Milan J. Math. Vol. 83 (2015) 199–209
DOI 10.1007/s00032-015-0238-x
Published online May 15, 2015

Milan Journal of Mathematics

Characterizations of Spacelike Hyperplanes in the Steady State Space via Generalized Maximum Principles

Cícero P. Aquino, Henrique F. de Lima,
Fábio R. dos Santos and Marco Antonio L. Velásquez

Abstract. We deal with complete spacelike hypersurfaces immersed in the half of the de Sitter space, which models the so-called *steady state space*. In this setting, under some appropriated constraints on the geometry of such a spacelike hypersurface, we apply suitable generalized maximum principles in order to guarantee that it must be isometric to the Euclidean space.

Mathematics Subject Classification. Primary 53C42; Secondary 53B30, 53C50.

Keywords. Steady state space, complete spacelike hypersurfaces, spacelike hyperplanes, mean curvature, Gauss map.

1. Introduction

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2}w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \left\{p \in \mathbb{L}^{n+2} : \langle p, p\rangle = 1\right\}.$$

The induced metric from $\langle , \rangle$ makes $\mathbb{S}_1^{n+1}$ a Lorentz manifold with constant sectional curvature one. Moreover, for all $p \in \mathbb{S}_1^{n+1}$, we have

$$T_p\left(\mathbb{S}_1^{n+1}\right) = \left\{v \in \mathbb{L}^{n+2} : \langle v, p\rangle = 0\right\}.$$

Birkhäuser

Let $a \in \mathbb{L}^{n+2} \setminus \{0\}$ be a past-pointing null vector, that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. Then, the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \left\{p \in \mathbb{S}_1^{n+1} : \langle p, a\rangle > 0\right\}$$

is the so-called *steady state space*.

The importance of considering $\mathcal{H}^{n+1}$ comes from the fact that, in Cosmology, $\mathcal{H}^4$ is the steady state model of the universe proposed by Bondi and Gold [3], and Hoyle [9], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times (cf. Section 5.2 of [8] or Section 14.8 of [18]).

On the other hand, apart from their physical meaning, the interest in the study of spacelike hypersurfaces immersed in a Lorentzian space is motivated by their nice Bernstein-type properties. In this direction, several authors have approached the problem of to characterize spacelike hyperplanes of $\mathcal{H}^{n+1}$, which are totally umbilical spacelike hypersurfaces that give a complete foliation of $\mathcal{H}^{n+1}$ and are isometric to the Euclidean space $\mathbb{R}^n$. We refer to readers, for instance, the works [2, 4, 5, 6, 15].

Proceeding into this branch, our aim in this paper is to apply some appropriated generalized maximum principles in order to establish new characterization results concerning these spacelike hyperplanes of the steady state space $\mathcal{H}^{n+1}$. This manuscript is organized in the following way: in Section 2 we recall some standard facts related to the foliation of $\mathcal{H}^{n+1}$ by spacelike hyperplanes. Afterwards, in Section 3 we establish our characterization results concerning spacelike hyperplanes of $\mathcal{H}^{n+1}$ (see Theorems 3.2, 3.5 and 3.10, and Corollaries 3.4, 3.7 and 3.11).

2. Foliating the steady state space by spacelike hyperplanes

As introduced before, the $(n+1)$-dimensional steady state space $\mathcal{H}^{n+1}$ is the hyperquadric

$$\mathcal{H}^{n+1} = \{p \in \mathbb{S}_1^{n+1} : \langle p, a\rangle > 0\},$$

where $a \in \mathbb{L}^{n+2} \setminus \{0\}$ is a fixed vector such that $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$.

We note that the steady state space $\mathcal{H}^{n+1}$ is extendible and, consequently, a noncomplete manifold, being only half of the de Sitter space $\mathbb{S}_1^{n+1}$ and having as boundary the null hypersurface

$$\mathcal{L}_0 = \left\{p \in \mathbb{S}_1^{n+1} : \langle p, a\rangle = 0\right\},$$

whose topology is $\mathbb{R} \times \mathbb{S}^{n-1}$ (cf. Section 2 of [15]).

