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Realizations of Slice Hyperholomorphic Generalized Contractive and Positive Functions

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Abstract. We introduce generalized Schur functions and generalized positive functions in the setting of slice hyperholomorphic functions and study their realizations in terms of associated reproducing kernel Pontryagin spaces. To this end, we also prove some results in quaternionic functional analysis like an invariant subspace theorem for contractions in a Pontryagin space. We also consider slice hyperholomorphic functions on the half space $\mathbb{H}_+$ of quaternions with positive real parts and we study the Hardy space $\mathbf{H}_2(\mathbb{H}_+)$ and Blaschke products in this framework.

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1. Introduction

In this paper we continue the study of Schur analysis in the hyperholomorphic setting, initiated in [4], and continued in [6, 5, 2]. To set the framework we first recall a few facts on the classical case.

1.1. Schur analysis

Functions analytic and contractive in the open unit disk, or in an open half-plane, play an important role in operator theory, signal processing and related fields. Their study, and the study of their counterparts in various settings, may be called Schur analysis; see [35, 1, 26]. In the case of matrix-valued, or operator-valued functions,

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contractivity is considered with respect to an indefinite metric. An important example is that of the characteristic operator function and associated operator models.

More precisely, let T be a (say, bounded, for the present discussion) self-adjoint operator in a Hilbert space such that $T - T^*$ has finite rank, say m. Let

$$T - T^* = \frac{CJC^*}{2i}$$

where J is a $m \times m$ matrix which is both self-adjoint and unitary (a *signature matrix*). Then, the *matrix-valued* function

$$\Theta(z) = I + 2iC^*(zI - T)^{-1}CJ$$

is such that

$$\Theta(z)J\Theta(z)^* \geq J, \tag{1.1}$$

for z in the intersection $\Omega(T)$ of the upper open half-plane and of the spectrum of T. The function Θ is the characteristic operator function of T.

Property (1.1) is called J-expansivity (or $-J$-contractivity), and is in fact equivalent to the fact that the kernel

$$K_\Theta(z,w) \stackrel{\text{def.}}{=} \frac{\Theta(z)J\Theta(w)^* - J}{-2i(z-\bar{w})} = C^*(zI-T)^{-1}(wI-T)^{-*}C \tag{1.2}$$

is positive definite in $\Omega(T)$.

The function Θ provides a functional model for T, see [24]. A key fact in the theory is the multiplicative structure of J-expansive functions, due to V. Potapov, see [55]. We also refer to the historical note of M. Livsic [53]. It is also worth mentioning the original papers of M. Livsic [51, 52], where the notion of characteristic operator function first appears.

Replacing J by $-J$ we obtain J-contractive, rather than J-expansive functions, and this is the choice we make in the sequel (see in particular the last section).

1.2. Negative squares

One can consider functions Θ such that the associated kernel K_Θ has a finite number of negative squares (see Definition 2.3), rather than being positive definite. Such classes of operator-valued functions were introduced and studied by Krein and Langer in a long series of papers, see for instance [44, 45, 46, 47, 48]. These works are set in the framework of the open upper half-plane. To make a better connection with the quaternionic case we consider here the open right half-plane. In the complex variable case, the two cases are equivalent via a conformal map. This differs from the quaternionic setting, as will be clear in the sequel. First recall that an operator J in a Hilbert space $\mathcal{H}$ is called a signature operator if it is self-adjoint and unitary. Its spectrum is then concentrated on ± 1. When -1 is an eigenvalue of finite order, we denote this multiplicity by $\nu_-(J)$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $J_1 \in \mathbf{L}(\mathcal{H}_1)$ and $J_2 \in \mathbf{L}(\mathcal{H}_2)$ be two signature operators, such that

$\nu_-(J_1) = \nu_-(J_2) < \infty$. The $\mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$-valued function S analytic in an open subset Ω of the open right half-plane is called a generalized Schur function if the kernel

$$\frac{J_2 - S(z)J_1S(w)^*}{z+\overline{w}} \tag{1.3}$$

has a finite number of negative squares in Ω.

For instance when $J_1 = J_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, functions in the corresponding class are introduced and used in [45] to describe the set of all generalized resolvents of a given Hermitian operator, see [45, Satz 3.5, p. 407 and Satz 3.9, p. 409].

Similarly given a Hilbert space $\mathcal{H}$ and a signature operator J (possibly with $\nu_-(J) = \infty$, see [42, p. 358, footnote]), a $\mathbf{L}(\mathcal{H})$-valued function Φ analytic in some open subset Ω of the right open half-plane Π_+ is called generalized positive if the kernel

$$\frac{J\Phi(z) + \Phi(w)^*J}{z+\bar{w}} \tag{1.4}$$

has a finite number of negative squares in Ω. The Q-function of an Hermitian operator in a Pontryagin space, introduced by Krein and Langer has such a property, see [43]. The function Φ will be called positive if the kernel (1.4) is positive definite.

In both cases, Krein and Langer proved in the above mentioned works, among numerous results, realization formulas which ensure the existence of a meromorphic extension to the whole of Π_+. It is worth mentioning that a key result to prove this extension is that the part of the spectrum of a contraction in a Pontryagin space which lies outside the closed unit disk consists only of a finite number of eigenvalues. One proof of this fact uses the Schauder-Tychonoff fixed point theorem (see the discussion [28, p. 248]). We also mention that a study of generalized Schur function of the open unit disk has been given in [8] and that unified formulas for a number of cases which include the line and circle case were developed in [3], based on an approach including both the disk and half-plane cases developed in [3, 10].

Finally we mention the works [17, 18, 38] to stress the interest of positive and generalized positive functions in linear system theory and operator theory.

1.3. The slice hyperholomorphic case

In previous papers we extended results of Schur analysis in the slice hyperholomorphic case, in the setting of the unit ball $\mathbb{B}_1$ of the quaternions. We considered in [4] the Schur algorithm, and the underlying counterpart of the Hardy space. Blaschke products and related interpolation problems in the Hardy space were studied in [5]. Nevanlinna-Pick interpolation for Schur functions is studied in [2] while the case of kernels having a number of negative squares was studied in [6].

In contrast to the above mentioned papers, we consider in this work functions which are slice hyperholomorphic in an open subset of the open half-space

$$\mathbb{H}_+ = \{p \in \mathbb{H}\,;\mathrm{Re}\,p > 0\}\,,$$

which intersects the positive real axis.

We define and study the counterparts of the kernel (1.3) and (1.4) in the setting of slice hyperholomorphic functions. Here we consider the case of operator-valued generalized positive functions and generalized Schur functions, rather than scalar or matrix-valued functions. The extension of realization of generalized positive functions to the slice hyperholomorphic setting, introduced in this work, calls upon a corresponding extension of the Kalman-Yakubovich-Popov lemma (also known as Positive Real Lemma; see the discussion for the classical case in the next paragraph). This will be addressed in another work.

A $\mathbb{C}^{p\times p}$-valued function $F(s)$, analytic in $\mathbb{C}_+$ is said to be *positive* if

$$F(s) + F(s)^* \geq 0, \quad s \in \mathbb{C}_+, \tag{1.5}$$

where the inequality sign means that the Hermitian matrix is non negative, and where $\mathbb{C}_+$ denotes the open right half of the complex plane. The study of rational positive functions has been motivated from the 1920's by (lumped) electrical networks theory, see e.g. [16], [20]. From the 1960's positive functions also appeared in books on absolute stability theory, see e.g. [54]. A $\mathbb{C}^{p\times p}$-valued function of bounded type in $\mathbb{C}_+$ (i.e. a quotient of two functions analytic and bounded in $\mathbb{C}_+$) is called generalized positive if

$$F(i\omega) + F(i\omega)^* \geq 0, \quad a.e.\ \omega \in \mathbb{R}, \tag{1.6}$$

where $F(i\omega)$ denotes the non-tangential limit[1] of F at the point $i\omega$. In the classical setting, generalized positive functions were introduced in the context of the Positive Real Lemma (PRL), see [15] and references therein[2]. For applications of generalized positive functions see [40]. The renowned Kalman-Yakubovich-Popov Lemma, which has been recognized as a fundamental result in System Theory, establishes a connection between two presentations of positive functions, as rational functions and the respective state space realization, see e.g. [16], [33]. For its extension to generalized positive functions, see [15], [27].

The paper consists of seven sections, besides the Introduction. In Section 2 we recall the notion of quaternionic Pontryagin spaces, and we discuss some preliminaries on negative squares, kernels and realizations; then we provide some preliminaries on slice hyperholomorphic functions, the class of functions that we use in this paper. Section 3 deals with operator-valued slice hyperholomorphic functions, their products and a useful property of extension, see Proposition 3.24. In Section 4 we study the Hardy space of the half-space $\mathbb{H}_+$ of quaternions with real positive part and Blaschke factors and products in this framework. Then, in Section 5 we provide the proof of the quaternionic version of the Schauder-Tychonoff fixed point theorem whose proof is not substantially different from the one in the complex case, but we insert it for the sake of completeness. This result is crucial to show an invariant

[1]This limit exists almost everywhere on $i\mathbb{R}$ because F is assumed of bounded type in $\mathbb{C}_+$, see e.g. [30].

[2]The original formulation was real. The case we address is in fact *generalized* positive and *complex*, but we wish to adhere to the commonly used term: Positive Real Lemma.

subspace theorem for contractions in Pontryagin spaces. Sections 6 and 7 deal with the study of kernels with a finite number of negative squares and associated with generalized Schur functions and we prove a realization theorem in this setting. We also give as an example the characteristic operator function of a quaternionic non anti-self-adjoint operator. Section 8 deals with realizations for generalized positive functions. We also define the positive function associated to a pair of anti-self-adjoint operators. The properties of this function will be presented in a future publication.

2. Preliminaries

In this section, which is divided into three subsections, we collect a number of facts respectively on Pontryagin spaces, slice hyperholomorphic functions and their realizations.

2.1. Negative squares and kernels

An important role in this paper is played by quaternionic Pontryagin spaces, and we first recall this notion. We refer to [7, 13] for more details. Let $\mathcal{V}$ be a right quaternionic vector space endowed with a Hermitian form (also called inner product) $[\cdot,\cdot]$ from $\mathcal{V}\times\mathcal{V}$ into $\mathbb{H}$, meaning that:

$$[ua+vb,w]=[u,w]a+[v,w]b,$$

$$[v,w]=\overline{[w,v]},$$

for all choices of $u,v,w\in\mathcal{V}$ and $a,b\in\mathbb{H}$. In particular the inner product $[\cdot,\cdot]$ satisfy

$$[va,wb]=\overline{b}[v,w]a.$$

When the space $\mathcal{V}$ is two-sided, we require that

$$[v,aw]=[\overline{a}v,w],\quad a\in\mathbb{H},\quad v,w\in\mathcal{V}. \tag{2.1}$$

Condition (2.1) is used in particular in the proof of formula (7.8).

Definition 2.1. The space $\mathcal{V}$ is called a right-quaternionic Pontryagin space if there exists two subspaces $\mathcal{V}_+$, $\mathcal{V}_-$ of $\mathcal{V}$ such that $\mathcal{V}=\mathcal{V}_++\mathcal{V}_-$ and:

(i) The space $\mathcal{V}_+$ endowed with $[\cdot,\cdot]$ is a right-quaternionic Hilbert space.
(ii) The space $\mathcal{V}_-$ endowed with $-[\cdot,\cdot]$ is a finite dimensional right-quaternionic Hilbert space.
(iii) The sum $\mathcal{V}_++\mathcal{V}_-$ is direct and orthogonal, meaning that $\mathcal{V}_+\cap\mathcal{V}_-=\{0\}$ and $[v_+,v_-]=0$ for every choice of $v_+\in\mathcal{V}_+$ and $v_-\in\mathcal{V}_-$.

We denote a direct and orthogonal sum by

$$\mathcal{V}=\mathcal{V}_+[\dot{+}]\mathcal{V}_-. \tag{2.2}$$

In general, such a decomposition will not be unique. The inner product

$$\langle v,w\rangle=[v_+,w_+]-[v_-,w_-]$$

where $v_\pm,w_\pm\in\mathcal{V}_\pm$, makes $\mathcal{V}$ into a Hilbert space. The inner product depends on the decomposition, but all the associated topologies are equivalent. We refer to [13] for more details on these facts in the quaternionic case, while the case of the field of complex numbers we refer to [21].

We now recall a few facts on matrices with quaternionic entries and on kernels, which we will need in the sequel. A matrix $A \in \mathbb{H}^{m\times m}$ can be written in a unique way as

$$A = A_1 + A_2 j,$$

where A_1 and A_2 belong to $\mathbb{C}^{m\times m}$. The map $\chi : \mathbb{C}^{m\times m} \to \mathbb{C}^{2m\times 2m}$ defined by

$$\chi(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \tag{2.3}$$

satisfies

$$\chi(AB) = \chi(A)\chi(B) \quad \text{and} \quad \chi(A^*) = (\chi(A))^*.$$

See for instance [64, Theorem 4.2, p. 29] (see also [13, Proposition 3.8, p. 439]). The result itself is due to Lee [50].

A key fact is that $A \in \mathbb{H}^{m\times m}$ is Hermitian (that is, $A = A^*$) if and only if it can be written as UDU^*, where $U \in \mathbb{H}^{m\times m}$ is unitary and $D \in \mathbb{R}^{m\times m}$ is diagonal. The matrix D is uniquely determined up to permutations, and one can define the signature of an Hermitian matrix with quaternionic entries as the signature of D, see [64, Corollary 6.2, p. 41] and the references therein. The following result follows from the properties of χ and can be found in [13, Proposition 3.16, p. 442].

Lemma 2.2. *Assume $A \in \mathbb{H}^{m\times m}$ Hermitian. Then A has signature (ν_+, ν_-, ν_0) if and only if $\chi(A)$ has signature $(2\nu_+, 2\nu_-, 2\nu_0)$.*

We now turn to the notion of kernels having a finite number of negative squares.

Definition 2.3. Let $\mathcal{H}$ be a two-sided quaternionic Hilbert space, with inner product $\langle\cdot,\cdot\rangle$, and let $K(z,w)$ be a $\mathbf{L}(\mathcal{H},\mathcal{H})$-valued function defined for z,w in some set Ω. The kernel is called Hermitian if

$$K(z,w) = K(w,z)^*, \quad z,w \in \Omega.$$

It is said to have a finite number (say κ) of negative squares if for every choice of $N \in \mathbb{N}$, of vectors $c_1,\ldots,c_N \in \mathcal{H}$ and of points $w_1,\ldots,w_N \in \Omega$, the $N\times N$ Hermitian matrix with (u,v) entry

$$[K(w_u,w_v)c_u, c_v]$$

has at most κ strictly negative eigenvalues, counted with multiplicities, and exactly κ strictly negative eigenvalues for some choice of $N, w_1,\ldots,w_N$ and $c_1,\ldots,c_N$.

When $\kappa = 0$ we have the classical notion of positive definite function. Given a set Ω, the one-to-one correspondence between positive definite functions on Ω and reproducing kernel Hilbert spaces of functions defined on Ω extends to a one-to-one correspondence between functions having a finite number of negative squares and reproducing kernel Pontryagin spaces (for more information on these spaces see [13]). This fact is due to P. Sorjonen [59] and L. Schwartz [58] in the complex case, and is proved in [13] in the quaternionic case.

We conclude by mentioning a result, [6, Proposition 5.3], which will be used in the sequel:

Proposition 2.4. *Assume that $K(p,q)$ is $\mathbb{H}^{N\times N}$-valued and has κ negative squares in V and let $\alpha(p)$ be a $\mathbb{H}^{N\times N}$-valued slice hyperholomorphic function and such that $\alpha(0)$ is invertible. Then the function*

$$B(p,q) = \alpha(p) \star K(p,q) \star_r \alpha(q)^* \tag{2.4}$$

has κ negative squares in V.

2.2. Slice hyperholomorphic functions

Let $\mathbb{H}$ be the real associative algebra of quaternions with respect to the basis $\{1, i, j, k\}$ satisfying the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. We will denote a quaternion p as $p = x_0 + ix_1 + jx_2 + kx_3$, $x_i \in \mathbb{R}$, its conjugate as $\bar{p} = x_0 - ix_1 - jx_2 - kx_3$, and $|p|^2 = p\bar{p}$. The real part x_0 of a quaternion will be denoted also by $\mathrm{Re}(p)$, $\mathbb{S}$ is the 2-sphere of purely imaginary unit quaternions, i.e.

$$\mathbb{S} = \{p = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Note that if $I \in \mathbb{S}$ then $I^2 = -1$ and a nonreal quaternion $p = x_0 + ix_1 + jx_2 + kx_3$ uniquely determines an element $I_p = ix_1 + jx_2 + kx_3/|ix_1 + jx_2 + kx_3| \in \mathbb{S}$. When p is real, then $p = p + I0$ for all $I \in \mathbb{S}$.

Definition 2.5. Given $p \in \mathbb{H}$, $p = p_0 + I_p p_1$ we denote by $[p]$ the set of all elements of the form $p_0 + Jp_1$ when J varies in $\mathbb{S}$.

The set $[p]$ is a 2-sphere (we will often write that $[p]$ is a sphere, for short) which is reduced to the point p when $p \in \mathbb{R}$. We now recall the definition of slice hyperholomorphic function.

Definition 2.6. Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on $\mathbb{C}_I$. We say that f is a left slice hyperholomorphic (or slice hyperholomorphic or slice regular) function in Ω if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0.$$

We say that f is a right slice hyperholomorphic function in Ω if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x+Iy) + \frac{\partial}{\partial y} f_I(x+Iy)I\right) = 0.$$

Slice hyperholomorphic functions have a nice behavior on the so called axially symmetric slice domains defined below.

Definition 2.7. Let Ω be a domain in $\mathbb{H}$. We say that Ω is a slice domain (s-domain for short) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$. We say that Ω is axially symmetric if, for all $q \in \Omega$, the sphere $[q]$ is contained in Ω.

Remark 2.8. Assume that $f: \ \Omega \subseteq \mathbb{C} \cong \mathbb{C}_I \to \mathbb{H}$ is a holomorphic map. Let U_Ω be the axially symmetric completion of Ω, i.e. $U_\Omega = \bigcup_{J\in\mathbb{S}, x+Iy\in\Omega}(x+Jy)$. Its left slice hyperholomorphic extension $\mathrm{ext}(f): \ U_\Omega \subseteq \mathbb{H} \to \mathbb{H}$ is computed as follows (see [25]):

$$\mathrm{ext}(f)(x+Jy) = \frac{1}{2}\left[f(x+Iy) + f(x-Iy) + JI(f(x-Iy) - f(x+Iy))\right]. \quad (2.5)$$

It is immediate that $\mathrm{ext}(f+g) = \mathrm{ext}(f) + \mathrm{ext}(g)$ and that if $f(z) = \sum_{n=0}^{\infty} f_n(z)$ then $\mathrm{ext}(f)(z) = \sum_{n=0}^{\infty} \mathrm{ext}(f_n)(z)$. It is also useful to recall that any function h slice hyperholomorphic on an axially symmetric s-domain Ω satisfies the formula, see [25, Theorem 4.3.2]

$$h(x+Jy) = \frac{1}{2}\left[h(x+Iy) + h(x-Iy) + JI(h(x-Iy) - h(x+Iy))\right]. \quad (2.6)$$

Let $f, g: \ \Omega \subseteq \mathbb{H}$ be slice hyperholomorphic functions. Their restrictions to the complex plane $\mathbb{C}_I$ can be decomposed as $f_I(z) = F(z)+G(z)J$, $g_I(z) = H(z)+L(z)J$ where $J \in \mathbb{S}$, $J \perp I$ where F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$, see [25], p. 117. The $\star_l$-product of f and g, see [25], p. 125, is defined as the unique left slice hyperholomorphic function whose restriction to the complex plane $\mathbb{C}_I$ is given by

$$\begin{aligned}(f_I \star_r g_I)(z) :&= (F(z) + G(z)J) \star_l (H(z) + L(z)J) \\ &= (F(z)H(z) - G(z)\overline{L(\bar{z})}) + (G(z)\overline{H(\bar{z})} + F(z)L(z))J.\end{aligned} \quad (2.7)$$

If f, g are right slice hyperholomorphic, then with the above notations we have $f_I(z) = F(z) + JG(z)$, $g_I(z) = H(z) + JL(z)$ and

$$\begin{aligned}(f_I \star_r g_I)(z) :&= (F(z) + JG(z)) \star_r (H(z) + JL(z)) \\ &= (F(z)H(z) - \overline{G(\bar{z})}L(z)) + J(G(z)H(z) + \overline{F(\bar{z})}L(z))J,\end{aligned} \quad (2.8)$$

and $f \star_r g = \mathrm{ext}(f_I \star_r g_I)$.

