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A Class of Quasilinear Dirichlet Problems with Unbounded Coefficients and Singular Quadratic Lower Order Terms

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Abstract. We study existence and regularity of positive solutions of problems like

$$\begin{cases} -\mathrm{div}([a(x)+u^q]\nabla u)+b(x)\dfrac{1}{u^\theta}|\nabla u|^2=f & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

depending on the values of $q>0$, $0<\theta<1$, and on the summability of the datum $f\geq 0$ in Lebesgue spaces.

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Dedicato a David Arcoya (maestro di tutti noi)
per i suoi cinquant'anni

1. Introduction

In this paper we are going to study the existence of solutions for the problem

$$\begin{cases} -\mathrm{div}([a(x)+u^q]\nabla u)+b(x)\dfrac{1}{u^\theta}|\nabla u|^2=f & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We assume that Ω is a bounded, open set of $\mathbb{R}^N$ $(N>2)$, that

$$q>0\,,\quad 0<\theta<1\,, \tag{1.2}$$

$$f\geq 0\,,\ f\not\equiv 0\,,\qquad f\in L^m(\Omega)\,,\ m\geq 1\,, \tag{1.3}$$

and that $a(x)$ and $b(x)$ are measurable functions such that

$$0<\alpha\leq a(x)\leq\beta, \tag{1.4}$$

$$0 < \mu \leq b(x) \leq \nu. \tag{1.5}$$

The boundary value problem (1.1) is a quasilinear elliptic problem having a lower order term with quadratic growth with respect to the gradient. The interest in the study of this kind of problems arises naturally since the Euler-Lagrange equations of some integral functionals of the Calculus of Variations are of this form. This is one of the reasons why the quadratic growth is also called “natural”. If the principal part is like a p-Laplace operator, the natural growth of the lower order term is of order p. A general theory of the existence and the motivation of the study in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ can be found in [12]. Furthermore, simple examples of integral functionals show that the assumption

“the quadratic lower order term has the same sign of the solution”

is natural, and it allows (see [11]) to prove existence of unbounded solutions (always in $W_0^{1,p}(\Omega)$). Such assumption was also used in [8] to prove the regularizing effect of the lower order term: i.e., existence of finite energy solutions even if the right hand side is only a summable function (if the datum is a measure, nonexistence results can be found in [9]). A complete study of these problems can be found in [5], [15] (see also the papers cited therein).

Recently, a problem introduced by D. Arcoya (see [1] and [2]) gave a strong impulse to the study of quasilinear problems having a quadratic lower order term which becomes singular where the solution is zero, since it depends on a negative power of the solution. This is the case of problem (1.1), which has the added difficulty of having an unbounded elliptic operator. Problems like (1.1) can be seen (at least formally) as Euler-Lagrange equations of functional integrals of the Calculus of Variations. For example, if f belongs to $L^2(\Omega)$, and

$$q = 1 - \theta\,, \quad \text{and} \quad b(x) \equiv \frac{1-\theta}{2}\,,$$

then solutions of (1.1) are minima of the functional

$$J(v) = \frac{1}{2}\int_\Omega [a(x) + |v|^{1-\theta}]|\nabla v|^2 - \int_\Omega f\, v\,, \tag{1.6}$$

defined on a suitable subset of $W_0^{1,2}(\Omega)$.

The study of problems with these features was developed in several recent papers (see [6], [3], and the references therein); here we follow the approach of [6] (see Section 2 for the details).

Our main result is the following.

Theorem 1.1. *Suppose that f belongs to $L^1(\Omega)$, and that (1.2), (1.4) and (1.5) hold true. Then there exists a solution u of (1.1), with $u > 0$ in Ω,*

$$[a(x) + u^q]|\nabla u| \in L^\rho(\Omega)\,,\ \forall \rho < \frac{N}{N-1}\,, \qquad b(x)|\nabla u|^2\, u^{-\theta} \in L^1(\Omega)\,,$$

and

$$\int_\Omega [a(x) + u^q]\, \nabla u \nabla \varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\, \varphi = \int_\Omega f\, \varphi\,, \tag{1.7}$$

for every φ in $W_0^{1,p}(\Omega)$, $p > N$. Furthermore, we have the following summability results for u:

$$\begin{cases} \text{if } 0 < q \le 1-\theta, \ u \text{ belongs to } W_0^{1,r}(\Omega), \text{ with } r = \frac{N(2-\theta)}{N-\theta}; \\ \text{if } 1-\theta < q \le 1, \ u \text{ belongs to } W_0^{1,r}(\Omega), \text{ for every } r < \frac{N(q+1)}{N+q-1}; \\ \text{if } q > 1, \text{ then } u \text{ belongs to } W_0^{1,2}(\Omega). \end{cases} \tag{1.8}$$

Remark 1.2. Remark that $\frac{N(2-\theta)}{N-\theta} < \frac{N(q+1)}{N+q-1}$ if $q > 1-\theta$.

We will prove Theorem 1.1 by approximating problem (1.1) with a sequence of nonsingular quasilinear quadratic problems with bounded data, and then proving both *a priori* estimates and convergence results on the sequence of approximating solutions (see Lemma 2.1 and Lemma 2.5). We will then prove regularity results on the solutions, depending on the summability of the datum f and on the possible values of q and θ. In the final sections, we will study the minimization of functionals like (1.6) (and the connections to (1.1)), and the correct assumptions on the datum f in order to have test functions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ instead of Lipschitz function as in (1.7).

We will make frequent use, in what follows, of the truncation function $T_k(s) = \max(-k, \min(s,k))$, defined for $k > 0$ and s in $\mathbb{R}$, and of its "companion" function $G_k(s) = s - T_k(s)$.

2. Proof of the main result

As stated in the Introduction, we approximate problem (1.1) by a sequence of non-singular, quadratic quasilinear problems with bounded data.

Take $0 < \varepsilon < 1$ belonging to a sequence converging to zero, and consider the following problems

$$\begin{cases} -\text{div}([a(x) + |u_\varepsilon|^q]\nabla u_\varepsilon) + b(x)\dfrac{u_\varepsilon |\nabla u_\varepsilon|^2}{(|u_\varepsilon| + \varepsilon)^{\theta+1}} = \dfrac{f}{1+\varepsilon f} & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

From the results of [8], [11], it follows the existence of a solution u_ε in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Moreover $u_\varepsilon \ge 0$ since the right hand side is positive (by the assumptions on f) and since the quadratic lower order term has the same sign of the solution. Therefore u_ε solves

$$\begin{cases} -\text{div}([a(x) + u_\varepsilon^q]\nabla u_\varepsilon) + b(x)\dfrac{u_\varepsilon |\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}} = \dfrac{f}{1+\varepsilon f} & \text{in } \Omega, \\ u_\varepsilon \ge 0, & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the sense that u_ε satisfies

$$\int_\Omega [a(x) + u_\varepsilon^q]\nabla u_\varepsilon \nabla\Phi + \int_\Omega \frac{b(x)\, u_\varepsilon\, |\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}}\, \Phi = \int_\Omega \frac{f}{1+\varepsilon f}\Phi \tag{2.2}$$

for every test function Φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

We are going to study the properties of the sequence $\{u_\varepsilon\}$ of solutions of (2.1) in the following lemmas, with the aim of passing to the limit in order to obtain a solution of (1.1). Note that *a priori* estimates are not enough due to the nonlinear nature of the equation, so that strong convergence results (in suitable spaces) will be necessary.

Our first result yields some *a priori* estimates on $\{u_\varepsilon\}$.