Now, we shall consider in $\mathcal{H}^{n+1}$ the timelike field

$$\mathcal{V} = -\langle p, a\rangle\, p + a. \tag{2.1}$$

From (2.1), it is not difficult to verify that, for all $V \in \mathfrak{X}(\mathcal{H}^{n+1})$,

$$\overline{\nabla}_V \mathcal{V} = -\langle p, a\rangle V,$$

where $\overline{\nabla}$ stands for the Levi-Civita connection of $\mathcal{H}^{n+1}$. In other words, $\mathcal{V}$ is conformal and closed (in the sense that its dual 1-form is closed; see Example 2 of Section 4 of [14]).

From Proposition 1 of [14], we have that the n-dimensional distribution $\mathcal{D}$ defined on $\mathcal{H}^{n+1}$ by

$$p \in \mathcal{H}^{n+1} \longmapsto \mathcal{D}(p) = \left\{ v \in T_p\mathcal{H}^{n+1} : \langle \mathcal{V}(p), v \rangle = 0 \right\}$$

determines a codimension one spacelike foliation $\mathcal{F}(\mathcal{V})$ which is oriented by $\mathcal{V}$.

Moreover, from Example 1 of [12], we conclude that the leaves of $\mathcal{F}(\mathcal{V})$ are given by

$$\mathcal{L}_\tau = \left\{ p \in \mathcal{H}^{n+1} : \langle p, a \rangle = \tau \right\}, \ \ \tau > 0,$$

which are totally umbilical spacelike hypersurfaces of $\mathcal{H}^{n+1}$ (that is, the metric on each $\mathcal{L}_\tau$ induced from its inclusion on $\mathcal{H}^{n+1}$ is positive definite), isometric to the Euclidean space $\mathbb{R}^n$ and having constant mean curvature 1 with respect to the timelike unit normal fields

$$N_\tau = -p + \frac{1}{\tau}a, \quad p \in \mathcal{L}_\tau. \tag{2.2}$$

An explicit isometry between the leaves $\mathcal{L}_\tau$ and $\mathbb{R}^n$ can be found at Section 2 of [2]. In this sense, along this work, each $\mathcal{L}_\tau$ will be called a *spacelike hyperplane* of $\mathcal{H}^{n+1}$.

It is also convenient to notice that these spacelike hyperplanes $\mathcal{L}_\tau$ approach to the boundary $\mathcal{L}_0$ of $\mathcal{H}^{n+1}$ when the parameter τ tends to zero and that, when τ tends to $+\infty$, they approach to the *spacelike future infinity* for timelike and null lines of $\mathbb{S}_1^{n+1}$, that, following [8], we will denote by $\mathcal{J}^+$.

3. Characterizing spacelike hyperplanes in the steady state space

In this section, we will deal with spacelike hypersurfaces $\psi : \Sigma^n \to \mathcal{H}^{n+1}$, namely, the induced metric via such an immersion ψ is a Riemannian metric on Σ^n. In this setting, ∇ will denote the Levi-Civita connection of Σ^n and we will choose the orientation N of ψ so that it is past-pointing, which means that N must be in the same half of the null cone of $\mathbb{L}^{n+2}$ as the nonzero null vector a is (in other words, $\langle N, a \rangle < 0$ along Σ^n). The mean curvature function of a spacelike hypersurface Σ^n is defined as $H = \frac{1}{n}\mathrm{tr}(A)$, where A stands for the shape operator (or second fundamental form) of Σ^n with respect to its past-pointing orientation N.

In the paper [19], Yau obtained the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold Σ^n:

If $\omega \in \Omega^{n-1}(\Sigma)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that $B_i \subset B_{i+1}$, $\Sigma^n = \bigcup_{i \geq 1} B_i$ and

$$\lim_{i \to +\infty} \int_{B_i} d\omega = 0.$$

By applying this result to $\omega = \iota_{\nabla f}$, where $f : \Sigma^n \to \mathbb{R}$ is a smooth function, ∇f denotes its gradient and $\iota_{\nabla f}$ the contraction in the direction of ∇f, Yau established an extension of Hopf's Theorem on a complete Riemannian manifold. In what follows, $\mathcal{L}^1(\Sigma)$ will stand for the space of Lebesgue integrable functions on Σ^n.