Remark 2.9. In the sequel, we will consider functions $k(p,q)$ left slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$. When taking the $\star$-product of a function $f(p)$ slice hyperholomorphic in the variable p with such a function $k(p,q)$, we will write $f(p) \star k(p,q)$ meaning that the $\star$-product is taken with respect to the variable p; similarly, the $\star_r$-product of $k(p,q)$ with functions right slice hyperholomorphic in the variable $\bar{q}$ is always taken with respect to $\bar{q}$.

Let Ω be an axially symmetric s-domain and let $p_0 \in \Omega$. Let us consider a function f slice hyperholomorphic in Ω and assume that, in a neighborhood of p_0 in Ω, it can be written in the form $f(p) = \sum_{n=-\infty}^{+\infty}(p-p_0)^{\star n} a_n$ where $a_n \in \mathbb{H}$.

Following the standard nomenclature and [60] we have:

Definition 2.10. A function f has a pole at the point p_0 if there exists $m \geq 0$ such that $a_{-k} = 0$ for $k > m$. The minimum of such m is called the order of the pole;
If p is not a pole then we call it an essential singularity for f;
f has a removable singularity at p_0 if it can be extended in a neighborhood of p_0 as a slice hyperholomorphic function.

Note the following important fact: a function f has a pole at p_0 if and only if its restriction to a complex plane has a pole. Note that there can be poles of order 0: let us consider for example the function $(p+I)^{-\star} = (p^2+1)^{-1}(p-I)$. It has a pole of order 0 at the point $-I$ which, however, is not a removable singularity, see [25, p.55] also for the definition of the $\star$-inverse.

Definition 2.11. Let Ω be an axially symmetric s-domain in $\mathbb{H}$. We say that a function $f:\Omega\to\mathbb{H}$ is slice hypermeromorphic in Ω if f is slice hyperholomorphic in $\Omega'\subset\Omega$ such that $\Omega\setminus\Omega'$ has no point limit in Ω and every point in $\Omega\setminus\Omega'$ is a pole.

The functions which are slice hypermeromorphic are called semi-regular in [60] and for these functions we have the following result, proved in [60, Proposition 7.1, Theorem 7.3]:

Proposition 2.12. *Let Ω be an axially symmetric s-domain in $\mathbb{H}$ and let $f,g:\Omega\to\mathbb{H}$ be slice hyperholomorphic. Then the function $f^{-\star}\star g$ is slice hypermeromorphic in Ω. Conversely, any slice hypermeromorphic function on Ω can be locally expressed as $f^{-\star}\star g$ for suitable f and g.*

Remark 2.13. Since $f^{-\star}=(f\star f^c)^{-1}f^c$ (see [25] for the notation) it is then clear that the poles of a slice hypermeromorphic function occur in correspondence to the zeros of the function $f\star f^c$ and so they are isolated spheres, possibly reduced to real points.

3. Slice hyperholomorphic operator-valued functions

We begin the section by characterizing slice hyperholomorphic functions as those functions which admit left derivative on each complex plane $\mathbb{C}_I$:

Definition 3.1. Let $f:\Omega\subseteq\mathbb{H}\to\mathbb{H}$ and let $p_0\in U$ be a nonreal point, $p_0=u_0+Iv_0$. Let f_I be the restriction of f to the plane $\mathbb{C}_I$. Assume that

$$\lim_{p\to p_0,\, p\in\mathbb{C}_I}(p-p_0)^{-1}(f_I(p)-f_I(p_0)) \tag{3.1}$$

exists. Then we say that f admits left slice derivative in p_0. If p_0 is real, assume that

$$\lim_{p\to p_0,\, p\in\mathbb{C}_I}(p-p_0)^{-1}(f_I(p)-f_I(p_0)) \tag{3.2}$$

exists, equal to the same value, for all $I\in\mathbb{S}$. Then we say that f admits left slice derivative in p_0. If f admits left slice derivative for every $p_0\in\Omega$ then we say that f admits left slice derivative in Ω or, for short, that f is *left slice differentiable* in Ω.

It is possible to give an analogous definition for right slice differentiable functions: it is sufficient to multiply $(p-p_0)^{-1}$ on the right. In this case we will speak of right slice hyperhomolomorphic functions. In this paper, we will speak of slice differentiable functions or slice hyperholomorphic functions when we are considering them on the left, while we will specify if we consider the analogous notions on the right.

We have the following result:

Proposition 3.2. *Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ be a real differentiable function. Then f is slice hyperholomorphic on Ω if and only if it admits slice derivative on Ω.*

Proof. Let f be a slice hyperholomorphic function on Ω. Then its restriction to the complex plane $\mathbb{C}_I$ can be written as $f_I(p) = F(p) + G(p)J$ where J is any element in $\mathbb{S}$ orthogonal to I, p belongs to $\mathbb{C}_I$ and $F, G : \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$ are holomorphic functions. Let p_0 be a nonreal quaternion and let $p_0 \in \Omega \cap \mathbb{C}_I$. Then we have

$$\begin{aligned}
&\lim_{p \to p_0,\, p \in \mathbb{C}_I} (p - p_0)^{-1}(f_I(p) - f_I(p_0)) \\
&= \lim_{p \to p_0,\, p \in \mathbb{C}_I} (p - p_0)^{-1}(F(p) + G(p)J - F(p_0) - G(p_0)J) \qquad (3.3) \\
&= F'(p_0) + G'(p_0)J
\end{aligned}$$

so the limit exists and f admits slice derivative at every nonreal point in Ω. If p_0 is real then the same reasoning shows that the limit in (3.3) exists on each complex plane $\mathbb{C}_I$. Moreover, since f is slice hyperholomorphic at p_0 we have

$$F'(p_0) + G'(p_0)J = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right)(F + GJ)(p_0) = \frac{\partial}{\partial x} f(p_0)$$

and so the limit exists on $\mathbb{C}_I$ for all $I \in \mathbb{S}$ equal to $\frac{\partial}{\partial x} f(p_0)$.

Conversely, assume that f admits slice derivative in Ω. By (3.1) and (3.2) f_I admits derivative on $\Omega \cap \mathbb{C}_I$ for all $I \in \mathbb{S}$. Decomposing f_I into complex components as $f_I(p) = F(p) + G(p)J$, where $F, G : \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$, $p = x + Iy$ and J is orthogonal to I, we deduce that both F and G admits complex derivative and thus they are in the kernel of the Cauchy Riemann operator $\partial_x + I\partial_y$ for all $I \in \mathbb{S}$ as well as f_I. Thus f is slice hyperholomorphic. □

Remark 3.3. The terminology of Definition 3.1 is consistent with the notion of slice derivative $\partial_s f$ of f, see [25], which is defined by:

$$\partial_s(f)(p) = \begin{cases} \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy)\right) & \text{if } p = x + Iy,\ \ y \neq 0, \\[2ex] \dfrac{\partial f}{\partial x}(p) & \text{if } p = x \in \mathbb{R}. \end{cases}$$

It is immediate that, analogously to what happens in the complex case, for any slice hyperholomorphic function we have $\partial_s(f)(x + Iy) = \partial_x(f)(x + Iy)$.

In the sequel, let $\mathcal{X}$ denote a left quaternionic Banach space and let $\mathcal{X}^*$ denote its dual, i.e. the set of bounded, left linear maps from $\mathcal{X}$ to $\mathbb{H}$. In order to have that $\mathcal{X}^*$ has a structure of quaternionic linear space, it is necessary to require that $\mathcal{X}$ is a two sided quaternionic vector space. In this case, $\mathcal{X}^*$ turns out to be a right vector space over $\mathbb{H}$.

Definition 3.4. Let $\mathcal{X}$ be a two sided quaternionic Banach space and let $\mathcal{X}^*$ be its dual. Let Ω be an open set in $\mathbb{H}$.

A function $f : \Omega \to \mathcal{X}$ is said to be *weakly slice hyperholomorphic* in Ω if Λf admits a slice derivative for every $\Lambda \in \mathcal{X}^*$.

A function $f:\Omega\to\mathcal{X}$ is said to be *strongly slice hyperholomorphic* in Ω if

$$\lim_{p\to p_0, p\in\mathbb{C}_I}(p-p_0)^{-1}(f_I(p)-f_I(p_0)) \tag{3.4}$$

exists in the topology of $\mathcal{X}$ in case $p_0\in\Omega$ is nonreal and $p_0\in\mathbb{C}_I$ and if

$$\lim_{p\to p_0, p\in\mathbb{C}_I}(p-p_0)^{-1}(f_I(p)-f_I(p_0)) \tag{3.5}$$

exists in the topology of $\mathcal{X}$ for every $I\in\mathbb{S}$, equal to the same value, in case $p_0\in\Omega$ is real.

Since the functionals $\Lambda\in\mathcal{X}^*$ are continuous, every strongly slice hyperholomorphic function is weakly slice hyperholomorphic. As it happens in the complex case, let us show that also the converse is true.

To this end, let us observe that the following lemma holds. We omit the proof since it works exactly as in the complex case (see e.g. [56], p. 189).

Lemma 3.5. *Let $\mathcal{X}$ be a two sided quaternionic Banach space. Then a sequence $\{v_n\}$ is Cauchy if and only if $\{\Lambda v_n\}$ is Cauchy uniformly for $\Lambda\in\mathcal{X}^*$, $\|\Lambda\|\le 1$.*

Theorem 3.6. *Every weakly slice hyperholomorphic function on $\Omega\subseteq\mathbb{H}$ is strongly slice hyperholomorphic on Ω.*

Proof. The proof will follow the lines of the proof in the complex case in [56], p. 189. Let f be a weakly slice hyperholomorphic function on Ω. Then, for any $\Lambda\in\mathcal{X}^*$ and any $I\in\mathbb{S}$, we can choose $J\in\mathbb{C}_I$ and write $(\Lambda f)_I(p)=(\Lambda f)_I(x+Iy)=F_\Lambda(x+Iy)+G_\Lambda(x+Iy)J$ where $F_\Lambda,G_\Lambda:\mathbb{C}_I\to\mathbb{C}_I$. By hypothesis, for any $p_0\in\Omega\cap\mathbb{C}_I$ the limit $\lim_{p\to p_0,\, p\in\mathbb{C}_I}(p-p_0)^{-1}((\Lambda f)_I(p)-(\Lambda f)_I(p_0))$ exists, and so the limits

$$\lim_{p\to p_0,\, p\in\mathbb{C}_I}(p-p_0)^{-1}(F_\Lambda(p)-F_\Lambda(p_0)),\qquad \lim_{p\to p_0,\, p\in\mathbb{C}_I}(p-p_0)^{-1}(G_\Lambda(p)-G_\Lambda(p_0))$$

exist. Thus the functions F_Λ and G_Λ are holomorphic on $\Omega\cap\mathbb{C}_I$ and so they admit a Cauchy formula on the plane $\mathbb{C}_I$, computed e.g. on a circle γ, contained in $\mathbb{C}_I$, whose interior contains p_0 and is contained in Ω. Note that $(\Lambda f)_I(x+Iy)=\Lambda f_I(x+Iy)$. Moreover, if p_0 is real we can pick any complex plane $\mathbb{C}_I$. For any increment h in $\mathbb{C}_I$ we compute

$$\begin{aligned}&h^{-1}((\Lambda f)_I(p_0+h)-(\Lambda f)_I(p_0))-\partial_s(\Lambda f)_I(p_0))\\&=\frac{1}{2\pi}\int_\gamma\left[h^{-1}\left(\frac{1}{p-(p_0+h)}-\frac{1}{p-p_0}\right)-\frac{1}{(p-p_0)^2}\right]dp_I(\Lambda f)_I(p)),\end{aligned}$$

where $dp_I=(dx+Idy)/I$. Then we observe that $(\Lambda f)_I(p)$ is continuous on γ which is compact, so $|(\Lambda f)_I(p)|\le C_\Lambda$ for all $p\in\gamma$. The family of maps $f(p):\mathcal{X}^*\to\mathbb{H}$ is pointwise bounded at each Λ, thus $\sup_{p\in\gamma}\|f_i(p)\|\le C$ by the uniform boundedness theorem, see [7]. Thus we have

$$\begin{aligned}&\left|\Lambda(h^{-1}(f_I(p_0+h)-f_I(p_0))-\partial_s(\Lambda f)_I(p_0)\right|\\&\le\frac{C}{2\pi}\|\Lambda\|\int_\gamma\left|\left(\frac{1}{p-(p_0+h)}-\frac{1}{p-p_0}\right)-\frac{1}{(p-p_0)^2}\right|dp_I,\end{aligned}$$

so $h^{-1}(f_I(p_0+h)-f_I(p_0))$ is uniformly Cauchy for $\|\Lambda\|\leq 1$ and by Lemma 3.5 it converges in $\mathcal{X}$. Thus f admits slice derivative at every $p_0\in\Omega$ and so it is strongly slice hyperholomorphic in Ω. □

Definition 3.7. Let $\mathcal{X}$ be a two-sided Banach space over $\mathbb{H}$. We say that a function $f:\Omega\to\mathcal{X}$ is (weakly) slice hypermeromorphic if for any $\Lambda\in\mathcal{X}^*$ the function $\Lambda f:\Omega\to\mathbb{H}$ is slice hypermeromorphic in Ω.

Remark 3.8. The previous definition means, in particular, that $f:\Omega'\to\mathcal{X}$ is slice hyperholomorphic, where the points in $\Omega\setminus\Omega'$ are the poles of f and $\Omega\setminus\Omega'$ has no point limit in Ω.

Our next task is to prove that weakly slice hyperholomorphic functions are those functions whose restrictions to any complex plane $\mathbb{C}_I$ are in the kernel of the Cauchy-Riemann operator $\partial_x+I\partial_y$.

Proposition 3.9. *Let $\mathcal{X}$ be a two sided quaternionic Banach space. A real differentiable function $f:\Omega\subseteq\mathbb{H}\to\mathcal{X}$ is weakly slice hyperholomorphic if and only if $(\partial_x+I\partial_y)f_I(x+Iy)=0$ for all $I\in\mathbb{S}$.*

Proof. If f is weakly slice hyperholomorphic, then, as it happens in the classical complex case, for every nonreal $p_0\in\Omega$, $p_0\in\mathbb{C}_I$, we can compute the limit (3.1) for the function Λf_I choosing $p=p_0+h$ with $h\in\mathbb{R}$ and for $p=p_0+Ih$ with $h\in\mathbb{R}$. We obtain, respectively, $\partial_x f_I\Lambda(p_0)$ and $-I\partial_y\Lambda f_I(p_0)$ which coincide. Thus we get $(\partial_x+I\partial_y)\Lambda f_I(p_0)=\Lambda(\partial_x+I\partial_y)f_I(p_0)=0$ for any $\Lambda\in\mathcal{X}^*$ and the statement follows by the Hahn-Banach theorem. If p_0 is real, then the statement follows by an analogous argument since the limit (3.2) exists for all $I\in\mathbb{S}$. Conversely, if f_I satisfies the Cauchy-Riemann on $\Omega\cap\mathbb{C}_I$ then $\Lambda((\partial_x+I\partial_y)f_I(x+Iy))=0$ for all $\Lambda\in\mathcal{X}^*$ and all $I\in\mathbb{S}$. Since Λ is linear and continuous we can write $(\partial_x+I\partial_y)\Lambda f_I(x+Iy)=0$ and thus the function $\Lambda f_I(x+Iy)$ is in the kernel of $\partial_x+I\partial_y$ for all $\Lambda\in\mathcal{X}^*$ or, equivalently by Proposition 3.2, it admits slice derivative. Thus at every $p_0\in\Omega\cap C_I$ we have

$$\lim_{p\to p_0, p\in\mathbb{C}_I}(p-p_0)^{-1}(\Lambda f_I(p)-\Lambda f_I(p_0))=\lim_{p\to p_0, p\in\mathbb{C}_I}\Lambda((p-p_0)^{-1}(f_I(p)-f_I(p_0))),$$

for all $\Lambda\in\mathcal{X}^*$. It follows that f is weakly slice hyperholomorphic. □

Since the class of weakly and strongly slice hyperholomorphic functions coincide and in view of Proposition 3.9, from now on we will refer to them simply as slice hyperholomorphic functions and we denote the set of $\mathcal{X}$-valued slice hyperholomorphic functions on Ω by $\mathscr{S}(\Omega,\mathcal{X})$.

The following result is immediate:

Proposition 3.10. *Let $\mathcal{X}$ be a two sided quaternionic Banach space. Then the set of slice hyperholomorphic functions defined on $\Omega\subseteq\mathbb{H}$ with values in $\mathcal{X}$ is a right quaternionic linear space.*

Proposition 3.11 (Identity Principle). *Let $\mathcal{X}$ be a two sided quaternionic Banach space, Ω be an s-domain and let $f,g:\Omega\subseteq\mathbb{H}\to\mathcal{X}$ be two slice hyperholomorphic* *functions. If $f = g$ on a set $Z \subseteq \Omega \cap \mathbb{C}_I$ having an accumulation point, for some $I \in \mathbb{S}$, then $f = g$ on Ω.*

Proof. The hypothesis implies $\Lambda f = \Lambda g$ on Z for every $\Lambda \in \mathcal{X}^*$ thus the slice hyperholomorphic function $\Lambda(f-g)$ is identically zero not only on Z but also on Ω by the Identity Principle for quaternionic-valued slice hyperholomorphic functions. By the Hahn-Banach theorem $f - g = 0$ on Ω. □

Remark 3.12. The Identity Principle implies that two slice hyperholomorphic functions defined on an s-domain and with values in a two sided quaternionic Banach space $\mathcal{X}$ coincide if their restrictions to the real axis coincide. More in general, any real analytic function $f : [a,b] \subseteq \mathbb{R} \to \mathcal{X}$ can be extended to a function $\mathrm{ext}(f)$ slice hyperholomorphic on an axially symmetric s-domain Ω containing $[a,b]$. The existence of the extension is assured by the fact that for any $x_0 \in [a,b]$ the function f can be written as $f(x) = \sum_{n\geq 0} x^n A_n$, $A_n \in \mathcal{X}$, and x such that $|x - x_0| < \varepsilon$ and thus $(\mathrm{ext}f)(p) = \sum_{n\geq 0} p^n A_n$ for $|p - x_0| < \varepsilon_{x_0}$. Thus the claim holds setting $B(x_0, \varepsilon_{x_0}) = \{p \in \mathbb{H} : \ |p - x_0| < \varepsilon_{x_0}\}$ and $\Omega = \cup_{x_0 \in I} B(x_0, \varepsilon_{x_0})$.