Lemma 2.1. *Suppose that* (1.2), (1.3), (1.4), *and* (1.5) *hold true. Then the sequence* $\{u_\varepsilon\}$ *satisfies the following estimates for every* $\varepsilon>0$, *and for every* $k>0$*:*

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\le\int_\Omega f\,, \tag{2.3}$$

$$\frac{1}{k}\int_\Omega[a(x)+u_\varepsilon^q]|\nabla T_k(u_\varepsilon)|^2\le\int_\Omega f\,\frac{T_k(u_\varepsilon)}{k}\,. \tag{2.4}$$

Furthermore, the sequence $\{T_k(u_\varepsilon)\}$ *is bounded in* $W_0^{1,2}(\Omega)$, *the sequence* $\{u_\varepsilon\}$ *is bounded in* $W_0^{1,r}(\Omega)$, *with* r *as in the statement of Theorem* 1.1, *and the sequence* $u_\varepsilon^q|\nabla u_\varepsilon|$ *is bounded in* $L^\rho(\Omega)$, *for every* $\rho<\frac{N}{N-1}$.

Remark 2.2. As a consequence of Lemma 2.1, there exists a subsequence (not relabeled) and a function $u\in W_0^{1,r}(\Omega)$ (with r as in the statement of Theorem 1.1) such that u_ε almost everywhere converges to u, and $T_k(u_\varepsilon)$ weakly converges to $T_k(u)$ in $W_0^{1,2}(\Omega)$ for every $k>0$.

Proof of Lemma 2.1. Take $k>0$, and choose $\frac{T_k(u_\varepsilon)}{k}$ as test function in (2.1). We obtain

$$\begin{aligned}\frac{1}{k}\int_\Omega[a(x)+u_\varepsilon^q]|\nabla T_k(u_\varepsilon)|^2+\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_k(u_\varepsilon)}{k}\\ \le\int_\Omega\frac{f}{1+\varepsilon f}\frac{T_k(u_\varepsilon)}{k}\,.\end{aligned} \tag{2.5}$$

Dropping the nonnegative first term, we obtain

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_k(u_\varepsilon)}{k}\le\int_\Omega\frac{f}{1+\varepsilon f}\frac{T_k(u_\varepsilon)}{k}\le\int_\Omega f\,.$$

Letting k tend to 0 we deduce (2.3) by Fatou's Lemma.

On the other hand, dropping the nonnegative second term of (2.5), we have

$$\frac{1}{k}\int_\Omega[a(x)+u_\varepsilon^q]|\nabla T_k(u_\varepsilon)|^2\le\int_\Omega\frac{f}{1+\varepsilon f}\,\frac{T_k(u_\varepsilon)}{k}\le\int_\Omega f\,\frac{T_k(u_\varepsilon)}{k}\,, \tag{2.6}$$

i.e., (2.4) holds true. As a consequence of (2.6) and using (1.4) it easily follows the boundedness (with respect to ε) of the sequence $\{T_k(u_\varepsilon)\}$ in $W_0^{1,2}(\Omega)$.

Next, we study the estimates of the sequence $\{u_\varepsilon\}$ in $W_0^{1,r}(\Omega)$. We split the proof in three parts according to the values of q and θ.

If $0<q\le1-\theta$, starting from (2.3), and using (1.5), we have

$$\frac{\mu}{2^{\theta+1}}\int_{\{u_\varepsilon\ge1\}}\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^\theta}\le\mu\int_{\{u_\varepsilon\ge1\}}\frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^{\theta+1}}\,|\nabla u_\varepsilon|^2\le\int_\Omega f\,.$$

Then, if $r < 2$, and thanks to Hölder inequality,

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^r = \int_\Omega \frac{|\nabla G_1(u_\varepsilon)|^r}{u_\varepsilon^{\theta\frac{r}{2}}} u_\varepsilon^{\theta\frac{r}{2}} \le C_1 \bigg(\int_{\{u_\varepsilon \ge 1\}} u_\varepsilon^{\frac{\theta r}{2-r}} \bigg)^{\frac{2-r}{2}} . \tag{2.7}$$

Choosing r such that $r^* = \frac{\theta r}{2-r}$, we obtain $r = \frac{N(2-\theta)}{N-\theta} < 2$ so that, by Sobolev inequality,

$$\bigg(\int_\Omega G_1(u_\varepsilon)^{r^*} \bigg)^{\frac{r}{r^*}} \le C_2 \bigg(\int_{\{u_\varepsilon \ge 1\}} u_\varepsilon^{r^*} \bigg)^{\frac{\theta}{r^*}} \le C_3 \bigg(\int_\Omega G_1(u_\varepsilon)^{r^*} \bigg)^{\frac{\theta}{r^*}} + C_3 .$$

Since $\theta < r$ (as it is easily seen), the last estimate implies that $G_1(u_\varepsilon)$, hence u_ε, is bounded in $L^{r^*}(\Omega)$. Using (2.7), we then have that $G_1(u_\varepsilon)$ is bounded in $W_0^{1,r}(\Omega)$. Since $T_1(u_\varepsilon)$ is bounded in $W_0^{1,2}(\Omega)$, hence in $W_0^{1,r}(\Omega)$, we have that u_ε is bounded in $W_0^{1,r}(\Omega)$, as desired.

If $1-\theta < q \le 1$, choose as test function $1-(1+u_\varepsilon)^{1-\lambda}$, with $\lambda > 1$. Dropping positive terms, and using (1.4), we obtain,

$$\int_\Omega \frac{\alpha + u_\varepsilon^q}{(1+u_\varepsilon)^\lambda} |\nabla u_\varepsilon|^2 \le \int_\Omega f ,$$

which then implies (since $q \le 1$)

$$\min(\alpha, 1) \int_\Omega \frac{(1+u_\varepsilon)^q}{(1+u_\varepsilon)^\lambda} |\nabla u_\varepsilon|^2 \le \int_\Omega f .$$

If $r < 2$, we then have, as before,

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^r &= \int_\Omega \frac{|\nabla u_\varepsilon|^r}{(1+u_\varepsilon)^{\frac{r(\lambda-q)}{2}}} (1+u_\varepsilon)^{\frac{r(\lambda-q)}{2}} \\ &\le \bigg(C_4 \int_\Omega f \bigg)^{\frac{2}{r}} \bigg(\int_\Omega (1+u_\varepsilon)^{\frac{r(\lambda-q)}{2-r}} \bigg)^{\frac{2-r}{r}} . \end{aligned}$$

Choosing r such that $r^* = \frac{r(\lambda-q)}{2-r}$, we have $r = \frac{N(2+q-\lambda)}{N+q-\lambda}$; since $\lambda > 1$, we have $r < \frac{N(q+1)}{N+q-1} < 2$. Thus,

$$\bigg(\int_\Omega u_\varepsilon^{r^*} \bigg)^{\frac{r}{r^*}} \le C_5 \bigg(\int_\Omega (1+u_\varepsilon)^{r^*} \bigg)^{\frac{\lambda-q}{r^*}} ,$$

which, since $\lambda - q < r$, implies the boundedness of u_ε in $L^{r^*}(\Omega)$. This boundedness then implies the boundedness of u_ε in $W_0^{1,r}(\Omega)$, as desired.

If $q > 1$, we choose as test function $1-(1+u_\varepsilon)^{1-q}$, which yields

$$\frac{\min(\alpha,1)}{2^{q-1}} \int_\Omega |\nabla u_\varepsilon|^2 \le \min(\alpha, 1) \int_\Omega \frac{1+u_\varepsilon^q}{(1+u_\varepsilon)^q} |\nabla u_\varepsilon|^2 \le \int_\Omega f ,$$

from which the boundness of $\{u_\varepsilon\}$ in $W_0^{1,2}(\Omega)$ follows.

