathcal{W}_\delta$ is not a product space as $\mathcal{V}_\delta$. Clearly, $\mathcal{W}_\delta \subset X_2$ densely since the trace operator acting on functions $H^1(\Omega)$ and into $H^{1/2}(\Gamma)$ is bounded and onto, and $\mathcal{W}_\delta$ is a Hilbert space with respect to the inner product inherited from $\mathcal{V}_\delta$, $\delta \geq 0$. Thus by definition we can identify

$$\mathcal{W}_\delta = \left\{(u, u_\Gamma) \in H^1(\Omega) \times H^1(\Gamma) : u_\Gamma = \mathrm{trace}\,(u)\right\},$$

for each $\delta > 0$, and

$$\mathcal{W}_0 = \left\{(u, u_\Gamma) \in H^1(\Omega) \times H^{1/2}(\Gamma) : u_\Gamma = \mathrm{trace}\,(u)\right\}. \tag{2.9}$$

We also recall the Stokes divergence theorem on Γ,

$$\int_\Gamma \Delta_\Gamma u v dS = -\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v dS,\ u \in H^2(\Gamma), v \in H^1(\Gamma) \tag{2.10}$$

and the notion of a weak normal derivative in the case when Γ is only Lipschitz continuous. Indeed, for functions $u \in H^1(\Omega)$ which satisfy $\Delta u \in L^2(\Omega)$, we say that u has a weak normal derivative if there exists a function $\zeta \in L^2(\Gamma)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega \Delta u v dx = \int_\Gamma \zeta v dS,\ \text{for all } v \in H^1(\Omega). \tag{2.11}$$

In this case, the function $\zeta \in L^2(\Gamma)$ verifying (2.11) is unique; we denote ζ by $\partial_{\mathbf{n}} u$.

We have the following generation result (cf. [34] in the case $\delta = 0$).

Theorem 2.2. *Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary Γ and $0 \leq q \in L^\infty(\Omega)$. For $\nu > 0$ and $\delta \geq 0$ define the linear operator $A_W^{\nu,\delta}$ on X_2, by*

$$A_W^{\nu,\delta}\begin{pmatrix} u \\ u_\Gamma \end{pmatrix} := \begin{pmatrix} -\nu\Delta u + q(x) u \\ -\delta\Delta_\Gamma u_\Gamma + \nu\partial_{\mathbf{n}} u \end{pmatrix}, \tag{2.12}$$

with domain

$$D(A_W^{\nu,\delta}) := \left\{U = (u, u_\Gamma) \in \mathcal{W}_\delta : \Delta u \in L^2(\Omega),\ -\delta\Delta_\Gamma u_\Gamma + \nu\partial_{\mathbf{n}} u \in L^2(\Gamma)\right\}. \tag{2.13}$$

Then, $A_W^{\nu,\delta}$ is self-adjoint and nonnegative on X_2. Moreover, the resolvent operator $(I + A_W^{\nu,\delta})^{-1} \in \mathcal{L}(X_2)$ is compact. Thus, $A_W^{\nu,\delta}$ generates a self-adjoint compact analytic (C_0)-semigroup $T_W^{\nu,\delta} = \{T_W^{\nu,\delta}(t) : t \geq 0\}$ on X_2.

Proof. We define the sesquilinear form a_δ with form domain $D(a_\delta) = \mathcal{W}_\delta$ on the Hilbert space X_2 by

$$a_\delta(U, V) = \int_\Omega (\nu\nabla u \cdot \nabla v + q(x) uv)\, dx + \int_\Gamma \delta\nabla_\Gamma u \cdot \nabla_\Gamma v dS, \tag{2.14}$$

for $U = (u, u_\Gamma), V = (v, v_\Gamma) \in \mathcal{W}_\delta$. Our next goal is to show that a_δ (which is sesquilinear, nonnegative and symmetric by definition, and densely defined) is associated with the self-adjoint operator $A_W^{\nu,\delta}$ on X_2. Clearly, the form a_δ is closed since the form norm $\|U\|_{a_\delta}^2 = a_\delta(U, U) + \|U\|_{X_2}^2$ is equivalent to $\|U\|_{\mathcal{W}_\delta}^2$ with respect to which $\mathcal{W}_\delta$ is complete. We claim that $A_W^{\nu,\delta}$ is the operator associated with the form a_δ. Denote by $\mathcal{B}$ the self-adjoint operator associated with a_δ. Let $U = (u, u_\Gamma) \in D(\mathcal{B}) \subset \mathcal{W}_\delta$, and let

$$\mathcal{B}U = F \overset{\text{def}}{=} \begin{pmatrix} f \\ g \end{pmatrix} \in X_2.$$

Then

$$\int_{\Omega}(\nu\nabla u\cdot\nabla v+quv)\,dx+\int_{\Gamma}\delta\nabla_{\Gamma}u_{\Gamma}\cdot\nabla_{\Gamma}v_{\Gamma}dS=a_{\delta}(U,V)=\langle F,V\rangle_{X_{2}},$$

for all $V=(v,v_{\Gamma})\in\mathcal{W}_{\delta}$. Choosing $v\in D(\Omega)$, we deduce that $-\nu\Delta u+q(x)u=f$ in Ω. Hence, also applying the surface divergence theorem (2.10), we have

$$\int_{\Omega}(\nu\nabla u\cdot\nabla v+quv)\,dx+\int_{\Omega}-\nu\Delta uvdx=\int_{\Gamma}(g+\delta\Delta_{\Gamma}u_{\Gamma})\,v_{\Gamma}dS,$$

for all $V\in\mathcal{W}_{\delta}$. In particular, by (2.11) $\zeta=\nu\partial_{n}u$ exists and

$$\nu\partial_{n}u=g+\delta\Delta_{\Gamma}u_{\Gamma}\text{ on }\Gamma.$$

Thus, we have proved that $U\in D(A_{W}^{\nu,\delta})$ and $A_{W}^{\nu,\delta}U=\mathcal{B}U$. In order to prove the converse, let $U\in D(A_{W}^{\nu,\delta})$. Then,

$$\begin{aligned}\int_{\Omega}(\nu\nabla u\cdot\nabla v+quv)\,dx-\int_{\Omega}\nu\Delta uvdx &= \int_{\Gamma}\nu\partial_{n}uv_{\Gamma}dS\\ &= \int_{\Gamma}(g+\delta\Delta_{\Gamma}u_{\Gamma})\,v_{\Gamma}dS,\end{aligned}$$

where $g=-\delta\Delta_{\Gamma}u_{\Gamma}+\nu\partial_{n}u$ on Γ, for all $V\in\mathcal{W}_{\delta}$. Hence,

$$a_{\delta}(U,V)=\int_{\Omega}(-\nu\Delta u+qu)\,vdx+\int_{\Gamma}gv_{\Gamma}dS,$$

for all $V\in\mathcal{W}_{\delta}$. By the definition of the operator $\mathcal{B}$ associated with the form a_{δ}, we deduce that $U\in D(\mathcal{B})$ and

$$\mathcal{B}U=\begin{pmatrix}-\nu\Delta u+qu\\ g\end{pmatrix}=A_{W}^{\nu,\delta}U.$$

To prove compactness, it suffices to show that the injection of $(D(a_{\delta}),\|\cdot\|_{a_{\delta}})$ into X_{2} is compact. But this is immediate since $D(a_{\delta})=\mathcal{W}_{\delta}$ and the injections $H^{1}(\Omega)\hookrightarrow L^{2}(\Omega)$ and $H^{1/2}(\Gamma)\hookrightarrow L^{2}(\Gamma)$ are both compact by the Sobolev embedding theorem. The rest of the claim follows. $\square$

From now on we shall refer to $A_{W}^{\nu,0}$ as the *unperturbed* Wentzell Laplacian and $A_{W}^{\nu,\delta}$ for $\delta>0$ as the *perturbed* Wentzell Laplacian. The eigenvalue problem associated with these operators is given by $A_{W}^{\nu,\delta}\varphi=\lambda\varphi$; this leads to the following spectral problem for the Laplacian

$$-\nu\Delta\varphi+q(x)\varphi=\lambda\varphi\text{ in }\Omega,\tag{2.15}$$

with a boundary condition that depends on the eigenvalue λ explicitly:

$$-\delta\Delta_{\Gamma}\varphi+\nu\partial_{n}\varphi=\lambda\varphi\text{ on }\Gamma.\tag{2.16}$$

The eigenvalue problem (2.15)-(2.16) can be then expressed in a weak form as

$$a_{\delta}(U,V)=\lambda\langle U,V\rangle_{X_{2}}=\lambda\left(\int_{\Omega}uvdx+\int_{\Gamma}u_{\Gamma}v_{\Gamma}dS\right),\tag{2.17}$$

for all $V=(v,v_{\Gamma})\in\mathcal{W}_{\delta}$, $\delta\geq0$. Such a function U will be called an eigenfunction associated with λ and the set of all eigenvalues λ of (2.15)-(2.16) will be denoted by $\lambda_W^{\nu,\delta}$, $\nu>0$ and $\delta\geq 0$. Let U_j and $\lambda_W^{\nu,\delta}$, $j\in J$, denote all the eigenfunctions and eigenvalues of (2.15)-(2.16). We will show that the index set J is countably infinite. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. From now, we also make the convention that if zero is an eigenvalue then it will be denoted by $\lambda_{X,0}$, X is a given self-adjoint operator.

Concerning the eigenvalue problem (2.15)-(2.16) we then have the following.

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold. Let $q\geq 0$ with $\int_\Omega q(x)\,dx>0$. Then, for each $\delta\geq 0$, the index set $J=\mathbb{N}$, that is, there exists a sequence of numbers*

$$0<\lambda_{W,1}^{\nu,\delta}\leq\lambda_{W,2}^{\nu,\delta}\leq\ldots\leq\lambda_{W,j}^{\nu,\delta}\leq\lambda_{W,j+1}^{\nu,\delta}\leq\ldots,\tag{2.18}$$

converging to $+\infty$, with the following properties:

(a) *The spectrum of $A_W^{\nu,\delta}$ is given by*

$$\sigma(A_W^{\nu,\delta})=\left\{\lambda_{W,j}^{\nu,\delta}\right\}_{j\in J},\ \ \delta\geq 0,\nu>0$$

and each number $\lambda_{W,j}^{\nu,\delta}$, $j\in J$, is an eigenvalue for $A_W^{\nu,\delta}$ of finite multiplicity.

(b) *For each $\delta\geq 0$, there exists a countable family of orthonormal eigenfunctions for $A_W^{\nu,\delta}$ which spans X_2. More precisely, there exists a collection of functions $\{U_j\}_{j\in J}$ with the following properties:*

$$\begin{aligned}U_j\in D(A_W^{\nu,\delta})\ \textit{and}\ A_W^{\nu,\delta}U_j&=\lambda_{W,j}^{\nu,\delta}U_j,\ j\in J,\\ \langle U_j,U_k\rangle_{X_2}&=\delta_{jk},\ j,k\in J,\\ X_2&=\oplus\overline{lin.span\,\{U_j\}_{j\in J}}\ \textit{(orthogonal direct sum).}\end{aligned}\tag{2.19}$$

(c) *If Γ is Lipschitz, then every eigenfunction $U_j\in\mathcal{W}_\delta$ is bounded in $L^\infty(\Omega,dx)\times L^\infty(\Gamma,dS)$ for $\delta\geq 0$, and in fact $U_j=(u_j,u_{\Gamma j})$ belongs to $C\left(\overline{\Omega}\right)\times C(\Gamma)$, $u_j\in C^\infty(\Omega)$, for every j provided that $q\equiv 0$. If Γ is also of class $\mathcal{C}^2$, then every eigenfunction $U_j\in\mathcal{W}_\delta\cap\left(C^2\left(\overline{\Omega}\right)\times C^2(\Gamma)\right)$, for every j.*

(d) *The following min-max principle holds:*

$$\lambda_{W,j}^{\nu,\delta}=\min_{\substack{Y_j^\delta\subset\mathcal{W}_\delta,\\ \dim Y_j^\delta=j}}\ \max_{0\neq U\in Y_j}R_W^\delta(U,U),\ j\in J,\tag{2.20}$$

where the Rayleigh quotient R_W^δ, $\delta\geq 0$, for the Wentzell operators $A_W^{\nu,\delta}$, is given by

$$R_W^\delta(U,U):=\frac{a_\delta(U,U)}{\|U\|_{X_2}^2},\ 0\neq U\in\mathcal{W}_\delta.\tag{2.21}$$

Proof. Let U be an eigenfunction associated with an eigenvalue λ, see (2.15)-(2.16). By definition, we can readily see that $\lambda_W^{\nu,\delta}\subset[0,\infty)$, for each $\delta\geq 0$ and $\nu>0$,

$$\frac{a_\delta(U,U)}{\|U\|_{X_2}^2}\geq 0,\ \text{if}\ q\equiv 0,$$

and $\lambda_W^{\nu,\delta} \subset (0,\infty)$, since

$$\frac{a_\delta(U,U)}{\|U\|_{X_2}^2} > 0, \text{ if } q \geq 0 \text{ with } \int_\Omega q(x)\,dx > 0.$$

Next, recall that $(I + A_W^{\nu,0})^{-1} \in \mathcal{L}(X_2)$, $(I + A_W^{\nu,\delta})^{-1} \in \mathcal{L}(X_2)$ are both self-adjoint and compact in X_2. Thus, the spectrum of $B_W^{\nu,\delta} := (I + A_W^{\nu,\delta})^{-1} \in \mathcal{L}(X_2)$ is given by

$$\sigma\left(B_W^{\nu,\delta}\right) = \left\{\mu_{W,j}^{\nu,\delta}\right\}_{j\in J} := \left\{\left(1 + \lambda_{W,j}^{\nu,\delta}\right)^{-1}\right\}_{j\in J}. \tag{2.22}$$

Now, from the spectral theory of compact, self-adjoint (injective) operators on Hilbert spaces (see, e.g., [37, Theorem 2.36]), it follows that there exists a family of functions $\{U_j\}_{j\in J}$ for which

$$\begin{aligned} U_j \in \mathcal{W}_\delta \text{ and } B_W^{\nu,\delta} U_j &= \mu_{W,j}^{\nu,\delta} U_j, \ j \in J, \\ \langle U_j, U_k\rangle_{X_2} &= \delta_{jk}, \ j,k \in J, \\ V &= \sum_{j=1}^\infty \langle V, U_j\rangle_{X_2} U_j, \ V \in X_2, \end{aligned} \tag{2.23}$$

with convergence in X_2. Obviously,

$$\mu \in \left\{\mu_{W,j}^{\nu,\delta} : j \in J\right\} \Longleftrightarrow \frac{1}{\mu} - 1 \in \lambda_W^{\nu,\delta}.$$

Thus, the set $\lambda_W^{\nu,\delta}$ can be arranged as an increasing sequence of numbers $\left\{\lambda_{W,j}^{\nu,\delta}\right\}_{j\in J}$,

$$\lambda_W^{\nu,\delta} = \left\{\lambda_{W,j}^{\nu,\delta} = \frac{1}{\mu_{W,j}^{\nu,\delta}} - 1 : j \in J\right\}, \tag{2.24}$$

for each $\delta \geq 0$. In the case $q \equiv 0$, we can easily see that $0 = \lambda_{W,0}^{\nu,\delta} \in \lambda_W^{\nu,\delta}$; in fact, $\lambda_{W,0}^{\nu,\delta} = 0$ is a simple eigenvalue, since an eigenfunction U associated with $\lambda_{W,0}^{\nu,\delta}$ is constant, owing to $V = U$ in (2.17). Finally, unraveling notation, (2.19) then readily follow from (2.23). In order to see that (d) holds, recall that $B_W^{\nu,\delta} = (I + A_W^{\nu,\delta})^{-1} \in \mathcal{L}(X_2)$ is a compact operator. Therefore, we can apply the Courant-Fischer principle, to write

$$\mu_{W,j}^{\nu,\delta} = \min_{\substack{Y_j^\delta \subset \mathcal{W}_\delta, \\ \dim Y_j^\delta = j}} \max_{0 \neq U \in Y_j} \frac{\|U\|_{X_2}^2}{a_\delta(U,U) + \|U\|_{X_2}^2}, \ j \in J. \tag{2.25}$$