Let us recall, see [25], that the Cauchy kernel to be used in the Cauchy formula for slice hyperholomorphic functions is

$$S_L^{-1}(s,p) = -(p^2 - 2p\mathrm{Re}(s) + |s|^2)^{-1}(p - \overline{s}).$$

It is a function slice hyperholomorphic on the left in the variable p and on the right in s. In the case of right regular functions the kernel is

$$S_R^{-1}(s,q) := -(q - \bar{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1},$$

which is slice hyperholomorphic on the right in the variable q and on the left in s. The Cauchy formula holds for slice hyperholomorphic functions with values in a quaternionic Banach space:

Theorem 3.13 (Cauchy formulas). *Let $\mathcal{X}$ be a two sided quaternionic Banach space and let W be an open set in $\mathbb{H}$. Let $\overline{\Omega} \subset W$ be an axially symmetric s-domain, and let $\partial(\Omega \cap \mathbb{C}_I)$ be the union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Set $ds_I = ds/I$. If $f : W \to \mathcal{X}$ is left slice hyperholomorphic, then, for $q \in \Omega$, we have*

$$f(p) = \frac{1}{2\pi} \int_{\partial(\Omega\cap\mathbb{C}_I)} S_L^{-1}(s,p)ds_I f(s), \tag{3.6}$$

if $f : W \to \mathcal{X}$ is right slice hyperholomorphic, then, for $q \in \Omega$, we have

$$f(q) = \frac{1}{2\pi} \int_{\partial(\Omega\cap\mathcal{C}_I)} f(s)ds_I S_R^{-1}(s,q), \tag{3.7}$$

and the integrals (3.6), (3.7) *do not depend on the choice of the imaginary unit $I \in \mathbb{S}$ and on $\Omega \subset W$.*

Proof. We have proved that weakly slice hyperholomorphic functions are strongly slice hyperholomorphic functions, so in particular they are continuous functions, so the validity of the formulas (3.6), (3.7) follows as in point (b) p. 80 [57]. □

We now show another description of the class $\mathscr{S}(\Omega, \mathcal{X})$ of slice hyperholomorphic functions on Ω with values in $\mathcal{X}$.

Definition 3.14. Consider the set of functions of the form $f(p) = f(x + Iy) = \alpha(x, y) + I\beta(x, y)$ where $\alpha, \beta : \Omega \to \mathcal{X}$ depend only on x, y, are real differentiable, satisfy the Cauchy-Riemann equations $\partial_x\alpha - \partial_y\beta = 0$, $\partial_y\alpha + \partial_x\beta = 0$ and assume that $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$. We will denote the class of function of this form by $\mathscr{H}(\Omega, \mathcal{X})$.

Observe that the conditions on α and β are required in order to have that the function f is well posed. Note also that if $p = x$ is a real quaternion, then I is not uniquely defined but the hypothesis that β is odd in the variable y implies $\beta(x, 0) = 0$.

Theorem 3.15. *Let Ω be an axially symmetric s-domain and let $\mathcal{X}$ be a two sided quaternionic Banach space. Then $\mathscr{S}(\Omega, \mathcal{X}) = \mathscr{H}(\Omega, \mathcal{X})$.*

Proof. The inclusion $\mathscr{H}(\Omega, \mathcal{X}) \subseteq \mathscr{S}(\Omega, \mathcal{X})$ is clear: any function $f \in \mathscr{H}(\Omega, \mathcal{X})$ is real differentiable and such that f_I satisfies $(\partial_x + I\partial_y)f_I = 0$ (note that this implication does not need any hypothesis on the open set Ω). Conversely, assume that $f \in \mathscr{S}(\Omega, \mathcal{X})$. Let us show that

$$f(x + Iy) = \frac{1}{2}(1 - IJ)f(x + Jy) + \frac{1}{2}(1 + IJ)f(x - Jy).$$

If we consider real quaternions, i.e. $y = 0$, then the formula holds trivially. For nonreal quaternions, set

$$\phi(x + Iy) = \frac{1}{2}(1 - IJ)f(x + Jy) + \frac{1}{2}(1 + IJ)f(x - Jy).$$

Then, using the fact that f is slice hyperholomorphic, it is immediate that $(\partial_x + I\partial y)\phi(x + Iy) = 0$ and so ϕ is slice hyperholomorphic. Since $\phi = f$ on $\Omega \cap \mathbb{C}_I$ then it coincides with f on Ω by the Identity Principle. By writing

$$f(x + Iy) = \frac{1}{2}\left[(f(x + Jy) + f(x - Jy) + IJ(f(x - Jy) - f(x + Jy))\right]$$

and setting $\alpha(x, y) = \frac{1}{2}(f(x+Jy)+f(x-Jy))$, $\beta(x, y) = \frac{1}{2}J(f(x-Jy)-f(x+Jy))$ we have that $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$. Reasoning as in [25, Theorem 2.2.18] we see that α, β do not depend on I. It is then an easy computation to verify that α, β satisfy the above assumptions. $\square$

Using this alternative description of slice hyperholomorphic functions with values in $\mathcal{X}$, we can now define a notion of product which is based on a suitable pointwise multiplication. To this end we need an additional structure on the two sided quaternionic Banach space $\mathcal{X}$. Assume that in $\mathcal{X}$ is defined a multiplication which is associative, distributive with respect to the sum in $\mathcal{X}$. Assume also that $q(x_1x_2) = (qx_1)x_2$ and $(x_1x_2)q = x_1(x_2q)$ for all $q \in \mathbb{H}$ and for all $x_1, x_2 \in \mathcal{X}$. Then we will say that $\mathcal{X}$ is a two sided quaternionic Banach algebra. As customary we will say that the algebra $\mathcal{X}$ is with unity is $\mathcal{X}$ possesses a unity with respect to the product.

Definition 3.16. Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \to \mathcal{X}$ be slice hyperholomorphic functions with values in a two sided quaternionic Banach algebra $\mathcal{X}$. Let $f(x+Iy) = \alpha(x,y) + I\beta(x,y)$, $g(x+Iy) = \gamma(x,y) + I\delta(x,y)$. Then we define

$$(f \star g)(x+Iy) := (\alpha\gamma - \beta\delta)(x,y) + I(\alpha\delta + \beta\gamma)(x,y). \tag{3.8}$$

By construction, the function $f \star g$ is slice hyperholomorphic, as it can be easily verified.

Remark 3.17. If Ω is a ball with center at a real point (let us assume at the origin for simplicity) then it is immediate that f, g admit power series expansion and thus if $f(p) = \sum_{n=0}^{\infty} p^n a_n$, $g(p) = \sum_{n=0}^{\infty} p^n b_n$, $a_n, b_n \in \mathcal{X}$ for all n. Then $f \star g(p) := \sum_{n=0}^{\infty} p^n (\sum_{r=0}^{n} a_r b_{n-r})$ where the series converges.

Remark 3.18. In case we consider right slice hyperholomorphic functions, the class $\mathscr{H}(\Omega, \mathcal{X})$ consists of functions of the form $f(x+Iy) = \alpha(x,y) + \beta(x,y)I$ where α, β satisfy the assumptions discussed above. We now give the notion of right slice product, denoted by $\star_r$. Given two right slice hyperholomorphic functions $f, g : \Omega \to \mathcal{X}$ with values in a two sided quaternionic Banach algebra $\mathcal{X}$ where $f(x+Iy) = \alpha(x,y) + \beta(x,y)I$, $g(x+Iy) = \gamma(x,y) + \delta(x,y)I$, we define

$$(f \star_r g)(x+Iy) := (\alpha\gamma - \beta\delta)(x,y) + (\alpha\delta + \beta\gamma)(x,y)I. \tag{3.9}$$

Remark 3.19. It is important to point out that if one is in need of considering slice hyperholomorphic functions on axially symmetric open sets U which are not necessarily s-domains, then it is more convenient to use the class $\mathcal{H}(\Omega, \mathcal{X})$ because they allow to have a notion of multiplication.

Remark 3.20. Consider the following case: let Ω be an axially symmetric s-domain in $\mathbb{H}$ and let $\mathcal{H}_i$; $i = 1,2,3$ be two sided quaternionic Hilbert spaces. Let $f : \Omega \to \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$, $g : \Omega \to \mathbf{L}(\mathcal{H}_2, \mathcal{H}_3)$ be slice hyperholomorphic and let

$$f(p) = f(x+Iy) = \alpha(x,y) + I\beta(x,y), \quad g(p) = g(x+Iy) = \gamma(x,y) + I\delta(x,y).$$

We define the $\star$-product as in (3.8) If f, g are right slice hyperholomorphic, then we define the $\star_r$-product as in (3.9). The product $\alpha(x,y)\gamma(x,y)$ (and the other three products appearing in $f \star g$) is an operator belonging to $\mathbf{L}(\mathcal{H}_1, \mathcal{H}_3)$, thus $f \star g : \Omega \to \mathbf{L}(\mathcal{H}_1, \mathcal{H}_3)$. In the special case in which

$$f(p) = \sum_{n=0}^{\infty} p^n A_n, \qquad A_n \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2),$$

$$g(p) = \sum_{n=0}^{\infty} p^n B_n, \qquad B_n \in \mathbf{L}(\mathcal{H}_2, \mathcal{H}_3),$$

then we have

$$f \star g(p) = \sum_{n=0}^{\infty} p^n (\sum_{r=0}^{n} A_r B_{n-r}),$$

as expected.

3.1. Realizations

The following notions of S-spectrum and of S-resolvent operator will be used in the sequel.

Definition 3.21. Let $\mathcal{X}$ be a two sided quaternionic Banach space and let A be a bounded operator on $\mathcal{X}$ into itself. We define the S-spectrum $\sigma_S(A)$ of A as:

$$\sigma_S(A) = \{p \in \mathbb{H} : \quad A^2 - 2\,\mathrm{Re}\,(p)A + |p|^2 I \quad \textit{is not invertible}\}.$$

The S-resolvent set $\rho_S(A)$ is defined by $\rho_S(A) = \mathbb{H} \setminus \sigma_S(A)$.

For $p \in \rho_S(A)$ the right S-resolvent operator is defined as

$$S_R^{-1}(p, A) := -(A - \overline{p}I)(A^2 - 2\mathrm{Re}\,(p)A + |p|^2 I)^{-1}. \tag{3.10}$$

Remark 3.22. It is useful to recall that when A is a matrix its (point) S-spectrum coincides with its right spectrum, see e.g. [6]. When $p \in \mathbb{R}$ or, more in general, when p commute with an operator A, then $S_R^{-1}(p, A) = (pI - A)^{-1}$, see Proposition 3.1.6 in [25].

Proposition 3.23. *Let $\mathcal{X}$ be a two sided quaternionic Banach space and let $f : \rho_S(A) \cap \mathbb{R} \setminus \{0\} \to \mathcal{X}$ be the function $f(x) = (I - xA)^{-1}$. Then*

$$p^{-1} S_R^{-1}(p^{-1}, A) = (I - \overline{p}A)(I - 2\mathrm{Re}(p)A + |p|^2 A^2)^{-1}$$

is the unique slice hyperholomorphic extension to $\rho_S(A)$.

Proof. The fact that $p^{-1}S_L^{-1}(p^{-1}, A)$ is slice hyperholomorphic in p outside the S-spectrum is trivial since it is the S-resolvent and it coincides with the function f on the real axis. The uniqueness follows from the identity principle. □

The notation $S_R^{-1}(p^{-1}, A)$ comes from [25] but we will also write

$$p^{-1} S_R^{-1}(p^{-1}, A) = (I - pA)^{-\star}.$$

This last expression makes sense when A acts on a two-sided quaternionic vector space. In a more general setting, we have the following result:

Proposition 3.24. *Let A be a bounded linear operator from a right-sided quaternionic Banach $\mathcal{P}$ space into itself, and let G be a bounded linear operator from $\mathcal{P}$ into $\mathcal{Q}$, where $\mathcal{Q}$ is a two sided quaternionic Banach space. The slice hyperholomorphic extension of $G(I - xA)^{-1}$, $1/x \in \sigma_S(A) \cap \mathbb{R}$, is*

$$(G - \overline{p}GA)(I - 2\mathrm{Re}(p)\, A + |p|^2 A^2)^{-1}.$$

Proof. First we observe that $G(I - xA)^{-1} = \sum_{n=0}^{\infty} x^n G A^n$ for $|x|\|A\| < 1$. It is immediate that, for $|p|\|A\| < 1$, the slice hyperholomorphic extension of the series $\sum_{n=0}^{\infty} x^n G A^n$ is $\sum_{n=0}^{\infty} p^n G A^n$ (as it is a converging power series with coefficients on the right). To show that

$$\sum_{n=0}^{\infty} p^n G A^n = (G - \overline{p}GA)(I - 2\mathrm{Re}(p)\, A + |p|^2 A^2)^{-1}$$

we prove instead the equality

$$(\sum_{n=0}^{\infty} p^n GA^n)(I - 2\mathrm{Re}(p)\, A + |p|^2 A^2) = (G - \overline{p}GA).$$

The left hand side gives

$$\begin{aligned}&\sum_{n=0}^{\infty} p^n GA^n - 2\sum_{n=0}^{\infty} \mathrm{Re}(p) p^n GA^{n+1} + \sum_{n=0}^{\infty} |p|^2 p^n GA^{n+2}\\&= G + (p - 2\mathrm{Re}(p))GA + (p^2 - 2p\mathrm{Re}(p) + |p|^2) \sum_{n=0}^{\infty} p^n GA^{n+2}\\&= G - \bar{p}GA\end{aligned}$$

where we have used the identity $p^2 - 2p\mathrm{Re}(p) + |p|^2 = 0$. This completes the proof. □

Remark 3.25. In analogy with the matrix case we will write, with an abuse of notation in this case, $G \star (I - pA)^{-\star}$ instead of the expression $(G - \overline{p}GA)(I - 2\mathrm{Re}(p)\, A + |p|^2 A^2)^{-1}$.

Proposition 3.26. *With the notation in Remark* 3.25 *it holds that*

$$D + pC \star (I - pA)^{-1} B = D^{-1} - pD^{-1} C \star (I - p(A - BD^{-1}C))^{-\star} BD^{-1}, \quad (3.11)$$

and

$$\begin{aligned}&(D_1 + pC_1 \star (I - pA_1)^{-\star} B_1) \star (D_2 + pC_2 \star (I - pA_2)^{-\star} B_2)\\&= D_1 D_2 + p \begin{pmatrix} C_1 & D_1 C_2 \end{pmatrix} \star \left(I - p \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix} \right)^{-\star} \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}.\end{aligned} \quad (3.12)$$

Proof. When p is real, the $\star$-product is replaced by the operator product (or matrix product in the finite dimensional case) and formulas (3.11) and (3.12) are then well known, see e.g. [19]. Slice-hyperholomorphic extensions lead then to the required result. □

4. The Hardy space of the half-space $\mathbb{H}_+$

Let Π_+ be the right open half-plane of complex numbers z such that $\mathrm{Re}(z) > 0$. The Hardy space $\mathbf{H}_2(\Pi_+)$ consists of functions f holomorphic in Π_+ such that

$$\sup_{x>0} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy < \infty. \quad (4.1)$$

We recall that a function $f \in \mathbf{H}_2(\Pi_+)$ has nontangential limit $f(iy)$ for almost all iy on the imaginary axis and $f(iy) \in L_2(\mathbb{R})$, see [36, Theorem 3.1], moreover

$$\sup_{x>0} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy = \int_{-\infty}^{\infty} |f(iy)|^2 dy. \quad (4.2)$$

Let us consider the kernel

$$k_{\Pi_+}(z, w) = \frac{1}{2\pi} \frac{1}{z + \overline{w}},$$

which is positive definite on Π_+. Then, the associated reproducing kernel Hilbert space is the Hardy space $\mathbf{H}_2(\Pi_+)$ endowed with the scalar product

$$\langle f, g\rangle_{\mathbf{H}_2(\Pi_+)} = \int_{-\infty}^{+\infty} \overline{g(iy)} f(iy) dy,$$

where $f, g \in \mathbf{H}_2(\Pi_+)$, and the norm in $\mathbf{H}_2(\Pi_+)$ is given by

$$\|f\|_{\mathbf{H}_2(\Pi_+)} = \left(\int_{-\infty}^{+\infty} |f(iy)|^2 dy \right)^{\frac{1}{2}}.$$

The kernel $k_{\Pi_+}(z, w)$ is reproducing in the sense that for every $f \in \mathbf{H}_2(\Pi_+)$

$$f(w) = \langle f(z), k_{\Pi_+}(z, w)\rangle_{\mathbf{H}_2(\Pi_+)} = \int_{-\infty}^{\infty} k_{\Pi_+}(w, iy) f(iy) dy,$$

Let us now consider the half-space $\mathbb{H}_+$ of the quaternions q such that $\mathrm{Re}(q) > 0$ and set $\Pi_{+,I} = \mathbb{H}_+ \cap \mathbb{C}_I$. We will denote by f_I the restriction of a function f defined on $\mathbb{H}_+$ to $\Pi_{+,I}$. We define

$$\mathbf{H}_2(\Pi_{+,I}) = \{f \text{ slice hyperholomorphic in } \mathbb{H}_+ : \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy < \infty\},$$

where $f(Iy)$ denotes the nontangential value of f at Iy. Note that these value exist almost everywhere, in fact any $f \in \mathbf{H}_2(\Pi_{+,I})$ when restricted to a complex plane $\mathbb{C}_I$ can be written as $f_I(x + Iy) = F(x + Iy) + G(x + Iy)J$ where J is any element in $\mathbb{S}$ orthogonal to I, and F, G are $\mathbb{C}_I$-valued holomorphic functions. Since the nontangential values of F and G exist almost everywhere at Iy, also the nontangential value of f exists at Iy a. e. on $\Pi_{+,I}$ and $f_I(Iy) = F(Iy) + G(Iy)J$ a.e.

Remark 4.1. In alternative, we could have defined $\mathbf{H}_2(\Pi_{+,I})$ as the set of slice hyperholomorphic functions f such that $\sup_{x>0} \int_{-\infty}^{+\infty} |f_I(x + Iy)|^2 dy < \infty$. However note that $f_I(x + Iy) = F(x + Iy) + G(x + Iy)J$, see the above discussion, and so $|f_I(x + Iy)|^2 = |F(x + Iy)|^2 + |G(x + Iy)|^2$. Thus, using (4.2), we have

$$\begin{aligned} &\sup_{x>0} \int_{-\infty}^{+\infty} |f_I(x + Iy)|^2 dy \\ &= \sup_{x>0} \int_{-\infty}^{+\infty} |F(x + Iy)|^2 dy + \sup_{x>0} \int_{-\infty}^{+\infty} |G(x + Iy)|^2 dy \\ &= \int_{-\infty}^{+\infty} |F(Iy)|^2 dy + \int_{-\infty}^{+\infty} |G(Iy)|^2 dy = \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy. \end{aligned} \tag{4.3}$$

In $\mathbf{H}_2(\Pi_{+,I})$ we define the scalar product

$$\langle f, g\rangle_{\mathbf{H}_2(\Pi_{+,I})} = \int_{-\infty}^{+\infty} \overline{g_I(Iy)} f_I(Iy) dy,$$

where $f_I(Iy)$, $g_I(Iy)$ denote the nontangential values of f, g at Iy on $\Pi_{+,I}$. This scalar product gives the norm

$$\|f\|_{\mathbf{H}_2(\Pi_{+,I})} = \left(\int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy \right)^{\frac{1}{2}}$$

(which is finite by our assumptions).