Finally, starting again from

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(1+u_\varepsilon)^{\lambda-q}} \le \frac{1}{\min(\alpha,1)} \int_\Omega f ,$$

which holds for every $\lambda > 1$, we have

$$\int_\Omega u_\varepsilon^{q\rho}|\nabla u_\varepsilon|^\rho \leq \int_\Omega \frac{|\nabla u_\varepsilon|^\rho}{(1+u_\varepsilon)^{\frac{\rho(\lambda-q)}{2}}}(1+u_\varepsilon)^{\frac{\rho(\lambda+q)}{2}}$$
$$\leq \left(\frac{1}{\min(\alpha,1)}\int_\Omega f\right)^{\frac{\rho}{2}}\left(\int_\Omega (1+u_\varepsilon)^{\frac{\rho(\lambda+q)}{2-\rho}}\right)^{\frac{2-\rho}{2}},$$

which then implies

$$\left(\int_\Omega u_\varepsilon^{(q+1)\rho^*}\right)^{\frac{\rho}{\rho^*}} \leq C_6\left(\int_\Omega u_\varepsilon^{\frac{\rho(\lambda+q)}{2-\rho}}\right)^{\frac{2-\rho}{2}}.$$

Choosing ρ so that $(q+1)\rho^* = \frac{\rho(\lambda+q)}{2-\rho}$ yields $\rho = \frac{N(2+q-\lambda)}{N(q+1)-(\lambda+q)}$. Since $\lambda > 1$, we have an estimate on $u_\varepsilon^q|\nabla u_\varepsilon|$ in $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, as desired. □

The next result yields the strict positivity of u. Before stating it, let us define for $t \geq 0$ the functions

$$H_\varepsilon(t) = \frac{(t+\varepsilon)^{1-\theta}-\varepsilon^{1-\theta}}{1-\theta}, \qquad H_0(t) = \frac{t^{1-\theta}}{1-\theta}, \tag{2.8}$$

and

$$\Phi_\varepsilon(t) = \mathrm{e}^{-\nu\frac{H_\varepsilon(t)}{\alpha}}, \qquad \Phi_0(t) = \mathrm{e}^{-\nu\frac{H_0(t)}{\alpha}}. \tag{2.9}$$

Lemma 2.3. *Suppose that* (1.2), (1.3) (1.4) *and* (1.5) *hold true. If u is given by Remark* 2.2, *then $u > 0$ in Ω.*

Proof. Let v be fixed in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, with $v \geq 0$, and choose $v\,\Phi_\varepsilon(u_\varepsilon)$ as a test function in (2.1), which can be done since it belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Hence, using that

$$\Phi_\varepsilon'(t) = -\frac{\nu}{\alpha}\frac{1}{(\varepsilon+t)^\theta}\Phi_\varepsilon(t),$$

we obtain

$$\int_\Omega [a(x)+u_\varepsilon^q]\nabla u_\varepsilon \nabla v\,\Phi_\varepsilon(u_\varepsilon) - \frac{\nu}{\alpha}\int_\Omega [a(x)+u_\varepsilon^q]\frac{|\nabla u_\varepsilon|^2}{(\varepsilon+u_\varepsilon)^\theta}\,\Phi_\varepsilon(u_\varepsilon)\,v$$
$$+\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\Phi_\varepsilon(u_\varepsilon)\,v = \int_\Omega \frac{f}{1+\varepsilon f}v\,\Phi_\varepsilon(u_\varepsilon).$$

Since $v \geq 0$, using (1.4) and (1.5), we have

$$\int_\Omega [a(x)+u_\varepsilon^q]\nabla u_\varepsilon \nabla v\,\Phi_\varepsilon(u_\varepsilon) - \frac{\nu}{\alpha}\int_\Omega \alpha\frac{|\nabla u_\varepsilon|^2}{(\varepsilon+u_\varepsilon)^\theta}\Phi_\varepsilon(u_\varepsilon)\,v$$
$$+\int_\Omega \nu\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^\theta}\Phi_\varepsilon(u_\varepsilon)\,v \geq \int_\Omega \frac{f}{1+f}v\,\Phi_\varepsilon(u_\varepsilon).$$

Hence,

$$\int_\Omega \{\Phi_\varepsilon(u_\varepsilon)[a(x)+u_\varepsilon^q]\}\nabla u_\varepsilon \nabla v \geq \int_\Omega \frac{f}{1+f}v\,\Phi_\varepsilon(u_\varepsilon), \tag{2.10}$$

for all v in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with $v \geq 0$.

Now, given $\delta > 0$, define the function

$$\psi_\delta(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ -\frac{1}{\delta}(t-1-\delta) & \text{if } 1 \le t < \delta+1, \\ 0 & \text{if } \delta+1 \le t, \end{cases}$$

and fix a function φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \ge 0$. Taking $v = \psi_\delta(u_\varepsilon)\varphi$ in (2.10) we have

$$\int_\Omega \psi_\delta(u_\varepsilon)\,\Phi_\varepsilon(u_\varepsilon)\,[a(x)+u_\varepsilon^q]\nabla u_\varepsilon \nabla\varphi \ge \int_\Omega \frac{f}{1+f}\Phi_\varepsilon(u_\varepsilon)\,\psi_\delta(u_\varepsilon)\,\varphi + \frac{1}{\delta}\int_{\{1\le u_\varepsilon(x)<\delta+1\}} \Phi_\varepsilon(u_\varepsilon)[a(x)+u_\varepsilon^q]|\nabla u_\varepsilon|^2\,\varphi\,.$$

and thus, dropping the positive term,

$$\int_\Omega \psi_\delta(u_\varepsilon)\,\Phi_\varepsilon(u_\varepsilon)\,[a(x)+u_\varepsilon^q]\nabla u_\varepsilon \nabla\varphi \ge \int_\Omega \frac{f\,\Phi_\varepsilon(u_\varepsilon)}{1+f}\psi_\delta(u_\varepsilon)\,\varphi\,.$$

Then, passing to the limit as δ tends to zero, we obtain

$$\int_\Omega \Phi_\varepsilon(T_1(u_\varepsilon))[a(x)+T_1(u_\varepsilon)^q]\nabla T_1(u_\varepsilon)\nabla\varphi \ge \int_{\{0\le u_\varepsilon<1\}} \frac{f\,\Phi_\varepsilon(T_1(u_\varepsilon))}{1+f}\,\varphi\,.$$

Since, by Remark 2.2, $\nabla T_1(u_\varepsilon)$ weakly converges in $(L^2(\Omega))^N$, we can pass to the limit in ε even if our original problem is nonlinear, to obtain

$$\int_\Omega \Phi_0(T_1(u))[a(x)+T_1(u)^q]\nabla T_1(u)\nabla\varphi \ge \int_{\{0\le u\le 1\}} \frac{f\,\Phi_0(T_1(u))}{1+f}\varphi\,,$$

for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\varphi \ge 0$.

If we define

$$w(x) = \int_0^{T_1(u(x))} \Phi_0(t)\,dt\,,$$

we have that w belongs to $W_0^{1,2}(\Omega)$; furthermore, since

$$\Phi_0(T_1(u)) \ge \Phi_0(1) = \mathrm{e}^{-\frac{\nu}{\alpha(1-\theta)}} > 0\,,$$

we deduce from the last inequality that

$$\int_\Omega [a(x)+T_1(u)^q]\nabla w\,\nabla\varphi \ge \int_\Omega \left[\frac{T_1(f)\,\mathrm{e}^{-\frac{\nu}{\alpha(1-\theta)}}}{1+f}\chi_{\{0\le u\le 1\}}\right]\varphi\,, \tag{2.11}$$

for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\varphi \ge 0$, and then, by density, for every nonnegative φ in $W_0^{1,2}(\Omega)$. Hence, w is a supersolution of a linear Dirichlet problem with a strictly positive and bounded, measurable coefficient, since

$$\alpha \le a(x) + T_1(u)^q \le \beta + 1\,,$$

and with right hand side a nonnegative function, not indentically zero. The strong maximum principle (see [14]) then implies that $w > 0$ in Ω. Since $T_1(u) \ge w$ (due to the fact that $\Phi_0(t) \le 1$), we conclude that $T_1(u) > 0$ in Ω, which then implies that $u > 0$ in Ω, since $u \ge T_1(u)$. □

Remark 2.4. The conclusion of Lemma 2.3 is a consequence of the strong maximum principle. Moreover, Harnack's inequality gives the stronger conclusion: if $\omega \subset\subset \Omega$, then there exists $c_\omega > 0$ such that $u \geq c_\omega > 0$.