The statement (d) of the theorem follows easily from (2.24). Finally, each eigenfunction U_j, $j \in J$, belongs to $\mathcal{W}_\delta$, since U_j is also a weak solution of (2.15)-(2.16). In fact each such weak solution $U_j \in X_\infty$ (see, for instance, [23]). By employing a series of bootstrap arguments for elliptic equations with inhomogeneous boundary conditions, the claim (c) also follows. Indeed, the case $\delta = 0$ is classical (see [2]). A variational approach to the elliptic boundary value problem (2.15)-(2.16) in the case $\delta > 0$ can be traced back as far as the work of Agmon et. al. [2], Hörmander [30], Peetre [38] and Visik [42]. These contain results on general elliptic operators with second-order derivatives in the boundary conditions. In this sense (cf. e.g., [42]), the elliptic boundary value problem (2.15)-(2.16) with $\delta > 0$, admits a unique solution $u \in H^{m+2}(\Omega)$, for each $f := \lambda u \in H^m(\Omega)$ and $g := \lambda u \in H^{m-1/2}(\Gamma)$, $m \in \mathbb{N}$, and the following a priori estimate holds:

$$\|u\|_{H^{m+2}(\Omega)} \leq C \left(\|f\|_{H^m(\Omega)} + \|g\|_{H^{m-1/2}(\Gamma)} \right), \tag{2.26}$$

for some $C > 0$ independent of u. It is easy to see that if each weak solution $U_j = (u_j, u_{\Gamma j})$ of problem (2.15)-(2.16) belongs to $\mathcal{W}_\delta$, then (2.26) yields $u_j \in H^3(\Omega)$ and a boot strap argument yields $u_j \in \bigcap_{k \geq 1} H^k(\Omega)$ provided that Γ is smooth enough and $q \equiv 0$. It is also worth mentioning that much of the classical existence theory, including Schauder type estimates $u_j \in C^{2,\varepsilon}(\overline{\Omega})$, $\varepsilon \in (0,1)$, for the linear problem (2.15)-(2.16) (recall that we have set $f = \lambda u$, $g = \lambda u$) was done by Luo and Trudinger in the early 1990s (see [32]). □

Concerning the case $q \equiv 0$, minor adaptations of the foregoing proof yield the following.

Theorem 2.4. *Let the assumptions of Theorem* 2.2 *hold. Let $q \equiv 0$. Then, for each $\delta \geq 0$, the index set $J = \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, that is, there exists a sequence of numbers*

$$0 = \lambda_{W,0}^{\nu,\delta} < \lambda_{W,1}^{\nu,\delta} \leq \lambda_{W,2}^{\nu,\delta} \leq \cdots \leq \lambda_{W,j}^{\nu,\delta} \leq \lambda_{W,j+1}^{\nu,\delta} \leq \cdots,$$

converging to $+\infty$, with the following properties:

(a) *The spectrum of $A_W^{\nu,\delta}$ is given by*

$$\sigma(A_W^{\nu,\delta}) = \left\{ \lambda_{W,j}^{\nu,\delta} \right\}_{j \in J}, \ \delta \geq 0,$$

and each number $\lambda_{W,j}^{\nu,\delta}$, $j \in J$, is an eigenvalue for A_W^δ of finite multiplicity.

(b) *For each $\delta \geq 0$, there exists a countable family of orthonormal eigenfunctions for $A_W^{\nu,\delta}$ which spans X_2. More precisely, the same conclusion* (b) *of Theorem* 2.3 *holds in this case as well. Finally, both conclusions* (c) *and* (d) *in Theorem* 2.3 *hold in the case $q \equiv 0$ as well.*

Remark 2.5. We note that both Theorems 2.3, 2.4 give the orthogonality of the eigenfunctions U_j in terms of the inner product for X_2 (cf. (2.19) above). Here we remark that the eigenfunctions U_j are also orthogonal with respect to the inner product of $\mathcal{W}_\delta$, for each $\delta \geq 0$. In fact, under the assumptions of Theorem 2.3,

$$\{Z_j\}_{j \in \mathbb{N}} := \left\{ U_j \left(\lambda_{W,j}^{\nu,\delta} \right)^{-1/2} \right\}_{j \in \mathbb{N}}$$

is an orthonormal subset of $\mathcal{W}_\delta$, when endowed with the new inner product of $a_\delta(\cdot,\cdot)$. We claim further that $\{Z_j\}_{j \in \mathbb{N}}$ is in fact an orthonormal basis for $\mathcal{W}_\delta$ with this new inner product. To see this, it suffices to verify that

$$a_\delta(U_j, U) = 0, \ j \in \mathbb{N},$$

implies that $U \equiv 0$. But this identity is clearly true, since both identities

$$a_\delta(U_j, U) = \lambda_{W,j}^{\nu,\delta} \langle U_j, U \rangle_{X_2} = 0, \ j \in \mathbb{N},$$

force $U \equiv 0$, as $\{U_j\}_{j \in \mathbb{N}}$ is a basis for X_2. Consequently, we have

$$V = \sum_{j=1}^{\infty} a_\delta(V, Z_j) Z_j, \ V \in \mathcal{W}_\delta,$$

with convergence in $\mathcal{W}_\delta$.

Consider now the map $\Lambda : L^2(\Gamma) \to L^2(\Omega)$, related to the homogeneous Dirichlet problem

$$\begin{cases} -\nu\Delta u + q(x) u = 0 \text{ in } \Omega, \\ u = f \text{ on } \Gamma, \end{cases}$$

for $f \in L^2(\Gamma)$. The map Λ is well-defined, linear and bounded from $L^2(\Gamma)$ (respectively, $H^{1/2}(\Gamma)$) to $L^2(\Omega)$ (respectively, $H^1(\Omega)$). As usual, we define the Dirichlet-to-Neumann operator $N_D^\nu : L^2(\Gamma) \to L^2(\Gamma)$, given by

$$N_D^\nu u = \nu \partial_n (\Lambda u),$$

with domain

$$D(N_D^\nu) = \left\{ u \in L^2(\Gamma) : N_D^\nu u \in L^2(\Gamma) \right\}.$$

The following result is generally known by experts. We include a proof taken from [29, Appendix C] for the sake of completeness.

Theorem 2.6. *Let $0 \leq q \in L^\infty(\Omega)$. The operator N_D^ν with domain $D(N_D^\nu)$ is nonnegative, self-adjoint and $(I + N_D^\nu)^{-1} \in \mathcal{L}\left(L^2(\Gamma)\right)$ is compact.*

Proof. We shall employ the form method when $q \equiv 0$ (the case $q \geq 0$ with $\int_\Omega q(x)\,dx > 0$ is analogous). Define a form on $H^{1/2}(\Gamma)$ by

$$q_N(f, g) := \nu \int_\Omega \nabla(\Lambda f) \cdot \nabla(\Lambda g)\, dx,$$

for all $f, g \in H^{1/2}(\Gamma)$. It is easy to see that q_N is sesquilinear, nonnegative, symmetric and bounded. Moreover, q_N is $L^2(\Gamma)$-elliptic in the sense that for all $\lambda > 0$ there exists a constant $C = C(\lambda)$ such that

$$q_N(f, f) + \lambda \|f\|_{L^2(\Gamma)}^2 \geq C \|f\|_{H^{1/2}(\Gamma)}^2,$$

for all $f \in D(q_N) = H^{1/2}(\Gamma)$. To see this even when $q \equiv 0$, fix $\lambda > 0$. By the Sobolev inequality (i.e., $\|f\|_{L^{2n/(n-2)}(\Omega)}^2 \leq C\left(\|\nabla f\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Gamma)}^2\right)$, $C > 0$; here the inequality is true when $n > 2$, and for $n = 1, 2$, one can take any L^p-norm on the left-hand side), and using the fact that trace$(\Lambda f) = f$, we have

$$\begin{aligned} q_N(f, f) + \lambda \|f\|_{L^2(\Gamma)}^2 &= \|\nabla(\Lambda f)\|_{L^2(\Omega)}^2 + \lambda \|f\|_{L^2(\Gamma)}^2 \\ &\geq C(\Omega, \lambda, \nu) \|\Lambda f\|_{L^{2n/(n-2)}(\Omega)}^2 \\ &\geq C(\Omega, \lambda, \nu) \|\Lambda f\|_{L^2(\Omega)}^2. \end{aligned}$$

In particular,

$$\begin{aligned} q_N(f,f) + \lambda \|f\|^2_{L^2(\Gamma)} &\geq C(\Omega,\lambda,\nu)\|\nabla(\Lambda f)\|^2_{L^2(\Omega)} + \lambda\|\Lambda f\|^2_{L^2(\Omega)} \\ &\geq C(\Omega,\lambda,\nu)\|f\|^2_{H^{1/2}(\Gamma)}, \end{aligned}$$

by the trace theorem. Next, we establish that N_D^ν is the operator associated with the form q_N. That operator, call it $\widetilde{N}_D$, is given by

$$\begin{aligned} &D(\widetilde{N}_D) = \left\{ f \in D(q_N) : \exists h \in L^2(\Gamma),\, q_N(f,g) = (h,g)_{L^2(\Gamma)},\ \forall g \in D(Q) \right\}, \\ &\widetilde{N}_D f = h. \end{aligned}$$

Suppose that $f \in D(N_D^\nu)$. We have $f \in D(q_N) = H^{1/2}(\Gamma)$, which implies that $N_D^\nu f = \nu\partial_n(\Lambda f) \in L^2(\Gamma)$, and $\Lambda f \in H^{3/2}(\Omega)$, by standard elliptic regularity theory. Then for any $g \in D(q_N)$,

$$\begin{aligned} \nu\int_\Omega \nabla(\Lambda f)\cdot\nabla(\Lambda g)\,dx &= \nu\int_\Omega \Lambda f\Delta(\Lambda g)\,dx + \nu\int_\Omega \nabla(\Lambda f)\cdot\nabla(\Lambda g)\,dx \\ &= \nu\int_\Gamma \partial_n(\Lambda f)\,g dS, \end{aligned}$$

that is, $q_N(f,g) = (\partial_n(\Lambda f), g)_{L^2(\Gamma)}$.This shows that if $f \in D(N_D^\nu)$ then $f \in D(\widetilde{N}_D)$ and $N_D^\nu \subset \widetilde{N}_D$ in the sense of operators. For the converse, let $f \in D(\widetilde{N}_D)$, write $\widetilde{N}_D f = h$, and for $v \in H^1(\Omega)$ arbitrary, write $v = u + \Lambda g$, where $u \in H_0^1(\Omega)$ and $g \in H^{1/2}(\Omega)$. Then $\Lambda f \in H^1(\Omega)$ and $\nu\Delta(\Lambda f) = 0$ in the sense of distributions; moreover,

$$\nu\int_\Omega \nabla(\Lambda f)\cdot\nabla u dx = 0,$$

since $\operatorname{trace}(u) = 0$. It follows that

$$\begin{aligned} &\nu\int_\Omega v\Delta(\Lambda f)\,dx + \nu\int_\Omega \nabla(\Lambda f)\cdot\nabla v dx \\ &= \nu\int_\Omega \nabla(\Lambda f)\cdot\nabla u dx + \nu\int_\Omega \nabla(\Lambda f)\cdot\nabla(\Lambda g)\,dx \\ &= \nu\int_\Gamma \operatorname{trace}(v)\,h dS, \end{aligned}$$

for all $v \in H^1(\Omega)$, where in the last step we have used the definition of $\widetilde{N}_D$. By definition, $h = \nu\partial_n(\Lambda f)$, that is, $f \in D(N_D^\nu)$ and $N_D^\nu f = h = \widetilde{N}_D f$. Hence, $N_D^\nu = \widetilde{N}_D$ is associated with the form q_N. Now it follows that N_D^ν has compact resolvent since the form domain $D(q_N) = H^{1/2}(\Gamma)$ embeds compactly into $L^2(\Gamma)$ by the Sobolev embedding theorem. The proof is finished. $\square$

Next, we define the sesquilinear form q_L with form domain $D(q_L) = H^1(\Gamma)$ on the Hilbert space $L^2(\Gamma)$ by

$$q_L(u,v) = \delta\int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma v dS,$$

for $u,v\in D(q_L)$. It is well-known that the operator $C_\delta u := -\delta\Delta_\Gamma u$,

$$u\in D(C_\delta)=\left\{u\in H^1(\Gamma):\delta\Delta_\Gamma u\in L^2(\Gamma)\right\}$$

is self-adjoint, nonnegative, having the opposite operator with compact resolvent, $(I+C_\delta)^{-1}\in\mathcal{L}\left(L^2(\Gamma)\right)$ provided that Γ is Lipschitz continuous and $\delta>0$ (see, e.g., [4, 5, 8, 30, 31]). This follows employing the form method and the surface divergence theorem (2.10). On the other hand, we let $B_\delta := C_\delta + N_D^\nu$ with domain $D(B_\delta)=D(C_\delta)$ if $\delta>0$ and let $B_0\equiv N_D^\nu$ in the operator sense. In the same fashion as in the proof of Theorem 2.6, for $\delta>0$ we have that B_δ is nonnegative, self-adjoint on $L^2(\Gamma)$, with $(I+B_\delta)^{-1}\in\mathcal{L}\left(L^2(\Gamma)\right)$ compact. Indeed, this operator is associated with the sesquilinear form

$$q_{LN}(u,u)\overset{\text{def}}{=}q_L(u,u)+q_N(u,u),\, u\in D(q_L)=H^1(\Gamma).$$

We have the following basic property concerning the two operators $B_0=N_D^\nu$ and $B_\delta=B_0+C_\delta$.

Proposition 2.7. *Let $\nu>0$, $\delta>0$. The form q_N is infinitesimally form bounded with respect to q_L on $L^2(\Gamma)$.*

Proof. By interpolation $\left[H^1(\Gamma),L^2(\Gamma)\right]_{2,1/2}=H^{1/2}(\Gamma)$ and Young's inequality, we have

$$\begin{aligned} q_N(u,u) &= \|f\|^2_{H^{1/2}(\Gamma)} \le C\|f\|_{L^2(\Gamma)}\|f\|_{H^1(\Gamma)}\\ &\le \frac{C}{\varepsilon}\|f\|^2_{L^2(\Gamma)}+\varepsilon\|f\|^2_{H^1(\Gamma)}\\ &= \frac{C}{\varepsilon}\|f\|^2_{L^2(\Gamma)}+\varepsilon q_L(u,u), \end{aligned}$$

for any $\varepsilon>0$, for some $C>0$ independent of ε,u. □

Remark 2.8. If Γ is of $\mathcal{C}^2$-class and $\delta>0$, then N_D^ν is relatively B_δ-bounded with null B_δ-bound owing to the fact that $D(C_\delta)=H^2(\Gamma)$ and interpolation (see, e.g., [29]).