Proposition 4.2. *Let f be slice hyperholomorphic in $\mathbb{H}_+$ and assume that $f \in \mathbf{H}_2(\Pi_{+,I})$ for some $I \in \mathbb{S}$. Then for all $J \in \mathbb{S}$ the following inequalities hold,*

$$\frac{1}{2}\|f\|_{\mathbf{H}_2(\Pi_{+,I})} \leq \|f\|_{\mathbf{H}_2(\Pi_{+,J})} \leq 2\|f\|_{\mathbf{H}_2(\Pi_{+,I})}.$$

Proof. Formula (2.6) implies the inequality

$$|f(x+Jy)| \leq |f(x+Iy)| + |f(x-Iy)|,$$

and also

$$|f(x+Jy)|^2 \leq 2(|f(x+Iy)|^2 + |f(x-Iy)|^2). \tag{4.4}$$

Using (4.3), (2.6) and (4.4) we deduce

$$\begin{aligned}
\|f\|^2_{\mathbf{H}_2(\Pi_{+,J})} = \int_{-\infty}^{+\infty} |f_J(Jy)|^2 dy &= \sup_{x>0} \int_{-\infty}^{+\infty} |f_J(x+Jy)|^2 dy \\
&\leq \sup_{x>0} \int_{-\infty}^{+\infty} 2(|f_I(x+Iy)|^2 + f_I(x-Iy)|^2) dy \\
&= 4\int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy
\end{aligned}$$

and so $\|f\|^2_{\mathbf{H}_2(\Pi_{+,J})} \leq 4\|f\|^2_{\mathbf{H}_2(\Pi_{+,I})}$. By changing J with I we obtain the reverse inequality and the statement follows. □

An immediate consequence of this result is:

Corollary 4.3. *A function $f \in \mathbf{H}_2(\Pi_{+,I})$ for some $I \in \mathbb{S}$ if and only if $f \in \mathbf{H}_2(\Pi_{+,J})$ for all $J \in \mathbb{S}$.*

We now introduce the Hardy space of the half space $\mathbb{H}_+$:

Definition 4.4. We define $\mathbf{H}_2(\mathbb{H}_+)$ as the space of slice hyperholomorphic functions on $\mathbb{H}_+$ such that

$$\sup_{I\in\mathbb{S}} \int_{-\infty}^{+\infty} |f(Iy)|^2 dy < \infty. \tag{4.5}$$

We have:

Proposition 4.5. *The function*

$$k(p,q) = (\bar{p}+\bar{q})(|p|^2 + 2\mathrm{Re}(p)\bar{q} + \bar{q}^2)^{-1} \tag{4.6}$$

is slice hyperholomorphic in p and $\bar{q}$ on the left and on the right, respectively in its domain of definition, i.e. for $p \notin [\bar{q}]$. The restriction of $\frac{1}{2\pi}k(p,q)$ to $\mathbb{C}_I \times \mathbb{C}_I$ coincides with $k_{\Pi_+}(z,w)$. Moreover we have the equality:

$$k(p,q) = (|q|^2 + 2\mathrm{Re}(q)p + p^2)^{-1}(p+q). \tag{4.7}$$

Proof. Some computations allow to obtain $k(p,q)$ as the left slice hyperholomorphic extension in z of $k_q(z) = k(z,q)$, by taking z on the same complex plane as q. The function we obtain turns out to be also right slice hyperholomorphic in $\bar{q}$. The second equality follows by taking the right slice hyperholomorphic extension in $\bar{q}$ and observing that it is left slice hyperholomorphic in p. □

Proposition 4.6. *The kernel $\frac{1}{2\pi}k(p,q)$ is reproducing, i.e. for any $f \in \mathbf{H}_2(\mathbb{H}_+)$,*

$$f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p, Iy) f(Iy) dy.$$

Proof. Let $q = u + I_q v$ and let $p = u + Iv$ be the point on the sphere determined by q and belonging to the plane $\mathbb{C}_I$. Then we have

$$f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p, Iy) f(Iy) dy, \quad f(\bar{p}) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(\bar{p}, Iy) f(Iy) dy.$$

The extension formula (2.5) applied to $k_{Iy}(p) = k(p, Iy)$ shows the statement. □

The following property will be useful in the sequel:

Proposition 4.7. *The kernel $k(p,q)$ satisfies*

$$pk(p,q) + k(p,q)\bar{q} = 1.$$

Proof. From the expression (4.6), and since q commutes with $(|p|^2+2\text{Re}(p)\bar{q}+\bar{q}^2)^{-1}$, we have

$$\begin{aligned} &p(\bar{p}+\bar{q})(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1} + (\bar{p}+\bar{q})(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1}\bar{q} \\ &= (|p|^2 + p\bar{q} + \bar{p}\bar{q} + \bar{q}^2)(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1} = 1. \end{aligned}$$ □

We know that if $\{\phi_n(z)\}$ is an orthonormal basis for $\mathbf{H}_2(\Pi_{+,I})$, for some $I \in \mathbb{S}$, then

$$k(z,w) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(w)}, \tag{4.8}$$

and so the kernel $k(z,w)$ is positive definite. We now prove the following:

Proposition 4.8. *Let $\{\phi_n(z)\}$ be an orthonormal basis for $\mathbf{H}_2(\Pi_{+,I})$, for some $I \in \mathbb{S}$, and let $\{\Phi_n(q)\} = \{\text{ext}(\phi_n(z))\}$ be the sequence of the slice hyperholomorphic extensions of its elements. Then $\{\Phi_n(q)\}$ is an orthonormal basis for $\mathbf{H}_2(\mathbb{H}_+)$, and*

$$k(p,q) = \sum_{n=1}^{\infty} \Phi_n(p)\overline{\Phi_n(q)}.$$

Proof. Let $\{\phi_n(z)\}$ be an orthonormal basis for $\mathbf{H}_2(\Pi_{+,I})$ and let $\{\Phi_n(q)\} = \{\text{ext}(\phi_n(z))\}$ be the sequence of the slice hyperholomorphic extensions of its elements. Then $\{\Phi_n(q)\}$ is a generating set for $\mathbf{H}_2(\mathbb{H}_+)$. In fact take any $f \in \mathbf{H}_2(\mathbb{H}_+)$ and consider its restriction to a complex plane $\mathbb{C}_I$, for some $I \in \mathbb{S}$. Then, by choosing $J \in \mathbb{S}$ such that I, J are orthogonal, and taking $q = x + Iy$ we have $f_I(x+Iy) = F(x+Iy) + G(x+Iy)J$ with F, G holomorphic on $\Pi_{+,I}$ and

$$\int_{-\infty}^{+\infty} |f(Iy)|^2 dy = \int_{-\infty}^{+\infty} (|F(Iy)|^2 + |G(Iy)|^2) dy < \infty$$

and, as a consequence,

$$\int_{-\infty}^{+\infty} |F(Iy)|^2 dy \leq \int_{-\infty}^{+\infty} (|F(Iy)|^2 + |G(Iy)|^2) dy < \infty.$$

We deduce that both

$$\int_{-\infty}^{+\infty} |F(Iy)|^2 dy \quad \text{and} \quad \int_{-\infty}^{+\infty} |G(Iy)|^2 dy$$

are finite and so F, G belong to $\mathbf{H}_2(\Pi_{+,I})$. We can write $F(x+Iy) = \sum_{n=1}^{\infty} \phi_n(z) a_n$ and $G(x+Iy) = \sum_{n=1}^{\infty} \phi_n(z) b_n$, thus $f_I(x+Iy) = \sum_{n=1}^{\infty} \phi_n(z)(a_n + b_n J)$. By taking the extension with respect to z we finally obtain $f(q) = \sum_{n=1}^{\infty} \Phi_n(q)(a_n + b_n J)$. The fact that $\{\Phi_n(p)\}$ is made by orthonormal elements (thus linearly independent) in $\mathbf{H}_2(\mathbb{H}_+)$ follows from

$$\begin{aligned} \langle \Phi_n(p), \Phi_m(p) \rangle_{\mathbf{H}_2(\Pi_{+,I_p})} &= \int_{-\infty}^{\infty} \overline{\Phi_m(I_p y)} \Phi_n(I_p y) dy \\ &= \int_{-\infty}^{\infty} \overline{\phi_m(I_p y)} \phi_n(I_p y) dy = \delta_{nm}. \end{aligned}$$

Then (4.8) yields

$$k(p, w) = \text{ext}_z k(z, w) = \sum_{n=1}^{\infty} \text{ext}_z(\phi_n)(z) \overline{\phi_n(w)} = \sum_{n=1}^{\infty} \Phi_n(p) \overline{\phi_n(w)},$$

where we have written ext_z to emphasize that we are taking the extension in the variable z (note that in this way we have obtained the kernel written in the form (4.6)). Now we observe that the function $\sum_{n=1}^{\infty} \Phi_n(p) \overline{\Phi_n(q)}$ is slice hyperholomorphic on the left and on the right with respect to p and $\bar{q}$, respectively, and coincides with $k(p, w)$ when restricted to the plane containing w. By the uniqueness of the extension we have $k(p, q) = \sum_{n=1}^{\infty} \Phi_n(p) \overline{\Phi_n(q)}$, and the statement follows. □

We now introduce the Blaschke factors in the half space $\mathbb{H}_+$.

Definition 4.9. For $a \in \mathbb{H}_+$ set

$$b_a(p) = (p + \bar{a})^{-\star} \star (p - a).$$

The function $b_a(p)$ is called Blaschke factor at a in the half space $\mathbb{H}_+$.

Remark 4.10. The function $b_a(p)$ is defined outside the sphere $[-a]$ as it can be easily seen by rewriting it as

$$b_a(p) = (p^2 + 2\text{Re}(a)p + |a|^2)^{-1}(p + a) \star (p - a) = (p^2 + 2\text{Re}(a)p + |a|^2)^{-1}(p^2 - a^2)$$

and it has a zero for $p = a$. Note in fact that $p = -a$ is not a zero since it is a pole (of order 0). When $a \in \mathbb{R}$ the function $b_a(p) = (p + a)^{-1}(p - a)$ has a pole at $p = -a$

and a zero at a. A Blaschke factor is slice hyperholomorphic where it is defined, by construction.

We have the following result which characterizes the convergence of a Blaschke product. We denote by $\Pi^\star$ the $\star$-product:

Theorem 4.11. *Let* $\{a_j\} \subset \mathbb{H}_+$, $j = 1, 2, \ldots$ *be a sequence of quaternions such that* $\sum_{j\geq 1} \mathrm{Re}(a_j) < \infty$. *Then the function*

$$B(p) := \Pi^\star_{j\geq 1}(p + \bar{a}_j)^{-\star} \star (p - a_j), \tag{4.9}$$

converges uniformly on the compact subsets of $\mathbb{H}_+$.

Proof. We reason as in the proof of the corresponding result in the complex case (but see also the proof of Theorem 5.6 in [5]). We note that, see Remark 5.4 in [5], we can write

$$(p + \bar{a}_j)^{-\star} \star (p - a_j) = (\tilde{p} + \bar{a}_j)^{-1}(\tilde{p} - a_j) \tag{4.10}$$

where $\tilde{p} = \lambda^c(p)^{-1} p \lambda^c(p)$ and $\lambda^c(p) = p + a_j$ (note that $\lambda^c(p) \neq 0$ for $p \notin [-a_j]$) and so

$$(p + \bar{a}_j)^{-\star} \star (p - a_j) = (\tilde{p} + \bar{a}_j)^{-1}(\tilde{p} - a_j) = 1 - 2\mathrm{Re}(a_j)(\tilde{p} + \bar{a}_j)^{-1}. \tag{4.11}$$

By taking the modulus of the right hand side of (4.9), using (4.11), and reasoning as in the complex case, we conclude that the Blaschke product converges if and only if $\sum_{j=1}^\infty \mathrm{Re}(a_j) < \infty$. □

As in the unit disk case, we have two kinds of Blaschke factors. In fact, products of the form

$$b_a(p) \star b_{\bar{a}}(p) = ((p + \bar{a})^{-\star} \star (p - a)) \star ((p + a)^{-\star} \star (p - \bar{a}))$$

can be written as

$$b_a(p) \star b_{\bar{a}}(p) = (p^2 + 2\mathrm{Re}(a)p + |a|^2)^{-1}(p^2 - 2\mathrm{Re}(a)p + |a|^2),$$

and they admit the sphere $[a]$ as set of zeros. Note that slice regular functions which vanish at two different points belonging to the same sphere in reality vanish on the whole sphere (see [25, Corollary 4.3.7]. Thus if we want to construct a Blaschke product vanishing at some prescribed points and spheres, it is convenient to introduce the following:

Definition 4.12. For $a \in \mathbb{H}_+$ set

$$b_{[a]}(p) = (p^2 + 2\mathrm{Re}(a)p + |a|^2)^{-1}(p^2 - 2\mathrm{Re}(a)p + |a|^2).$$

The function $b_a(p)$ is called Blaschke factor at the sphere $[a]$ in the half space $\mathbb{H}_+$.

Note that the definition is well posed since it does not depend on the choice of the point a. As a consequence of Theorem 4.11 we have:

Corollary 4.13. *Let* $\{c_j\} \subset \mathbb{H}_+$, $j = 1, 2, \ldots$ *be a sequence of quaternions such that* $\sum_{j\geq 1} \mathrm{Re}(c_j) < \infty$. *Then the function*

$$B(p) := \Pi_{j\geq 1}(p^2 + 2\mathrm{Re}(c_j)p + |c_j|^2)^{-1}(p^2 - 2\mathrm{Re}(c_j)p + |c_j|^2), \tag{4.12}$$

converges uniformly on the compact subsets of $\mathbb{H}_+$.

Proof. It is sufficient to write $B(p) = \prod_{j\geq 1} b_{[c_j]}(p) = \prod_{j\geq 1} b_{c_j}(p) \star b_{\bar{c}_j}(p)$ and to observe that $\sum_{j\geq 1} \mathrm{Re}(c_j) < \infty$ by hypothesis. □

To state the next result, we need to repeat the notion of multiplicity of a sphere of zeros and of a point which is an isolated zero.

We say that the *multiplicity of the spherical zero* $[c_j]$ of a function $Q(p)$ is m_j if m_j is the maximum of the integers m such that $(p^2 + 2\mathrm{Re}(c_j)p + |c_j|^2)^m$ divides $Q(p)$.

Let $\alpha_j \in \mathbb{H} \setminus \mathbb{R}$ and let

$$Q(p) = (p-\alpha_1)\star\ldots\star(p-\alpha_n)\star g(p), \quad \alpha_{j+1} \neq \bar{\alpha}_j,\ \ j = 1, \ldots, n-1,\ \ g(p) \neq 0. \tag{4.13}$$

We say that α_1 is a zero of Q of *multiplicity* 1 if $\alpha_j \notin [\alpha_1]$ for $j = 2, \ldots, n$.

We say that α_1 is a zero of Q of *multiplicity* $n \geq 2$ if $\alpha_j \in [\alpha_1]$ for all $j = 2, \ldots, n$.

If $\alpha_j \in \mathbb{R}$ we can repeat the notion of multiplicity of α_1 where (4.13) holds by removing the assumption $\alpha_{j+1} \neq \bar{\alpha}_j$. This definition coincides with the standard notion of multiplicity since, in this case, the $\star$-product reduces to the pointwise product. Note that if a function has a sphere of zeros at $[\alpha]$ with multiplicity m, at most one point on $[\alpha]$ can have higher multiplicity; in fact if there are two such points it means that the sphere $[\alpha]$ of zeros has higher multiplicity.

Thus we can prove the following:

Theorem 4.14. *A Blaschke product having zeros at the set*

$$Z = \{(a_1, \mu_1), (a_2, \mu_2), \ldots, ([c_1], \nu_1), ([c_2], \nu_2), \ldots\}$$

where $a_j \in \mathbb{H}_+$, a_j *have respective multiplicities* $\mu_j \geq 1$, $[a_i] \neq [a_j]$ *if* $i \neq j$, $c_i \in \mathbb{H}_+$, *the spheres* $[c_j]$ *have respective multiplicities* $\nu_j \geq 1$, $j = 1, 2, \ldots$, $[c_i] \neq [c_j]$ *if* $i \neq j$ *and*

$$\sum_{i,j\geq 1} \Big(\mu_j(1 - |a_j|) + 2\nu_i(1 - |c_i|)\Big) < \infty$$

is given by

$$\prod_{i\geq 1} (b_{[c_i]}(p))^{\nu_i} \prod_{j\geq 1}^{\star} \prod_{k=1}^{\star\mu_j} (b_{a_{jk}}(p))^{\star\mu_j},$$

where $a_{11} = a_1$ *and* $a_{jk} \in [a_j]$ *are such that* $\alpha_{j+1} \neq \bar{\alpha}_j$, $j = 1, \ldots, n-1$, *if* $\alpha_j \in \mathbb{H}\setminus\mathbb{R}$, $k = 1, 2, 3, \ldots, \mu_j$.

Proof. The Blaschke product converges and defines a slice hyperholomorphic function by Theorem 4.11 and its Corollary 4.13. Let us consider the product:

$$\prod_{i=1}^{\star\mu_1} (B_{a_{i1}}(p)) = B_{a_{11}}(p) \star B_{a_{12}}(p) \star \ldots \star B_{a_{1\mu_1}}(p). \tag{4.14}$$

As we already observed in the proof of Proposition 5.10 in [5] this product admits a zero at the point $a_{11} = a_1$ and it is a zero of multiplicity 1 if $n_1 = 1$; if $n_1 \geq 2$, the other zeros are $\tilde{a}_{12}, \ldots, \tilde{a}_{1n_1}$ where $\tilde{a}_{1j}$ belong to the sphere $[a_{1j}] = [a_1]$. Thus $\tilde{a}_{12}, \ldots, \tilde{a}_{1n_1}$ all coincide with a_1 which is the only zero of the product (4.14) and it has multiplicity μ_1. Let us now consider $r \geq 2$ and

$$\prod_{j=1}^{\star\mu_r}(B_{a_{rj}}(p)) = B_{a_{r1}}(p) \star \ldots \star B_{a_{rn_r}}(p), \tag{4.15}$$

and set

$$B_{r-1}(p) := \prod_{i\geq 1}^{\star(r-1)} \prod_{k=1}^{\star\mu_j}(B_{a_{jk}}(p)).$$

Then from the formula that relates the $\star$-product to the pointwise product (see Proposition 4.3.22 in [25]) we have that:

$$B_{r-1}(p) \star B_{a_{r1}}(p) = B_{r-1}(p)B_{a_{r1}}(B_{r-1}(p)^{-1}pB_{r-1}(p))$$

has a zero at a_r if and only if $B_{a_{r1}}(B_{r-1}(a_r)^{-1}a_rB_{r-1}(a_r)) = 0$, i.e. if and only if $a_{r1} = B_{r-1}(a_r)^{-1}a_rB_{r-1}(a_r)$. If $n_r = 1$ then a_r is a zero of multiplicity 1 while if $\mu_r \geq 2$, all the other zeros of the product (4.15) belongs to the sphere $[a_r]$ thus the zero a_r has multiplicity μ_r. □

We conclude this section by proving that the operator of multiplication by a Blaschke factor is an isometry. In the proof we are in need of the notion of conjugate of a function f. Given a slice hyperholomorphic function f consider its restriction to a complex plane $\mathbb{C}_I$ and write it, as customary, in the form $f_I(z) = F(z)+G(z)J$ where J is an element in $\mathbb{S}$ orthogonal to I and F, G are $\mathbb{C}_I$-valued holomorphic functions. Define $f^c(p) = \mathrm{ext}(\overline{F(\bar{z})} - G(z)J)$ where the extension operator is defined in (2.5). Note that if $f(p) = \sum_{n\geq 0} p^n a_n$ then $f^c(p) \sum_{n\geq 0} p^n \bar{a}_n$. We have the following:

Lemma 4.15. *Let $f \in \mathbf{H}_2(\mathbb{H}_+)$. Then $\|f\|_{\mathbf{H}_2(\mathbb{H}_+)} = \|f^c\|_{\mathbf{H}_2(\mathbb{H}_+)}$.*

Proof. By definition we have

$$\|f\|^2_{\mathbf{H}_2(\Pi_{+,I})} = \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy = \int_{-\infty}^{+\infty} (|F(Iy)|^2 + |G(Iy)|^2)dy$$

and

$$\begin{aligned}\|f^c\|^2_{\mathbf{H}_2(\Pi_{+,I})} &= \int_{-\infty}^{+\infty} |f^c_I(Iy)|^2 dy = \int_{-\infty}^{+\infty} (|\overline{F(-Iy)}|^2 + |G(Iy)|^2)dy \\ &= \int_{-\infty}^{+\infty} (|F(-Iy)|^2 + |G(Iy)|^2)dy.\end{aligned}$$

Thus $\|f\|^2_{\mathbf{H}_2(\Pi_{+,I})} = \|f^c\|^2_{\mathbf{H}_2(\Pi_{+,I})}$ and taking the supremum for $I \in \mathbb{S}$ the statement follows. □

Theorem 4.16. *Let b_a be a Blaschke factor. The operator*

$$M_{b_a} \,:\, f \mapsto b_a \star f$$

is an isometry from $\mathbf{H}_2(\mathbb{H}_+)$ into itself.