Now we prove that the gradients of the approximating solutions u_ε almost everywhere converge in Ω. Due to the nonlinearity of the equation, this result will be crucial in order to pass to the limit in the approximate equations. Related results can be found in [4] and [10].

Lemma 2.5. *Suppose that* (1.2), (1.3), (1.4), *and* (1.5) *hold true. If u is given by Remark* 2.2, *then there exists a subsequence (not relabelled) such that $\{\nabla u_\varepsilon\}$ converges to ∇u almost everywhere in Ω. Furthermore, u is such that $b(x)|\nabla u|^2 u^{-\theta}$ belongs to $L^1(\Omega)$, $[a(x)+u^q]|\nabla u|$ belongs to $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, and for every $k > 0$ we have*

$$\frac{1}{k}\int_\Omega [a(x)+u^q]|\nabla T_k(u)|^2 \leq \int_\Omega f\,\frac{T_k(u)}{k}\,. \tag{2.12}$$

Proof. Given h, $k > 0$, we choose $T_h[u_\varepsilon - T_k(u)]$ as a test function in (2.1) to obtain, using (2.3), that

$$\alpha\int_\Omega |\nabla T_h[u_\varepsilon - T_k(u)]|^2 \leq 2h\,\|f\|_{L^1(\Omega)} - \int_\Omega [a(x)+u_\varepsilon^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)]\,.$$

Setting $M = h+k$, we remark that $\nabla T_h[u_\varepsilon - T_k(u)] \neq 0$ implies $u_\varepsilon \leq M$. Hence,

$$\begin{aligned}&\int_\Omega [a(x)+u_\varepsilon^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)]\\ &\qquad = \int_\Omega [a(x)+T_M(u_\varepsilon)^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)]\,.\end{aligned}$$

Since $T_h[u_\varepsilon - T_k(u)]$ weakly converges to $T_h[u - T_k(u)]$ in $(L^2(\Omega))^N$, while $[a(x) + T_M(u_\varepsilon)^q]\nabla T_k(u)$ strongly converges to $[a(x)+T_M(u)^q]\nabla T_k(u)$ in the same space, we have

$$\lim_{\varepsilon\to 0^+}\int_\Omega [a(x)+u_\varepsilon^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)] = 0\,,$$

since $\nabla T_k(u)\nabla T_h[u - T_k(u)] \equiv 0$. Therefore,

$$\alpha\limsup_{\varepsilon\to 0^+}\int_\Omega |\nabla T_h[u_\varepsilon - T_k(u)]|^2 \leq 2h\,\|f\|_{L^1(\Omega)}\,. \tag{2.13}$$

Now, let $s < r \leq 2$, where r is as in the statement of Theorem 1.1. If R is such that the norm of $\{u_\varepsilon\}$ in $W_0^{1,r}(\Omega)$ is bounded by R (see Lemma 2.1), we have

$$\begin{aligned}\int_\Omega |\nabla(u_\varepsilon - u)|^s &= \int_{\{|u_\varepsilon - u|\leq h,\, u\leq k\}} |\nabla(u_\varepsilon - u)|^s\\ &\quad + \int_{\{|u_\varepsilon - u|\leq h,\, u>k\}} |\nabla(u_\varepsilon - u)|^s + \int_{\{|u_\varepsilon - u|>h\}} |\nabla(u_\varepsilon - u)|^s\\ &\leq \int_\Omega |\nabla T_h[u_\varepsilon - T_k(u)]|^s + 2^s R^s \text{meas}(\{u>k\})^{1-\frac{s}{r}}\\ &\quad + 2^s R^s \text{meas}(\{|u_\varepsilon - u|>h\})^{1-\frac{s}{r}}\,.\end{aligned}$$

Since u_ε converges to u in measure, using Hölder's inequality and (2.13), we deduce for every $h > 0$ and $k > 0$, that

$$\limsup_{\varepsilon\to 0^+}\int_\Omega |\nabla(u_\varepsilon - u)|^s \le \left(\frac{2\,h\,\|f\|_{L^1(\Omega)}}{\alpha}\right)^{\frac{s}{2}} \mathrm{meas}(\Omega)^{1-\frac{s}{2}} + 2^{s-1}R^s \mathrm{meas}(\{u > k\})^{1-\frac{s}{r}}\,.$$

Letting h tend to zero, and then k tends to infinity, we obtain

$$\lim_{\varepsilon\to 0^+}\int_\Omega |\nabla(u_\varepsilon - u)|^s = 0\,, \tag{2.14}$$

which then implies that (up to a subsequence) ∇u_ε almost everywhere converges to ∇u in Ω.

Using the almost everywhere convergence of both ∇u_ε and u_ε, Fatou lemma and Lebesgue theorem, we can pass to the limit in (2.4) to have that

$$\frac{1}{k}\int_\Omega [a(x) + u^q]|\nabla T_k(u)|^2 \le \int_\Omega f\,\frac{T_k(u)}{k}\,,$$

which is exactly (2.12).

Furthermore, the fact that ∇u_ε converges to ∇u almost everywhere in Ω, (2.3), and Fatou Lemma imply

$$\int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta} \le \int_\Omega f\,,$$

which is what we wanted to prove.

Finally, using the almost everywhere convergence of the sequence ∇u_ε, the boundedness of $u_\varepsilon^q|\nabla u_\varepsilon|$ in $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, and Fatou Lemma we obtain that $u^q|\nabla u|$ belongs to $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, as desired. □

We are now ready to prove the main result of this paper.

Proof of Theorem 1.1. We are going to prove that the weak limit u given by Remark 2.2 is a solution of the singular problem (1.1). By Remark 2.2 and Remark 2.4 we recall that u belongs to $W_0^{1,r}(\Omega)$, and is such that $u > 0$ in Ω. Moreover, $[a(x) + u^q]|\nabla u|$ and $b(x)u^{-\theta}|\nabla u|^2$ both belong to $L^1(\Omega)$ by Lemma 2.5.

In order to prove the result, we have to pass to the limit in (2.2). To this aim, let $0 \le B(s) \le 1$ be a function in $C^1(\mathbb{R})$ such that

$$B(s) = \begin{cases} 1 & \text{if } 0 \le s \le \frac{1}{2}, \\ 0 & \text{if } s \ge 1. \end{cases}$$

Furthermore, if $k > 0$, and u as in Remark 2.2, we define

$$Q(k) = \int_\Omega f\,\frac{T_k(u)}{k}\,.$$

Remark that by Lebesgue theorem, and the assumptions on f, one has

$$\lim_{k\to+\infty} Q(k) = 0\,. \tag{2.15}$$

The proof of the result will be achieved in two steps.

Step 1. The first inequality. We fix $\psi \in W_0^{1,p}(\Omega)$, $p > N$, with $\psi \geq 0$ and take

$$\Phi = \psi\, B\Big(\frac{u_\varepsilon}{k}\Big)$$

as a test function in (2.2). Since

$$\nabla\Phi = B\Big(\frac{u_\varepsilon}{k}\Big)\nabla\psi + \frac{\psi}{k}B'\Big(\frac{u_\varepsilon}{k}\Big)\nabla u_\varepsilon = \nabla\psi B\Big(\frac{u_\varepsilon}{k}\Big) + \frac{\psi}{k}B'\Big(\frac{u_\varepsilon}{k}\Big)\nabla T_k(u_\varepsilon)\,,$$

by the assumptions on B, we have

$$\begin{aligned}&\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\nabla T_k(u_\varepsilon)\nabla\psi B\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad+\frac{1}{k}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]|\nabla T_k(u_\varepsilon)|^2\psi\, B'\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad+\int_\Omega \frac{b(x)\,T_k(u_\varepsilon)\,|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^{\theta+1}}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big) = \int_\Omega \frac{f}{1+\varepsilon f}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big)\,.\end{aligned}$$

Hence, using (2.4), we have

$$\begin{aligned}&\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\nabla T_k(u_\varepsilon)\nabla\psi B\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad+\int_\Omega \frac{b(x)\,T_k(u_\varepsilon)\,|\nabla T_k(u_\varepsilon)|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad\leq \int_\Omega \frac{f}{1+\varepsilon f}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big) + \|B'\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\int_\Omega f\,\frac{T_k(u_\varepsilon)}{k}\,.\end{aligned}$$

Using the weak convergence of the truncates in $W_0^{1,2}(\Omega)$, and the almost everywhere convergence of both ∇u_ε and u_ε, we can use Fatou lemma and Lebesgue theorem to pass to the limit in the above inequality as ε tends to zero to obtain

$$\begin{aligned}&\int_\Omega [a(x) + T_k(u)^q]\nabla T_k(u)\nabla\psi B\Big(\frac{u}{k}\Big) + \int_\Omega \frac{b(x)\,|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\psi\, B\Big(\frac{u}{k}\Big)\\ &\quad\leq \int_\Omega f\,\psi\, B\Big(\frac{u}{k}\Big) + \|B'\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\,Q(k)\,,\end{aligned}$$

for all $\psi \in W_0^{1,p}(\Omega)$, $p > N$, with $\psi \geq 0$.