In the remainder of this section we devote our attention to the asymptotic behavior of the eigenvalue sequence $\lambda^{\nu,\delta}_{W,j}$, $j\in J$, of the perturbed and unperturbed Wentzell Laplacians $A^{\nu,\delta}_W$ as well-as the behavior of the eigenvalue sequence associated with the operator B_δ, $\delta\ge 0$. In order to do so, several other self-adjoint versions of the Laplacian, subject to standard boundary conditions, will become important. For a bounded domain Ω with Lipschitz boundary Γ, denote by

$$0<\lambda^\nu_{D,1}\le\lambda^\nu_{D,2}\le\ldots\le\lambda^\nu_{D,j}\le\lambda^\nu_{D,j+1}\le\ldots$$

the collection of the eigenvalues for the Dirichlet Laplacian $A_D=-\nu\Delta_D$ (again, listed according to their multiplicity). Then, if $q\ge 0$, $q\in L^\infty(\Omega)$, we have a known formula (cf., e.g., [11], for $q\equiv 0$),

$$\lambda^\nu_{D,j}=\min_{\substack{Y_j\subset V_0,\\ \dim Y_j^\delta=j}}\ \max_{0\neq U\in Y_j} R_D(U,U),\ j\in J,$$

where $R_D(U,U)$, the Rayleigh quotient for the perturbed Dirichlet Laplacian, is given after a suitable isomorphic identification of

$$u \in H_0^1(\Omega) \simeq V_0 \overset{\text{def}}{=} \left\{ U = (u, u_\Gamma) : u \in H_0^1(\Omega) : u_\Gamma = \text{trace}(u) = 0 \right\},$$

by

$$R_D(U,U) := \frac{\nu \|\nabla u\|^2_{L^2(\Omega)} + \left\| q^{1/2} u \right\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}, \; U \in V_0 \text{ with } u \neq 0. \tag{2.27}$$

Similarly, we denote by

$$0 < \lambda^\nu_{N,1} \leq \lambda^\nu_{N,2} \leq \ldots \leq \lambda^\nu_{N,j} \leq \lambda^\nu_{N,j+1} \leq \ldots \tag{2.28}$$

the collection of the eigenvalues for the perturbed Neumann Laplacian $A_N = -\nu\Delta_N$ (again, listed according to their multiplicity, and which converge to $+\infty$). Then, if $q \geq 0$, $q \in L^\infty(\Omega)$ and since $\mathcal{W}_0$ is topologically isomorphic to $H^1(\Omega)$ in the sense of (2.9), we have a known formula

$$\lambda^\nu_{N,j} = \min_{\substack{Y_j \subset \mathcal{W}_0, \\ \dim Y_j^\delta = j}} \max_{0 \neq U \in Y_j} R_N(U,U), \; j \in J, \tag{2.29}$$

where the Rayleigh quotient $R_N(U,U)$ for the Neumann Laplacian A_N coincides exactly with the right-hand side of (2.27).

The following result shows that the nonzero eigenvalues of the Wentzell Laplacian $A_W^{\nu,\delta}$, $\delta \geq 0$, are at most as large as the corresponding eigenvalues of the Dirichlet and Neumann Laplacian, respectively.

Theorem 2.9. *Let the assumptions of Theorem* 2.2 *hold. Then, the non-zero eigenvalues of the Wentzell Laplacians* $A_W^{\nu,\delta}$ *satisfy*

$$\lambda^{\nu,0}_{W,j} \leq \lambda^{\nu,\delta}_{W,j} \leq \lambda^\nu_{D,j} \text{ and } \lambda^{\nu,0}_{W,j} \leq \lambda^\nu_{N,j}, \text{ for all } j \in J. \tag{2.30}$$

Proof. For $U \in H_0^1(\Omega) \simeq V_0 \subset \mathcal{W}_\delta$, we have

$$\frac{a_\delta(U,U)}{\|U\|^2_{X_2}} \leq R_D(U,U), \tag{2.31}$$

whenever $U = (u, u_\Gamma) \in V_0$ with $0 \neq u \in H_0^1(\Omega)$ (note that $u_\Gamma = 0$ in the trace sense). With this at hand, the second inequality in (2.30) follow from (2.31) and (2.20). The first inequality in (2.30) is a simple consequence of the fact that,

$$\frac{a_0(U,U)}{\|U\|^2_{X_2}} \leq \frac{a_\delta(U,U)}{\|U\|^2_{X_2}},$$

for all $U \in \mathcal{W}_\delta \subset \mathcal{W}_0$ with $U \neq 0$, owing to (2.20). The last inequality in (2.30) is also immediate. $\square$

The next result establishes another upper bound for the eigenvalue sequence $\lambda^{\nu,\delta}_{W,j}$, $j \in J$. But first, we want to recall some known facts. For a bounded domain Ω with Lipschitz continuous boundary Γ, denote by

$$0 < \lambda^{\nu,\delta}_{S,1} \leq \lambda^{\nu,\delta}_{S,2} \leq \ldots \leq \lambda^{\nu,\delta}_{S,j} \leq \lambda^{\nu,\delta}_{S,j+1} \leq \ldots$$

the collection of the eigenvalues for the operators B_δ (again, listed according to their multiplicity). The eigenvalue problem associated with the operator B_δ is to the following *Steklov eigenvalue problem:*

$$\begin{cases} -\nu\Delta\varphi + q(x)\varphi = 0 \text{ in } \Omega, \\ -\delta\Delta_\Gamma\varphi + \nu\partial_n\varphi = \lambda\varphi \text{ on } \Gamma. \end{cases} \tag{2.32}$$

Then, if $q \geq 0$, $q \in L^\infty(\Omega)$, we have (cf., e.g., [40])

$$\lambda_{S,j}^{\nu,\delta} = \min_{\substack{Y_j^\delta \subset \mathcal{W}_\delta, \\ \dim Y_j^\delta = j}} \max_{0 \neq U \in Y_j} R_S^\delta(U,U), \ j \in J, \tag{2.33}$$

where $R_S^\delta(U,U)$, the Rayleigh quotient for the (un)perturbed Steklov operator B_δ, is given by

$$R_S^\delta(U,U) := \frac{\nu \|\nabla u\|_{L^2(\Omega)}^2 + \delta\|\nabla_\Gamma u_\Gamma\|_{L^2(\Gamma)}^2 + \left\|q^{1/2}u\right\|_{L^2(\Omega)}^2}{\|u_\Gamma\|_{L^2(\Gamma)}^2},$$

for all $U \in \mathcal{W}_\delta$, $U \neq 0$ such that $\nu > 0$ and $\delta \geq 0$.

Theorem 2.10. *Under the assumptions of Theorem* 2.9*, for each fixed $\nu > 0$ and $\delta \geq 0$ the non-zero eigenvalues of the Wentzell Laplacian $A_W^{\nu,\delta}$ satisfy*

$$\lambda_{W,j}^{\nu,\delta} \leq \lambda_{S,j}^{\nu,\delta}, \ \textit{for all } j \in J. \tag{2.34}$$

Moreover, $\lambda_{S,j}^{\nu,0} \leq \lambda_{S,j}^{\nu,\delta}$ for all $j \in J$.

Proof. Indeed, the proof follows from (2.20) and (2.33), and the fact

$$R_W^\delta(U,U) \leq R_S^\delta(U,U),$$

for all $0 \neq U \in \mathcal{W}_\delta$. The final claim is also immediate on account of the variational characterization (2.33) and the fact that $\mathcal{W}_\delta \subset \mathcal{W}_0$ for $\delta > 0$. □

We will now study the asymptotic behavior of the eigenvalue sequence $\lambda_{W,j}^{\nu,\delta}$, $j \in J$, in detail. Let us recall a classical result concerning the asymptotic behavior of the eigenvalues $\left\{\lambda_{D,j}^\nu\right\}_{j\in J}$ for the Dirichlet operator $A_D = -\nu\Delta_D$. It is well known (see, e.g., [3]), for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with Lipschitz boundary Γ, that

$$\lambda_{D,j}^\nu = \nu C_D(\Omega) j^{2/n} + o\left(j^{2/n}\right), \text{ as } j \to +\infty, \tag{2.35}$$

where

$$C_D(\Omega) \stackrel{\text{def}}{=} \frac{(2\pi)^2}{(v_n|\Omega|)^{2/n}}.$$

Here v_n denotes the volume of the unit ball in $\mathbb{R}^n$, and we recall that $|\Omega|$ stands for the n-dimensional Lebesgue measure of Ω. From Theorem 2.9, one might expect an analogous asymptotic behavior for the eigenvalues $\lambda_{W,j}^{\nu,0}$ of the Wentzell Laplacian $A_W^{\nu,0}$. But this turns out to be true only in one space dimension; in dimensions $n \geq 2$, the eigenvalues $\lambda_{W,j}^{\nu,0}$ grow like $j^{1/(n-1)}$. By Theorem 2.10, this growth coincides with the growth order of the unperturbed ($\delta = 0$) Steklov eigenvalues $\lambda_{S,j}^{\nu,0}$, $j \in J$. The eigenvalues for the Dirichlet-to-Neumann map $B_0 = D_N^\nu = \nu\partial_n(\Lambda)$ behave according to the following asymptotic formula:

$$\lambda_{S,j}^{\nu,0} = \nu C_S(\Gamma)\, j^{1/(n-1)} + o\left(j^{1/(n-1)}\right), \text{ as } j \to +\infty, \tag{2.36}$$

where the Steklov constant $C_S(\Gamma)$ is defined as

$$C_S(\Gamma) \stackrel{\text{def}}{=} \frac{2\pi}{\left(v_{n-1}|\Gamma|\right)^{1/(n-1)}},$$

where we recall that $|\Gamma| := S(\Gamma)$ stands for the usual $(n-1)$-dimensional Lebesgue surface measure on Γ (see [40, Section 4]).

The eigenvalue sequence associated with $B_\delta \equiv C_\delta + B_0$, $\delta > 0$ has the following asymptotic behavior.

Theorem 2.11. *Let $\nu, \delta > 0$ be fixed. The eigenvalue sequence $\lambda_{S,j}^{\nu,\delta}$ obeys the following Weyl law:*

$$\lambda_{S,j}^{\nu,\delta} = \delta\widetilde{C}_S(\Gamma)\, j^{2/(n-1)} + o(j^{2/(n-1)}), \text{ as } j \to \infty, \tag{2.37}$$

where

$$\widetilde{C}_S(\Gamma) \stackrel{\text{def}}{=} \frac{(2\pi)^2}{\left(v_{n-1}|\Gamma|\right)^{2/(n-1)}}.$$

Proof. We know how the spectrum of C_δ for $\delta > 0$ behaves asymptotically. This is the classical result due to Hörmander [31]. We have

$$\lambda_{C_\delta,j} = \delta\widetilde{C}_S(\Gamma)\, j^{2/(n-1)} + o\left(j^{2/(n-1)}\right), \text{ as } j \to \infty$$

while $\lambda_{B_0,j} = \lambda_{S,j}^{\nu,0}$ obeys the Weyl-like law (2.36). The statement of Theorem 2.11 follows then from Proposition 2.7 and Lemma 2.1. □

In the case $\delta = 0$, we have the following result for $\lambda_{W,j}^{\nu,0}$ (cf., [16]).

Theorem 2.12. *Let the assumptions of Theorem 2.2 hold. The eigenvalue sequence $\left\{\lambda_{W,j}^{\nu,0}\right\}_{j\in J}$ of the Wentzell Laplacian $A_W^{\nu,0}$ satisfies:*

(i) *For $n \geq 2$, we have*

$$\lambda_{W,j}^{\nu,0} = \nu C_W(\Omega,\Gamma)\, j^{1/(n-1)} + o\left(j^{1/(n-1)}\right), \text{ as } j \to +\infty, \tag{2.38}$$

for some

$$C_W(\Omega,\Gamma) \in \begin{cases} C_S(\Gamma)\left[2^{-1/(n-1)}, 1\right], & \text{for } n \geq 3, \\ \left[\frac{C_D(\Omega)C_S(\Gamma)}{2(C_D(\Omega)+C_S(\Gamma))}, \min\left\{C_D(\Omega), C_S(\Gamma)\right\}\right], & \text{for } n = 2. \end{cases} \tag{2.39}$$

(ii) *For $n = 1$, we have*

$$\lambda_{W,j}^{\nu,0} = \nu C_D(\Omega)\, j^2 + o\left(j^2\right), \text{ as } j \to +\infty. \tag{2.40}$$

Remark 2.13. In the case $n = 1$, $A_W^{\nu,0} = A_W^{\nu,\delta}$ in the operator sense, since the Laplace-Beltrami operator Δ_Γ does not appear in the boundary condition (2.16).

It remains to investigate the asymptotic behavior of the eigenvalue sequence associated with the Wentzell Laplacian $A_W^{\nu,\delta}$ for $\delta > 0$ and in any space dimension $n \geq 2$. It turns out that the asymptotic behavior of the eigenvalue sequences associated with C_δ and a *"gentle"* perturbation of the classical Neumann Laplacian operator for $\delta > 0$ is crucial. For the latter, the perturbation occurs in the homogeneous boundary condition. That is, for a bounded domain Ω with Lipschitz boundary Γ, denote by

$$0 < \lambda_{N,1}^{\nu,\delta} \leq \lambda_{N,2}^{\nu,\delta} \leq \ldots \leq \lambda_{N,j}^{\nu,\delta} \leq \lambda_{N,j+1}^{\nu,\delta} \leq \ldots$$

the collection of the eigenvalues for the *perturbed* ($\delta > 0$) Neumann Laplacian A_N^δ (again, listed according to their multiplicity). More precisely, the eigenvalue problem associated with this operator is

$$\begin{cases} -\nu\Delta\varphi + q(x)\varphi = \lambda\varphi \text{ in } \Omega, \\ -\delta\Delta_\Gamma\varphi + \nu\partial_n\varphi = 0 \text{ on } \Gamma, \end{cases} \tag{2.41}$$

where $q \geq 0$, $q \in L^\infty(\Omega)$. In particular, by (2.28) we note that $\lambda_{N,j}^{\nu,0} \equiv \lambda_{N,j}^{\nu}$ for all $j \in J$. With the domain $D\left(A_N^\delta\right)$, consisting of functions $U \in \mathcal{W}_\delta$, which satisfy $\Delta u \in L^2(\Omega)$, $\delta\Delta u_\Gamma \in L^2(\Gamma)$ and the boundary condition $-\delta\Delta_\Gamma\varphi + \nu\partial_n\varphi = 0$ on Γ, $A_N^\delta = -\nu\Delta$ is nonnegative, selfadjoint on $L^2(\Omega)$ and with compact resolvent $\left(I + A_N^\delta\right)^{-1} \in \mathcal{L}\left(L^2(\Omega)\right)$. This operator is naturally associated with the sesquilinear form $a_\delta(U,U)$, $U \in D(a_\delta) = \mathcal{W}_\delta$. Moreover, analogous to the proof of Theorem 2.3 we have

$$\lambda_{N,j}^{\nu,\delta} = \min_{\substack{Y_j^\delta \subset \mathcal{W}_\delta, \\ \dim Y_j^\delta = j}} \max_{0 \neq U \in Y_j} R_N^\delta(U,U), \ j \in J, \tag{2.42}$$

where $R_N^\delta(U,U)$, the Rayleigh quotient for the operator A_N^δ, is given by

$$R_N^\delta(U,U) := \frac{\nu\|\nabla u\|_{L^2(\Omega)}^2 + \delta\|\nabla_\Gamma u_\Gamma\|_{L^2(\Gamma)}^2 + \left\|q^{1/2}u\right\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2},$$

for all $U \in \mathcal{W}_\delta$, $U \neq 0$ such that $\nu > 0$ and $\delta \geq 0$.

We have the following basic comparison result for the eigenvalue sequence $\lambda_{N,j}^{\nu,\delta}$. For this result, we assume that $q \equiv 0$ without loss of generality (so that $\lambda_{N,0}^{\nu,\delta} = 0$ is an eigenvalue of (2.41); in this case, the eigenvalue sequence is arranged as $0 = \lambda_{N,0}^{\nu,\delta} < \lambda_{N,1}^{\nu,\delta} \leq \lambda_{N,2}^{\nu,\delta} \leq \ldots \leq \lambda_{N,j}^{\nu,\delta} \leq \lambda_{N,j+1}^{\nu,\delta} \leq \ldots$, and converges to $+\infty$).