Lemma 3.1. *Let Σ^n be a complete Riemannian manifold and let $f : \Sigma^n \to \mathbb{R}$ be a smooth function. If f is a subharmonic (or superharmonic) function with $|\nabla f| \in \mathcal{L}^1(\Sigma)$, then f must actually be harmonic.*

Now, we are in position to state and prove our first result.

Theorem 3.2. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with mean curvature $H \geq 1$, which is contained in the closure of the interior domain enclosed by a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by the nonzero null vector $a \in \mathbb{L}^{n+2}$. If one of the following conditions is satisfied:*

(a) *$n = 2$ and the Gaussian curvature of Σ^2 is nonnegative,*
(b) $|\, a^\top | \in \mathcal{L}^1(\Sigma)$,

then Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a.

Proof. Let us consider the support functions $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. We observe that l_a is always a positive function, while, from our choice of orientation N of ψ, f_a will be a negative function. Moreover, a direct computation allows us to conclude that the gradients of such functions are given by $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ stands for the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^\top = a + f_a N - l_a \psi. \tag{3.1}$$

Using Gauss and Weingarten formulas, it is not difficult to verify that

$$\nabla_X \nabla l_a = -f_a AX - l_a X, \tag{3.2}$$

for all $X \in \mathfrak{X}(\Sigma)$. Consequently, from (3.2) we obtain

$$\Delta l_a = -nHf_a - nl_a. \tag{3.3}$$

From (3.1) we get

$$f_a^2 - l_a^2 = |\nabla l_a|^2. \tag{3.4}$$

In particular, from (3.4) we have that

$$0 < l_a \leq -f_a. \tag{3.5}$$

Moreover, from equation (3.3) we also have that

$$\begin{aligned} \frac{1}{2}\Delta l_a^2 &= l_a \Delta l_a + |\nabla l_a|^2 \\ &= -nHl_a f_a - nl_a^2 + |\nabla l_a|^2. \end{aligned} \tag{3.6}$$

Considering (3.5) into (3.6), we obtain

$$\frac{1}{2}\Delta l_a^2 \geq n(H-1)l_a^2 + |\nabla l_a|^2. \tag{3.7}$$

Consequently, since we are supposing that $H \geq 1$, from (3.7) we have that

$$\frac{1}{2}\Delta l_a^2 \geq |\nabla l_a|^2. \tag{3.8}$$

On the other hand, according to [15], we observe that our hypothesis that Σ^n is contained in the closure of the interior domain enclosed by a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a means that $l_a \leq \tau$, for some $\tau > 0$.

Hence, from (3.8) we conclude that l_a^2 is a bounded subharmonic function. However, a classical result due to Huber [10] assures that complete surfaces of non-negative Gaussian curvature must be parabolic. Therefore, if $n = 2$ and the Gaussian curvature of Σ^2 is nonnegative, we get that l_a is constant on Σ^2, that is, Σ^2 is a spacelike plane of $\mathcal{H}^3$ determine by a.

Furthermore, since

$$|\nabla l_a^2| = 2l_a|\nabla l_a| \leq \tau|\nabla l_a|,$$

the hypothesis $|\nabla l_a| = |\,a^\top| \in \mathcal{L}^1(\Sigma)$ implies that $|\nabla l_a^2| \in \mathcal{L}^1(\Sigma)$. Thus, in this case, we can apply Lemma 3.1 in order to conclude that l_a^2 is harmonic. Therefore, taking into account (3.8) once more, we have that l_a is constant on Σ^n.