Now we let k tend to infinity; using the fact that B is bounded, that $[a(x) + u^q]|\nabla u|$ belongs to $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, and that $b(x)u^{-\theta}|\nabla u|^2$ belongs to $L^1(\Omega)$ by Lemma 2.5, and (2.15), we obtain

$$\int_\Omega [a(x) + u^q]\,\nabla u\nabla\psi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\psi \leq \int_\Omega f\,\psi\,,$$

for every $\psi \in W_0^{1,p}(\Omega)$, $p > N$, with $\psi \geq 0$; i.e., u is a subsolution of problem (1.1).

Step 2. The second inequality. Let ψ be in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, with $\psi \leq 0$, let H_ε be given by (2.8), and choose

$$\phi = \psi\,\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,B\Big(\frac{u_\varepsilon}{k}\Big)$$

as test function in (2.2). Thus, recalling that both $B(\frac{u_\varepsilon}{k})$ and $B'(\frac{u_\varepsilon}{k})$ are zero on the set $\{u_\varepsilon > k\}$, we obtain

$$\begin{aligned}&\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\nabla T_k(u_\varepsilon)\nabla\psi \mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, B\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad -\frac{\nu}{\alpha}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\frac{|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^\theta}\,\psi\,\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, B\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad +\frac{1}{k}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]|\nabla T_k(u_\varepsilon)|^2\,\psi\,\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, B'\Big(\frac{u_\varepsilon}{k}\Big)\\ &\quad +\int_\Omega \frac{b(x)\,T_k(u_\varepsilon)\,|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^{\theta+1}}\,\psi\,\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, B\Big(\frac{u_\varepsilon}{k}\Big)\\ &= \int_\Omega \frac{f}{1+\varepsilon f}\,\psi\,\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, B\Big(\frac{u_\varepsilon}{k}\Big)\,.\end{aligned}$$

Remark now that, by the assumptions on a and b, and since $\psi \le 0$, we have

$$\frac{|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^\theta}\psi\,\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, B\Big(\frac{u_\varepsilon}{k}\Big)\Big[\frac{b(x)T_k(u_\varepsilon)}{T_k(u_\varepsilon)+\varepsilon} - \frac{\nu}{\alpha}[a(x)+u_\varepsilon^q]\Big] \ge 0\,.$$

Therefore, using the almost everywhere convergence of both ∇u_ε and u_ε, the weak convergence of $T_k(u_\varepsilon)$ in $W_0^{1,2}(\Omega)$, Fatou lemma and both (2.4) and (2.12), we obtain, letting ε tend to zero,

$$\begin{aligned}&\int_\Omega [a(x) + T_k(u)^q]\nabla T_k(u)\nabla\psi \mathrm{e}^{-\frac{\nu H_0(u)}{\alpha}}\, B\Big(\frac{u}{k}\Big)\\ &\quad -\frac{\nu}{\alpha}\int_\Omega [a(x) + T_k(u)^q]\frac{|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\psi\,\mathrm{e}^{-\frac{\nu H_0(u)}{\alpha}}\, B\Big(\frac{u}{k}\Big)\\ &\quad +\int_\Omega \frac{b(x)\,|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\psi\,\mathrm{e}^{-\frac{\nu H_0(u)}{\alpha}}\, B\Big(\frac{u}{k}\Big)\\ &\le \int_\Omega f\,\psi\,\mathrm{e}^{-\frac{\nu H_0(u)}{\alpha}}\, B\Big(\frac{u}{k}\Big) + \|B'\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\,Q(k)\,.\end{aligned} \tag{2.16}$$

The idea now is to take a particular function ψ and pass to the limit as k tends to infinity. Let $k > 0$ be large enough such that

$$\sigma(k) = \Big(-\frac{\alpha(1-\theta)}{2\nu}\log(Q(k))\Big)^{\frac{1}{1-\theta}}$$

is well defined (see (2.15)), and note that

$$\lim_{k\to+\infty}\sigma(k) = +\infty\,,$$

since the argument of the logarithm tends to zero as k diverges. Note also that, by definition,

$$\mathrm{e}^{\frac{\nu\,H_0(\sigma(k))}{\alpha}} = \frac{1}{\sqrt{Q(k)}}\,. \tag{2.17}$$

Let φ belong to $C^1_{\mathrm{c}}(\Omega)$, with $\varphi \le 0$; since u is strictly positive on compact subsets of Ω (see Remark 2.4) we have that $u^{-\theta}\varphi$ belongs to $L^\infty(\Omega)$, so that the negative function

$$\psi = \mathrm{e}^{\frac{\nu H_0(u)}{\alpha}}\, B\Big(\frac{u}{\sigma(k)}\Big)\,\varphi$$

belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ (also because both B and B' have compact support in $\mathbb{R}$). Hence, it can be chosen as test function in (2.16) to obtain, after cancelling equal terms, and using (2.12) and (2.17),

$$
\begin{aligned}
&\int_\Omega [a(x)+T_k(u)^q]\nabla T_k(u)\nabla\varphi\, B\Big(\frac{u}{k}\Big)\, B\Big(\frac{u}{\sigma(k)}\Big)\\
&\quad+\int_\Omega \frac{b(x)\,|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\varphi\, B\Big(\frac{u}{k}\Big)\, B\Big(\frac{u}{\sigma(k)}\Big)\\
&\quad\le \int_\Omega f\,\varphi\, B\Big(\frac{u}{k}\Big)\, B\Big(\frac{u}{\sigma(k)}\Big)+\|B'\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\sqrt{Q(k)}\\
&\quad+\frac{1}{\sigma(k)}\|B'\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\int_\Omega [a(x)+T_{\sigma(k)}(u)^q]|\nabla T_{\sigma(k)}(u)|^2\\
&\quad\le \int_\Omega f\,\varphi\, B\Big(\frac{u}{k}\Big)\, B\Big(\frac{u}{\sigma(k)}\Big)+\|B'\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\sqrt{Q(k)}\\
&\quad+\|B'\|_{L^\infty(\Omega)}\,\|\varphi\|_{L^\infty(\Omega)}\,Q(\sigma(k))\,.
\end{aligned}
$$

To finish, we pass to the limit as k tends to infinity. Using once again that B is bounded, that both $[a(x)+u^q]|\nabla u|$ and $b(x)u^{-\theta}|\nabla u|^2$ belong to $L^1(\Omega)$ by Lemma 2.5, and (2.15), we have

$$\int_\Omega [a(x)+u^q]\nabla u\nabla\varphi+\int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi\le\int_\Omega f\,\varphi\,,$$

for all φ in $C_c^1(\Omega)$, with $\varphi\le 0$. Using the results of Lemma 2.5, we conclude by density that

$$\int_\Omega [a(x)+u^q]\nabla u\nabla\varphi+\int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi\le\int_\Omega f\,\varphi$$

for all φ in $W_0^{1,p}(\Omega)$, $p>N$, with $\varphi\le 0$.