Lemma 2.14. *Let $\nu > 0$ and $\delta > 0$ be fixed. There holds $\lambda_{N,j}^{\nu,0} = \lambda_{N,j}^{\nu} \leq \lambda_{N,j}^{\nu,\delta}$, for all $j \in \mathbb{N}_0$, and $\lambda_{N,j-1}^{\nu,\delta} \leq \lambda_{D,j}^{\nu}$, for all $j \in \mathbb{N}$. Moreover, we have the following Weyl law*

$$\lambda_{N,j}^{\nu,\delta} = \nu C_D(\Omega)\, j^{2/n} + o(j^{2/n}), \ \text{as } j \to \infty. \tag{2.43}$$

Proof. The first inequality follows from the fact that $R_N^0(U,U) = R_N(U,U) \leq R_N^\delta(U,U)$, for all $0 \neq U \in \mathcal{W}_\delta$, and the variational characterizations (2.42), (2.29). The second inequality is also a consequence of the min-max characterization of the corresponding eigenvalue problems since $H_0^1(\Omega) \simeq V_0 \subset \mathcal{W}_\delta$ for $\delta > 0$. The final claim then follows from this comparison and the fact that each one of eigenvalue sequences $\lambda_{N,j}^{\nu}$ and $\lambda_{D,j}^{\nu}$, respectively, obeys the same Weyl asymptotic law (2.35). □

Remark 2.15. The result of Lemma 2.14 carries to the case $q \geq 0$ with $\int_{\Omega} q(x)\,dx > 0$ with minor modifications. Thus the behavior of the eigenvalue sequences associated with the perturbed and unperturbed operators A_N^{δ} is the same.

We are now ready to give the full asymptotic behavior for eigenvalue sequences associated with the perturbed Wentzell Laplacian $A_W^{\nu,\delta}$ in the case $\delta > 0$.

Theorem 2.16. *Let the assumptions of Theorem* 2.2 *hold. The eigenvalue sequence* $\left\{\lambda_{W,j}^{\nu,\delta}\right\}_{j\in J}$ *of the Wentzell Laplacian* $A_W^{\nu,\delta}$ *satisfies*

$$\lambda_{W,j}^{\nu,\delta} = \nu\widetilde{C}_W(\Omega)\, j^{2/n} + o\left(j^{2/n}\right), \ \text{as } j \to +\infty, \tag{2.44}$$

for some

$$\widetilde{C}_W(\Omega) \in \left[2^{-2/n} C_D(\Omega), C_D(\Omega)\right].$$

Proof. Fix $\nu, \delta > 0$. We first observe that by Theorem 2.9 we have

$$\limsup_{j\to\infty} \frac{\lambda_{W,j}^{\nu,\delta}}{j^{2/n}} \leq \nu C_D(\Omega). \tag{2.45}$$

In order to determine a lower bound for $\left\{\lambda_{W,j}^{\nu,\delta}\right\}_{j\in J}$, we use the variational formulation in the statement of Theorem 2.3 (cf. also Theorem 2.4). Indeed, we notice that the quadratic form for the inverse of the Rayleigh quotient for $I + A_W^{\nu,\delta}$ is given by

$$\begin{aligned}
&\frac{\|U\|_{X_2}^2}{a_\delta(U,U) + \|U\|_{X_2}^2} \\
&\leq \frac{\|u\|_{L^2(\Omega)}^2}{\nu\|\nabla u\|_{L^2(\Omega)}^2 + \delta\|\nabla_\Gamma u_\Gamma\|_{L^2(\Gamma)}^2 + \left\|q^{1/2}u\right\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2} \\
&\quad + \frac{\|u_\Gamma\|_{L^2(\Gamma)}^2}{\nu\|\nabla u\|_{L^2(\Omega)}^2 + \delta\|\nabla_\Gamma u_\Gamma\|_{L^2(\Gamma)}^2 + \left\|q^{1/2}u\right\|_{L^2(\Omega)}^2 + \|u_\Gamma\|_{L^2(\Gamma)}^2},
\end{aligned} \tag{2.46}$$

for all $U = (u, u_\Gamma) \in \mathcal{W}_\delta$, such that $U \neq 0$. In particular, from (2.46) we observe that we can estimate this form in terms of the quadratic forms for the inverses of the Rayleigh quotient for $I + A_N^{\delta}$ and $I + B_\delta$, respectively. Since the variation of these Rayleigh quotients take place in the same space $\mathcal{W}_\delta$, we see that the left-hand side of (2.46) can be estimated in terms of the compact operator $\left(I + A_N^{\delta}\right)^{-1} + (I + B_\delta)^{-1}$. Thus, by a well known spectral estimate for sums of compact operators (see, e.g., [12]), we have

$$\mu_{W,j+m}^{\nu,\delta} = \frac{1}{1 + \lambda_{W,j+m}^{\nu,\delta}} \leq \frac{1}{1 + \lambda_{N,j}^{\nu,\delta}} + \frac{1}{1 + \lambda_{S,m}^{\nu,\delta}},$$

for all $j, m \in J$. This implies for $j \in J$,

$$\lambda_{W,2j}^{\nu,\delta} \geq \frac{\left(1+\lambda_{N,j}^{\nu,\delta}\right)\left(1+\lambda_{S,j}^{\nu,\delta}\right)}{2+\lambda_{N,j}^{\nu,\delta}+\lambda_{S,j}^{\nu,\delta}} - 1$$

and

$$\lambda_{W,2j+1}^{\nu,\delta} \geq \frac{\left(1+\lambda_{N,j+1}^{\nu,\delta}\right)\left(1+\lambda_{S,j}^{\nu,\delta}\right)}{2+\lambda_{N,j+1}^{\nu,\delta}+\lambda_{S,j}^{\nu,\delta}} - 1,$$

which yields for all $j \in J$,

$$\frac{\lambda_{W,2j}^{\nu,\delta}}{(2j)^{2/n}} \geq \frac{1}{2^{2/n}} \frac{\left[\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+1\right]\left(j^{-2/n}+\lambda_{S,j}^{\nu,\delta} j^{-2/n}\right)}{2\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+\lambda_{S,j}^{\nu,\delta}\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+1} - \frac{1}{(2j)^{2/n}} \tag{2.47}$$

and

$$\begin{aligned}\frac{\lambda_{W,2j+1}^{\nu,\delta}}{(2j+1)^{2/n}} \geq{}& \frac{\left[\left(\lambda_{N,j+1}^{\nu,\delta}\right)^{-1}+1\right]\left(j^{-2/n}+\lambda_{S,j}^{\nu,\delta} j^{-2/n}\right)}{(2+j^{-1})^{2/n}\left[2\left(\lambda_{N,j+1}^{\nu,\delta}\right)^{-1}+\lambda_{S,j}^{\nu,\delta}\left(\lambda_{N,j+1}^{\nu,\delta}\right)^{-1}+1\right]} \\ &- \frac{1}{(2j+1)^{2/n}}.\end{aligned} \tag{2.48}$$

Both the right-hand sides of the preceding inequalities have the same limit as j goes to infinity, and this limit equals precisely $2^{-2/n}\nu C_D(\Omega)$. Indeed, setting

$$\begin{aligned} Q_j &\overset{\text{def}}{=} \frac{\lambda_{S,j}^{\nu,\delta} j^{-2/n}}{2\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+\lambda_{S,j}^{\nu,\delta}\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+1}, \\ \widetilde{Q}_j &\overset{\text{def}}{=} \frac{j^{-2/n}}{2\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+\lambda_{S,j}^{\nu,\delta}\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+1}, \end{aligned}$$

we have

$$\frac{\lambda_{W,2j}^{\nu,\delta}}{(2j)^{2/n}} \geq \frac{1}{2^{2/n}}\left(\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1}+1\right)\left(Q_j+\widetilde{Q}_j\right) - \frac{1}{(2j)^{2/n}}. \tag{2.49}$$

Exploiting to the fact that $\lambda_{N,j}^{\nu,\delta} \to +\infty$, $\lambda_{S,j}^{\nu,\delta} \to +\infty$, as $j \to \infty$, together with the asymptotic laws (2.37), (2.43) and

$$\lim_{j\to\infty} j^{2/n}\left(\lambda_{N,j+1}^{\nu,\delta}\right)^{-1} = \lim_{j\to\infty} j^{2/n}\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1} = \left(\nu C_D(\Omega)\right)^{-1},$$

we see that

$$Q_j = \frac{1}{2\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1} j^{2/n}\left(\lambda_{S,j}^{\nu,\delta}\right)^{-1}+\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1} j^{2/n}+\left(\lambda_{S,j}^{\nu,\delta}\right)^{-1} j^{2/n}} \to \nu C_D(\Omega)$$

as $j \to \infty$ while

$$\widetilde{Q}_j = \frac{1}{2\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1} j^{2/n} + \lambda_{S,j}^{\nu,\delta}\left(\lambda_{N,j}^{\nu,\delta}\right)^{-1} j^{2/n} + j^{2/n}} \to 0.$$

Thus, from (2.49) we obtain

$$\liminf_{j\to\infty} \frac{\lambda_{W,2j}^{\nu,\delta}}{(2j)^{2/n}} \geq \frac{\nu C_D(\Omega)}{2^{2/n}}. \tag{2.50}$$

Analogously for the subsequence $\lambda_{W,2j+1}^{\nu,\delta}$ we deduce

$$\liminf_{j\to\infty} \frac{\lambda_{W,2j+1}^{\nu,\delta}}{(2j+1)^{2/n}} \geq \frac{\nu C_D(\Omega)}{2^{2/n}}. \tag{2.51}$$

Thus the claim (2.44) follows from (2.50)-(2.51) and (2.45). □

3. Parabolic equations with dynamic boundary conditions

We consider the parabolic equation

$$\partial_t u = \nu\Delta u - f(u) + \lambda u, \ \ (t,x) \in (0,+\infty)\times\Omega, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with boundary Γ of class $\mathcal{C}^2$ and ν, λ are positive constants. The function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be $C^{1,1}_{\text{loc}}$ and satisfies

$$f^{'}(y) \geq -c_f, \text{ for all } y \in \mathbb{R}, \text{ for some } c_f > 0. \tag{3.2}$$

We recall that (3.1) is subject to dynamic boundary conditions of *pure-reactive* ($\delta = 0$) and *reactive-diffusive* type ($\delta > 0$), of the form

$$\partial_t u - \delta\Delta_\Gamma u + \nu\partial_{\mathbf{n}} u = 0, \quad \text{on } (0,\infty)\times\Gamma. \tag{3.3}$$

Our main goal in this section is to investigate the dependence in $\delta \geq 0$ of the dimension of the global attractor for the system (3.1)-(3.3). But first, we briefly explain how to adapt the results of [17] to prove that the system (3.1), (3.3) generates a dynamical system on X_2, possessing a finite dimensional global attractor $\mathcal{G}_W^{\nu,\delta}$. We begin by assuming that, in addition to (3.2), the following condition for f holds:

$$\eta_1 |y|^p - C_f \leq f(y)\, y \leq \eta_2 |y|^p + C_f, \tag{3.4}$$

for some η_1, $\eta_2 > 0$, $C_f \geq 0$ and $p > 2$.

We have the following rigorous notion of weak solution to (3.1), (3.3), with initial condition $u(0) = u_0$, as in [19] (cf. also [17, 18]).

Definition 3.1. *Let $\delta \geq 0$. The pair $U(t) = (u(t), u_\Gamma(t))$ is said to be a weak solution if $u_\Gamma(t)$ =trace(u) for almost all $t \in (0,T)$, for any $T > 0$, and U fulfills*

$$\begin{aligned} &U \in C([0,T]; X_2) \cap L^2(0,T;\mathcal{W}_\delta), \ \ u \in L^p(\Omega\times(0,T)), \qquad (3.5)\\ &u \in H^1_{loc}(0,\infty; L^2(\Omega)), \ \ u_\Gamma \in H^1_{loc}(0,\infty; L^2(\Gamma)),\\ &\partial_t U \in L^2(0,T;\mathcal{W}_\delta^* + X_q), \end{aligned}$$

such that the identity

$$\langle \partial_t U, \Xi \rangle_{X_2} + \nu \langle \nabla u, \nabla \sigma \rangle_{L^2(\Omega)} + \delta \langle \nabla_\Gamma u_\Gamma, \nabla_\Gamma \sigma_\Gamma \rangle_{L^2(\Gamma)} + \langle f(u) - \lambda u, \sigma \rangle_{L^2(\Omega)} = 0,$$

holds almost everywhere in $(0,T)$, *for all* $\Xi = (\sigma, \sigma_\Gamma) \in \mathcal{W}_\delta$, $\sigma \in L^p(\Omega)$. *Here* q *denotes the dual conjugate of* p, $1/q + 1/p = 1$. *Moreover, we have, in the space* X_2,

$$U(0) = (u_0, v_0) =: U_0, \tag{3.6}$$

where $u(0) = u_0$ *almost everywhere in* Ω, *and* $u_\Gamma(0) = v_0$ *almost everywhere in* Γ. *Note that in this setting,* v_0 *need not be the trace of* u_0 *at the boundary.*

The following result is a direct consequence of results contained in [17, 18] (cf. also [20, 21]). Indeed, the linear term $\delta \Delta_\Gamma u$ in the boundary condition (3.3) is coercive in the sense

$$\begin{aligned} -\delta \left\langle \Delta_\Gamma u_\Gamma, |u|^{r-1} u \right\rangle_{L^2(\Gamma)} &= \delta \left\langle \nabla_\Gamma u_\Gamma, \nabla_\Gamma \left(|u|^{r-1} u \right) \right\rangle_{L^2(\Gamma)} \\ &= \frac{4\delta r}{(r+1)^2} \left\| \nabla_\Gamma |u_\Gamma|^{\frac{r+1}{2}} \right\|^2_{L^2(\Gamma)}, \end{aligned} \tag{3.7}$$

for any $r \geq 1$, so that mathematically speaking this term is of no real significance to the energy estimates and only enhances the boundary regularity of the solution.

Theorem 3.2. *Let the assumptions of* (3.2), (3.4) *be satisfied. For any given initial data* $U_0 \in X_2$, *the problem* (3.1), (3.3), (3.6) *has a unique weak solution* U *in the sense of Definition* 3.1 *which depends continuously on the initial data in a Lipschitz way. Moreover, this problem defines a (nonlinear) continuous semigroup* $\mathcal{S}_t^{\nu,\delta}$ *acting on the phase-space* X_2,

$$\mathcal{S}_t^{\nu,\delta} : X_2 \to X_2, \ t \geq 0,$$

given by

$$\mathcal{S}_t^{\nu,\delta} U_0 = U(t).$$

Next, we first set $\mathbb{V}_0 = (H^2(\Omega) \times H^{3/2}(\Gamma)) \cap \mathcal{W}_0$ and $\mathbb{V}_\delta = (H^2(\Omega) \times H^2(\Gamma)) \cap \mathcal{W}_\delta$ for $\delta > 0$. It follows from the proof of [17, Theorem 2.3] (cf. also [18, Section 3.3]) and the elementary observation (3.7) that we have the following.