To finish the proof, we note that from the definition of l_a, if $l_a \equiv \widetilde{\tau}$ on a complete hypersurface Σ^n, then $\Sigma^n \subset \mathcal{L}_{\widetilde{\tau}}$. Therefore, by completeness, we must have $\Sigma^n = \mathcal{L}_{\widetilde{\tau}}$. $\square$

Remark 3.3. Taking into account once more Example 1 of [12], it follows from the description of the totally umbilical hypersurfaces of the steady state space given in Section 3 of [15] that there exists no totally umbilical complete spacelike hypersurfaces with mean curvature $|H| < 1$ in $\mathcal{H}^{n+1}$. So, since our aim in this paper is to obtain characterizations of spacelike hyperplanes of $\mathcal{H}^{n+1}$, it is natural to restrict our attention to complete spacelike hypersurfaces immersed with mean curvature function $H \geq 1$ in $\mathcal{H}^{n+1}$.

We observe that any timelike unit vector field N normal to a spacelike immersion $\psi : \Sigma^n \to \mathcal{H}^{n+1} \subset \mathbb{L}^{n+2}$ can be viewed as a map

$$N : \Sigma^n \to \{p \in \mathbb{L}^{n+2} : \langle p, p\rangle = -1\},$$

where each one of the two sheets of the hyperboloid on the right side are isometric, with the induced metric, to the hyperbolic space $\mathbb{H}^{n+1}$ with constant sectional curvature -1. In this setting, N is said the *hyperbolic Gauss map* of Σ^n.

So, with our choice of orientation of ψ, the hyperbolic Gauss map N of Σ^n takes values in the lower sheet of the corresponding hyperboloid, which will be simply denoted by $\mathbb{H}^{n+1}$. Furthermore, in a similar way of Section 4 of [11], we note that the level sets

$$L_\rho = \left\{p \in \mathbb{H}^{n+1} : \langle p, a\rangle = \rho\right\}, \ \ \rho < 0,$$

give foliate all $\mathbb{H}^{n+1}$ by means of parallel horospheres.

In [15], Montiel have proved that if a complete spacelike hypersurface Σ^n in the de Sitter space $\mathbb{S}_1^{n+1}$ with constant mean curvature $H \geq 1$ is such that the image of its hyperbolic Gauss map is contained in the closure of the interior domain enclosed by a horosphere, then its mean curvature is, in fact, equal to 1. When $n = 2$, from [1] or [16], it follows that Σ^2 is also an umbilical surface. From Theorem 3.2 we obtain a sort of extension of this Montiel's result. More precisely,

Corollary 3.4. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with mean curvature $H \geq 1$. Suppose that the image of the hyperbolic Gauss map of Σ^n is contained in the closure of the interior domain enclosed by a horosphere of $\mathbb{H}^{n+1}$ determined by the nonzero null vector $a \in \mathbb{L}^{n+2}$. If one of the following conditions is satisfied:*

(a) *$n = 2$ and the Gaussian curvature of Σ^2 is nonnegative,*
(b) $|a^\top| \in \mathcal{L}^1(\Sigma)$,

then Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by the null vector a.

Proof. We observe that if $N(\Sigma)$ is contained in the closure of the interior domain enclosed by a horosphere L_ρ of $\mathbb{H}^{n+1}$ determined by a, then we have that $0 > f_a \geq \rho$. Hence, from (3.5) we get that $0 < l_a \leq -\rho$, which means that Σ^n is contained in the closure of the interior domain enclosed by the spacelike hyperplane $\mathcal{L}_{-\rho}$ of $\mathcal{H}^{n+1}$ determined by the null vector a. Therefore, the result follows from Theorem 3.2. □

From (2.2), we see that the support functions $l_a = \langle \psi, a \rangle$ and $f_a = \langle N, a \rangle$ of a spacelike hyperplane $\mathcal{L}_\tau$ of $\mathcal{H}^{n+1}$ satisfy the relation $l_a = -f_a$. This fact allows us to consider complete spacelike hypersurfaces of $\mathcal{H}^{n+1}$ whose support functions are linearly related. In this context, we get the following characterization result:

Theorem 3.5. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface whose support functions with respect to a nonzero null vector $a \in \mathbb{L}^{n+2}$ satisfy the relation $l_a = \lambda f_a$, where $\lambda : \Sigma^n \to \mathbb{R}$ is a smooth function. Suppose that Σ^n is contained in the closure of the interior domain enclosed by a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by the null vector a and that its mean curvature H is such that $H \geq -\lambda$. If one of the following conditions is satisfied:*

(a) *$n = 2$ and the Gaussian curvature of Σ^2 is nonnegative,*
(b) $|a^\top| \in \mathcal{L}^1(\Sigma)$,

then Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a and $\lambda \equiv -1$.

Proof. Initially, from the signs of the functions l_a and f_a, we observe that the sign of the function λ is strictly negative on Σ^n. Thus, from (3.4) we conclude that λ takes its values in the interval $[-1, 0)$.

On the other hand, from equation (3.6) and our hypothesis on the support functions of Σ^n, we have that

$$\Delta l_a^2 = -2n \left(\frac{H}{\lambda} + 1 \right) l_a^2 + 2|\nabla l_a|^2. \tag{3.9}$$

Since Σ^n is contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{L}_\tau$ of $\mathcal{H}^{n+1}$ determined by the null vector a, we have that $l_a \leq \tau$. Consequently, from (3.9) we conclude that l_a^2 is a bounded subharmonic function on Σ^n.

In the case that $n = 2$, if Σ^2 has nonnegative Gaussian curvature, by using once more Huber's result [10], we have that l_a is constant and, hence, Σ^2 must be a spacelike plane of $\mathcal{H}^3$ determined by a.

Now, suppose that $|\,a^\top| \in \mathcal{L}^1(\Sigma)$. Then, ∇l_a^2 has integrable norm on Σ^n. So, from equation (3.9) we conclude, by applying once more Lemma 3.1, that l_a^2 is a harmonic function and, returning to equation (3.9), we get that $|\nabla l_a|^2 = 0$ on Σ^n. Therefore, we also conclude that l_a is constant and, hence, Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a and, from (2.2), we get that $\lambda \equiv -1$. $\square$

Remark 3.6. According to Section 3 of [15] (see also Example 2 of [12] or Section 2 of [13]), the so-called hyperbolic cylinder of the de Sitter space $\mathbb{S}_1^{n+1}$, which are defined by

$$\{p \in \mathbb{S}_1^{n+1} : p_1^2 + \cdots + p_{k+1}^2 = \cosh^2 r\},$$

where $1 \le k \le n-1$ and $r > 0$, has two connected components which are isometric to $\mathbb{S}^k(\cosh r) \times \mathbb{H}^{n-k}(\sinh r)$. Moreover, one of the components of the hyperbolic cylinder is contained in the steady state space $\mathcal{H}^{n+1}$, and it will be denote by $\mathcal{C}_{k,r}$.

It is not difficult to verify that $\mathcal{C}_{k,r}$ has the following past-pointing (that is, contained in the same time cone of the null vector a) hyperbolic Gauss map

$$N(p) = \frac{1}{\cosh r \sinh r}(\xi(p) - \cosh^2 r\, p), \tag{3.10}$$

where $\xi : \mathcal{C}_{k,r} \to \mathbb{L}^{n+2}$ is given by $\xi(p) = (p_1, \ldots, p_{k+1}, 0, \ldots, 0)$. Consequently, from (3.10) we conclude that $\mathcal{C}_{k,r}$ is a isoparametric spacelike hypersurface of $\mathcal{H}^{n+1}$, whose mean curvature H with respect to N is given by

$$H = \frac{1}{n}(k \tanh r + (n-k)\coth r). \tag{3.11}$$

In particular, from (3.11) we get that, for $1 \le k \le \frac{n}{2}$ and all $r > 0$, $H \ge 1$. Moreover, from (3.10) we see that the support functions of $\mathcal{C}_{k,r}$ satisfy the following relation

$$l_a = -\tanh r\, f_a.$$

On the other hand, taking into account once more Lemma 1 of [2], we have that $\mathcal{C}_{k,r}$ cannot be contained in the closure of the interior domain enclosed by any spacelike hyperplane of $\mathcal{H}^{n+1}$.