Putting together the results of both steps we conclude that

$$\int_\Omega [a(x)+u^q]\nabla u\nabla\varphi+\int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi\le\int_\Omega f\,\varphi\,,$$

for all $\varphi\in W_0^{1,p}(\Omega)$, $p>N$ and then (exchanging φ with $-\varphi$)

$$\int_\Omega [a(x)+u^q]\nabla u\nabla\varphi+\int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi=\int_\Omega f\,\varphi$$

for every φ in $W_0^{1,p}(\Omega)$, $p>N$. □

3. Summability results

As stated in the Introduction, in this Section we prove some regularity results on the solution u given by Theorem 1.1, depending on summability assumptions on f, and on the values of both q and θ.

Theorem 3.1. *Let $\delta=\min(\theta,1-q)$, and let $1<m<\frac{N}{2}$. Then the solution u belongs to $L^s(\Omega)$, where $s=m^{**}(2-\delta)$. Furthermore, if $q<1$, then*

1) *if $1<m<\left(\frac{2^*}{\delta}\right)'$, then u belongs to $W_0^{1,r}(\Omega)$, with $r=\frac{Nm(2-\delta)}{N-m\delta}$;*

2) *if* $m \geq \left(\frac{2^*}{\delta}\right)'$, *and* $m > 1$, *the* u *belongs to* $W_0^{1,2}(\Omega)$.

If $q = 1$, *then* $m > 1$ *implies* u *in* $W_0^{1,2}(\Omega)$, *while if* $q > 1$ *then* u *belongs to* $W_0^{1,2}(\Omega)$ *by the results of Theorem* 1.1.

Remark 3.2. Note that, by definition, $\delta < 1$.

Before giving the proof of Theorem 3.1, we need a lemma.

Lemma 3.3. *Let* $\delta = \min(\theta, 1-q)$, *and let* $\gamma > 0$*; then there exists* $C_0 > 0$ *such that*

$$\gamma(t+\varepsilon)^{\gamma-1}(\alpha+t^q) + \mu t(t+\varepsilon)^{\gamma-1-\theta} \geq C_0(t+\varepsilon)^{\gamma-\delta}\,, \tag{3.1}$$

for every $t \geq 0$.

Proof. Multiplying (3.1) by $(t+\varepsilon)^{\delta-\gamma}$, we have to prove that

$$\gamma(t+\varepsilon)^{\delta-1}(\alpha+t^q) + \mu t(t+\varepsilon)^{\delta-1-\theta} \geq C_0 > 0\,.$$

If $\delta = \theta$, we have to prove that

$$\gamma\frac{\alpha+t^q}{(t+\varepsilon)^{1-\theta}} + \mu\frac{t}{t+\varepsilon} \geq C_0 > 0\,.$$

Clearly, if $t \geq \varepsilon$ we have $\frac{t}{t+\varepsilon} \geq \frac{1}{2}$, while if $t < \varepsilon$ we have $\frac{\alpha+t^q}{(t+\varepsilon)^{1-\theta}} \geq \frac{\alpha}{(2\varepsilon)^{1-\theta}} \geq \frac{\alpha}{2^{1-\theta}}$, since $\varepsilon < 1$; therefore, the claim is proved.

If, instead, $\delta = 1-q$, we have to prove that

$$\gamma\frac{\alpha+t^q}{(t+\varepsilon)^{q}} + \mu\frac{t}{(t+\varepsilon)^{q+\theta}} \geq C_0 > 0\,,$$

which is true since the first term is greater than $\frac{\gamma}{2^q}$ if $t \geq \varepsilon$, and is greater than $\frac{\gamma\alpha}{2^q}$ if $t \leq \varepsilon$. □

Proof of Theorem 3.1. The key point is to prove an *a priori* estimate on the sequence $\{u_\varepsilon\}$ since the compactness has been proved in Theorem 1.1.

Let $\gamma > 0$, and choose, following [7], $(u_\varepsilon+\varepsilon)^\gamma - \varepsilon^\gamma$ as test function in (2.1); we obtain, using the assumptions on a and b, and dropping a negative term,

$$\begin{aligned}\gamma\int_\Omega(\alpha+u_\varepsilon^q)(u_\varepsilon+\varepsilon)^{\gamma-1}|\nabla u_\varepsilon|^2 + \mu\int_\Omega\frac{u_\varepsilon(u_\varepsilon+\varepsilon)^\gamma}{(u_\varepsilon+\varepsilon)^{\theta+1}}|\nabla u_\varepsilon|^2 \\ \leq \int_\Omega f\,(u_\varepsilon+\varepsilon)^\gamma + \varepsilon^\gamma\int_\Omega\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} \leq \int_\Omega f\,(u_\varepsilon+\varepsilon)^\gamma + C_1\varepsilon^\gamma\,,\end{aligned}$$

where in the last passage we have used (2.3). In the left hand side we have

$$\int_\Omega|\nabla u_\varepsilon|^2[\gamma(\alpha+u_\varepsilon^q)(u_\varepsilon+\varepsilon)^{\gamma-1} + \mu\,u_\varepsilon(u_\varepsilon+\varepsilon)^{\gamma-1-\theta}]\,.$$

Recalling Lemma 3.3, we have, if $\delta = \min(\theta, 1-q)$,

$$\int_\Omega(u_\varepsilon+\varepsilon)^{\gamma-\delta}|\nabla u_\varepsilon|^2 \leq C_2\int_\Omega f\,(u_\varepsilon+\varepsilon)^\gamma + C_2\varepsilon^\gamma\,. \tag{3.2}$$

Now we rewrite

$$\int_\Omega(u_\varepsilon+\varepsilon)^{\gamma-\delta}|\nabla u_\varepsilon|^2 = \frac{4}{(\gamma-\delta+2)^2}\int_\Omega|\nabla[(u_\varepsilon+\varepsilon)^{\frac{\gamma-\delta+2}{2}} - \varepsilon^{\frac{\gamma-\delta+2}{2}}]|^2\,,$$

and use Sobolev and Hölder inequalities to obtain

$$\left(\int_\Omega [(u_\varepsilon+\varepsilon)^{\frac{(\gamma-\delta+2)}{2}}-\varepsilon^{\frac{(\gamma-\delta+2)}{2}}]^{2^*}\right)^{\frac{2}{2^*}} \leq C_3\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma m'}\right)^{\frac{1}{m'}}+C_3\,.$$

Since $[(t+\varepsilon)^\beta-\varepsilon^\beta]^{2^*}\geq C_4(t+\varepsilon)^{2^*\beta}-C_4$, for every $t>0$ (and for a suitable C_4 independent on ε) we then have

$$\left(\int_\Omega [C_4(u_\varepsilon+\varepsilon)^{\frac{2^*(\gamma-\delta+2)}{2}}-C_4]\right)^{\frac{2}{2^*}} \leq C_3\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma m'}\right)^{\frac{1}{m'}}+C_3\,.$$

Choosing γ such that $\frac{2^*}{2}(\gamma-\delta+2)=\gamma m'$ yields

$$\gamma=\frac{N(m-1)(2-\delta)}{N-2m}\,,$$

and then $\gamma>0$ (since $m>1$), and $\gamma m'=m^{**}(2-\delta)=s$. Therefore, we have

$$\left(\int_\Omega [C_4(u_\varepsilon+\varepsilon)^s-C_4]\right)^{\frac{2}{2^*}} \leq C_3\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^s\right)^{\frac{1}{m'}}+C_3\,,$$

which then yields, since $\frac{2}{2^*}>\frac{1}{m'}$ being $m<\frac{N}{2}$, that

$$\|u_\varepsilon\|_{L^s(\Omega)}^{2-\delta}\leq C_5\|f\|_{L^m(\Omega)}\,.$$

By Fatou lemma, and the almost everywhere convergence of u_ε to u, we obtain that u belongs to $L^s(\Omega)$, as desired.