Proof. Recall that, by (4.10), we can write $b_a(p) = (\tilde{p} + \bar{a})^{-\star}(\tilde{p} - a)$ for $\tilde{p} = \lambda^c(p)^{-1}p\lambda(p)$. Let us set $\tilde{p} = Iy$ where $I \in \mathbb{S}$. We have

$$|b_a(Iy)| = |(Iy + \bar{a})^{-1}(Iy - a)| = |-(Iy + \bar{a})^{-1}\overline{(Iy + a)}| = 1.$$

Similarly, $|b_a^c(Iy)| = 1$. We now observe that for any two functions f and g we have $(f\star g)^c = g^c \star f^c$. We prove this equality by showing that the two functions $(f\star g)^c$ and $g^c \star f^c$ coincide on a complex plane (so the needed equality follows from the identity principle). Using the notation introduced above, let us write $f_I(z) = F(z) + G(z)J$ and $g_I(z) = H(z) + L(z)J$. We have

$$(f \star g)_I(z) = f_I(z) \star g_I(z) = (F(z)H(z) - G(z)\overline{L(\bar{z})}) + (F(z)L(z) + G(z)\overline{H(\bar{z})})J$$

so, by definition of $(f \star g)^c$, we have

$$(f \star g)_I^c(z) = (\overline{F(\bar{z})}\,\overline{H(\bar{z})} - \overline{G(\bar{z})}L(z)) - (F(z)L(z) + G(z)\overline{H(\bar{z})})J$$

and

$$\begin{aligned}(g^c \star f^c)_I(z) &= (\overline{H(\bar{z})} - L(z)J) \star (\overline{F(\bar{z})} - G(z)J)\\ &= (\overline{H(\bar{z})}\,\overline{F(\bar{z})} - L(z)\overline{G(\bar{z})}) - (\overline{H(\bar{z})}G(z) + L(z)F(z))J\end{aligned}$$

the two expressions coincide since the functions F, G, H, L are $\mathbb{C}_I$-valued and thus they commute. To compute $\|b_a \star f\|_{\mathbf{H}_2(\mathbb{H}_+)}$, where $f \in \mathbf{H}_2(\mathbb{H}_+)$, we follow an idea used in [2] and we compute $\|(b_a \star f)^c\|^2_{\mathbf{H}_2(\mathbb{H}_+)}$. Note that $(f^c \star b_a^c)(x + Iy) = 0$ where $f^c(x + Iy) = 0$, i.e. on a set of isolated points on $\Pi_{+,I}$ while, if $q = f^c(x + Iy) \neq 0$, $(f^c \star b_a^c)(x + Iy) = f^c(x + Iy)b_a^c(q^{-1}(x + Iy)q)$, see [25, Proposition 4.3.22], where $q^{-1}(x + Iy)q = x + I'y$, see [37, Proposition 2.22]. Thus we have $(f^c \star b_a^c)(Iy) = f^c(Iy)b_a^c(I'y)$ almost everywhere and

$$\begin{aligned}\|b_a \star f\|^2_{\mathbf{H}_2(\mathbb{H}_+)} &= \|(b_a \star f)^c\|^2_{\mathbf{H}_2(\mathbb{H}_+)}\\ &= \sup_{I\in\mathbb{S}} \int_{-\infty}^{+\infty} |(f^c \star b_a^c)(Iy)|^2 dy\\ &= \sup_{I\in\mathbb{S}} \int_{-\infty}^{+\infty} |f^c(Iy)b_a^c(I'y)|^2 dy\\ &= \sup_{I\in\mathbb{S}} \int_{-\infty}^{+\infty} |f^c(Iy)|^2|b_a^c(I'y)|^2 dy\\ &= \sup_{I\in\mathbb{S}} \int_{-\infty}^{+\infty} |f^c(Iy)|^2 dy\\ &= \|f^c\|^2_{\mathbf{H}_2(\mathbb{H}_+)}.\end{aligned}$$

By the previous lemma, we have $\|f^c\|^2_{\mathbf{H}_2(\mathbb{H}_+)} = \|f\|^2_{\mathbf{H}_2(\mathbb{H}_+)}$ and this concludes the proof. □

Blaschke factors will provide a concrete example of the functions studied in Section 7, see Example 7 there.

5. The Schauder-Tychonoff fixed point theorem

In this section we extend the Schauder-Tychonoff fixed point theorem to the quaternionic setting. The proof repeats that of the classical case given in [29], in fact it is readily seen that the arguments hold also in the quaternionic case, but we include it for the reader's convenience. This results is crucial to prove an invariant subspace theorem for contractions in a Pontryagin spaces.

5.1. The Schauder-Tychonoff fixed point theorem

In the sequel we will use a consequence of the Ascoli-Arzelà theorem that we state in this corollary.

Lemma 5.1 (Corollary of Ascoli-Arzelá theorem). *Let $\mathcal{G}_1$ be a compact subset of a topological group $\mathcal{G}$ and let $\mathcal{K}$ be a bounded subset of the space of quaternionic-valued continuous functions $\mathcal{C}(\mathcal{G}_1)$. Then $\mathcal{K}$ is conditionally compact if and only if for every $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ such that $|f(t) - f(s)| < \varepsilon$ for every $f \in \mathcal{K}$ and every pair $s, t \in \mathcal{G}_1$ with $t \in \mathcal{U}s$.*

Proof. It is Corollary 9 p. 267 in [29] and its proof can be obtained in the same arguments. □

Definition 5.2. We say that a quaternionic topological vector space $\mathcal{V}$ has the fixed point property if for every continuous mapping $T : \mathcal{V} \to \mathcal{V}$ there exists $u \in \mathcal{V}$ such that $u = T(u)$.

To show our result we need the following Lemmas:

Lemma 5.3. *Let $\mathfrak{C}$ be the subspace of $\ell^2(\mathbb{H})$ defined by*

$$\mathfrak{C} = \{\{\xi_n\} \in \ell^2(\mathbb{H}) : \ |\xi_n| \leq 1/n, \quad \forall n \in \mathbb{N}\}.$$

Then $\mathfrak{C}$ has the fixed point property.

Proof. Let $P_n : \ \mathfrak{C} \to \mathfrak{C}$ be the map

$$P_n(\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1} \ldots) = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots).$$

Then $\mathfrak{C}_n = P_n(\mathfrak{C})$ is homeomorphic to the closed sphere in $\mathcal{H} \cong \mathbb{R}^{4n}$. Let now $T : \ \mathfrak{C} \to \mathfrak{C}$ be a continuous map. Then $P_nT : \mathfrak{C}_n \to \mathfrak{C}_n$ is continuous. Brower theorem implies that there is a fixed point $\zeta_n \in \mathfrak{C}_n \subseteq \mathfrak{C}$ and so

$$|\zeta_n - T(\zeta_n)| \leq \left(\sum_{i=n+1}^{\infty} \frac{1}{i^2} \right)^{\frac{1}{2}}.$$

Since $\mathfrak{C}$ is compact, then $\{\zeta_n\}$ contains a subsequence converging to a point which is a fixed point of T. □

Lemma 5.4. *Let $\mathcal{K}$ be a compact convex subset of a locally convex linear quaternionic space $\mathcal{V}$ and let $T : \mathcal{K} \to \mathcal{K}$ be continuous. If $\mathcal{K}$ contains at least two points, then there exists a proper closed convex subset $\mathcal{K}_1 \subset \mathcal{K}$ such that $T(\mathcal{K}_1) \subseteq \mathcal{K}_1$.*

Proof. It is possible to assume that $\mathcal{K}$ has the $\mathcal{V}^*$ topology.

We will say that a set of continuous linear functionals F is determined by another set G, if for every $f \in F$ and $\varepsilon > 0$ there exists a neighborhood

$$\mathcal{N}(0;\gamma,\delta) = \{v \in \mathcal{V} : |g(v)| < \delta, \quad g \in \gamma\},$$

where γ is a finite subset of G with the property that if $u, v \in \mathcal{K}$ and $u-v \in \mathcal{N}(0;\gamma,\delta)$ then $|f(Tu) - f(Tv)| < \varepsilon$. It is clear that if F is determined by G, then $g(u) = g(v)$ for $g \in G$ implies that $f(Tu) = f(Tv)$ for $f \in F$. Each continuous linear functional f is determined by some countable set of functional $G = \{g_m\}_{m\in\mathbb{N}}$.

Thanks to Lemma 5.1 the scalar function $f(Tu)$ is uniformly continuous on the compact set $\mathcal{K}$. Hence for every integer n there is a neighborhood $\mathcal{N}(0;\gamma_n,\delta_n)$ of the origin in $\mathcal{V}$, given by a set of linear continuous functionals γ_n and a $\delta_n > 0$, such that if u, $v \in \mathcal{K}$ and $u - v \in \mathcal{N}(0;\gamma_n,\delta_n)$ then $|f(Tu) - f(Tv)| < 1/n$. Let $G = \bigcup_{n=1}^{\infty} \gamma_n$ then f is determined by G. It follows that if F is a countable subset of $\mathcal{V}^*$, there exists a countable subset G_F of $\mathcal{V}^*$ such that each $f \in F$ is determined by G_F. We claim that each continuous linear functional f can be included in a countable self-determined set G of functionals. In fact, if f is determined by the countable set G_1, let each functional in G_1 be determined by the countable set G_2; then let each functional in G_2 be determined by the countable set G_3, and so on. We obtain a sequence $\{G_i\}$ and we set $G = \{f\} \cup \cup_{i=1}^{\infty} G_i$. Assume now that $\mathcal{K}$ contains two points u, v, $u \neq v$ and let $f \in \mathcal{V}^*$ be such that $f(u) \neq f(v)$. Let $G = \{g_i\}$ be a countable self-determined set of continuous linear functionals containing f. Since $\mathcal{K}$ is compact, $g_i(\mathcal{K})$ is a bounded set of scalars for every i and since we can multiply g_i by a suitable constant we may suppose that $g_i(\mathcal{K}) \leq 1/i$. In this case the mapping $H : \mathcal{K} \to \ell^2(\mathbb{H})$, defined by

$$H(k) := [g_i(k)]$$

is a continuous mapping of $\mathcal{K}$ onto a compact convex subset $\mathcal{K}_0$ of the subspace $\mathfrak{C}$ of $\ell^2(\mathbb{H})$. Then $\mathfrak{C}$ contains trivially at least two points since there are at least two points in $\ell^2(\mathbb{C})$, see [29]. Consider the mapping

$$T_0 = HTH^{-1} : \mathcal{K}_0 \to \mathcal{K}_0$$

since G is self determined T_0 is single-valued. To see that T_0 is continuous, let $b_0 \in K_0$ and $\varepsilon \in (0, 1)$. Choose N such that $\sum_{i=N+1}^{\infty} 1/i^2 < \varepsilon$. Then G is self-determined, there exists a $\delta > 0$ and an m such that if $|g_j(u) - g_j(v)| < \delta$, $j = 1, \ldots, m$ then

$$|g_i(Tu) - g_i(Tv)| < \sqrt{\varepsilon/N}, \quad i = 1, \ldots, N. \tag{5.1}$$

Thus if $|b-b_0|<\delta$ and u and v are point in K with $b=[g_i(u)]$ and $b_0=[g_i(v)]$ then (5.1) holds and

$$\begin{aligned}|T_0(b)-T_0(b_0)|^2 &= |HTH^{-1}(b)-HTH^{-1}(b_0)|^2\\ &\le \sum_{i=1}^{N}|g_i(Tu)-g_i(Tv)|^2+2\sum_{i=N+1}^{\infty}1/i^2\\ &< 3\varepsilon.\end{aligned}$$

So T_0 is a continuous mapping of $\mathcal{K}_0$ into itself. From the fixed point property of $\mathfrak{C}$, see Lemma 5.3, it follows that T_0 has a fixed point k_0. Thus

$$TH(k_0)\subseteq H^{-1}T_0(k_0)=H^{-1}(k_0).$$

Setting $\mathcal{K}_1=H^{-1}(k_0)$ we note that $\mathcal{K}_1$ is a proper closed subset of $\mathcal{K}$, and that $T(\mathcal{K}_1)\subseteq\mathcal{K}_1$ The linearity of H implies that $\mathcal{K}_1$ is convex. This concludes the proof. □

Theorem 5.5 (Schauder-Tychonoff). *A compact convex subset of a locally convex quaternionic linear space has the fixed point property.*

Proof. By the Zorn lemma there exists a minimal convex subset of $\mathcal{K}_1$ of $\mathcal{K}$ with the property that $T\mathcal{K}_1\subseteq\mathcal{K}_1$. By Lemma 5.4 this minimal subset contains only one point. □

5.2. An invariant subspace theorem

As we explained at the beginning of the section, the Schauder-Tychonoff theorem is now used to prove an invariant subspace theorem for contractions in quaternionic Pontryagin spaces. This theorem is used in the realization theorems to prove the existence of slice hyperholomorphic extensions of certain functions defined in a neighborhood of a point on the positive axis. In the complex numbers case, this theorem can be found in [28, Theorem 1.3.11]. We also refer to [28, Notes on chapter 1] for historical notes on the theorem.

Theorem 5.6. *A contraction in a quaternionic Pontryagin space has a unique maximal invariant negative subspace, and it is one-to-one on it.*

Proof. The proof of [28] carries up to the quaternionic setting, and we recall the main lines for the convenience of the reader. Let A be a contraction in the Pontryagin space $\mathcal{P}$. To prove that A has a maximal negative invariant subspace we first recall a well known fact in the theory of linear fractional transformations (see for instance [31] for more details). Let $\mathcal{P}=\mathcal{P}_+[\dot{+}]\mathcal{P}_-$ be a fundamental decomposition of $\mathcal{P}$. Let

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$$

be the block decomposition of A along $\mathcal{P}_+[\dot{+}]\mathcal{P}_-$. Since A is a contraction, and hence a bicontraction (see [6][Theorem 7.2]) we have

$$A_{21}A_{21}^*-A_{22}A_{22}^*\le -I,$$

and it follows that A_{22}^{-1} and $A_{22}^{-1}A_{21}$ are strict contractions. Thus the map

$$L(X) = (A_{11}X + A_{12})(A_{21}X + A_{22})^{-1}$$

is well defined, and sends in fact the closed unit ball $\mathcal{B}_1$ of $\mathbf{L}(\mathcal{P}_,\mathcal{P}_+)$ into itself. The main point in the proof of the theorem is to show that the map L is continuous in the weak operator topology from $\mathcal{B}_1$ into itself. Since $\mathcal{B}_1$ is compact in this topology (and of course convex) the Schauder-Tychonoff theorem implies that L has a unique fixed point, say X. To conclude one notes (see Theorem [28, 1.3.10]) that the space spanned by the elements

$$f + Xf, \quad f \in \mathcal{P}_- \tag{5.1}$$

is then negative. It is maximal negative because X cannot have a kernel (any f such that $Xf = 0$ will lead to a strictly positive element of (5.1)). □

6. The spaces $\mathcal{P}(S)$

We now introduce the counterparts of the kernels (1.3) in the slice hyperholomorphic setting. In the quaternionic case signature operators are defined as in the complex case. Here we consider real signature operators, that is, which are unitarily equivalent to an operator of the form

$$\begin{pmatrix} I_+ & 0 \\ 0 & -I_- \end{pmatrix}.$$

It is clear that the S-spectrum is concentrated on ± 1, so if J is a signature operator we define $\nu_-(J)$ as in the complex case. This follows by simple computations, that is $1 \pm 2\mathrm{Re}(s_0) + |s|^2 = 0$ which give ± 1.

In next result we set $\mathbf{L}(\mathcal{H}) \stackrel{\text{def.}}{=} \mathbf{L}(\mathcal{H}, \mathcal{H})$ where $\mathcal{H}$ is a two sided quaternionic Hilbert space.

Definition 6.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two quaternionic two-sided Hilbert spaces and let $J_1 \in \mathbf{L}(\mathcal{H}_1)$ and $J_2 \in \mathbf{L}(\mathcal{H}_2)$ be two real signature operators such that $\nu_-(J_1) = \nu_-(J_2) < \infty$. The $\mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$-valued function S slice hypermeromorphic in an axially symmetric s-domain Ω which intersects the positive real line belongs to the class $\mathcal{S}_\kappa(J_1, J_2)$ if the kernel

$$K_S(p, q) = J_2 k(p, q) - S(p) \star k(p, q) \star_r J_1 S(q)^*$$

has κ negative squares in Ω, where $k(p, q)$ is defined in (4.6).

We do not mention the dependence of the class on Ω. As we will see, every element of these classes has a unique meromorphic extension to $\mathbb{H}_+$.

To reduce the case of arbitrary signature operators (with same number of negative squares) to the case of the identity, we define the Potapov-Ginzburg transform in the present setting. We refer to the book [24] for the classical case, even though some formulas are also recalled in [1].

We begin with a lemma. A proof in the classical case can be found in [8, Lemma 4.4.3, p. 164] (the argument there is based on [9, Lemma 2.1, p. 20]) but we repeat the argument for completeness. First a remark: a matrix $A \in \mathbb{H}^{m\times m}$ is not invertible if and only if there exists $c \neq 0 \in \mathbb{H}^m$ such that $c^*A = 0$. This fact can be seen for instance from [49, Theorem 7, p. 202], where it is shown that a matrix over a division ring has row rank equal to the column rank, or [63, Corollary 1.1.8].

Lemma 6.2. *Let T be a $\mathbb{H}^{m\times m}$-valued function slice hyperholomorphic in an axially symmetric s-domain Ω which intersect the positive real line, and such that the kernel*

$$T(p) \star k(p,q) \star_r T(q)^* - k(p,q)I_m$$

has a finite number of negative squares, say κ, in Ω. Then T is invertible in Ω, with the possible exception of a countable number of spheres.

