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q-Fourier Transform and its Inversion-Problem

A. Plastino and M.C. Rocca

Abstract. Tsallis' q-Fourier transform is not generally one-to-one. It is shown here that, if we eliminate the requirement that q be fixed, and let it instead "float", a simple extension of the F_q-definition, this procedure restores the one-to-one character.

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1. Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a current generalization of the BoltzmannGibbs (BG) one, is actively studied in diverse areas of Science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index q (with q = 1 recovering the standard BG entropy). It has been applied to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spinglasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker-Planck equation's applications [16], etc.

NEXT can be advantageously expressed via q-generalizations of standard mathematical concepts (the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [17, 22, 25]). The q-Fourier transform F_q exhibits the nice property of transforming q-Gaussians into q-Gaussians [17]. Recently, plane waves, and the representation of the Dirac delta into plane waves have been also generalized [18, 19, 20, 21].

A serious problem afflicts F_q. It is not generally one-to-one. A detailed example is discussed below. In this work we show that by recourse to a rather simple but efficient stratagem that consists in

- eliminating the requirement that q be fixed and instead

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- let it "float",

one restores the one-to-one character.

2. Generalizing the q-Fourier transform

We define, following [17], a q-Fourier transform of $f(x) \in L^1(\mathbb{R})$, $f(x) \geq 0$ as

$$F(k,q) = [H(q-1) - H(q-2)] \times \int_{-\infty}^{\infty} f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\, dx \tag{2.1}$$

where $H(x)$ is the Heaviside step function.

The only difference between this definition and that given in [17] *is that q is not fixed and varies within the interval* $[1,2)$. Herein lies the hard-core of our presentation. This simple change of perspective makes it is easy to find the inversion-formula for (2.1) by recourse to the inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\lim_{\epsilon \to 0^+} \int_1^2 F(k,q)\delta(q-1-\epsilon)\, dq \right] e^{-ikx}\, dk. \tag{2.2}$$

As a consequence, we see that this q-Fourier transform is one-to-one, unlike what happens in [23],[24]. The link between Eqs. (2.1)- (2.2) is discussed in more detail in the illustrative example presented below (next Section).

3. Example

As an illustration we discuss the example given by Hilhorst in Ref. ([22]). Let $f(x)$ be

$$f(x) = \begin{cases} \left(\frac{\lambda}{x}\right)^{\beta} \ ; \ x \in [a,b] \ ; \ 0 < a < b \ ; \ \lambda > 0 \\ 0 \ ; \ x \text{ outside [a, b].} \end{cases} \tag{3.1}$$

The corresponding q-Fourier transform is

$$F(k,q) = \lambda^{\beta} \int_a^b x^{-\beta}\{1 + i(1-q)k\lambda^{\beta(q-1)}x^{1-\beta(q-1)}\}^{\frac{1}{1-q}}\, dx. \tag{3.2}$$

Effecting the change of variables

$$y = x^{1-\beta(q-1)},$$

we have for (3.2)

$$F(k,q) = [H(q-1) - H(q-2)] \times \frac{\lambda^{\beta}}{1-\beta(q-1)} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}}\{1 + i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy. \tag{3.3}$$

Now, (3.3) can be rewritten in the useful form

$$F(k,q) = [H(q-1) - H(q-2)] \times \left\{ \left\{ H(q-1) - H\left[q - \left(1+\frac{1}{\beta}\right)\right]\right\} \times \frac{\lambda^\beta}{1-\beta(q-1)} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1-q)k\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\, dy + \left\{ H\left[q - \left(1+\frac{1}{\beta}\right)\right] - H(q-2)\right\} \times \frac{\lambda^\beta}{\beta(q-1)-1} \int\limits_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1-q)k\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\, dy \right\}. \quad (3.4)$$

Taking into account that the involved integrals are defined in a finite interval, we can cast (3.4) as

$$F(k,q) = [H(q-1) - H(q-2)] \times \left\{ \left\{ H(q-1) - H\left[q - \left(1+\frac{1}{\beta}\right)\right]\right\} \times \frac{\lambda^\beta}{1-\beta(q-1)} \lim_{\epsilon\to 0^+} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\, dy + \left\{ H\left[q - \left(1+\frac{1}{\beta}\right)\right] - H(q-2)\right\} \times \frac{\lambda^\beta}{\beta(q-1)-1} \lim_{\epsilon\to 0^+} \int\limits_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\, dy \right\}. \quad (3.5)$$