Remark that once γ is chosen, (3.2) becomes

$$\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma-\delta}|\nabla u_\varepsilon|^2\leq C_2\int_\Omega f(u_\varepsilon+\varepsilon)^\gamma+C_2\varepsilon^\gamma\leq C_6\,. \tag{3.3}$$

Now we turn to gradient estimates. If $q<1$ and $\gamma\geq\delta$, that is if $m\geq\left(\frac{2^*}{\delta}\right)'$, from (3.3) we obtain

$$\int_{\{u_\varepsilon>1\}}|\nabla u_\varepsilon|^2\leq C_7\,,$$

which, together with the boundedness of $T_1(u_\varepsilon)$ in $W_0^{1,2}(\Omega)$, yields that u_ε is bounded in the same space. Therefore, u belongs to $W_0^{1,2}(\Omega)$.

If, instead, $1<m<\left(\frac{2^*}{\delta}\right)'$, i.e., if $\gamma<\delta$, let $r<2$ and write

$$\int_\Omega |\nabla u_\varepsilon|^r=\int_\Omega \frac{|\nabla u_\varepsilon|^r}{(u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2}}}(u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2}}\,.$$

Then, by Hölder inequality, and (3.3)

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^r&\leq\left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\delta-\gamma}}\right)^{\frac{r}{2}}\left(\int_\Omega (u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2-r}}\right)^{\frac{2-r}{2}}\\&\leq C_7\left(\int_\Omega (u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2-r}}\right)^{\frac{2-r}{2}}\,.\end{aligned}$$

We now choose r such that $\frac{(\delta-\gamma)r}{2-r} = \frac{Nm(2-\delta)}{N-2m}$, with $\gamma = \frac{N(m-1)(2-\delta)}{N-2m}$. This yields $r = \frac{Nm(2-\delta)}{N-\delta m}$. Therefore, u_ε is bounded in $W_0^{1,r}(\Omega)$, so that u belongs to the same space.

If $q = 1$, and $m > 1$, we choose $\log(1 + u_\varepsilon)$ as test function in (2.1), to obtain, after dropping nonnegative terms, that

$$\min(\alpha, 1) \int_\Omega |\nabla u_\varepsilon|^2 \leq \int_\Omega \frac{\alpha + u_\varepsilon}{1 + u_\varepsilon} |\nabla u_\varepsilon|^2 \leq \int_\Omega f \log(1 + u_\varepsilon)\,,$$

and this gives an *a priori* estimate of u_ε in $W_0^{1,2}(\Omega)$ since $\log(1 + u_\varepsilon)$ is bounded in $L^{m'}(\Omega)$. □

Remark 3.4. If we assume that f belongs to $L^m(\Omega)$ with $m > \frac{N}{2}$, we can prove that the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ (so that $u \in L^\infty(\Omega)$ as well). Indeed, taking $G_k(u_\varepsilon)$ as a test function in (2.1), using the sign condition on the quadratic lower order term and dropping the positive terms we obtain that

$$\alpha \int_\Omega |\nabla G_k(u_\varepsilon)|^2 \leq \int_\Omega f\, G_k(u_\varepsilon)$$

which implies the result by the classical Stampacchia boundedness theorem (see [16]). Once we have proved that boundedness of u_ε in $L^\infty(\Omega)$, the boundedness of u_ε in $W_0^{1,2}(\Omega)$ easily follows (choosing for example u_ε as test function).

Remark 3.5. If $q = 1$, it is enough to assume that $f \log(1 + f)$ belongs to $L^1(\Omega)$ to obtain that u belongs to $W_0^{1,2}(\Omega)$.

4. Minimization

In this Section we deal with the minimization problem for a functional of the Calculus of Variations whose Euler-Lagrange equation is of the type of (1.1), with $q = 1 - \theta$; note that this case is the "dividing range" in every result on the solution u of (1.1) proved so far (see (1.8) in Theorem 1.1 and Theorem 3.1).

Let us define the functional

$$J(v) = \frac{1}{2} \int_\Omega [a(x) + |v|^{1-\theta}] |\nabla v|^2 - \int_\Omega f\, v\,, \qquad v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

We have the following result.

Theorem 4.1. *Let $f \geq 0$, f in $L^m(\Omega)$, with $m \geq \frac{2N+(1-\theta)N}{N+2+(1-\theta)N}$. Then there exists a function u in $W_0^{1,2}(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$, with $u \geq 0$, such that*

$$\frac{1}{2} \int_\Omega [a(x) + u^{1-\theta}] |\nabla u|^2 - \int_\Omega f\, u \leq J(v)\,, \qquad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,. \tag{4.1}$$

Furthermore, u is a solution of the equation

$$\begin{cases} -\operatorname{div}([a(x) + u^{1-\theta}] \nabla u) + \dfrac{1-\theta}{2} \dfrac{|\nabla u|^2}{u^\theta} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Remark 4.2. We point out that the result is interesting if

$$\frac{2N+(1-\theta)N}{N+2+(1-\theta)N}\le m<\frac{2N}{N+2}\,,$$

since in this case the functional cannot be defined on $W_0^{1,2}(\Omega)$ (both terms may be unbounded).

Proof. Let $\varepsilon>0$ and define

$$g_\varepsilon(t)=(1-\theta)\int_0^t\frac{s}{(|s|+\varepsilon)^{\theta+1}}\,ds\,,$$

and note that, for $t\ge 0$, we have

$$0\le g_\varepsilon(t)\le t^{1-\theta}\,. \tag{4.3}$$

Define, for v in $W_0^{1,2}(\Omega)$, the functional

$$J_\varepsilon(v)=\begin{cases}\dfrac12\displaystyle\int_\Omega[a(x)+g_\varepsilon(v)]|\nabla v|^2-\int_\Omega\frac{f\,v}{1+\varepsilon\,f}\,, & \text{if } |v|^{1-\theta}|\nabla v|^2\in L^1(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

Note that, thanks to (4.3), the first integral in the definition of J_ε is finite if $|v|^{1-\theta}|\nabla v|^2$ belongs to $L^1(\Omega)$.

We claim that there exists u_ε in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $u_\varepsilon\ge 0$, minimum of J_ε on $W_0^{1,2}(\Omega)$. Indeed, it is easy to see that the functional is coercive, since (recalling (1.4))

$$J_\varepsilon(v)\ge\frac{\alpha}{2}\int_\Omega|\nabla v|^2-\frac{1}{\varepsilon}\int_\Omega v\,,$$

while weak lower semicontinuity in $W_0^{1,2}(\Omega)$ follows from a classical result by De Giorgi (see [13]). Thus the functional has a minimum u_ε in $W_0^{1,2}(\Omega)$, and one can prove that u_ε belongs to $L^\infty(\Omega)$ using standard techniques by Stampacchia (see [16]), and starting from the inequalities $J_\varepsilon(u_\varepsilon)\le J_\varepsilon(T_k(u_\varepsilon))$, $k\ge 0$. The fact that $u_\varepsilon\ge 0$ easily follows from the assumption $f\ge 0$, using that $J_\varepsilon(u_\varepsilon)\le J_\varepsilon(u_\varepsilon^+)$. Furthermore, starting from the inequality $J_\varepsilon(u_\varepsilon)\le J_\varepsilon(u_\varepsilon+t\varphi)$, with φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, one can prove that u_ε is a solution of

$$\begin{cases}-\mathrm{div}([a(x)+g_\varepsilon(u_\varepsilon)]\nabla u_\varepsilon)+\dfrac{1-\theta}{2}\dfrac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}=\dfrac{f}{1+\varepsilon\,f} & \text{in } \Omega,\\ u_\varepsilon=0 & \text{on } \partial\Omega,\end{cases} \tag{4.4}$$

in the sense that

$$\int_\Omega[a(x)+g_\varepsilon(u_\varepsilon)]\nabla u_\varepsilon\nabla\varphi+\frac{1-\theta}{2}\int_\Omega\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\varphi=\int_\Omega\frac{f\,\varphi}{1+\varepsilon\,f}\,,$$

for every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Note now that problem (4.4) is essentially problem (2.1), thanks to inequality (4.3). Therefore, starting from (4.4) and using the assumptions on m, one has that u_ε is bounded in $W_0^{1,2}(\Omega)$ (since $m>(\frac{2^*}{\theta})'$) and in $L^s(\Omega)$, with $s=m^{**}(2-\theta)$ (see Theorem 3.1). Therefore, and up to subsequences, it converges, weakly in $W_0^{1,2}(\Omega)$

and weakly in $L^s(\Omega)$, to a function u. Furthermore, ∇u_ε almost everywhere converges to ∇u in Ω (see Lemma 2.5), and u is a solution of (4.2) (see Theorem 1.1).