Proof. We first show that T is invertible on $\Omega \cap \mathbb{R}_+$ with the possible exception of a countable number of points. Let $x_1, \ldots, x_M$ be zeros of T. Then, there exist vectors $c_1, \ldots, c_M$ such that

$$c_j^* T(x_j) = 0, \quad j = 1, \ldots, M.$$

Thus

$$m_{jk} = c_j^* k(x_j, x_k) c_k = -\frac{c_j^* c_k}{x_j + x_k}.$$

To conclude we apply [9, Lemma 2.1, p. 20] to the matrix with block entries $\chi(m_{jk})$ (which is unitarily equivalent to the matrix $\chi((m_{jk})_))$) to see that the $M \times M$ matrix with jk entry m_{jk} is strictly negative, and so $M \leq k$. The result in [9, Lemma 2.1, p. 20] is proved for the case of complex numbers, but extends to the quaternionic case, as is seen by using the map χ defined in (2.3) and Lemma 2.2. □

Let now $S \in \mathcal{S}_\kappa(J_1, J_2)$ and let

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{6.1}$$

be the decomposition of S according to fundamental decompositions of the coefficient spaces. In the statement of the following theorem, we denote by I_{2+} the identity of the positive space in the fundamental decomposition of $\mathcal{H}_2$.

Theorem 6.3. *Let $S \in \mathcal{S}_\kappa(J_1, J_2)$, defined in an axially symmetric s-domain Ω intersecting the real positive axis, and with decomposition* (6.1). *Then the function S_{22} is $\star$-invertible in Ω, with the possible exception of a countable number of spheres. Let*

$$A(p) = \begin{pmatrix} I_{2+} & S_{12}(p) \\ 0 & S_{22}(p) \end{pmatrix} \text{ and } \Sigma(p) = \begin{pmatrix} S_{11} - S_{12} \star S_{22}^{-\star} \star S_{21} & S_{12} \star S_{22}^{-\star} \\ S_{22}^{-\star} \star S_{21} & S_{22}^{-\star} \end{pmatrix}(p). \tag{6.2}$$

Then,

$$\begin{aligned} &J_2 k(p,q) - S(p) \star k(p,q) \star_r J_1 S(q)^* \\ &= A(p) \star (k(p,q) - \Sigma(p) \star k(p,q) \star_r \Sigma(q)^*) \star_r A(q)^*, \end{aligned} \tag{6.3}$$

and the kernel

$$k(p,q) - \Sigma(p) \star k(p,q) \star_r \Sigma(q)^* \tag{6.4}$$

has a finite number of negative squares on the domain of definition of Σ in Ω and hence has a slice hyperholomorphic extension to the whole of the right half-space, with the possible exception of a finite number of spheres.

The function Σ is called the Potapov-Ginzburg transform of S, see e.g. [9, (i), p. 25].

Proof of Theorem 6.3. To show that S_{22} is $\star$-invertible, we note that

$$\begin{aligned}&\begin{pmatrix}0 & I\end{pmatrix}(J_2k(p,q) - S(p)\star k(p,q)\star_r J_1S(q)^*)\begin{pmatrix}0\\ I\end{pmatrix}\\ &= S_{22}(p)\star k(p,q)\star_r S_{22}(q)^* - k(p,q)I_m.\end{aligned}$$

This last kernel has therefore a finite number of negative squares, and Lemma 6.2 allows to conclude that S_{22} is $\star$-invertible, and the definition of the Potapov-Ginzburg transform makes sense.

When $p\in\Omega\cap\mathbb{R}_+$, the star product is replaced by the pointwise product and the (6.3) then follow from [8, p. 156]. The case of $p\in\Omega$ follows by slice hyperholomorphic extension. The claim on the number of negative squares of (6.4) follows

$$\begin{aligned}&k(p,q) - \Sigma(p)\star k(p,q)\star_r \Sigma(q)^*\\ &= A(p)^{-\star}\star(J_2k(p,q) - S(p)\star k(p,q)\star_r J_1S(q)^*)\star_r (A(q)^*)^{-\star_r},\end{aligned}\tag{6.5}$$

and from an application of Proposition 2.4. □

Definition 6.4. Let $S\in\mathcal{S}_\kappa(J_1,J_2)$. We denote by $\mathcal{P}(S)$ the associated reproducing kernel Pontryagin space of $\mathcal{H}_2$-valued functions defined in Ω and with reproducing kernel $K_S(p,q)$.

7. Realization for elements in $\mathcal{S}_\kappa(J_1,J_2)$

In this section we present a realization theorem for elements in $\mathcal{S}_\kappa(J_1,J_2)$, where the state space is the reproducing kernel Pontryagin space $\mathcal{P}(S)$ (see Definition 6.4 for the latter). In the case $\kappa=0$ one could get the existence of a realization using a Cayley transform in the variable and use our previous results in [4]. Here we give a direct proof to get a realization defined in $\mathcal{P}(S)$, taking into account that κ may be strictly positive. We begin with a definition:

Definition 7.1. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two quaternionic right Pontryagin spaces. A pair of operators $(G,A)\in\mathbf{L}(\mathcal{P}_1,\mathcal{P}_2)\times\mathbf{L}(\mathcal{P}_1)$ is called observable (or closely outer connected) if

$$\bigcap_{n=0}^{\infty}\ker GA^n = \{0\}\,.$$

The terminology *observable* is the one from the theory of linear systems, while *closely outer connected* has been used in operator theory in particular by Krein and Langer, see [8].

Theorem 7.2. *Let x_0 be a strictly positive real number. A function S slice hyperholomorphic in an axially symmetric s-domain Ω containing x_0 is the restriction to Ω of an element of $\mathcal{S}_\kappa(J_1,J_2)$ if and only if it can be written as*

$$S(p) = H - (p-x_0)\left(G - (\overline{p}-x_0)(\overline{p}+x_0)^{-1}GA\right) \times \left(\frac{|p-x_0|^2}{|p+x_0|^2}A^2 - 2\mathrm{Re}\,\left(\frac{p-x_0}{p+x_0}\right)A + I\right)^{-1} F, \tag{7.1}$$

where A is a linear bounded operator in a right-sided quaternionic Pontryagin space Π_κ of index κ, and, with $B = -(I + x_0A)$, the operator matrix

$$\begin{pmatrix} B & F \\ G & H \end{pmatrix} : \begin{pmatrix} \Pi_k \\ \mathcal{H}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_k \\ \mathcal{H}_2 \end{pmatrix}$$

is co-isometric. In particular S has a unique slice hypermeromorphic extension to $\mathbb{H}_+$. Furthermore, when the pair (G,A) is observable, the realization is unique up to a unitary isomorphism of Pontryagin right quaternionic spaces.

Remark 7.3. When the operators are finite matrices we note that formula (7.1) can be rewritten as:

$$S(p) = H - (p-x_0)G \star ((x_0+p)I + (p-x_0)B)^{-\star}F.$$

Sometimes, and by abuse of notation, we will use this expression also for the infinite dimensional case, see Proposition 3.24 for more information.

Proof of Theorem 7.2. We proceed in a number of steps, and first prove in Steps 1-8 that a realization of the asserted type exists with $\Pi_k = \mathcal{P}(S)$. We denote by $\mathcal{H}_2(J_2)$ the space $\mathcal{H}_2$ endowed with the indefinite inner product

$$[u,v]_{J_2} = [u, J_2v]$$

and similarly $\mathcal{H}_1(J_1)$. Both $\mathcal{H}_1(J_1)$ and $\mathcal{H}_2(J_2)$ are quaternionic Pontryagin spaces, and they have the same index.

Following [3, pp. 51-52] we introduce a relation R in $(\mathcal{P}(S)\oplus\mathcal{H}_2(J_2))\times(\mathcal{P}(S)\oplus\mathcal{H}_1(J_1))$ by the linear span of the vectors of the form (U,V) where

$$U = \begin{pmatrix} K_S(\cdot,q)(x_0-\overline{q})u \\ (x_0-\overline{q})v \end{pmatrix}$$

$$V = \begin{pmatrix} K_S(\cdot,q)(x_0+\overline{q})u - 2x_0K_S(\cdot,x_0)u + \sqrt{2x_0}K_S(\cdot,x_0)(x_0-\overline{q})v \\ \sqrt{2x_0}(S(q)^* - S(x_0)^*)u + S(x_0)^*(x_0-\overline{q})v \end{pmatrix}.$$

STEP 1: *The relation R is isometric.*

Indeed, let (F_1,G_1) and (F_2,G_2) be two elements in the relation, corresponding to $q_1\in\Omega, u_1,v_1\in\mathcal{H}_1$ and to $q_2\in\Omega, u_2,v_2\in\mathcal{H}_2$ respectively. On the one hand we have

$$[F_2,F_1] = [(x_0-q_1)K_S(q_1,q_2)(x_0-\overline{q_2})u_2,u_1] + [(x_0-q_1)J_2(x_0-\overline{q_2})v_2,v_1].$$

On the other hand, with $G_1 = \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}$ where

$$g_1(\cdot) = K_S(\cdot, q_1)(x_0 + \overline{q_1})u_1 - 2x_0K_S(\cdot, x_0)u_1 + \sqrt{2x_0}K_S(\cdot, x_0)(x_0 - \overline{q_1})v_1,$$
$$h_1 = \sqrt{2x_0}(S(q_1)^* - S(x_0)^*)u_1 + S(x_0)^*(x_0 - \overline{q_1})v_1$$

(and similarly for G_2) we have

$$[G_2, G_1] = [g_2, g_1] + [h_2, h_1].$$

We want to show that

$$[F_2, F_1] = [g_2, g_1] + [h_2, h_1]. \tag{7.2}$$

In the computations of these inner products, there are terms which involve only u_1, u_2, terms which involve only v_1, v_2 and similarly for u_1, v_2 and v_1, u_2. We now write these inner terms separately:

Terms involving u_1, u_2. To show that these terms are the same on both sides of (7.2) we have to check that

$$\begin{aligned}
&[(x_0 - q_1)K_S(q_1, q_2)(x_0 - \overline{q_2})u_2, u_1] \\
&= [(x_0 + q_1)\left(K_S(q_1, q_2)(x_0 + \overline{q_2}) - 2x_0K_S(x_0, q_2)(x_0 + \overline{q_2})\right. \\
&\quad \left. -2x_0(x_0 + q_1)K_S(q_1, x_0) + 4x_0^2K_S(x_0, x_0)\right) u_2, u_1] \\
&\quad + 2x_0[(S(q_1) - S(x_0))J_1(S(q_2)^* - S(x_0))u_2, u_1].
\end{aligned}$$

Using

$$k(x_0, x_0) = \frac{1}{2x_0} \quad \text{and} \quad K_S(x_0, x_0) = \frac{1}{2x_0}\left(J_2 - S(x_0)J_1S(x_0)^*\right), \tag{7.3}$$

we see that this is equivalent to proving that

$$\begin{aligned}
&(x_0 - q_1)J_2k(q_1, q_2)(x_0 - \overline{q_2}) - (x_0 - q_1)S(q_1)J_1k(q_1, q_2)S(q_2)^*(x_0 - \overline{q_2}) \\
&= (x_0 + q_1)J_2k(q_1, q_2)(x_0 + \overline{q_2}) - (x_0 + q_1)S(q_1)J_1k(q_1, q_2)S(q_2)^*(x_0 + \overline{q_2}) \\
&\quad - 2x_0(J_2 - S(q_1)J_1S(x_0)^*) - 2x_0(J_2 - S(x_0)J_1S(q_2)^*) \\
&\quad + 2x_0\left(J_2 - S(x_0)J_1S(x_0)^*\right) \\
&\quad + 2x_0(S(q_1) - S(x_0)J_1(S(q_2)^* - S(x_0)^*).
\end{aligned}$$

But this amounts to checking that

$$q_1k(q_1, q_2) + k(q_1, q_2)\overline{q_2} = 1,$$

which has been seen to hold in Proposition 4.7.

Terms involving v_1, v_2. To show that these terms are the same on both sides of (7.2) we have to check that

$$\begin{aligned}
(x_0 - q_1)J_2(x_0 - \overline{q_2}) = &[(x_0 - q_1)S(x_0)J_1S(x_0)^*(x_0 - \overline{q_2})v_2, v_1] \\
&+ 2x_0[(x_0 - q_1)K_S(x_0, x_0)(x_0 - \overline{q_2})v_2, v_1].
\end{aligned}$$

This follows directly from the formula for $K_S(x_0, x_0)$, see (7.3).

Terms involving u_2, v_1. There are no such terms on the left side of (7.2) and so we need to show that the terms on the right add up to 0. This is the case since

$$\begin{aligned}
&\sqrt{2x_0}[(x_0 - q_1)S(x_0)J_1(S(q_2)^* - S(x_0)^*)u_2, v_1] \\
&\quad + \sqrt{2x_0}[(x_0 - q_1)\left(K_S(x_0, q_2)(x_0 + \overline{q_2}) - 2x_0 K_S(x_0, x_0)\right) u_2, v_1] = \\
&= [Xu_2, v_1] \\
&= 0
\end{aligned}$$

with

$$\begin{aligned}
X = &\sqrt{2x_0}(x_0 - q_1)S(x_0)J_1(S(q_2)^* - S(x_0)^*) \\
&+ \sqrt{2x_0}(x_0 - q_1) \times (J_2 - S(x_0)J_1S(q_2)^*) \\
&- \sqrt{2x_0}(x_0 - q_1)\left(J_2 - S(x_0)J_1S(x_0)^*\right) = 0
\end{aligned}$$

since

$$K_S(x_0, q_2)(x_0 + \overline{q_2}) = J_2 - S(x_0)J_1S(q_2)^*.$$

Terms involving u_1, v_2. These form a symmetric expression to the previous one, and will not be written down.

STEP 2: *The domain of R is dense.*

To prove this step, let $\begin{pmatrix} f \\ w \end{pmatrix} \in (\mathcal{P}(S) \oplus \mathcal{H}_2(J_2))$ be orthogonal to $\operatorname{Dom} R$. Then, for all $q \in \Omega$ and $u, v \in \mathcal{H}_2$ we have

$$[(x_0 - q)f(q), u] + [(x_0 - q)w, v]_{J_2} = 0.$$

It follows that $w = 0$ and that

$$(x_0 - q)f(q) \equiv 0, \quad q \in \Omega,$$

and so $f \equiv 0$ in Ω.

STEP 3: *The relation R extends to the graph of an isometry*

Indeed, the spaces $\mathcal{P}(S) \oplus \mathcal{H}_2(J_2)$ and $\mathcal{P}(S) \oplus \mathcal{H}_1(J_1)$ are Pontryagin spaces with same index. By the quaternionic version of a theorem of Shumlyan (see [8, Theorem 1.4.1, p. 27] for the classical case and [5, Theorem 7.2] for the quaternionic case) a densely defined contractive relation defined on a pair of Pontryagin spaces with same index extends to the graph of a contraction.

In preparation to the next step we introduce an operator R_{x_0} as follows. Let $\mathcal{H}$ be a two-sided quaternionic Hilbert space. A $\mathcal{H}$-valued function slice hyperholomorphic in a neighborhood of $x_0 > 0$ can be written as a convergent power series

$$f(p) = \sum_{n=0}^{\infty} (p - x_0)^n f_n,$$

where the coefficients $f_n \in \mathcal{H}$. We define

$$(R_{x_0} f)(p) = (p - x_0)^{-1}(f(p) - f(x_0)) \overset{\text{def.}}{=} \begin{cases} \sum_{n=1}^{\infty} (p - x_0)^{n-1} f_n, & p \neq x_0, \\ f_1, & p = x_0. \end{cases} \tag{7.4}$$

STEP 4: *Let V denote the isometry in the previous step. We compute V^* and show that, with*

$$V^* = \begin{pmatrix} B & F \\ G & H \end{pmatrix} : \quad \mathcal{P}(S) \oplus \mathcal{H}_2(J_2) \implies \mathcal{P}(S) \oplus \mathcal{H}_2(J_1), \tag{7.5}$$

we have $H = S(x_0)$ and

$$Bf = -(I + 2x_0 R_{x_0})f, \tag{7.6}$$

$$Fu = -\sqrt{2x_0} R_{x_0} Su, \tag{7.7}$$

$$Gf = \sqrt{2x_0} f(x_0). \tag{7.8}$$

To compute (7.6) let $f \in \mathcal{P}(S)$ and $(p, u) \in \Omega \times \mathcal{H}_2$. We have

$$\begin{aligned}
[(x_0 - p)(Bf(p)), u] &= [Bf, K_S(\cdot, p)(x_0 - \overline{p})u] \\
&= [f, B^*(K_S(\cdot, p)(x_0 - \overline{p})u)] \\
&= [f, K_S(\cdot, p)(x_0 + \overline{p})u - 2x_0 K_S(\cdot, x_0)u] \\
&= [(p + x_0)f(p) - 2x_0 f(x_0), u],
\end{aligned}$$

and so

$$(x_0 - p)(Bf(p)) = (p + x_0)f(p) - 2x_0 f(x_0), \quad p \in \Omega,$$

which can be rewritten as (7.6).

Similarly, to compute (7.7) let $v \in \mathcal{H}_2$. We have:

$$\begin{aligned}
[(x_0 - p)((Fv)(p)), u] &= [Fv, K_S(\cdot, p)(x_0 - \overline{p})u] \\
&= [v, \sqrt{2x_0}(S(p)^* - S(x_0)^*)u] \\
&= [\sqrt{2x_0}(S(p) - S(x_0))v, u],
\end{aligned}$$

and so

$$(x_0 - p)(Fv(p)) = \sqrt{2x_0}(S(p) - S(x_0))v, \quad p \in \Omega.$$

Finally, we have:

$$\begin{aligned}
[(x_0 - p)Gf, v] &= [Gf, (x_0 - \overline{p})v] \\
&= [f, G^*(x_0 - \overline{p})v)] \\
&= [f, \sqrt{2x_0} K_S(\cdot, x_0)(x_0 - \overline{p})v] \\
&= \sqrt{2x_0}[(x_0 - p)f(x_0), v].
\end{aligned}$$

where we have used (2.1) to get the first equality.

STEP 5: *We prove* (7.1) *for real p near x_0.*

The operator $I + 2x_0 R_{x_0}$ is bounded and so is the operator R_{x_0} (with $x_0 > 0$). Let $f \in \mathcal{P}(S)$, with power series expansion

$$f(p) = \sum_{n=0}^{\infty} (p - x_0)^n f_n, \quad f_0, f_1, \ldots \in \mathcal{H}_2,$$

around x_0. We have for real $p = x$ near x_0:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}(x-x_0)^n f_n \\ &= \frac{1}{\sqrt{2x_0}}\sum_{n=0}^{\infty}(x-x_0)^n G R_{x_0}^n f \\ &= \frac{1}{\sqrt{2x_0}} G(I-(x-x_0)R_{x_0})^{-1} f. \end{aligned}$$

Applying this formula to $f = R_{x_0}Su = -\frac{1}{\sqrt{2x_0}}Fu$ where $u \in \mathcal{H}_1$ we have

$$(R_{x_0}Su)(x) = -G(2x_0 I - 2(x-x_0)x_0 R_{x_0})^{-1}Fu$$

and so, since $B = -I - 2x_0 R_{x_0}$,

$$\begin{aligned} S(x)u &= S(x_0)u + (x-x_0)(R_{x_0}Su)(x) \\ &= S(x_0)u - (x-x_0)G(2x_0 I - 2(x-x_0)x_0 R_{x_0})^{-1}Fu \\ &= S(x_0)u - (x-x_0)G(2x_0 I + (x-x_0)(B+I))^{-1}Fu \\ &= S(x_0)u - (x-x_0)G((x+x_0)I + (x-x_0)B)^{-1}Fu. \end{aligned}$$

STEP 6: *Assume that $J_1 = I_{\mathcal{H}_1}$ and $J_2 = I_{\mathcal{H}_2}$. Then, the operator $(x_0+x)I+(x-x_0)B$ is invertible for all real x, with the possible exception of a finite set in $\mathbb{R}$.*

Assume first the kernel K_S to be positive definite. Then, the operator matrix (7.5) is a contraction between Hilbert spaces and so B is a Hilbert space contraction, and the operator

$$I - \frac{x_0 - x}{x_0 + x}B$$

is invertible for all $x > 0$, with the possible exception of a finite set, since $|\frac{x_0-x}{x_0+x}| < 1$ for such x.