Proceeding, we rewrite Theorem 3.5 in the following way:

Corollary 3.7. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface whose support functions satisfy the relation $l_a = \lambda f_a$, where $\lambda : \Sigma^n \to \mathbb{R}$ is a smooth function. Suppose that the image of the hyperbolic Gauss map of Σ^n is contained in the closure of the interior domain enclosed by a horosphere of $\mathbb{H}^{n+1}$ determined by the nonzero null vector $a \in \mathbb{L}^{n+2}$, and that its mean curvature H is such that $H \ge -\lambda$. If one of the following conditions is satisfied:*

(a) *$n = 2$ and the Gaussian curvature of Σ^2 is non-negative,*
(b) *$|\,a^\top| \in \mathcal{L}^1(\Sigma)$,*

then Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a and $\lambda \equiv -1$.

Remark 3.8. It is worth to make a brief discussion on the meaning of our assumption in the previous results concerning the integrability of $|a^\top|$ on the spacelike hypersurface Σ^n, both from geometric and physical viewpoints. From the first viewpoint, Lemma 1 of [2] asserts that if a complete spacelike hypersurface is contained in the closure of the interior domain enclosed by a spacelike hyperplane of $\mathcal{H}^{n+1}$, then it must be diffeomorphic to $\mathbb{R}^n$. In particular, it follows that there is no compact (without boundary) spacelike hypersurfaces in $\mathcal{H}^{n+1}$. In this sense, our assumption of $|a^\top| \in \mathcal{L}^1(\Sigma)$ in our previous results comes to supply the fact that Σ^n is noncompact.

On the other hand, some physical interpretation is now in order. In fact, assume $n=3$ and use then the orthogonal decomposition $T_{\psi_p}\mathcal{H}^4 = \mathrm{Span}\{N_p\} \oplus N_p^\perp$, where $p \in \Sigma^3$, $\psi : \Sigma^3 \to \mathcal{H}^4$ is a spacelike hypersurface and $N_p^\perp = d\psi_p(T_p\Sigma)$. Since from (2.1) it follows that $\mathcal{V}^\top = a^\top$, we can write $\mathcal{V}_p = e_p N_p + a_p^\top$, where $e_p = -\langle \mathcal{V}_p, N_p\rangle > 0$ and $a_p^\top$ are, respectively, the energy and the 3-momentum that the instantaneous observer N_p measures for $\mathcal{V}_p$.

Furthermore, the quantity $\dfrac{1}{e_p}a_p^\top$ is the relative velocity (and, hence, $\dfrac{1}{e_p}|a_p^\top|$ is the relative speed) of $\mathcal{V}_p$ with respect to N_p (for more details, we refer Section 2.1.3 of [17]). Note that $|a_p^\top| = \sqrt{-\langle \mathcal{V}_p, \mathcal{V}_p\rangle}\sinh\theta_p$, where θ_p is the hyperbolic angle between $\mathcal{V}_p$ and N_p. Thus, we get $|a_p^\top| = e_p \tanh\theta_p \le e_p$. Consequently, the integrability of $|a^\top|$ on Σ^3 can be regarded as been the 3-momentum of N having integrable norm along Σ^3 and, in particular, such condition is satisfied when the total energy $\displaystyle\int_\Sigma e_p d\Sigma$ is finite.