Since u_ε is a minimum of J_ε, and

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega [a(x) + g_\varepsilon(v)]|\nabla v|^2 - \int_\Omega \frac{f\,v}{1+\varepsilon\, f}\,,$$

if v belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, we have

$$\frac{1}{2}\int_\Omega [a(x) + g_\varepsilon(u_\varepsilon)]|\nabla u_\varepsilon|^2 - \int_\Omega \frac{f\,u_\varepsilon}{1+\varepsilon\, f} \le \frac{1}{2}\int_\Omega [a(x) + g_\varepsilon(v)]|\nabla v|^2 - \int_\Omega \frac{f\,v}{1+\varepsilon\, f}\,,$$

for every v in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. The weak convergence of u_ε to u in $L^s(\Omega)$, with $s = m^{**}(2-\theta)$, and the assumptions on m imply that

$$\frac{1}{m} + \frac{1}{m^{**}(2-\theta)} \le 1\,,$$

so that

$$\lim_{\varepsilon\to 0^+}\int_\Omega \frac{f\,u_\varepsilon}{1+\varepsilon\, f} = \int_\Omega f\,u\,.$$

Furthermore, the almost everywhere convergence of u_ε and ∇u_ε, and Fatou lemma, imply that

$$\int_\Omega [a(x) + u^q]|\nabla u|^2 \le \liminf_{\varepsilon\to 0^+}\int_\Omega [a(x) + g_\varepsilon(u_\varepsilon)]|\nabla u_\varepsilon|^2\,.$$

Thus,

$$J(u) \le \liminf_{\varepsilon\to 0^+} J_\varepsilon(u_\varepsilon)\,.$$

On the other hand, if v belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, one also has

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(v) = J(v)\,,$$

and so, passing to the limit in the inequalities $J_\varepsilon(u_\varepsilon) \le J_\varepsilon(v)$ one obtains that

$$\frac{1}{2}\int_\Omega [a(x) + u^q]|\nabla u|^2 - \int_\Omega f\,u \le \frac{1}{2}\int_\Omega [a(x) + v^q]|\nabla v|^2 - \int_\Omega f\,v\,,$$

for every v in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Hence, (4.1) holds. □

5. "Finite energy" solutions

In this Section we give the precise assumptions on the datum f (depending on the values of q and θ) that allow to widen the class of test function from $W_0^{1,p}(\Omega)$, $p > N$, to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, which is the "standard" set of test functions for quadratic quasilinear equations. In order to do that, we only need to have $u^q|\nabla u|$ in $L^2(\Omega)$, since this assumption (together with the fact that $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every k) yields that u belongs to $W_0^{1,2}(\Omega)$, and since the lower order term $b(x)\,u^{-\theta}|\nabla u|^2$ always belongs to $L^1(\Omega)$ for any value of q and θ, and for every f in $L^1(\Omega)$. In analogy with the "standard" quasilinear case, we will call these functions "finite energy" solutions.

In order to have $u^q|\nabla u|$ in $L^2(\Omega)$, we can either choose u^{q+1} as test function and use the higher order part of the equation, or choose $u^{2q+\theta}$ as test function and use the lower order term. Clearly, in order to do that one has to work on the approximate equations (2.1), choosing either u_ε^{q+1} or $u_\varepsilon^{2q+\theta}$, and proving *a priori* estimates which then pass to the limit thanks to the results proved in Section 2. Since it is better to choose the power having the lower exponent, if we define $\sigma = \min(2q+\theta, q+1)$, the choice of u_ε^σ yields, after dropping nonnegative terms,

$$\int_\Omega u_\varepsilon^{2q}|\nabla u_\varepsilon|^2 \le C \int_\Omega f\, u_\varepsilon^\sigma + C\,.$$

Therefore, if we assume that f belongs to $L^m(\Omega)$, an *a priori* estimate on $u_\varepsilon^q|\nabla u_\varepsilon|$ in $L^2(\Omega)$ will follow if the summability of u_ε^σ is larger than m', the Hölder conjugate of m.

We now recall that, setting $\delta = \min(\theta, 1-q)$, one has by Theorem 3.1 that u_ε is bounded in $L^s(\Omega)$, with $s = m^{**}(2-\delta)$. Therefore, the desired *a priori* estimate will hold true if

$$\frac{\sigma}{m^{**}(2-\delta)} \le 1 - \frac{1}{m}\,.$$

We now remark that $\sigma = \min(2q+\theta, q+1) = 2q + \min(\theta, 1-q) = 2q+\delta$. Therefore, the previous inequality can be rewritten as

$$\frac{2q+\delta}{m^{**}(2-\delta)} \le 1 - \frac{1}{m}\,.$$

Recalling that $\frac{1}{m^{**}} = \frac{1}{m} - \frac{2}{N}$, the previous inequality becomes

$$m \ge \frac{2N(q+1)}{N(2-\delta)+4q+2\delta} = \frac{2N(q+1)}{(N+2)(q+1)+(N-2)(1-q-\delta)}\,.$$

If $\delta = 1-q$, the above inequality is

$$m \ge \frac{2N}{N+2}\,;$$

in other words, the "standard" assumption on the datum which yields finite energy solutions for uniformly elliptic and bounded operators, yields solutions such that $u^q|\nabla u|$ belongs to $L^2(\Omega)$. Since $\delta = 1-q$ implies that the principal part of the equation gives a better estimate than the lower order term, this was somehow to be expected.

If $\delta = \theta$, the situation is different: in this case, the lower order term is "dominant" with respect to the differential operator, and the assumption on m becomes

$$m \ge \frac{2N(q+1)}{N(2-\theta)+4q+2\theta} = \frac{2N(q+1)}{(N+2)(q+1)+(N-2)(1-q-\theta)}\,,$$

with $\theta < 1-q$. Note that this assumption implies that

$$\frac{2N(q+1)}{(N+2)(q+1)+(N-2)(1-q-\theta)} < \frac{2N}{N+2}\,,$$

so that if the lower order term is "dominant", one needs less summability on f in order to have "finite energy" solutions. Note that, in this case, we have a condition depending on both q and θ since we want to use the lower order term (where $u^{-\theta}$ appears) to obtain an estimate on u^{2q}. Furthermore, since

$$\frac{2N(q+1)}{N(2-\theta)+4q+2\theta} > 1 \qquad \Longleftrightarrow \qquad (2q+\theta)(N-2) > 0\,,$$

which is always true, the lower bound on m is always strictly larger than 1. In other words, if the lower order term is "dominant", one never has finite energy solutions in the case of $L^1(\Omega)$ data: a fact which is in contrast with well-known results on quasilinear equations having a quadratic lower order term which does not vanish as the solution u tends to infinity.

We therefore have the following result.

Theorem 5.1. *Suppose that* (1.2), (1.4) *and* (1.5) *hold true, and let*

$$m_0 = \begin{cases} \dfrac{2N}{N+2} & \text{if } \theta \geq 1-q, \\ \dfrac{2N(q+1)}{N(2-\theta)+4q+2\theta} & \text{if } \theta < 1-q. \end{cases}$$

If f belongs to $L^m(\Omega)$, with $m \geq m_0$, and u is a solution of (1.1) *given by Theorem* 1.1*, then $u^q|\nabla u|$ belongs to $L^2(\Omega)$ and one has*

$$\int_\Omega [a(x)+u^q]\nabla u \nabla\varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi = \int_\Omega f\,\varphi\,,$$

for every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

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