Assume now that $\mathcal{P}(S)$ is a Pontryagin space. The operator V^* is a contraction between Pontryagin spaces of same index, and so its adjoint V is a contraction (see [6, Theorem 7.2]). So it holds that

$$B^*B + G^*G \leq I.$$

But

$$\langle G^*Gf, f\rangle = \langle Gf, Gf\rangle_{\mathcal{H}_2} \geq 0$$

since $J_2 = I_{\mathcal{H}_2}$ and so B is a contraction. It admits a maximal strictly negative invariant subspace, say $\mathcal{M}$ (see [28, Theorem 1.3.11] for the complex case and Theorem 5.6 for the quaternionic case). Writing

$$\mathcal{P}(S) = \mathcal{M}[+]\mathcal{M}^{[\perp]},$$

the operator matrix representation of B is upper triangular with respect to this decomposition where

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

The operator B_{22} is a contraction from the Hilbert space $\mathcal{M}^{[\perp]}$ into itself, and so $I - \frac{x_0-x}{x_0+x}B_{22}$ is invertible for every $x > 0$, with the possible exception of a finite set. The operator B_{11} is a contraction from the finite dimensional anti-Hilbert space $\mathcal{M}$ onto itself, and so has right eigenvalues *outside the open unit ball*. So the operator $I - \frac{x_0-x}{x_0+x}B_{11}$, is invertible in $x > 0$, except the points $x \neq x_0$ such that $\frac{x+x_0}{x-x_0}$ is a real eigenvalue of B_{11} of modulus greater or equal to 1. There is a finite number of such points since, see [64, Corollary 5.2, p. 39], a $n \times n$ quaternionic matrix has exactly n right eigenvalues (counting multiplicity) up to equivalence (in other words, it has exactly n spheres of eigenvalues).

It follows that the operator

$$I - \frac{x_0 - x}{x_0 + x}B = \begin{pmatrix} I - \frac{x_0-x}{x_0+x}B_{11} & -\frac{x_0-x}{x_0+x}B_{12} \\ 0 & I - \frac{x_0-x}{x_0+x}B_{22} \end{pmatrix}$$

is invertible for all $x > 0$, with the possible exception of a finite number of points.

STEP 7: *Assume that $J_1 = I_{\mathcal{H}_1}$ and $J_2 = I_{\mathcal{H}_2}$. The function S admits a slice hypermeromorphic extension to $\mathbb{H}_+$, with the possible exception of a finite number of spheres.*

We note that, for $p \in \mathbb{H}$ near x_0 we can extend $S(x)u$ computed in STEP 5 to a slice hyperholomorphic function:

$$\begin{aligned} S(p)u &= S(x_0)u + \frac{x_0 - p}{x_0 + p}G \star \left(I - \frac{x_0 - p}{x_0 + p}B\right)^{-\star} Fu \\ &= S(x_0)u \\ &\quad + \frac{p - x_0}{p + x_0} \star \left(G - \frac{x_0 - \bar{p}}{x_0 + \bar{p}}GB\right)\left(\frac{|x_0 - p|^2}{|x_0 + p|^2}B^2 - 2\mathrm{Re}\left(\frac{x_0 - p}{x_0 + p}\right)B + I\right)^{-1} Fu. \end{aligned}$$

Let $t = \frac{\mathrm{Re}\, q}{|q|^2}$ where $q = \frac{x_0-p}{x_0+p}$. We have

$$\begin{aligned} &|q|^2B^2 - 2(\mathrm{Re}\, q)B + I \\ &= |q|^2 \begin{pmatrix} B_{11}^2 - 2tB_{11} + \frac{1}{|q|^2} & B_{11}B_{12} + B_{12}B_{22} - 2tB_{12} + \frac{1}{|q|^2} \\ 0 & B_{22}^2 - 2tB_{22} + \frac{1}{|q|^2} \end{pmatrix}. \end{aligned}$$

By the property of the resolvent, the operator $B_{22}^2 - 2tB_{22} + \frac{1}{|q|^2}$ is invertible for q such that $\frac{1}{|q|^2}$ is in the resolvent set of B_{22}. Since B_{22} is a Hilbert space contraction, this happens in particular when $|q| < 1$, see [25], proof of Theorem 4.8.11. Similarly the operator $B_{11}^2 - 2tB_{11} + \frac{1}{|q|^2}$ is invertible if and only if $\frac{1}{|q|^2}$ is in the resolvent set of B_{11}. Since B_{11} is a finite dimensional Hilbert space expansion, it has just point S-spectrum which is inside the closed unit ball. The point S-spectrum coincides with the set of right eigenvalues, see Remark 3.22, and it consists of a finite number of (possibly degenerate) spheres.

We now consider the case of arbitrary signature matrices, with same negative index.

STEP 8: *We use the Potapov-Ginzburg transform to show that S has a meromorphic extension.*

This follows from computing S from its Potapov-Ginzburg transform.

STEP 9: *Any S with a realization of the form* (7.1) *is in a class $\mathcal{S}_\kappa(J_1, J_2)$.*

Indeed, for real $p = x$ and $q = y$ near x_0, the existence of the realization leads to

$$\frac{J_2 - S(x)J_1S(y)^*}{x+y} = G(I(x_0+x) - (x+x_0)B)^{-1}(I(y+x_0) - (y-x_0)B)^{-*}G^*,$$

where $B = -(I + x_0A)$. Thus, with $K(x,y) = G(I(x_0+x) - (x+x_0)B)^{-1}(I(y+x_0) - (y-x_0)B)^{-*}G^*$,

$$J_2 - S(x)J_1S(y)^* = xK(x,y) + K(x,y)y$$

and the result follows by observing that (7.1) is the hyperholomorphic extension.

STEP 10: *The realization is unique up to isomorphism when it is observable.*

We follow [8]. Let p be a real number and set $x = \frac{p-x_0}{p+x_0}$. When p varies in a real neighborhood of x_0 then x varies in a real neighborhood $\mathcal{I}_0$ of the origin. For $x, y \in \mathcal{I}_0$ we have

$$\begin{aligned}\frac{J_2 - S(x)J_1S(y)^*}{1-xy} &= G_1(I_{\mathcal{P}_1} - xB_1)^{-1}(I_{\mathcal{P}_1} - yB_1)^{-*}G_1^*\\ &= G_2(I_{\mathcal{P}_2} - xB_2)^{-1}(I_{\mathcal{P}_2} - yB_2)^{-*}G_2^*,\end{aligned}$$

where the indices 1 and 2 correspond to two observable and coisometric realizations, with state spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Then the domain and range of the relation R spanned by the pairs

$$((I_{\mathcal{P}_1} - yB_1)^{-*}G_1^*h, (I_{\mathcal{P}_2} - yB_2)^{-*}G_2^*k), \quad h, k \in \mathcal{H}_2,$$

are dense. By the quaternionic version of a theorem of Shmulyan (see [5, Theorem 7.2]) R is the graph of a unitary map, which provides the desired equivalence. The arguments are as in [8]. □

Remark 7.4. In the case $x_0 \to 0$, we can rewrite the computations with

$$\begin{aligned}G_0f &= f(x_0)\\ F_0u &= R_{x_0}Su\\ I + B &= -x_0R_{x_0},\\ I - B &= 2I + x_0R_{x_0};\end{aligned}$$

we have that

$$\begin{aligned}S(p)u &= S(x_0)u + (p-x_0)G \star \left(\mathcal{M}^\ell_{x_0+p} + \mathcal{M}^\ell_{p-x_0}B\right)^{-1} F_0u\\ &= S(x_0) + (p-x_0)2x_0G_0 \star (x_0(I-B) + p(I+B))^{-1} F_0u\\ &= S(x_0) + (p-x_0)G_0 \star (I + x_0R_{x_0} - pR_{x_0})^{-1} F_0u\end{aligned}$$

which tends formally to the backward shift realization as $x_0 \to 0$.

Example. We now show how to obtain a realization for a Blaschke factor $b_a(p)$. For real $p = x$, using formula (4.10) we obtain that $b_a(x) = (x-\bar{a})^{-1}(x-a)$, moreover

$$\begin{aligned} b_a(x) &= b_a(1) + b_a(x) - b_a(1) \\ &= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{(x+\overline{a})(1+\overline{a})} \\ &= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{(x+\frac{1-B}{1+B})(1+\overline{a})}, \quad \text{where } B = \frac{1-\overline{a}}{1+\overline{a}} \\ &= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)(1+B)}{(x(1+B)+(1-B))(1+\overline{a})} \\ &= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{(x(1+B)+(1-B))}\frac{2}{(1+\overline{a})^2} \\ &= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{1+\overline{a}}((x+1)+(x-1)B)^{-1}\frac{2}{1+\overline{a}} \end{aligned}$$

since

$$\frac{1+B}{1+\bar{a}} = \frac{2}{(1+\overline{a})^2}.$$

Now note that

$$b_a(p) = H - (p-1)G \star ((p+1)+(p-1)B)^{-\star}F$$

is slice hyperholomorphic, extends $S(x)$, and

$$\begin{pmatrix} B & F \\ G & H \end{pmatrix} = \begin{pmatrix} \dfrac{1-\overline{a}}{1+\overline{a}} & \dfrac{2\sqrt{\mathrm{Re}\, a}}{1+\overline{a}} \\ \\ -\dfrac{2\sqrt{\mathrm{Re}\, a}}{1+\overline{a}} & \dfrac{1-a}{1+\bar{a}} \end{pmatrix}.$$

This matrix is unitary.

We now present an example of functions in a class $\mathcal{S}_0(J,J)$. Consider a linear bounded operator A in a right quaternionic Hilbert space $\mathcal{H}$, and assume that $A+A^*$ is finite dimensional, say of rank m. We can thus write:

$$A + A^* = -CJC^*,$$

where $J \in \mathbb{H}^{m\times m}$ is a real signature matrix, and where C is linear bounded operator from $\mathbb{H}^m$ into $\mathcal{H}$. We will assume (C,A) observable.

Remark 7.5. The pair (C,A) is observable if and only if there is no non trivial invariant subspace of A on which $A + A^* = 0$.

The proof of this lemma is as in the complex case, and will be omitted.

We conclude this section with an example of a function in $\mathcal{S}_0(J,J)$, which, by analogy with the classical case, we call the characteristic operator function of the operator. Connections with operator models will be considered elsewhere, but we remark here that the function S in (7.9) defined uniquely A when the pair (C,A) is observable.

Definition 7.6. The function

$$S(p) = I - pC^* \star (I - pA)^{-\star} CJ \tag{7.9}$$

is called the characteristic operator function of the operator A.

Theorem 7.7. *The characteristic operator function belongs to* $\mathcal{S}_0(J, J)$.

Proof. Let

$$K(p, q) = C^* \star (I - pA)^{-\star}(I - \overline{q}A^*)^{-\star_r} \star_r C.$$

Then it holds that

$$J - S(p)JS(q)^* = pK(p, q) + K(p, q)\overline{q}.$$

This formula is proved by first considering the case of real p and q, and taking the slice hyperholomorphic extension, and proves that $S \in \mathcal{S}_0(J, J)$. □

We note that formula (7.9) corresponds to a realization centered at 0, as in Remark 7.4, and not to a realization of the form (7.1). It would be interesting to find a functional model for the operator A in terms of S. The special case where S is a (possibly infinite convergent) Blaschke product is of special interest. The case of general S leads to the question of finding the $\star$-multiplicative structure of elements in $\mathcal{S}_0(J, J)$, that is the counterpart of the paper [55] in the present setting.

8. The space $\mathcal{L}(\Phi)$ and realizations for generalized positive functions

In the present section we give realization for a generalized positive function with $\mathcal{L}(\Phi)$ as state space. Note that a Cayley transform (with real coefficients) will map a generalized positive function into a generalized Schur function, and even more a Cayley trasform on the variable will reduce the problem to the case of a Schur function of the quaternionic unit ball. But this procedure will not lead an intrinsic realization in the natural space associated to generalized positive function.

8.1. The indefinite case

Definition 8.1. Let $\mathcal{H}$ be a quaternionic Hilbert space, and let $J \in \mathbf{L}(\mathcal{H})$ be a real signature operator. A $\mathbf{L}(\mathcal{H})$-valued function Φ slice hyperholomorphic in an axially symmetric s-domain Ω which intersects the positive real line belongs to the class $\mathrm{GP}_\kappa(J)$ if the kernel

$$K_\Phi(p, q) = J\Phi(p) \star k(p, q) + k(p, q) \star_r \Phi(q)^* J \tag{8.1}$$

has κ negative squares in Ω.

Lemma 8.2. *The kernel* K_Φ *satisfies*

$$pK_\Phi(p, q) + K_\Phi(p, q)\overline{q} = J\Phi(p) + \Phi(q)^* J. \tag{8.2}$$

Proof. It follows with immediate computations from Proposition 4.7. □

As in the case of generalized Schur functions, we do not mention the dependence of the class on Ω since, as we prove later, every element of a class $\mathrm{GP}_\kappa(J)$ has a unique slice hypermeromorphic extension to $\mathbb{H}_+$.

We note that J does not play a role, as noted in [42, p. 358, footnote], and could be set to be the identity. We denote by $\mathcal{H}$ a two sided quaternionic Hilbert space, and recall that $\mathbf{L}(\mathcal{H})$=$\mathbf{L}(\mathcal{H},\mathcal{H})$.

Theorem 8.3. *A $\mathbf{L}(\mathcal{H})$-valued function Φ slice hyperholomorphic in an axially symmetric s-domain Ω containing $x_0 > 0$ is in the class $\mathrm{GP}_\kappa(J)$ if and only if there exists a right quaternionic Pontryagin space Π_κ of index κ and operators*

$$\begin{pmatrix} B & F \\ G & H \end{pmatrix} : \begin{pmatrix} \Pi_k \\ \mathcal{H} \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_k \\ \mathcal{H} \end{pmatrix}$$

verifying

$$(I + 2x_0B)(I + 2x_0B)^* = I$$

and such that Φ can be written as

$$\Phi(p) = H - (p - x_0)G \star ((p + x_0)I + (p - x_0)B)^{-\star}F. \tag{8.3}$$

Furthermore, Φ has a unique slice hypermeromorphic extension to $\mathbb{H}_+$. Finally, when the pair (G, B) is observable, the realization is unique up to a unitary isomorphism of Pontryagin right quaternionic spaces.

Proof. Given $\Phi \in \mathrm{GP}_\kappa(J)$, we denote by $\mathcal{L}(\Phi)$ associated right reproducing kernel Pontryagin space of $\mathcal{H}$-valued functions with reproducing kernel K_Φ. We proceed in a number of steps to prove the theorem.

STEP 1: *The formula*

$$(p - x_0)(Bh(p)) = (p + x_0)h(p) - 2x_0h(x_0), \quad h \in \mathcal{L}(\Phi). \tag{8.4}$$

defines a (continuous) coisometry in $\mathcal{L}(\Phi)$.

Indeed, define a relation $\mathcal{R}_{x_0}$ on $\mathcal{L}(\Phi) \times \mathcal{L}(\Phi)$ generated by the linear span of the pairs

$$\mathcal{R}_{x_0} = (K_\Phi(\cdot, p)(\overline{p} - x_0)u, (K_\Phi(\cdot, p) - K_\Phi(\cdot, x_0))u)\,. \tag{8.5}$$

Then the following holds:

$$(f, g) \in \mathcal{R}_{x_0} \quad \Longrightarrow \quad [f, f] = [f + 2x_0g, f + 2x_0g]. \tag{8.6}$$

We first prove that

$$[f, g] + [g, f] + 2x_0[g, g] = 0. \tag{8.7}$$

An element in $\mathcal{R}_{x_0}$ can be written as (f, g) with

$$\begin{aligned} f(p) &= \sum_{j=1}^m K_\Phi(p, p_j)(\overline{p_j} - x_0)u_j \\ g(p) &= \sum_{j=1}^m K_\Phi(p, p_j)u_j - K_\Phi(p, x_0)d, \quad \text{where} \quad d = \sum_{j=1}^m u_j. \end{aligned} \tag{8.8}$$

With f and g as in (8.8) we have:

$$[f,g]=\left(\sum_{i,j=1}^m u_i^*K_\Phi(p_i,p_j)(\overline{p_j}-x_0)u_j\right)-d^*\left(\sum_{j=1}^m K_\Phi(x_0,p_j)(\overline{p_j}-x_0)u_j\right),$$

$$[g,f]=\left(\sum_{i,j=1}^m u_i^*(p_i-x_0)K_\Phi(p_i,p_j)u_j\right)-\left(\sum_{i=1}^m u_i^*(p_i-x_0)K_\Phi(p_i,x_0)\right)d.$$

Thus

$$\begin{aligned}[f,g]+[g,f]=-2x_0&\left(\sum_{i,j=1}^m u_i^*K_\Phi(p_i,p_j)u_j\right)\\&+\sum_{i,j=1}^m u_i^*\left\{p_iK_\Phi(p_i,p_j)+K_\Phi(p_i,p_j)\overline{p_j}\right\}u_j-d^*\left(\sum_{j=1}^m K_\Phi(x_0,p_j)\overline{p_j}u_j\right)\\&+x_0d^*\left(\sum_{j=1}^m K_\Phi(x_0,p_j)u_j\right)-\left(\sum_{i=1}^m u_i^*p_iK_\Phi(p_i,x_0)\right)d\\&+x_0\left(\sum_{j=1}^m u_j^*K_\Phi(p_j,x_0)\right)d.\end{aligned}$$

Taking into account (8.2) we have

$$\begin{aligned}[f,g]+[g,f]=-2x_0&\left(\sum_{i,j=1}^m u_i^*K_\Phi(p_i,p_j)u_j\right)\\&+\left(\sum_{i=1}^m u_i^*J\Phi(p_i)\right)d+d^*\left(\sum_{j=1}^m \Phi(p_j)^*Ju_j\right)\\&-d^*\left(\sum_{j=1}^m K_\Phi(x_0,p_j)\overline{p_j}u_j\right)+x_0d^*\left(\sum_{j=1}^m K_\Phi(x_0,p_j)u_j\right)\\&-\left(\sum_{i=1}^m u_i^*p_iK_\Phi(p_i,x_0)\right)d+x_0\left(\sum_{i=1}^m u_i^*K_\Phi(p_i,x_0)\right)d.\end{aligned}$$

We now turn to $[g,g]$. We have:

$$\begin{aligned}[g,g]=&\left(\sum_{i,j=1}^m u_i^*(K_\Phi(p_i,p_j)u_j\right)-d^*\left(\sum_{j=1}^m K_\Phi(x_0,p_j)u_j\right)\\&-\left(\sum_{i=1}^m u_i^*K_\Phi(p_i,x_0)\right)d+d^*K_\Phi(x_0,x_0)d.\end{aligned}$$