By considering a complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ as being the boundary of a suitable domain of $\mathcal{H}^{n+1}$, from Theorem 1 of [7] we get the following tangency principle (see also Theorem 2 of [15], for a version corresponding to constant mean curvature spacelike hypersurfaces):

Proposition 3.9. *Let Σ_1 and Σ_2 be complete spacelike hypersurfaces immersed in $\mathcal{H}^{n+1}$ with mean curvatures H_1 and H_2, respectively. Suppose that Σ_1 lies above Σ_2. If, in a neighbourhood of a common tangent point, we have that $H_1 \le \alpha \le H_2$, for some real number α, then Σ_1 and Σ_2 must coincide.*

We refer to the hyperbolic angle θ of a spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ as being the hyperbolic angle between its hyperbolic Gauss map N and the timelike vector field $\mathcal{V}$ defined in (2.1). In other words, $\cosh\theta = -\langle N, \nu\rangle$, where $\nu = \dfrac{\mathcal{V}}{\sqrt{-\langle \mathcal{V}, \mathcal{V}\rangle}}$.

In our next result, we will use Proposition 3.9 to revisit Theorem 3.2 of [6] and present a different and more simple proof of it, without asking that the mean curvature of the spacelike hypersurface be bounded from above.

Theorem 3.10. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with mean curvature $H \ge 1$ and contained in the closure of the interior domain enclosed by a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a nonzero null vector $a \in \mathbb{L}^{n+2}$. If the* *hyperbolic angle θ of Σ^n satisfies $\cosh\theta \le \inf_\Sigma H$, then Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determine by the null vector a.*

Proof. Suppose, by contradiction, that $H_0 = \inf_\Sigma H > 1$. Thus, we consider the family of totally umbilical spacelike hypersurfaces of $\mathcal{H}^{n+1}$ with a given common axis of rotation, having constant mean curvature H_0 and such that their corresponding Gauss map are past-pointing , coming from the future infinity $\mathcal{J}^+$. We observe that, according to the description of the totally umbilical spacelike hypersurfaces of $\mathcal{H}^{n+1}$ due to Montiel in Section 3 of [15], such spacelike hypersurfaces are isometric to suitable hyperbolic spaces. So, in analogy with the context of the hyperbolic geometry, we will call such spacelike hypersurfaces of equidistant hypersurfaces of $\mathcal{H}^{n+1}$.

By a rigid motion of this family, we arrive until the first contact point of Σ^n with one of such equidistant hypersurfaces, which occurs in some common interior point of both hypersurfaces. From Proposition 3.9, we have that Σ^n must be one of these equidistant hypersurfaces. But, taking into account once more the study made by Montiel in Section 3 of [15], we see that such equidistant hypersurfaces cannot be contained in the closure of the interior domain enclosed by a spacelike hyperplane of $\mathcal{H}^{n+1}$. Hence, we arrive at a contradiction and, consequently, $H_0 = 1$.

Therefore, we use the hypothesis $\cosh\theta \le H_0$ to conclude that $\cosh\theta = 1$ on Σ^n, that is, Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$. □

From Theorem 3.10 we obtain the following

Corollary 3.11. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface with mean curvature $H \ge 1$ and contained in the closure of the exterior domain enclosed by a spacelike hyperplane $\mathcal{L}_\tau$ of $\mathcal{H}^{n+1}$ determined by a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that the hyperbolic image $N(\Sigma)$ is contained in the closure of the interior domain enclosed by a horosphere L_ρ. If $-\frac{\rho}{\tau} \le \inf_\Sigma H$, then Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by the null vector a.*

Proof. Initially, we observe that the hyperbolic angle θ of Σ^n is such that

$$\cosh\theta = -\langle N, \nu\rangle = -\langle N, -\psi + \frac{1}{\langle \psi, a\rangle} a\rangle = -\frac{1}{\langle \psi, a\rangle}\langle N, a\rangle. \tag{3.12}$$

Consequently, since we are supposing that Σ^n is contained in the closure of the exterior domain enclosed by the spacelike hyperplane $\mathcal{L}_\tau$, from (3.12) we get

$$\cosh\theta \le -\frac{1}{\tau}\langle N, a\rangle. \tag{3.13}$$

Hence, taking into account our hypothesis on the image of the hyperbolic Gauss map of Σ^n, (3.13) amounts to

$$\cosh\theta \le -\frac{\rho}{\tau} \le \inf_\Sigma H.$$

Therefore, since inequality (3.5) guarantees that Σ^n is also contained in the closure of the interior domain enclosed by the spacelike hyperplane $\mathcal{L}_{-\rho}$, we can apply Theorem 3.10 in order to conclude that Σ^n is a spacelike hyperplane of $\mathcal{H}^{n+1}$ determined by a. □