Theorem 3.3. *Let* f *satisfy assumptions* (3.2), (3.4) *and let* $\nu > 0$, $\delta \geq 0$. *Then,* $\mathcal{S}_t^{\nu,\delta}$ *possesses a connected global attractor* $\mathcal{G}_W^{\nu,\delta}$, *which is a bounded subset of* $\mathbb{V}_\delta \cap X_\infty$. *Moreover,* $\mathcal{S}_t^{\nu,\delta}$ *is uniformly differentiable on* $\mathcal{G}_W^{\nu,\delta}$ *with differential*

$$\mathbf{L}(t, U(t)) : \Theta = (\xi_1, \xi_2) \in X_2 \mapsto V = (v, v_\Gamma) \in X_2, \tag{3.8}$$

where V *is the unique solution to*

$$\begin{aligned} &\partial_t v = \nu \Delta v - f'(u(t)) v + \lambda v, \ \ (\partial_t v_\Gamma - \delta \Delta_\Gamma v_\Gamma + \nu \partial_n v)_{|\Gamma} = 0, \\ &V(0) = \Theta. \end{aligned} \tag{3.9}$$

Finally, $\mathbf{L}(t, U(t))$ *is compact for all* $t > 0$.

Proof. The existence of an absorbing set in $\mathcal{W}_\delta \cap L^p(\Omega)$ and, hence, the existence of the global attractor $\mathcal{G}_W^{\nu,\delta} \subset \mathcal{W}_\delta$ follows exactly as in the proofs of [19, Theorem 2.8 and Corollary 3.11], owing to (3.7). The boundedness of $\mathcal{G}_W^{\nu,\delta}$ in X_∞ is also a consequence of the proof of [17, Theorem 2.3] and (3.7) since the Laplace-Beltrami operator Δ_Γ is coercive. Indeed, there holds

$$\sup_{t\geq t_0} \|U(t)\|_{X_\infty} \leq C_*, \tag{3.10}$$

for some positive constant C_* independent of t, U and initial data, and some positive time $t_0 = t_0(\|U_0\|_{X_2})$. We briefly explain the reason why $\mathcal{G}_W^{\nu,\delta}$ is bounded in $\mathbb{V}_\delta$ for $\delta > 0$ (The case $\delta = 0$ is essentially different and is contained in [17, Proposition 2.6]; in fact, the case $\delta > 0$ is simpler). We claim that there is a positive constant C_1, independent of time and initial data, and there exists $\tau_0 = \tau_0(t_0) > 0$ such that

$$\|U(t)\|_{\mathbb{V}_\delta} \leq C_1, \qquad \text{for all } t \geq \tau_0. \tag{3.11}$$

Before we prove (3.11), let us recall the following estimate:

$$\begin{aligned} &\sup_{t\geq\tau_0}\left(\|U(t)\|^2_{\mathcal{W}_\delta} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t u_\Gamma(t)\|^2_{L^2(\Gamma)}\right) \\ &+ \sup_{t\geq\tau_0}\int_t^{t+1}\left(\nu\|\nabla\partial_t u(s)\|^2_{L^2(\Omega)} + \delta\|\nabla_\Gamma\partial_t u_\Gamma(s)\|^2_{L^2(\Gamma)}\right) ds \\ &\leq C_2, \end{aligned} \tag{3.12}$$

for some positive constant C_2 that is independent of time and the initial data (see [19, Theorems 3.5, 3.10]). We recall that in order to deduce (3.12), it suffices to differentiate (3.1) and (3.3) with respect to time and to exploit the uniform estimate (3.10). For $\delta > 0$, we observe that U is also a strong solution of the elliptic boundary value problem

$$\begin{cases} \nu\Delta u = j_1 \overset{\text{def}}{=} \partial_t u + f(u) - \lambda u, & \text{in } \Omega\times(\tau_0,\infty), \\ \delta\Delta_\Gamma u_\Gamma - \nu\partial_n(u) = j_2 \overset{\text{def}}{=} \partial_t u, & \text{on } \Gamma\times(\tau_0,\infty). \end{cases}$$

Since $j_1 \in L^\infty(\tau_0,\infty;L^2(\Omega))$ and $j_2 \in L^\infty(\tau_0,\infty;L^2(\Gamma))$ owing to (3.10) and (3.12), we can now apply elliptic regularity (e.g., [20, Lemma 2.2]) to infer that $U \in L^\infty(\tau_0,\infty;\mathbb{V}_\delta)$. This yields the first claim of the theorem. The uniform differentiability of $\mathcal{S}_t^{\nu,\delta}$ on $\mathcal{G}_W^{\nu,\delta}$ is also a consequence of the boundedness of $\mathcal{G}_W^{\nu,\delta}$ into $\mathbb{V}_\delta \cap X_2$ and [17, Proposition 2.6] (see also [6]). $\square$

Even though surface diffusion has no qualitative impact on the energy estimates for problem (3.1)–(3.3), (3.6), it does have a significant qualitative impact from the *dynamical* point of view.

Theorem 3.4. *Let the assumptions of Theorem* 3.3 *be satisfied.*

(i) **Pure-reactive ($\delta = 0$) dynamic boundary conditions.** *The fractal dimension of* $\mathcal{G}_W^{\nu,0}$ *admits the two-sided estimate*

$$c_0\left(\frac{\lambda}{C_W(\Omega,\Gamma)\nu}\right)^{n-1} \leq \dim_F(\mathcal{G}_W^{\nu,0}, X_2) \leq c_0\left(1 + \frac{c_f+\lambda}{C_W(\Omega,\Gamma)\nu}\right)^{n-1}. \tag{3.13}$$

(ii) **Reactive-diffusive ($\delta > 0$) dynamic boundary conditions.** *The fractal dimension of $\mathcal{G}_W^{\nu,\delta}$ admits the two-sided estimate*

$$c_0 \left(\frac{\lambda}{\widetilde{C}_W (\Omega) \nu} \right)^{\frac{n}{2}} \leq \dim_F (\mathcal{G}_W^{\nu,\delta}, X_2) \leq c_0 \left(1 + \frac{c_f + \lambda}{\widetilde{C}_W (\Omega) \nu} \right)^{\frac{n}{2}}. \tag{3.14}$$

Here c_0 depends on the shape of Ω and $n \geq 2$ only, and the positive constants C_W, $\widetilde{C}_W$ depend only on n, $|\Omega|$, $|\Gamma|$ and are given in Section 2.

Proof. The case (i) is proved in [17, Theorem 2.7 and Theorem 3.1] while the case (ii) follows in the same fashion after some minor modifications. The crucial piece of information is found in the asymptotic behavior of the eigenvalue sequence associated with the perturbed Wentzell Laplacian $A_W^{\nu,\delta}$ (see Theorem 2.16). Following the same procedure in [17, Theorem 2.7], we consider $\mathcal{S}_t^{\nu,\delta} U_0 = U(t)$, U is the solution of (3.1)-(3.3), (3.6), $U_1, \ldots, U_m$ are m solutions of (3.8)-(3.9) corresponding to $\Theta_1, \ldots, \Theta_m$ and let Q_m be the orthogonal projector in X_2 onto the space spanned by $U_1, \ldots, U_m$. At any given time t, let now $\varphi_j = \varphi_j(t)$, $j \in \mathbb{N}$ be an orthonormal basis in X_2 with $\varphi_1, \ldots, \varphi_m$ spanning $Q_m X_2 =$Span$(U_1, \ldots, U_m)$, with $\varphi_j \in \mathcal{W}_\delta$. We have

$$\begin{aligned}
\operatorname{Tr}(\mathbf{L}(t, U(t)) Q_m) &= \sum_{j=1}^{m} \left\langle \mathbf{L}(t, U(t)) \varphi_j, \varphi_j \right\rangle_{X_2} \\
&= -\nu \sum_{j=1}^{m} \left\| \nabla \varphi_j \right\|_{L^2(\Omega)}^2 - \delta \sum_{j=1}^{m} \left\| \nabla_\Gamma \varphi_j \right\|_{L^2(\Gamma)}^2 \\
&\quad - \sum_{j=1}^{m} \left\langle f'(u(t)) \varphi_j, \varphi_j \right\rangle_{L^2(\Omega)} + \sum_{j=1}^{m} \lambda \left\langle \varphi_j, \varphi_j \right\rangle_{L^2(\Omega)}.
\end{aligned}$$

Using assumption (3.2) on f (i.e., $f'(y) \geq -c_f$, for all $y \in \mathbb{R}$), we find

$$\operatorname{Tr}(\mathbf{L}(t, U) Q_m) \leq -\nu \sum_{j=1}^{m} \left\| \nabla \varphi_j \right\|_{L^2(\Omega)}^2 - \delta \sum_{j=1}^{m} \left\| \nabla_\Gamma \varphi_j \right\|_{L^2(\Gamma)}^2 + (c_f + \lambda) m.$$

Arguing in a similar fashion as in the proof of [17, Proposition 5.5], we obtain

$$\begin{aligned}
\operatorname{Tr}(\mathbf{L}(t, U) Q_m) &\leq -\nu c_1 \widetilde{C}_W m^{\frac{2}{n}+1} + \left(c_1 \nu \widetilde{C}_W + c_f + \lambda \right) m \\
&=: \rho(m),
\end{aligned}$$

since for the perturbed Wentzell Laplacian $A_W^{\nu,\delta}$ we have

$$\nu \sum_{j=1}^{m} \left\| \nabla \varphi_j \right\|_{L^2(\Omega)}^2 + \delta \sum_{j=1}^{m} \left\| \nabla_\Gamma \varphi_j \right\|_{L^2(\Gamma)}^2 \geq c_1 \widetilde{C}_W (\Omega) \nu \left(m^{\frac{2}{n}+1} - m \right),$$

for some $c_1 > 0$ depending only on the shape of Ω and n. The function $\rho(y)$ is concave. The root of the equation $\rho(d) = 0$ is

$$d^* = \left(1 + \frac{c_f + \lambda}{\nu c_1 \widetilde{C}_W (\Omega)} \right)^{\frac{n}{2}}.$$

Thus, we can apply [10, Corollary 4.2 and Remark 4.1] to deduce that $\dim_F \mathcal{A}_W \leq d^*$, from which the right-hand side of the inequality (3.14) follows.

The left-hand side of inequality (3.14) is obtained in the same spirit of [17, Theorem 3.1], owing to the asymptotic behavior of the eigenvalue sequence associated with $A_W^{\nu,\delta}$ and relies on the fact that, owing to the boundedness of $U \in X_\infty \cap \mathbb{V}_\delta$, the semigroup $\mathcal{S}_t^{\nu,\delta}$ is uniformly differentiable with derivative of Hölder class $\mathcal{C}^\alpha$, $\alpha \in (0,1)$. More precisely, there exists a smooth manifold $W_{\nu,\delta}^{loc}(U_*)$ (of class $\mathcal{C}^{1,\alpha}$) localized in an open neighborhood of a fixed constant solution $U_* = (c,c)$ with finite instability dimension $\dim X_*^{\nu,\delta}(U_*) < \infty$. In particular, $X_*^{\nu,\delta}(U_*)$ is the unstable subspace of

$$L(U_*)W = \begin{pmatrix} \nu\Delta w - f'(U_*)w + \lambda w \\ -\nu\partial_n w + \delta\Delta_\Gamma w \end{pmatrix},$$

which is tangent to $W_{\nu,\delta}^{loc}(U_*)$ at the point U_* and we recall that the global attractor always contains localized unstable manifolds [6, 10]. As in the proof of [17, Theorem 3.1], by virtue of Theorem 2.16, it follows

$$\dim_F(\mathcal{G}_W^{\nu,\delta}, X_2) \geq \dim X_*^{\nu,\delta}(U_*) \geq c_0 \left(\frac{\lambda}{\widetilde{C}_W(\Omega)\,\nu} \right)^{\frac{n}{2}}.$$

The proof is finished. $\square$

Remark 3.5. Condition (3.2) on f is not necessary for the validity of both statements (i)–(ii) of Theorem 3.4. The same result can be essentially proven without this assumption and in a more general context, allowing for nonlinear boundary conditions as well (see [18]).

4. Elliptic equations with dynamic boundary conditions

In this section we consider the following elliptic-parabolic initial-boundary value problems of the form

$$\begin{cases} \nu\Delta u - \lambda u = 0, & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \nu\partial_{\mathbf{n}} u = \delta\Delta_\Gamma u - g(u), & \text{on } (0,\infty)\times\Gamma, \\ u|_{t=0} = \psi_0, & \text{on } \Gamma. \end{cases} \tag{4.1}$$

Once again we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary Γ of class $\mathcal{C}^2$, $\lambda \in \mathbb{R}$, $\nu > 0$, $\delta \geq 0$ and $g \in C^{1,1}_{\text{loc}}(\mathbb{R})$.

4.1. Solvability in the class of weak and strong solutions

Generally speaking there are several ways to deal with (4.1) in order to show well-posedness in various Banach spaces. It is worth emphasizing that the linear case can be directly solved by the Fourier method (see [22, 43, 44]) in terms of the eigenfunctions of the operator B_δ (see Section 2.2). On the other hand, the solvability of the nonlinear problem (4.1) was investigated by J.L. Lions in the late 60s in the case $\delta = 0$, using a Galerkin truncation method and compactness arguments, or by

Escher [13], by means of fixed point theorems when dealing say with global well-posedness of classical solutions for smooth initial data. We refer the reader to the recent contribution [22] where a detailed and extensive description of the pertinent literature for a slightly more general problem than (4.1) can be found. In this section, we develop a new and more interesting approach to handle the well-posedness of our system, of that in which (4.1) can be viewed as a *singular* perturbation of a sequence of fully *parabolic* problems, of the form

$$\begin{cases} \varepsilon\partial_t u - \nu\Delta u + \lambda u = 0, & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \nu\partial_{\mathbf{n}} u = \delta\Delta_\Gamma u - g(u), & \text{on } (0,\infty)\times\Gamma, \\ u|_{t=0} = u_0, & \text{on } \overline{\Omega}, \end{cases} \tag{4.2}$$

where $\varepsilon\in(0,1]$ is a given relaxation parameter. Indeed, if we formally set $\varepsilon=0$ in the first equation of (4.2), then we can easily deduce (4.1).

It turns out that (4.2) possesses a unique strong solution. The following result is standard and follows from a series of results proven in [18].

Theorem 4.1. *Let $\varepsilon\in(0,1]$ and $\lambda\in\mathbb{R}$. Assume that*

$$-g(y)\,y \le c_g\left(y^2+1\right), \text{ for all } y\in\mathbb{R}, \tag{4.3}$$

for some $c_g\ge 0$. Then for any initial datum $U_0 \overset{def}{=} (u_0,u_{0\Gamma})\in\mathcal{W}_\delta\cap X_\infty$, the parabolic system (4.2) possesses a unique solution $U(t)=(u(t),u_\Gamma(t))$, a.e. $t\in(0,T)$, for any $T>0$, with the properties

$$\begin{cases} U\in L^\infty\left(0,T;\mathcal{W}_\delta\cap X_\infty\right)\cap L^2(0,T;D(A_W^{\nu,\delta})), \\ \partial_t u_\Gamma\in L^2\left((0,T)\times\Gamma\right), \ \sqrt{\varepsilon}\partial_t u\in L^2\left((0,T)\times\Omega\right), \end{cases} \tag{4.4}$$

such that $u(t)_{|\Gamma}=u_\Gamma(t)$, a.e. on $(0,T)$. The solution satisfies the equations in a strong sense, i.e., a.e. in $(0,T)\times\Omega$ and $(0,T)\times\Gamma$, respectively. Moreover, the following estimates hold:

$$\sup_{t\in(0,T)}\|U(t)\|_{X_\infty}\le Q_\varepsilon\left(e^{C_\varepsilon T},\|U_0\|_{X_\infty}\right), \tag{4.5}$$

$$\begin{aligned} &\sup_{t\in(0,T)}\|U(t)\|^2_{\mathcal{W}_\delta}+\int_0^T\left(\|\partial_t u_\Gamma(s)\|^2_{L^2(\Gamma)}+\varepsilon\|\partial_t u(s)\|^2_{L^2(\Omega)}\right)ds \\ &\le Q_\varepsilon\left(\|U_0\|^2_{\mathcal{W}_\delta\cap X_\infty},e^{C_\varepsilon T}\right). \end{aligned} \tag{4.6}$$

The function $Q_\varepsilon:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is monotone (in each of its variables) and is independent of t,T and the initial data.