We now use results of the Integral's table [26] to evaluate (3.5) and get

$$\lim_{\epsilon\to 0^+} \int\limits_{a^{1-\beta(q-1)}}^{\infty} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\, dy = \frac{(q-1)[1-\beta(q-1)]a^{\frac{q-2}{q-1}}}{(2-q)[(1-q)i(k+i0)\lambda^\beta]^{\frac{1}{q-1}}} \times F\left(\frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; -\frac{1}{(1-q)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right), \quad (3.6)$$

and

$$\lim_{\epsilon\to 0^+}\int\limits_0^{a^{1-\beta(q-1)}} y^{\frac{\beta(2-q)}{\beta(q-1)-1}}\{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy$$
$$=\frac{[\beta(q-1)-1]a^{1-\beta}}{\beta-1}$$
$$\times F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right), \quad (3.7)$$

where $F(a,b,c;z)$ is the hypergeometric function. Thus we obtain for $F(k,q)$

$$F(k,q)=[H(q-1)-H(q-2)]\times\left\{\left\{H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\right\}\right.$$
$$\times\frac{(q-1)\lambda^{\beta}}{(2-q)[(1-q)i(k+i0)\lambda^{\beta}]^{\frac{1}{q-1}}}$$
$$\times\left\{a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.\right.$$
$$\left.\frac{1}{(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right)$$
$$-b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.$$
$$\left.\left.\frac{1}{(q-1)i(k+i0)\lambda^{\beta(q-1)}b^{1-\beta(q-1)}}\right)\right\}$$
$$+\left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\}\frac{\lambda^{\beta}}{\beta-1}$$
$$\times\left\{a^{1-\beta}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};\right.\right.$$
$$\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right)$$
$$-b^{1-\beta}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};\right.$$
$$\left.\left.\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}b^{1-\beta(q-1)}\right)\right\}\right\}. \quad (3.8)$$

As we can see from (3.8), $F(k,q)$ *depends on* a *and* b, and, as consequence, is one-to-one, as shown in Section 2.

However, and *this is the crucial issue*, if we **fix** q and select $\beta=1/(q-1)$ (3.8) simplifies and adopts the appearance

$$F(k,q) = \lambda^{\frac{1}{q-1}} \frac{q-1}{2-q} \left[H(q-1) - H(q-2)\right] \times \left[a^{\frac{q-2}{q-1}} F\left(\frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)i(k+i0)\lambda\right) - b^{\frac{q-2}{q-1}} F\left(\frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)i(k+i0)\lambda\right)\right]. \tag{3.9}$$

With the help of the result given in [27] for

$$F(-a,b,b,-z) = (1+z)^a,$$

we obtain for (3.9):

$$F(k,q) = \lambda^{\frac{1}{q-1}} \frac{q-1}{2-q} \left[H(q-1) - H(q-2)\right] \left(a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}}\right) \left[1 + (1-q)ik\lambda\right]^{\frac{1}{1-q}}. \tag{3.10}$$

Using now the expression for λ of [22], i.e.,

$$\lambda = \left[\left(\frac{q-1}{2-q}\right)\left(a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}}\right)\right]^{1-q},$$

we have, finally,

$$F(k,q) = \left[H(q-1) - H(q-2)\right] \left[1 + (1-q)ik\lambda\right]^{\frac{1}{1-q}}, \tag{3.11}$$

which is the result given by Hilhorst in [22], that is *independent* of the values adopted by a, b. Such independence is evidence that $F(k,q)$ is not one-to-one. All infinite $F(k,q,a,b)$ associated to each possible pair a, b coalesce now in a infinitely degenerate solution $F(k,q)$. As a conclusion we can say that for fixed q the q-Fourier transform is NOT one-to-one for fixed q. On the contrary, as we have shown in section 2, when q is NOT fixed, the q-Fourier transform is indeed one-to-one.

Conclusions

In the present communication we have discussed the NOT one-to-one nature of the q-Fourier transform F_q. We have shown that, if we eliminate the requirement that q be fixed and let it"float" instead, such simple extension of the F_q-definition restores the desired one-to-one character.

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A. Plastino
Instituto de Física (IFLP-CCT-Conicet)
Universidad Nacional de La Plata (UNLP)
C.C. 67 (1900)
La Plata
Argentina
e-mail: plastino@fisica.unlp.edu.ar
angeloplastino@gmail.com

M.C. Rocca
Instituto de Física (IFLP-CCT-Conicet)
Universidad Nacional de La Plata (UNLP)
C.C. 67 (1900)
La Plata
Argentina
and
Departamento de Física
Fac. de Ciencias Exactas
Universidad Nacional de La Plata (UNLP)
La Plata
Argentina
e-mail: rocca@fisica.unlp.edu.ar
mariocarlosrocca@gmail.com

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