Remark 3.12. As it was already observed in Remark 3.4 of [6], when Σ^n is a compact spacelike hypersurface immersed with constant mean curvature $H > 1$ in $\mathcal{H}^{n+1}$, whose boundary $\partial\Sigma$ is contained in a spacelike hyperplane of $\mathcal{H}^{n+1}$, Theorem 7 of [15] assures that the hyperbolic angle θ of Σ^n satisfies the estimate $\cosh\theta \leq H$. In this sense, our restriction on the hyperbolic angle in Theorem 3.10 is a mild hypothesis.

Acknowledgements

The first author is partially supported by CNPq, Brazil, grant 302738/2014-2. The second author is partially supported by CNPq, Brazil, grant 300769/2012-1. The third author is partially supported by CAPES, Brazil. The second and fourth authors are partially supported by CAPES/CNPq, Brazil, grant Casadinho/Procad 552.464/2011-2. The authors would like to thank the referee for giving some valuable suggestions which improved the paper.

References

[1] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z., **196** (1987), 13–19.

[2] A.L. Albujer and L.J. Alías, *Spacelike hypersurfaces with constant mean curvature in the steady state space*, Proc. American Math. Soc., **137** (2009), 711-721.

[3] H. Bondi and T. Gold, *On the generation of magnetism by fluid motion*, Monthly Not. Roy. Astr. Soc., **108** (1948), 252-270.

[4] A. Caminha and H. F. de Lima, *Complete vertical graphs with constant mean curvature in semi-Riemannian warped products*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), 91–105.

[5] A.G. Colares and H.F. de Lima, *Spacelike hypersurfaces with constant mean curvature in the steady state space*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 287–302.

[6] A.G. Colares and H.F. de Lima, *On the rigidity of spacelike hypersurfaces immersed in the steady state space $\mathcal{H}^{n+1}$*, Publ. Math. Debrecen **81** (2012), 103–119.

[7] J.-H. Eschenburg, *Maximum principles for hypersurfaces*, Manuscripta Math. **64** (1989), 55–75.

[8] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge Univ. Press, Cambridge (1973).

[9] F. Hoyle, *A new model for the expanding universe*, Monthly Not. Roy. Astr. Soc., **108** (1948), 372-382.

[10] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., **32** (1957), 13–72.

[11] R. López and S. Montiel, *Existence of constant mean curvature graphs in hyperbolic space*, Calc. Var. and PDE **8** (1999), 177–190.

[12] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), 909–917.

[13] S. Montiel, *A characterization of hyperbolic cylinders in the de Sitter space*, Tôhoku Math. J. **48** (1996), 23–31.

[14] S. Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), 711–748.

[15] S. Montiel, *Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces*, J. Math. Soc. Japan **55** (2003), 915–938.

[16] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana Univ. Math. J., **36** (1987), 349–359.

[17] R.K. Sachs and H. Wu, *General Relativity for Mathematicians*, Graduate Texts in Mathematics Vol. 48, Springer-Verlag, New York (1977).

[18] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, New York (1972).

[19] S.T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 659–670.

Cícero P. Aquino
Departamento de Matemática
Universidade Federal do Piauí
64.049-550 Teresina, Piauí, Brazil
e-mail: `cicero.aquino@ufpi.edu.br`

Henrique F. de Lima, Fábio R. dos Santos and Marco Antonio L. Velásquez
Departamento de Matemática
Universidade Federal de Campina Grande
58.409-970 Campina Grande, Paraíba, Brazil
e-mail: `henrique@dme.ufcg.edu.br`
`fabio@dme.ufcg.edu.br`
`marco.velasquez@pq.cnpq.br`

Received: October 13, 2014.