Proof. We fix $\varepsilon>0$. A basic approach for a proof is to truncate the nonlinearity g in problem (4.2) in such a way that $|g_h^{'}|\le c_h\sim h^{-\beta}$ ($\beta\ge 1$), i.e., g_h is globally Lipschitz but g_h still obeys (4.3) with a constant $c_g\in\mathbb{R}$ independent of $h>0$. We view the new sequence of truncated problems as an abstract Cauchy problem

$$\begin{pmatrix}\varepsilon\partial_t u\\ \partial_t u\end{pmatrix}+A_W^{\nu,\delta}U=G_h(U),$$

for the Wentzell operator $A_W^{\nu,\delta}$ (see Section 2.2) with a globally Lipschitz perturbation $G_h(U) \overset{\text{def}}{=} \binom{-\lambda u}{-g_h(u)}$. In this case, it is standard to show by semigroup methods that the latter problem is also globally well-posed for each $h > 0$ (see, e.g., [24, 25]). Our point is to observe that g_h satisfies the same condition (4.3) and that the various constants involved in the estimates performed [18, Theorem 3.2 and Remark 3.3] are actually independent of t, T, and h. This procedure allows us to obtain an estimate like (4.5) uniformly in $h > 0$. For the last uniform estimate (4.6), we refer the reader to [18, Proposition 3.7] for a proof which can be easily adapted on account of (4.5) (cf. also Proposition 4.4 below). It is then standard procedure to pass to the limit as $h \to 0^+$. In order to keep the presentation light we refrain from showing all these constructions in detail, and so we leave the details to the interested reader. $\square$

The first goal of this subsection is to prove the existence of at least one strong solution to (4.1) by passing to the limit as $\varepsilon \to 0$ in the parabolic system (4.2). However, our elliptic system is a singular perturbation (i.e., $\varepsilon = 0$) of a parabolic problem, since when we collapse (4.2) into (4.1), we lose the information on the initial datum u_0 in Ω. Indeed, (4.1) requires knowledge of *only* the initial value of $u_\Gamma(t)$ =trace$(u(t))$ at the initial time $t = 0$. Thus, we must proceed very carefully. First, we briefly recall how to solve a linear elliptic problem with inhomogeneous Dirichlet data. More precisely, we consider the following system

$$\begin{cases} \lambda u - \nu\Delta u = 0, & \text{in } \Omega, \\ u|_\Gamma = g & \text{on } \Gamma, \end{cases} \tag{4.7}$$

for $\lambda \in \mathbb{R}$, and a given $g \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. We prove the following elementary lemma.

Lemma 4.2. *Assume $\lambda > \lambda_* \overset{def}{=} -\nu\lambda_{D,1}$, where $\lambda_{D,1} > 0$ is the first eigenvalue of $A_D = -\Delta$ with null Dirichlet boundary conditions. Then, there exists a unique weak solution $u = \mathcal{D}_\lambda(g)$ of (4.7),*

$$\mathcal{D}_\lambda : H^{1/2}(\Gamma) \cap L^\infty(\Gamma) \to H^1(\Omega) \cap L^\infty(\Omega) \tag{4.8}$$

such that u satisfies the following estimates:

$$\|u\|_{H^1(\Omega)} \le C \|g\|_{H^{1/2}(\Gamma)}, \tag{4.9}$$

$$\|u\|_{L^\infty(\Omega)} \le C\left(\|g\|_{H^{1/2}(\Gamma)}, \|g\|_{L^\infty(\Gamma)}\right), \tag{4.10}$$

for some constant $C > 0$ independent of λ and u.

Definition 4.3. *The precise notion of a weak solution to problem* (4.7) *is the following:*

$$\nu \langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega)} + \lambda \langle u, \varphi\rangle_{L^2(\Omega)} = 0,$$

for all $\varphi \in H_0^1(\Omega)$ with $u_\Gamma = u|_\Gamma = g$.

Proof. Let $g \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. For the solvability of (4.7), we can exploit, for instance, [22, Lemma 4.1]. It follows that there exists a unique solution $w \in H^1(\Omega)$ such that $\|w\|_{H^1(\Omega)} \le C \|g\|_{H^{1/2}(\Gamma)}$. Moreover, $\|w\|_{L^\infty(\Omega)} \le C\left(\|g\|_{H^{1/2}(\Gamma)}, \|g\|_{L^\infty(\Gamma)}\right)$

by application of [39, Theorem 7.1]. In order to show uniqueness, let u_1, u_2 be any two weak solutions of (4.7) such that $u_{1|\Gamma} = u_{2|\Gamma} = g$, for the same given g. Setting $\xi := u_1 - u_2$, we see that ξ is a weak solution of the following elliptic problem

$$\lambda\xi - \nu\Delta\xi = 0 \text{ in } \Omega,\ \xi_{|\Gamma} = 0.$$

Testing the first equation by $\xi \in H_0^1(\Omega)$ and exploiting the Poincare inequality $\|\nabla\xi\|^2_{L^2(\Omega)} \geq \lambda_{D,1} \|\xi\|^2_{L^2(\Omega)}$, yields $(\lambda + \nu\lambda_{D,1}) \|\xi\|^2_{L^2(\Omega)} \leq 0$, from which the desired conclusion follows. □

Step 1. **Data reconstruction.** We set $\mathcal{X}_0 := H^{1/2}(\Gamma)$ and $\mathcal{X}_\delta := H^1(\Gamma)$ if $\delta > 0$, and then let $\psi_0 (= u_\Gamma(0)) \in \mathcal{X}_\delta \cap L^\infty(\Gamma)$ be any (but given) initial datum for (4.1) with $\delta \geq 0$. We will now apply Lemma 4.2 to reconstruct an initial datum u_0 in the domain Ω in a canonical way. Indeed, $u_0 = \mathcal{D}_\lambda(\psi_0)$ has the required properties: $U_0 = (u_0, \psi_0) \in \mathcal{W}_\delta \cap X_\infty$ since the solution operator $\mathcal{D}_\lambda$ obeys the estimates (4.9), (4.10). Moreover, we have

$$\|(u_0, \psi_0)\|_{\mathcal{W}_\delta} \leq C(\|\psi_0\|_{\mathcal{X}_\delta}) \text{ and } \|(u_0, \psi_0)\|_{X_\infty} \leq C(\|\psi_0\|_{L^\infty(\Gamma)}), \tag{4.11}$$

for some constant $C > 0$ which is independent of $\varepsilon > 0$ and the initial datum U_0. By Lemma 4.2, $u_0 = \mathcal{D}_\lambda(\psi_0)$ is also uniquely determined by the boundary data ψ_0 for (4.1). Therefore, for each such initial datum $U_0 \in \mathcal{W}_\delta \cap X_\infty$ constructed above we can infer from Theorem 4.1 that there exists a unique strong solution $U_\varepsilon(t) = (u^\varepsilon(t), u_\Gamma^\varepsilon(t))$, $t \in (0, T)$, to the parabolic problem (4.2) for any $T > 0$. This solution belongs to the class of functions (4.4) and satisfies estimates (4.5)-(4.6).
Step 2. **Uniform estimates in $\varepsilon > 0$.** We aim to provide sufficiently strong estimates for U_ε that are also uniform in $\varepsilon \in (0, 1]$. We proceed with this program in several propositions.

Proposition 4.4. *Let $\lambda \geq 0$ and assume (4.3). The following estimates hold:*

$$\sup_{t\in(0,T)} \|u_\Gamma^\varepsilon(t)\|_{L^\infty(\Gamma)} \leq C(\|\psi_0\|_{L^\infty(\Gamma)}), \tag{4.12}$$

$$\begin{aligned}&\sup_{t\in(0,T)} \|U_\varepsilon(t)\|^2_{\mathcal{W}_\delta} + \int_0^T \left(\|\partial_t u_\Gamma^\varepsilon(s)\|^2_{L^2(\Gamma)} + \varepsilon \|\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega)}\right) ds \\ &\leq C(\|\psi_0\|_{\mathcal{X}_\delta\cap L^\infty(\Gamma)}),\end{aligned} \tag{4.13}$$

for some function $C > 0$ which is independent of ε and the initial data.

Proof. To show (4.12), we modify the arguments of [22, Proposition 5.11] and [33, Lemma 5.5.3] slightly. For the sake of notational convenience we drop ε from the solution $U_\varepsilon = (u^\varepsilon, u_\Gamma^\varepsilon)$. For each $\varepsilon > 0$, we define

$$Y_{k,\varepsilon}(t) := \left(\varepsilon \int_\Omega |u(t)|^{m_k} dx + \int_\Gamma |u_\Gamma(t)|^{m_k} dS\right)^{1/m_k},$$

where the sequence $\{m_k\}_{k\in\mathbb{N}}$ is such that $m_k := 2^k$, $k \geq 1$. We claim that

$$\begin{aligned}
&\partial_t Y_{k,\varepsilon}^{m_k}(t) + \frac{4\nu(m_k-1)}{m_k}\int_\Omega \left|\nabla |u(t)|^{\frac{m_k}{2}}\right|^2 dx \\
&+ \frac{4\delta(m_k-1)}{m_k}\int_\Gamma \left|\nabla_\Gamma |u_\Gamma(t)|^{\frac{m_k}{2}}\right|^2 dS \\
&\leq m_k\left(-\lambda \|u(t)\|_{L^{m_k}(\Omega)}^{m_k} + C\|u_\Gamma(t)\|_{L^{m_k}(\Gamma)}^{m_k} + 1\right),
\end{aligned} \tag{4.14}$$

for all $t \geq 0$. Indeed, testing each equation of (4.2) with $|u|^{m_k-2}u$, and integrating by parts over Ω and Γ, respectively, (4.14) is immediate from [22, Proposition 5.11, (5.14)]. Note that the first term on the right-hand side of (4.14) will be dropped out since $\lambda \geq 0$. Next, performing a Moser iteration scheme exactly as in [22, Proposition 5.11, (5.15)], then applying Growall's inequality [33, Lemma 5.5.3] (see also [9, Proposition 9.3.1, (9.3.10)-(9.3.11)]), and exploiting the basic inequality $\sqrt[p]{a+b} \leq \sqrt[p]{a} + \sqrt[p]{b}$ $(a, b \geq 0)$, we obtain

$$Y_{k,\varepsilon}(t) \leq \max\left\{Y_{k,\varepsilon}(0), \left(C2^{k\tau}Y_{k-1,\varepsilon}^{m_k}(t) + C\right)^{1/m_k}\right\},$$

where C and τ are positive constants independent of $k \geq 1$ and $\varepsilon > 0$. On the other hand, freezing $\varepsilon \in (0,1]$ and noting that

$$Y_{k,\varepsilon}(0) \leq C(\sqrt{\varepsilon}\|u_0\|_{L^\infty(\Omega)} + \|\psi_0\|_{L^\infty(\Gamma)}) \leq C(\|\psi_0\|_{L^\infty(\Gamma)}),$$

owing to (4.11), we infer from [33, Lemma 5.5.3, Steps (III)-(IV)] that

$$\sup_{t\in(0,T)}\|u_\Gamma(t)\|_{L^\infty(\Gamma)} \leq \lim_{k\to\infty}\sup Y_{k,\varepsilon}(t) \leq C\max\left\{\|\psi_0\|_{L^\infty(\Gamma)}, \sup_{t\in(0,T)} Y_{1,\varepsilon}(t)\right\}. \tag{4.15}$$

It is left to show that $Y_{1,\varepsilon} \in L^\infty(0,T)$ uniformly in $\varepsilon > 0$. To this end, we test both equations of (4.2) with u itself and get

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\|u(t)\|_{L^2(\Omega)}^2 + \|u_\Gamma(t)\|_{L^2(\Gamma)}^2\right) \\
&+ 2\nu\|\nabla u(t)\|_{L^2(\Omega)}^2 + \delta\|\nabla_\Gamma u_\Gamma(t)\|_{L^2(\Gamma)}^2 + \lambda\|u(t)\|_{L^2(\Omega)}^2 \\
&= 2\langle g(u_\Gamma(t)), u_\Gamma(t)\rangle_{L^2(\Gamma)}.
\end{aligned} \tag{4.16}$$

Exploiting the assumption (4.3), we easily derive

$$\frac{d}{dt}\left(\varepsilon\|u(t)\|_{L^2(\Omega)}^2 + \|u_\Gamma(t)\|_{L^2(\Gamma)}^2\right) \leq C\left(\|u_\Gamma(t)\|_{L^2(\Gamma)}^2 + 1\right),$$

for some $C > 0$ independent of $\varepsilon > 0$, time and the initial data. Thus, by Gronwall's inequality,

$$Y_{1,\varepsilon}(t) = \varepsilon\|u(t)\|_{L^2(\Omega)}^2 + \|u_\Gamma(t)\|_{L^2(\Gamma)}^2 \leq (tC + Y_{1,\varepsilon}(0))\,e^{Ct},\ t \in (0,T). \tag{4.17}$$

Owing once more to (4.11), we have $Y_{1,\varepsilon}(0) \leq C(\|\psi_0\|_{L^\infty(\Gamma)})$ uniformly in $\varepsilon \in (0,1]$. The desired inequality (4.12) follows then by combining this estimate with (4.15).

The proof of (4.13) is standard now that we have the first uniform bound (4.12). Indeed, multiply the first equation of (4.2) by $\partial_t u(t)$, then integrate over Ω, and multiply the second equation of (4.2) by $\partial_t u_\Gamma(t)$ and integrate over Γ. Setting

$$\begin{aligned}\mathcal{E}(t) &:= \lambda \|u(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + \delta \|\nabla_\Gamma u_\Gamma(t)\|_{L^2(\Gamma)}^2 \\ &\quad + 2\langle G(u_\Gamma(t)), 1\rangle_{L^1(\Gamma)} + C_{\mathcal{E}},\end{aligned}$$

we obtain

$$2\varepsilon \|\partial_t u\|_{L^2(\Omega)}^2 + 2\|\partial_t u_\Gamma\|_{L^2(\Gamma)}^2 + \partial_t \mathcal{E}(t) = 0, \text{ a.e. } t \in (0,T). \tag{4.18}$$

Here the constant $C_{\mathcal{E}} > 0$ is taken large enough, depending only the initial data ψ_0, in order to ensure that $\mathcal{E}(t)$ is nonnegative (recall that $G(y)$ is bounded for $|y| \le r$). Furthermore, one can easily check

$$C(\|U\|_{\mathcal{W}_\delta} - \|\psi_0\|_{L^\infty(\Gamma)}) \le \mathcal{E}(t) \le Q\left(\|U\|_{\mathcal{W}_\delta}\right) + C(\|\psi_0\|_{L^\infty(\Gamma)}),$$

for some positive function Q and $C > 0$, both independent of ε. Integrating (4.18) over time with $t \in (0,T)$, then exploiting (4.12) together with the fact that $Y_{1,\varepsilon} \in L^\infty(0,T)$ yields the desired bound in (4.13). The proof is finished. $\square$

We now exploit the preceding result in order to derive additional uniform estimates for the solutions of (4.2) as $\varepsilon \to 0$. First, a comparison in (4.2) for every $\varepsilon \le 1$ shows that when $\delta \ge 0$ it holds

$$\int_0^T \|\Delta u^\varepsilon(s)\|_{L^2(\Omega)}^2 ds \le C(\|\psi_0\|_{\mathcal{X}_\delta \cap L^\infty(\Gamma)}), \tag{4.19}$$

on account of estimates (4.12)-(4.13). Moreover, by comparison in the second equation of (4.2) we have

$$\int_0^T \|\partial_{\mathbf{n}} u^\varepsilon(s)\|_{L^2(\Gamma)}^2 ds \le C(\|\psi_0\|_{\mathcal{X}_\delta \cap L^\infty(\Gamma)}), \tag{4.20}$$

when $\delta = 0$, while in the case $\delta > 0$, the application of [20, Lemma 2.2] with $j_1 := -\lambda u^\varepsilon - \varepsilon \partial_t u^\varepsilon$ and $j_2 := g(u_\Gamma^\varepsilon) - \partial_t u_\Gamma^\varepsilon$, entails from (4.12)-(4.13), that

$$\int_0^T \left(\|u^\varepsilon(s)\|_{W^{2,2}(\Omega)}^2 + \|u_\Gamma^\varepsilon(s)\|_{W^{2,2}(\Omega)}^2\right) ds \le C(\|\psi_0\|_{\mathcal{X}_\delta \cap L^\infty(\Gamma)}). \tag{4.21}$$

Summing up, these estimates entail

$$\int_0^T \left\|A_W^{\nu;\delta} U_\varepsilon(s)\right\|_{X_2}^2 ds \le C(\|\psi_0\|_{\mathcal{X}_\delta \cap L^\infty(\Gamma)}), \tag{4.22}$$

for any $\delta \ge 0$, $\nu > 0$.