Thus

$$
\begin{aligned}
[f,g]+[g,f]+2x_0[g,g] &= \left(\sum_{i=1}^m u_i^* J\Phi(p_i)\right) d + d^* \left(\sum_{j=1}^m \Phi(p_j)^* J u_j\right)\\
&\quad - d^* \left(\sum_{j=1}^m K_\Phi(x_0,p_j)\overline{p_j} u_j\right) - x_0 d^* \left(\sum_{j=1}^m K_\Phi(x_0,p_j) u_j\right)\\
&\quad - \left(\sum_{i=1}^m u_i^* p_i K_\Phi(p_i,x_0)\right) d - x_0 \left(\sum_{i=1}^m u_i^* K_\Phi(p_i,x_0)\right) d\\
&\quad + 2x_0 d^* K_\Phi(x_0,x_0) d\\
&= \left(\sum_{i}^m u_i^* J\Phi(p_i)\right) d + d^* \left(\sum_{j=1}^m \Phi(p_j)^* J u_j\right)\\
&\quad - d^* \left(\sum_{j=1}^m K_\Phi(x_0,p_j)(\overline{p_j}+x_0) u_j\right)\\
&\quad - \left(\sum_{j=1}^m u_i^* (p_i+x_0) K_\Phi(p_i,x_0)\right) d + 2d^* x_0 K_\Phi(x_0,x_0) d
\end{aligned}
$$

using

$$
K_\Phi(x_0,x_0) = \frac{1}{2x_0}\left(J\Phi(x_0)+\Phi(x_0)^*J\right),
$$

we obtain

$$
\begin{aligned}
[f,g]+[g,f]+2x_0[g,g] &= \left(\sum_{i}^m u_i^* J\Phi(p_i)\right) d + d^* \left(\sum_{j=1}^m \Phi(p_j)^* J u_j\right)\\
&\quad - d^* \left(\sum_{j=1} \left(J\Phi(x_0)+\Phi(p_j)^*J\right) u_j\right)\\
&\quad - \left(\sum_{j=1}^m u_i^* \left(J\Phi(p_i)+\Phi(x_0)^*J\right)\right) d + 2d^* x_0 K_\Phi(x_0,x_0) d\\
&= 0
\end{aligned}
$$

and so we have proved (8.7). Equation (8.6) follows since

$$
[f+2x_0g, f+2x_0g] = [f,f] + 2x_0\left([f,g]+[g,f]+2x_0[g,g]\right).
$$

Equation (8.7) expresses that the linear space of functions $(f, f+2x_0g)$ with f, g as in (8.8) define an isometric relation $\mathcal{R}$ from the Pontryagin space $\mathcal{L}(\Phi)$ into itself. Let now $h \in \mathcal{L}(\Phi)$ be such that

$$
[h, K_\Phi(\cdot,p)(\overline{p}-x_0)u] = 0 \quad \forall p \in \Omega \quad \text{and} \quad u \in \mathcal{H}.
$$

Then

$$(p-x_0)h(p)=0,\quad \forall p\in\Omega$$

and $h\equiv 0$ in Ω (recall that the elements of $\mathcal{L}(\Phi)$ are slice hyperholomorphic in Ω). Thus the domain of this relation is dense. By the quaternionic version of Shmulyan's theorem (see [5, Theorem 7.2]), $\mathcal{R}$ extends to the graph of a (continuous) isometry, say B^*, on $\mathcal{L}(\Phi)$. We have for $h\in\mathcal{L}(\Phi)$

$$\begin{aligned}
u^*(p-x_0)((Bh(p)) &= [Bh, K_\Phi(\cdot,p)(\overline{p}-x_0)u]\\
&= [h, B^*(K_\Phi(\cdot,p)(\overline{p}-x_0)u)]\\
&= [h, K_\Phi(\cdot,p)(\overline{p}-x_0)u+2(K_\Phi(\cdot,p)-K_\Phi(\cdot,p))u]\\
&= u^*\left((p-x_0)h(p)+2h(x_0)-2h(x_0)\right)\\
&= u^*\left((p+x_0)h(p)-2h(x_0)\right).
\end{aligned}$$

We note that $\mathcal{R}_{x_0}$ extends to the graph of $R_{x_0}^*$.

STEP 2: *The function* $p\mapsto R_{x_0}\Phi\eta$ *belongs to* $\mathcal{L}(\Phi)$ *for every* $\eta\in\mathcal{H}$ *and the operator* F *from* $\mathcal{H}$ *into* $\mathcal{L}(\Phi)$ *defined by*

$$F\eta=R_{x_0}\Phi\eta$$

is bounded.

We note that $B=I+2x_0R_{x_0}$ and so R_{x_0} is a bounded operator in $\mathcal{L}(\Phi)$. From (8.2) we have for $\xi\in\mathcal{H}$

$$J\Phi(p)\xi+\Phi(x_0)^*J\xi=pK_\Phi(p,x_0)\xi+K_\Phi(p,x_0)\xi x_0. \tag{8.9}$$

Apply R_{x_0} on both sides (as an operator on slice hyperholomorphic functions; the two sides of (8.9) will no belong to $\mathcal{L}(\Phi)$ in general). Note that

$$R_{x_0}(pf(p))=f(p)+x_0(R_{x_0}f)(p),$$

and so we obtain

$$R_{x_0}\Phi J\xi=K_\Phi(p,x_0)\xi+x_0(R_{x_0}K_\Phi(\cdot,x_0)\xi)(p)+(R_{x_0}K_\Phi(p,x_0)\xi x_0)(p),$$

and this expresses $R_{x_0}\Phi J\xi$ as an element of $\mathcal{L}(\Phi)$ since R_{x_0} is bounded in $\mathcal{L}(\Phi)$ and so the elements on the right side of the above equality belong to $\mathcal{L}(\Phi)$. This ends the proof of the first claim since J is invertible. Finally, to see that the operator F is bounded we remark that it is closed and everywhere defined.

We remark that the argument is different from the one for the corresponding operator F (defined by (7.7)) in the spaces $\mathcal{P}(S)$. In the classical case, the argument we are aware of, uses a Cayley transform to go back to the case of generalized Schur functions. The argument we presented here is probably known in the classical case, but we are not aware of any reference for it.

STEP 3: *The realization formula* (8.3) *holds.*

The proof is the same as the one in STEP 6 for S.

STEP 4: *The function* Φ *admits a slice hypermeromorphic extension to* $\mathbb{H}_+$*.*

Recall that $T = I + 2x_0B$ is co-isometric. Using Theorem 5.6 we can thus write T as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

where T_{11} is a bijective contraction from a anti-Hilbert space onto itself, and T_{22} is a contraction from a Hilbert space into itself. By scaling we can reduce to the case $x_0 = 1$. Thus for $x > 0$ in a neighborhood of 1,

$$\begin{aligned}(1+x)I + (x-1)B &= (1+x)I + (x-1)\left(\frac{T-I}{2}\right)\\ &= (3+x) + (x-1)T\\ &= (3+x)\left(I + \begin{pmatrix} \frac{x-1}{3+x}T_{11} & \frac{x-1}{3+x}T_{12} \\ 0 & \frac{x-1}{3+x}T_{22} \end{pmatrix}\right)\end{aligned}$$

and hence the result by slice hyperholomorphic extension since $\frac{q-1}{3+q}$ sends $\mathbb{H}_+$ into $\mathbb{B}_1$.

STEP 5: A function Φ admitting a realization of the form (8.3) is in a class $\mathrm{GP}_\kappa(J)$.

The proof is as in the case of the functions S and is based on the identity

$$J\Phi(x) + \Phi(y)^*J = (x+y)G(I(x_0+x) - (x_0-x)B)^{-1}(I(x_0+y) - (x_0-y)B)^{-*},$$

where x, y are real and in a neighborhood of x_0.

STEP 6: *An observable realization of the form* (8.3) *is unique up to a isomorphism of quaternionic Pontryagin spaces.* □

We note that the relation (8.5) is inspired from [14, p. 708] and more generally, by the constructions of the "ϵ-method" developed in the papers of Krein and Langer; see for instance [42, 43] for the latter.

Corollary 8.4. *In $\mathcal{L}(\Phi)$ it holds that*

$$R_{x_0} + R_{x_0}^* = -2x_0R_{x_0}^*R_{x_0}. \tag{8.10}$$

Proof. This is a rewriting of (8.6). □

We note that (8.10) is a special case of the structural identity characterizing $\mathcal{L}(\Phi)$ spaces in the complex case, see [22]. To ease the notation we consider the case $J = I$.

Theorem 8.5. *Let the $\mathbf{L}(\mathcal{H})$-valued function Φ be slice hyperholomorphic in an axially symmetric s-domain Ω containing $p = 0$, such that the associated space does not contain non zero constants, and has its elements slice hyperholomorphic in a neighborhood of the origin. Assume that $\Phi \in \mathrm{GP}_\kappa(I)$. Then there exists a right quaternionic Hilbert space $\mathcal{H}_1$ and operators*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \ \mathcal{H}_1 \oplus \mathcal{H} \ \longrightarrow \ \mathcal{H}_1 \oplus \mathcal{H}$$

such that

$$\Phi(p) = D + pC \star (I_{\mathcal{H}_1} - pA)^{-\star}B \tag{8.11}$$

and

$$\mathrm{Re}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix} = 0.$$

Proof. We first define a linear relation R_Φ in $(\mathcal{L}(\Phi) \oplus \mathcal{H}) \times (\mathcal{L}(\Phi) \oplus \mathcal{H})$ via the formulas

$$\left(\begin{pmatrix} -K_\Phi(\cdot, q)\overline{q}u \\ u \end{pmatrix}, \begin{pmatrix} K_\Phi(\cdot, q)u \\ \Phi(q)^* u \end{pmatrix}\right). \tag{8.12}$$

STEP 1: *The relation R_Φ satisfies*

$$\mathrm{Re}\,\langle \begin{pmatrix} f \\ -g \end{pmatrix}, \begin{pmatrix} F \\ G \end{pmatrix} \rangle = 0. \tag{8.13}$$

Furthermore, it has dense domain since the space $\mathcal{L}(\Phi)$ contains no non zero constant functions.

Let

$$\begin{pmatrix} f \\ -g \end{pmatrix} = -\sum_{n=1}^{t} \begin{pmatrix} K_\Phi(\cdot, q_n)\overline{q}_n u_n \\ u_n \end{pmatrix},$$

and

$$\begin{pmatrix} F \\ G \end{pmatrix} = \sum_{n=1}^{t} \begin{pmatrix} K_\Phi(\cdot, q_n)u_n \\ \Phi(q_n)^* u_n \end{pmatrix}.$$

Then

$$\langle \begin{pmatrix} f \\ -g \end{pmatrix}, \begin{pmatrix} F \\ G \end{pmatrix} \rangle = -\sum_{n,m=1}^{t} u_m^* K_\Phi(q_m, q_n)\overline{q_n} u_n + u_m^* \Phi(q_n) u_m$$

so that, using (8.2), we obtain

$$\mathrm{Re}\,\langle \begin{pmatrix} f \\ -g \end{pmatrix}, \begin{pmatrix} F \\ G \end{pmatrix} \rangle = 0.$$

STEP 2: *The relation R_Φ is the graph of a densely defined operator which has a continuous extension, and its adjoint is the backward-shift realization*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{aligned} pAf(p) &= f(p) - f(0), \\ pBu(p) &= (\Phi(p) - \Phi(0))u, \\ Cf &= f(0), \\ Du &= \Phi(0)u. \end{aligned}$$

We only have to consider the operator B. Consider a family T of pairs $(q, u) \in \Omega \times \mathcal{H}$ such that the functions $K_\Phi(\cdot, q)u$ are linearly independent and span the space of all the functions $K_\Phi(\cdot, p)v$, where p runs through all of Ω and v runs through all of $\mathcal{H}$. Define a densely defined operator from $\mathcal{L}(\Phi)$ into $\mathcal{H}$ by

$$X(K_\Phi(\cdot, p)u) = (\Phi(p)^* - \Phi(0)^*)u, \quad (p, u) \in A.$$

We claim that X has an adjoint which is the operator B above. To see that, we remark that (8.13) can be rewritten as

$$\langle \begin{pmatrix} f \\ -g \end{pmatrix}, \begin{pmatrix} A^* & C^* \\ X & D^* \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \rangle + \langle \begin{pmatrix} A^* & C^* \\ X & D^* \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ -g \end{pmatrix} \rangle = 0. \tag{8.14}$$

Using the quaternionic polarization formula it follows that for any

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$$

in the domain of R_Φ we have

$$\langle \begin{pmatrix} f_1 \\ -g_1 \end{pmatrix}, \begin{pmatrix} A^* & C* \\ X & D^* \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \rangle + \langle \begin{pmatrix} A^* & C^* \\ X & D^* \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ -g_2 \end{pmatrix} \rangle = 0 \tag{8.15}$$

and so R_Φ has an adjoint and so does X. It is then clear that $X^* = B$ on a dense set, from the definition of X. □

It is useful to note that the operator B appearing in the previous theorem is the opposite of the operator in (7.6).

Example. As an illustration of the previous theorem consider the function

$$\varphi(p) = (p + a)^{-\star}, \tag{8.16}$$

where $a \in \mathbb{H}$ is such that $\operatorname{Re} a = 0$. Set

$$M(p, q) = (p + a)^{-\star}\overline{(q + a)^{-\star}}.$$

Since $a + \overline{a} = 0$ we have

$$pM(p, q) + M(p, q)\overline{q} = \varphi(p) + \overline{\varphi(q)},$$

and so φ is a positive function. For $p = x > 0$ we have

$$\varphi(x) = a^{-1} - \frac{x}{(1 + xa^{-1})a^2},$$

which leads to the realization (8.11) with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -a^{-1} & a^{-1} \\ -a^{-1} & a^{-1} \end{pmatrix}$$

So

$$2\text{Re} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -(a^{-1} + \overline{a^{-1}}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

since $a + \overline{a} = 0$.

8.2. The positive case

In this section we prove results in the case $\kappa = 0$ and $J = I$. We say that the function Φ is positive rather that writing $\Phi \in \mathrm{GP}_0(I)$. The proof uses the existence of a squareroot of a positive operator in a quaternionic Pontryagin space. In the indefinite case, such a result still exists in the complex case (this is called the Bognar-Kramli theorem, see [21, Theorem 2.1 p. 149], [28, Theorem 1.1.2]). A quaternionic version of this factorization theorem is not available at present.

Theorem 8.6. *Let Φ be slice-hyperholomorphic in an axially symmetric s-domain of the origin with realization* (8.11) *such that*

$$\mathrm{Re}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix} \leq 0.$$

Then Φ is positive.

Proof. We first note that a positive operator T, in a quaternionic Hilbert space has a squareroot, that is, there exists a positive operator X such that $X^2 = T$. The proof uses the spectral theorem, which holds for Hermitian operators in quaternionic Hilbert spaces. The theorem is mentioned without proof in a number of papers (see for instance [34], [61], [62]). The spectrum used in these works is not the S-spectrum, see [25, p. 141]); a proof is given in the preprint [7]. Another way to prove the existence of a squareroot is to define (assuming first $\|T\| \leq 1$), as in the complex case, a sequence of operators $X_0, X_1, \ldots$ by $X_0 = 0$ and

$$X_{n+1} = \frac{1}{2}((I - T) + X_n^2), \quad n = 0, 1, \ldots,$$

(see for instance [39, p. 64]) and check that:

(1) A weakly convergent increasing sequence of positive operators converges strongly.

(2) An increasing family $(X_n)_{n\in\mathbb{N}}$ of bounded positive operators such that

$$\lim_{n\to\infty} \langle X_n f, f\rangle < \infty, \quad \forall f \in \mathcal{H}$$

converges strongly to a positive operator. Since the arguments do not differ from the complex case we omit them.

Let X be the squareroot of $-\mathrm{Re}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix}$. We write

$$X = \begin{pmatrix} L \\ K \end{pmatrix}$$

where L is a linear operator from $\mathcal{H} \times \mathcal{H}_1$ into $\mathcal{H}$ and K is a a linear operator from $\mathcal{H} \times \mathcal{H}_1$ into $\mathcal{H}_1$.

Let now

$$\mathrm{Re}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix} = -\begin{pmatrix} L \\ K \end{pmatrix}\begin{pmatrix} L \\ K \end{pmatrix}^*,$$

Then

$$\begin{aligned}\Phi(x)+\Phi(y)^* &= D+D^*+xC(I-xA)^{-1}B+yB^*(I-yA)^{-*}C^*\\ &= KK^*+xC(I-xA)^{-1}(C^*-LK^*)\\ &\quad +y(C-KL^*)(I-yA)^{-*}C^*\\ &= (K-xC(I-xA)^{-1}L)(K-yC(I-yA)^{-1}L)^*\\ &\quad +xC(I-xA)^{-1}C^*+yC(I-yA)^{-*}C^*\\ &\quad -xyC(I-xA)^{-1}LL^*(I-yA)^{-*}C^*.\end{aligned}$$

But, using $A+A^*+LL^*=0$, we have

$$\begin{aligned}&xC(I-xA)^{-1}C^*+yC(I-yA)^{-*}C^*-xyC(I-xA)^{-1}LL^*(I-yA)^{-*}C^*\\ &\qquad\qquad = (x+y)C(I-xA)^{-1}(I-yA)^{-*}C^*\end{aligned}$$

The claim follows by slice hyperholomorphic extension. □

We note that the computations are classical, see for instance [33], [32, Theorem 3.3, p. 26].

We conclude this section with an example of elements of $\mathrm{GP}_0(I)$ (that is, positive functions) which play an important role in models for pairs of anti self-adjoint operators. This originates with the paper of de Branges and Rovnyak [23]. We refer to [11, 12, 41] for examples and applications of the model of de Branges and Rovnyak. In this section, we briefly outline how a positive function also appears in the present setting. We follow the approach of [12], and consider bounded operators for the sake of illustration. The proof of the following lemma is as in [12, p. 18] and is omitted.

Lemma 8.7. *Let T_+ and T_- be two anti-self-adjoint operators in the quaternionic space $\mathcal{H}$. Then:*

(1) *The space*

$$\cap_{u=1}^{\infty}\ker(T_+^u-T_-^u) \tag{8.17}$$

is the largest subspace, invariant under T_+ and T_- and on which they coincide.

(2) *Assume that* $\operatorname{rank} T_+-T_-=n<\infty$. *Then there exists a $n\times n$ matrix $J\in\mathbb{H}^{n\times n}$ such that $J^2=-I_n$ and $J^*=-J$, and a linear bounded operator C from $\mathcal{H}$ into $\mathbb{H}^n$ such that*

$$T_+-T_-=-C^*JC.$$

Theorem 8.8. *Using the notation of the preceding lemma, the function*

$$\Phi(p)=J+C\star(pI-T_+)^{-\star}C^*$$

is positive and its inverse is equal to

$$\Phi^{-\star}(p)=-J-JC\star(pT-T_-)^{-\star}C^*J.$$

Proof. For x, y on the positive real axis we have (recall that $T_+^* = -T_+$)

$$\begin{aligned}\Phi(x) + \Phi(y)^* &= J + J^* + C(xI - T_+)^{-1}C^* + C(yI - T_+)^{-*}C^*) \\ &= C(xI - T_+)^{-1}C^* + C(yI + T_+)^{-1}C^* \\ &= xK(x,y) + K(x,y)y,\end{aligned}$$

where $K(x,y) = C(xI - T_+)^{-1}(yI - T_+)^{-*}C^*$. The result follows then by slice hyperholomorphic extension.

Still for positive x and using for instance formula (3.11) we have

$$\begin{aligned}\Phi(x)^{-1} &= J^{-1} - J^{-1}C(xI - (T_+ - C^*J^{-1}C^*)^{-1}C^*J^{-1} \\ &= -J - JC(xI - T_-)^{-1}C^*J.\end{aligned}$$

The formula for Φ^{-1} follows then by slice hyperholomorphic extension. □

When the space (8.7) in Lemma 8.7 is trivial the function Φ characterizes the pair (T_+, T_-). Models for pairs of (possibly unbounded) anti-self-adjoint operators in a quaternionic Hilbert space in terms of the reproducing kernel Hilbert spaces $\mathcal{L}(\Phi)$ and $\mathcal{L}(\Phi^{-1})$, and related trace formulas similar to the ones presented in the papers [23, 12] will be considered elsewhere.

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