Step 3. **Passage to limit as** $\varepsilon \to 0$. We are now ready to pass to the limit, as $\varepsilon \to 0$, in the parabolic problem (4.2), using the uniform estimates (4.19)-(4.22) and (4.12)-(4.13). Indeed, on account of these uniform inequalities, we can find u and u_Γ such that, up to subsequences,

$$\begin{cases} u_\Gamma^\varepsilon \to u_\Gamma \quad \text{weakly-}* \text{ in } L^\infty\left(0,T;L^\infty\left(\Gamma\right)\right), \\ u^\varepsilon \to u \quad \text{weakly-}* \text{ in } L^\infty\left(0,T;\mathcal{X}_\delta\right), \\ \partial_t u_\Gamma^\varepsilon \to \partial_t u_\Gamma \quad \text{weakly in } L^2\left(\left(0,T\right)\times\Gamma\right), \\ \varepsilon\partial_t u^\varepsilon \to 0 \quad \text{strongly in } L^2\left(\left(0,T\right)\times\Omega\right), \\ \Delta u^\varepsilon \to \Delta u \text{ weakly in } L^2\left(0,T;L^2\left(\Omega\right)\right), \text{ if } \delta=0, \\ \partial_{\mathbf{n}} u^\varepsilon \to \partial_{\mathbf{n}} u^\varepsilon \text{ weakly in } L^2\left(0,T;L^2\left(\Omega\right)\right), \text{ if } \delta=0, \\ \left(u^\varepsilon,u_\Gamma^\varepsilon\right) \to \left(u,u_\Gamma\right) \text{ weakly in } L^2\left(0,T;\mathbb{V}_\delta\right), \text{ if } \delta>0. \end{cases} \tag{4.23}$$

Note that the first and third convergences of (4.23) implies that u_Γ belongs to $C\left(\left[0,T\right];L^2\left(\Gamma\right)\right)$ such that $u_\Gamma\left(0\right)=\psi_0$ a.e. on Γ. The second and third of (4.23), and a classical compactness theorem (see, e.g., [10, Theorem 1.4]) yields

$$u_\Gamma^\varepsilon \to u_\Gamma \quad \text{strongly in } L^2\left(0,T;L^2\left(\Gamma\right)\right). \tag{4.24}$$

This strong convergences entails that, up to subsequences, u_Γ^ε converges also to u_Γ almost everywhere on Γ, a.e. $t\in(0,T)$. Thus, we can control the nonlinear boundary term. More precisely, using the fact that $g\in C^1$, we have

$$g\left(u_\Gamma^\varepsilon\right) \to g\left(u_\Gamma\right) \text{ strongly in } L^2\left(0,T;L^2\left(\Gamma\right)\right), \tag{4.25}$$

thanks to (4.24), the first convergence of (4.23), and estimate (4.12). By means of the above convergence properties (4.23), (4.25), we can now pass to the limit in both equations of (4.2) to deduce that $U=(u,u_\Gamma)$ solves the elliptic-parabolic system (4.1). Moreover, due to the arbitrariness of $T>0$, passing to limit as $\varepsilon\to 0$ in (4.19)-(4.22) and (4.12)-(4.13), and recalling (4.23), we also deduce that the limit solution (u,u_Γ) satisfies these inequalities with a constant $C>0$ independent of $\varepsilon>0$.

In other words, we have proved the following.

Theorem 4.5. *Let $\lambda\geq 0$ and assume (4.3). Then, for any initial datum satisfying $\psi_0\in\mathcal{X}_\delta\cap L^\infty\left(\Gamma\right)$, the nonlinear elliptic system (4.1) possesses a unique strong solution with the properties*

$$\begin{cases} \left(u,u_\Gamma\right)\in L^\infty\left(0,T;\mathcal{X}_\delta\cap L^\infty\left(\Gamma\right)\right), & u\in L^\infty\left(0,T;H^1\left(\Omega\right)\right), \\ \partial_t u_\Gamma\in L^2\left(\left(0,T\right)\times\Gamma\right), & U\in L^2(0,T;D(A_W^{\nu,\delta})), \end{cases}$$

such that $u\left(t\right)|_\Gamma=u_\Gamma\left(t\right)$, a.e. $t\in\left(0,T\right)$, for any $T>0$.

Proof. The existence argument is provided in the Steps 1-3 above. As usual to show uniqueness, we set $U\left(t\right):=U_1\left(t\right)-U_2\left(t\right)$, where $U_1=(u_1,u_{1\Gamma})$ and $U_2=(u_2,u_{2\Gamma})$ are any two strong solutions of (4.1) corresponding to the initial data ψ_{0i}, $i=1,2$. We see that U solves

$$\lambda u-\nu\Delta u=0 \text{ a.e. in } \left(0,T\right)\times\Omega,\ u|_\Gamma=u_\Gamma,$$

and the boundary condition:

$$\partial_t u_\Gamma-\delta\Delta_\Gamma u_\Gamma+\nu\partial_n u=g\left(u_{1\Gamma}\right)-g\left(u_{2\Gamma}\right), \text{ a.e. on } \left(0,T\right)\times\Gamma.$$

Testing the first and last equations by u and u_Γ, respectively, and exploiting the bound (4.12) yields

$$\begin{aligned}\frac{d}{dt}\left\|u_\Gamma\left(t\right)\right\|_{L^2(\Gamma)}^2 &\leq \left\langle g\left(u_{1\Gamma}\left(t\right)\right)-g\left(u_{2\Gamma}\left(t\right)\right),u_\Gamma\left(t\right)\right\rangle_{L^2(\Omega)}\\ &\leq C(\left\|\psi_0\right\|_{L^\infty(\Gamma)})\left\|u_\Gamma\left(t\right)\right\|_{L^2(\Gamma)}^2,\end{aligned}$$

since $u_{i\Gamma}\in L^\infty\left(0,T;L^\infty\left(\Gamma\right)\right), i=1,2$. Thus, if $\psi_{01}\equiv\psi_{02}$ on Γ, Gronwall's inequality gives the desired uniqueness $u_{1\Gamma}\left(t\right)\equiv u_{2\Gamma}\left(t\right)$, on $\left(0,T\right)\times\Gamma$. Moreover, in view of Lemma 4.2 there also holds $u_1\left(t\right)\equiv u_2\left(t\right)$ in $\left(0,T\right)\times\Omega$. The proof is finished. □

In the final part of this subsection, we briefly explain how to solve (4.1) in the class of weak L^2-energy solutions for a polynomial nonlinearity g. We first give the following rigorous notion of weak solution to problem (4.1), with initial condition $u\left(0\right)=\psi_0$ in $L^2\left(\Gamma\right)$.

Definition 4.6. *Let $\lambda\geq 0$, $\delta\geq 0$. The pair $U\left(t\right)=\left(u\left(t\right),u_\Gamma\left(t\right)\right)$ is said to be a weak solution of* (4.1) *if $u_\Gamma\left(t\right)=u|_\Gamma$, in the trace sense, for a.e. $t\in\left(0,T\right)$, for any $T>0$, and U fulfills*

$$\begin{cases} u_\Gamma\left(t\right)\in L^\infty\left(0,T;L^2\left(\Gamma\right)\right),\\ U\left(t\right)\in L^2\left(0,T;\mathcal{W}_\delta\right),\ u_\Gamma\in L^q\left(\left(0,T\right)\times\Gamma\right),\\ \partial_t u_\Gamma\left(t\right)\in L^2\left(0,T;\mathcal{X}_\delta^*\right)\oplus L^{\widetilde{q}}\left(\left(0,T\right)\times\Gamma\right),\end{cases}\tag{4.26}$$

such that the following identity

$$\begin{aligned}&\left\langle\partial_t u_\Gamma\left(t\right),\sigma_\Gamma\right\rangle+\nu\left\langle\nabla u\left(t\right),\nabla\sigma\right\rangle_{L^2(\Omega)}\\ &+\delta\left\langle\nabla_\Gamma u_\Gamma\left(t\right),\nabla_\Gamma\sigma_\Gamma\right\rangle_{L^2(\Omega)}+\left\langle\lambda u\left(t\right),\sigma\right\rangle\\ &=\left\langle g\left(u_\Gamma\left(t\right)\right),\sigma_\Gamma\right\rangle,\end{aligned}\tag{4.27}$$

holds for all $\Xi=\left(\sigma,\sigma_\Gamma\right)\in\mathcal{W}_\delta$, $\sigma_\Gamma\in L^q\left(\Gamma\right)$, a.e. $t\in\left(0,T\right)$. Here, $\widetilde{q}$ denotes the dual conjugate of q, i.e., $1/\widetilde{q}+1/q=1$.

Remark 4.7. As usual from (4.26) it holds $u_\Gamma\left(t\right)\in C_w\left(\left[0,T\right];L^2\left(\Gamma\right)\right)$, for any $T>0$. Hence, the initial datum $u_\Gamma\left(0\right)=\psi_0$ makes sense.

We state the following result.

Theorem 4.8. *Assume $\lambda\geq 0$ and the following conditions:*

$$\begin{cases} g^{'}\left(y\right)\geq -c_g, & \text{for all } y\in\mathbb{R},\\ \eta_1\left|y\right|^q-C_g\leq g\left(y\right)y\leq\eta_4\left|y\right|^q+C_g, & \text{for all } y\in\mathbb{R},\end{cases}\tag{4.28}$$

for some η_1, $\eta_2>0$, $C_g,c_g\geq 0$ and $q>2$. Then, for any initial data $\psi_0\in L^2\left(\Gamma\right)$, there exists exactly one global weak solution to problem (4.1) *in the sense of Definition* 4.6*. Moreover, $u_\Gamma\left(t\right)\in C\left(\left[0,T\right];L^2\left(\Gamma\right)\right)$ such that $\left\|u\left(t\right)\right\|_{L^2(\Gamma)}^2$ is absolutely continuous on $\left(0,T\right)$, and $U=\left(u,u_\Gamma\right)$ satisfies the following* energy identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u_\Gamma\left(t\right)\right\|_{L^2(\Gamma)}^2+\nu\left\langle\nabla u\left(t\right),\nabla u\left(t\right)\right\rangle_{L^2(\Omega)}+\delta\left\langle\nabla_\Gamma u_\Gamma\left(t\right),\nabla_\Gamma u_\Gamma\left(t\right)\right\rangle_{L^2(\Gamma)}\\ &=\left\langle-\lambda u\left(t\right),u\left(t\right)\right\rangle_{L^2(\Omega)}+\left\langle g\left(u_\Gamma\left(t\right)\right),u_\Gamma\left(t\right)\right\rangle,\end{aligned}\tag{4.29}$$

for almost all $t\in\left(0,T\right)$.

Proof. The proof carries over essentially with only minor modifications of the proof of [18, Theorem 2.2 and Proposition 2.3] (see also [19]), owing to (4.28), with the exception that we take advantage of a different (but natural) approximation scheme which is now based on the existence of strong solutions for (4.1), see Theorem 4.5. Indeed, we can choose a sequence of data $\psi_{0\varepsilon} \in \mathcal{X}_\delta \cap L^\infty(\Gamma)$ such that $u^\varepsilon(0) = \psi_{0\varepsilon}$ and $u_\Gamma^\varepsilon(0) \to u_\Gamma(0) = \psi_0$ in $L^2(\Gamma)$. For each $\psi_{0\varepsilon}$ there is a unique bounded strong solution U_ε of problem (4.1). Another advantage of this construction is that now every weak solution of (4.1) can be approximated by regular ones U_ε and the justification of our subsequent asymptotic estimates for such solutions is also immediate. The proof of the energy identity (4.29) follows along the lines of [18, Proposition 2.5] while the uniqueness argument is exactly the same as in the proof of Theorem 4.5, except that we employ the first condition of (4.28). This completes the proof of the theorem. □

We conclude with the following. Recall that $\nu > 0$ and $\delta \geq 0$.

Proposition 4.9. *Let the assumptions of Theorem* 4.8 *be satisfied. Then* (4.1) *defines a (nonlinear) continuous semigroup*

$$\mathcal{T}_{\nu,\delta}(t) : L^2(\Gamma) \to L^2(\Gamma)$$

given by

$$\mathcal{T}_{\nu,\delta}(t)\,\psi_0 = u_\Gamma(t),$$

where $U(t) = (u(t), u_\Gamma(t))$ is the (unique) weak solution in the sense of Theorem 4.8.

4.2. Finite dimensional attractors

In this subsection, we wish to investigate the question of regularity and long-time behavior of the weak solutions constructed in the previous subsection. In particular, we show that each weak solution becomes a strong solution after some time in the sense of Theorem 4.5. We begin with the following important result which says that under assumption (4.28) on g, all such L^2-energy solutions become ultimately bounded and sufficiently smooth for all positive times. To this end, we state the following straight-forward proposition.

Proposition 4.10. *Let the assumptions of Theorem* 4.8 *be satisfied. Then, every weak $U = (u, u_\Gamma)$ of* (4.1) *satisfies*

$$\begin{aligned}
&\|u_\Gamma(t)\|_{L^2(\Gamma)}^2 + \int_t^{t+1} \left(\nu \|\nabla u(s)\|_{L^2(\Omega)}^2 + \delta \|\nabla_\Gamma u_\Gamma(s)\|_{L^2(\Gamma)}^2\right) ds \\
&+ \int_t^{t+1} \|u_\Gamma(s)\|_{L^q(\Gamma)}^q\, ds \\
&\leq C \|\psi_0\|_{L^2(\Gamma)}^2 e^{-\rho t} + C,
\end{aligned} \tag{4.30}$$

a.e. $t \geq 0$, for some positive constants ρ, C independent of the initial data and time.

Proof. Estimate (4.30) is a direct consequence of the energy identity (4.29), the second assumption of (4.28) and the Gronwall inequality. □

In particular, this result implies that the semigroup $\mathcal{T}_{\nu,\delta}$ associated with (4.1) possesses an absorbing ball $\mathcal{B}_0$ in $L^2(\Gamma)$-topology. More precisely, for each subset $B \subset L^2(\Gamma)$, there exists a positive time $t_0 = t_0(\|B\|_{L^2(\Gamma)})$ such that $\mathcal{T}_{\nu,\delta}(t)(B) \subseteq \mathcal{B}_0$, for each $t \geq t_0$.

Theorem 4.11. *There exists a time $t_1 > 0$ depending only on t_0 and the other structural parameters of the problems, such that*

$$\sup_{t\geq t_1}\left(\|u_\Gamma(t)\|_{L^\infty(\Gamma)} + \|U(t)\|^2_{\mathcal{W}_\delta} + \int_t^{t+1} \|\partial_t u_\Gamma(s)\|^2_{L^2(\Gamma)}\, ds\right) \leq \mathcal{C}, \tag{4.31}$$

for some positive constant $\mathcal{C}$ independent of time and the initial data.

Proof. The $L^\infty(\Gamma)$-estimate in (4.31) is a consequence of the same Moser iteration scheme performed in [17, Theorem 2.3] (see also [18, Theorem 3.2]), owing to the inequality (4.14) and the existence of an absorbing ball $\mathcal{B}_0$ for $\mathcal{T}_{\nu,\delta}$. Next, recall the energy identity (4.18) which holds a.e. on (t_0, ∞), for the strong solutions U_ε of the elliptic-parabolic problem (4.1). The application of the uniform Gronwall lemma, together with assumption (4.28) and the energy inequality (4.30), gives

$$\sup_{t\geq t_1}\left(\|U_\varepsilon(t)\|^2_{\mathcal{W}_\delta} + \int_t^{t+1} \|\partial_t u^\varepsilon_\Gamma(s)\|^2_{L^2(\Gamma)}\, ds\right) \leq C, \tag{4.32}$$

for some $C > 0$ independent of $\varepsilon > 0$, time and the initial data. Henceforth, passing to the limit as $\varepsilon \to 0$ in (4.32) in a standard way, it is not difficult to realize that (4.32) also holds for the limit solution U of the elliptic system (4.1). The proof is complete. □

Consequently, from Lemma 4.2 we also have

Theorem 4.12. *Any weak solution $U = (u, u_\Gamma)$ of (4.1) satisfies*

$$\sup_{t\geq t_1} \|u(t)\|_{L^\infty(\Omega)} \leq \mathcal{C}. \tag{4.33}$$

Next, let $\mathbb{L} : H^{1/2}(\Gamma) \to X_2$ be the lifting map

$$\mathbb{L}(u_\Gamma) \overset{\text{def}}{=} (u, u_\Gamma),\ u = D_\lambda(u_\Gamma).$$

Concerning the long-time behavior of the elliptic-parabolic problem (4.1), we have proved the following.

Theorem 4.13. *Let all the assumptions of Theorem 4.8 be satisfied. The dynamical system $(\mathcal{T}_{\nu,\delta}(t), L^2(\Gamma))$ possesses a connected global attractor $\mathcal{E}_{\nu,\delta} \subset L^2(\Gamma)$ such that $\mathbb{L}(\mathcal{E}_{\nu,\delta})$ is bounded in $\mathcal{W}_\delta \cap X_\infty$. Moreover, $\mathcal{E}_{\nu,\delta}$ contains only strong solutions and is of finite fractal dimension,*

$$\dim_F\left(\mathcal{E}_{\nu,\delta}, L^2(\Gamma)\right) < \infty.$$

Proof. The first part of the statement of theorem follows by virtue of the compact embedding $H^{1/2}(\Gamma) \subset L^2(\Gamma)$, and from the statements of Theorems 4.11, 4.12. The last part is a consequence of the proof of Theorem 4.5 and the fact that $\mathbb{L}(\mathcal{E}_{\nu,\delta})$ is a bounded in $\mathcal{W}_\delta \cap X_\infty$, which entails that $\mathcal{T}_{\nu,\delta}$ is also uniformly differentiable on $\mathcal{E}_{\nu,\delta}$ (see, e.g., [17, Proposition 2.6]). $\square$

Remark 4.14. Based on the estimates (4.31), (4.33), it is possible to exploit the ideas contained in [22] and a bootstrap argument to show that each strong solution on the global attractor is in fact a classical solution $u \in C^2\left((t_1,\infty)\times\overline{\Omega}\right)$.

Our final goal of this section is to obtain two-sided sharp estimates for the fractal dimension of the global attractor associated with the elliptic-parabolic system (4.1).

Theorem 4.15. (i) **Pure-reactive ($\delta = 0$) dynamic boundary conditions.** *The fractal dimension of $\mathcal{E}_{\nu,0}$ admits the one-sided estimate*

$$\dim_F(\mathcal{E}_{\nu,0}, L^2(\Gamma)) \leq \max\left\{1, c_0\left(\frac{c_g}{\nu C_S(\Gamma)}\right)^{n-1}\right\}. \tag{4.34}$$

(ii) **Reactive-diffusive ($\delta > 0$) dynamic boundary conditions.** *The fractal dimension of $\mathcal{E}_{\nu,0}$ admits the one-sided estimate*

$$\dim_F(\mathcal{E}_{\nu,\delta}, L^2(\Gamma)) \leq \max\left\{1, c_0\left(\frac{c_g}{\delta\widetilde{C}_S(\Gamma)}\right)^{\frac{n-1}{2}}\right\}. \tag{4.35}$$

Here c_0 depends on the shape of Ω and $n \geq 2$ only, and the positive constants C_S, $\widetilde{C}_S$ depend only on n, $|\Gamma|$ and are given in Section 2.2.

Proof. We shall employ a volume contraction argument as in the proof of Theorem 3.4. The first variation of the elliptic-parabolic system (4.1) is given by the compact operator for $t > 0$,

$$\mathbf{\Lambda}(t, U(t)) : \xi \in L^2(\Gamma) \mapsto v_\Gamma \in L^2(\Gamma)$$

where $V = (v, v_\Gamma)$ is the unique strong solution to

$$\lambda v - \nu\Delta v = 0,\ \partial_t v_\Gamma + \nu\partial_n v = \delta\Delta_\Gamma v_\Gamma - g'(u_\Gamma(t))\, v, \tag{4.36}$$

subject to $v_\Gamma(0) = \xi$. Following [10, Chapter III, Definition 4.1], it suffices to estimate the j-trace of the operator $\mathbf{\Lambda}(t, U(t))$ as follows:

$$\begin{aligned}
\mathrm{Tr}\left(\mathbf{\Lambda}(t, U(t))\, Q_m\right) &= \sum_{j=1}^m \left\langle \mathbf{\Lambda}(t, U(t))\,\varphi_j, \varphi_j\right\rangle_{L^2(\Gamma)} \\
&= \sum_{j=1}^m \left\langle B_\delta\varphi_j, \varphi_j\right\rangle_{L^2(\Gamma)} - \lambda\sum_{j=1}^m \left\langle \varphi_j, \varphi_j\right\rangle_{L^2(\Omega)} \\
&\quad + \sum_{j=1}^m \left\langle g'(u_\Gamma(t))\,\varphi_j, \varphi_j\right\rangle_{L^2(\Gamma)}
\end{aligned}$$

where the set of real-valued functions $\varphi_j \in \mathcal{W}_\delta \cap X_\infty$, $j \in \mathbb{N}$, is an orthonormal basis of $L^2(\Gamma)$ with $\{\varphi_1, \ldots, \varphi_m\}$ spanning $Q_m\left(L^2(\Gamma)\right)$. Here Q_m corresponds to an orthogonal projector in $L^2(\Gamma)$ onto the space spanned by m-solutions $V_1, \ldots, V_m$ of (4.36) corresponding to some data $\xi = \xi_1, \ldots, \xi_m \in L^2(\Gamma)$. Furthermore, recall the definition of the operator B_δ from Section 2.2 and that the associated Steklov eigenvalue problem for $\delta \geq 0$ yields a sequence $\lambda_{S,j}^{\nu,\delta}$ converging to $+\infty$, obeying an appropriate Weyl asymptotic law (cf. (2.36) and (2.37)). The Courant-Fischer principle for the operator B_δ further yields

$$\operatorname{Tr}\left(\mathbf{\Lambda}\left(t,U\right)Q_m\right) \leq -\sum_{j=1}^{m} \lambda_{S,j}^{\nu,\delta} + c_g m \tag{4.37}$$

owing to $\lambda \geq 0$ and the first condition of (4.28). On the other hand, the application of [45, Chapter VI, Lemma 2.1] together with the fact that as $j \to \infty$, $\lambda_{S,j}^{\nu,0} \sim \nu C_S\left(\Gamma\right) j^{1/(n-1)}$ and $\lambda_{S,j}^{\nu,\delta} \sim \delta \widetilde{C}_S\left(\Gamma\right) j^{2/(n-1)}$ for $\delta > 0$, respectively, gives

$$\begin{cases} \sum_{j=1}^{m} \lambda_{S,j}^{\nu,0} \geq c_1 \nu C_S\left(\Gamma\right) m^{1/(n-1)+1}, \\ \sum_{j=1}^{m} \lambda_{S,j}^{\nu,\delta} \geq c_1 \delta \widetilde{C}_S\left(\Gamma\right) m^{2/(n-1)+1} \text{ for } \delta > 0, \end{cases} \tag{4.38}$$

for some absolute constant $c_1 > 0$ which depends only on the shape of Ω and $n \geq 2$. From (4.37) in the case $\delta > 0$, we deduce

$$\operatorname{Tr}\left(\mathbf{\Lambda}\left(t,U\right)Q_m\right) \leq -c_1 \delta \widetilde{C}_S\left(\Gamma\right) m^{2/(n-1)+1} + c_g m =: \rho\left(s\right).$$

The function $\rho\left(s\right)$ is concave. The root of the equation $\rho\left(s\right) = 0$ is

$$s^* = \left(\frac{c_g/c_1}{\delta \widetilde{C}_S\left(\Gamma\right)}\right)^{\frac{n-1}{2}}.$$

Thus, we can apply [10, Corollary 4.2 and Remark 4.1] to deduce that

$$\dim_F(\mathcal{E}_{\nu,\delta}, L^2\left(\Gamma\right)) \leq \max\left\{1, s^*\right\},$$

from which (4.35) follows. The proof of (4.34) when $\delta = 0$ is based instead on the first eigenvalue inequality of (4.38) and so the proof is similar. The proof of the theorem is complete. $\square$

To derive a lower bound for $\mathcal{E}_{\nu,\delta}$ it suffices to analyze the dimension of the unstable manifold associated with a constant equilibrium z for (4.1); let $\lambda = 0$ and observe that steady-state solutions satisfy

$$\Delta u = 0 \text{ in } \Omega, \quad -\delta \Delta_\Gamma u + \nu \partial_{\mathbf{n}} u = g\left(u_\Gamma\right) \text{ on } \Gamma.$$

We seek a constant solution $z \in \mathbb{R}$ of this system: such a solution should satisfy $g\left(z\right) = 0$ and obey $g^{'}\left(z\right) > 0$. By the second assumption of (4.28), we have $g\left(z\right) z < 0$ on the interval $I_R = \left(-R, R\right)$, if R is large enough. It follows that $g\left(z\right) = 0$ has at least one solution $z = z_*$ (see, e.g., [10, Chapter III]). Now fix a nonlinearity g and a constant solution $z = z_*$ such that $\zeta_* := g^{'}\left(z_*\right) > 0$. In order to find a (sharp) lower bound on the dimension of the global attractor $\mathcal{E}_{\nu,\delta}$, it suffices to establish a lower bound for $\dim E_+\left(z\right)$, where $E_+\left(z\right)$ is an invariant subspace of $\mathbf{\Lambda}_\delta\left(z\right)$, which corresponds to $\mathbf{\Lambda}_\delta\left(z\right) w = -B_\delta w + g^{'}\left(z\right) w$, with spectrum $\sigma\left(\mathbf{\Lambda}_\delta\left(z\right)\right) \subset \left\{\xi : \xi > 0\right\}$. Next, let $\left\{\varphi_j\left(x\right)\right\}_{j\in\mathbb{N}}$ be an orthonormal basis in $L^2\left(\Gamma\right)$ consisting of eigenfunctions of the operator

$$B_\delta \varphi_j = \lambda_{S,j}^{\nu,\delta} \varphi_j, \ j \in \mathbb{N}, \ \varphi_j \in \mathcal{X}_\delta \cap C^2\left(\Gamma\right), \tag{4.39}$$

where we recall that $\left\{\lambda_{S,j}^{\nu,\delta}\right\}$ is the (real) sequence associated with the eigenvalue problem (2.32) (see Section 2.2).

Theorem 4.16. *Let* $\lambda = 0$, $\nu > 0$, $\delta \geq 0$ *and assume* g *satisfies* (4.28) *such that* $g^{'}(z_*) = \zeta_* > 0$.

(i) *The global attractor* $\mathcal{E}_{\nu,0}$ *admits the estimate*

$$\dim_F\left(\mathcal{E}_{\nu,0}, L^2(\Gamma)\right) \geq \dim_H\left(\mathcal{E}_{\nu,0}, L^2(\Gamma)\right) \geq \widetilde{c}_0\left(\frac{\zeta_*}{\nu}\right)^{n-1}|\Gamma|. \tag{4.40}$$

(ii) *For* $\delta > 0$, *the global attractor* $\mathcal{E}_{\nu,\delta}$ *admits the estimate*

$$\dim_F\left(\mathcal{E}_{\nu,\delta}, L^2(\Gamma)\right) \geq \dim_H\left(\mathcal{E}_{\nu,\delta}, L^2(\Gamma)\right) \geq \widetilde{c}_0\left(\frac{\zeta_*}{\delta}\right)^{\frac{n-1}{2}}|\Gamma|. \tag{4.41}$$

Here $\widetilde{c}_0$ *is an absolute constant depending only on* n *and the shape of* Ω, *but is independent of the size of* Ω *and* Γ.

Proof. We shall seek for eigenvectors $w_j \in L^2(\Gamma)$ of the form $w_j(x) = \varphi_j(x)p_j$, $p_j \in \mathbb{R}$, satisfying equation

$$\mathbf{\Lambda}_\delta(z)w_j = \zeta_j w_j,\ \ w_j \in D(\mathbf{\Lambda}_\delta(z)) = D(B_\delta). \tag{4.42}$$

Substituting such w_j into (4.42), taking into account (4.39) and the fact that

$$\mathbf{\Lambda}_\delta(z)w_j = -B_\delta w_j + g^{'}(z_*)w_j,$$

we obtain the equation

$$(-\lambda_{S,j}^{\nu,\delta} + g^{'}(z_*))p_j = \zeta_j p_j. \tag{4.43}$$

We prove (i). A nonzero p_j exists if $\nu = 0$ and $\zeta = \zeta_j = g^{'}(z_*) > 0$ (indeed, this follows by taking the inner product in $L^2(\Gamma)$ of (4.42) with φ_j). Therefore, for sufficiently small $\nu \ll 1$, there exists $\gamma > 0$ such that when $\lambda_{S,j}^{\nu,0} < \gamma < g^{'}(z_*)$, the equation (4.43) has a root $\zeta_j = \zeta_j(\nu)$ with $\zeta_j > 0$. Therefore, to any such root ζ_j, we can assign a nontrivial p_j, which is a solution of (4.43), and thus an eigenvector $w_j = \varphi_j p_j$. Let us now compute how many j's satisfy the inequality $\lambda_{S,j}^{\nu,0} < \gamma$. In light of the asymptotic behavior of $\left\{\lambda_{S,j}^{\nu,0}\right\}$ from (2.36), this certainly holds when

$$1 \leq j \leq c_1\gamma^{n-1}(C_S(\Gamma)\nu)^{1-n} = c_1\left(\frac{\gamma}{C_S(\Gamma)\nu}\right)^{n-1}.$$

The constant $c_1 > 0$ is independent of $g^{'}(z_*)$. It follows that

$$\dim_H E_+(z) \geq c_1\left(\frac{\gamma}{C_S(\Gamma)\nu}\right)^{n-1}$$

and since $\dim_H(\mathcal{E}_{\nu,0}) \geq \dim_H E_+(z)$, the claim in (i) follows. The case (ii) is similar and is left to the reader (we only note that it is instead based on the asymptotic behavior of the Steklov sequence $\lambda_{S,j}^{\nu,\delta}$, $\delta > 0$, from (2.37)). The proof is complete. $\square$

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